Multiple Activities in Networks*

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Abstract

We consider a network model where individuals exert efforts in two types of activities that are interdependent. These activities can be either substitutes or complements. We provide a full characterization of the Nash equilibrium of this game for any network structure. We show, in particular, that quadratic games with linear best-reply functions aggregate nicely to multiple activities because equilibrium efforts obey similar formulas to that of the one-activity case. We then derive some comparative statics results showing how own productivity affects equilibrium efforts and how network density impacts equilibrium outcomes.

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1 Introduction

Peer decisions have been shown to be important in predicting different outcomes of individuals, ranging from education[1], alcohol consumption[2], drug use and cigarette smoking[3], and crime[4] to labor-market outcomes.[5] Most of this literature has, however, considered the effect of peers on one specific choice. In reality, individuals make a multitude of choices, many of which of are interdependent. As a result, peers can have multiple and sometimes opposing influences on their friends.

In the current paper, we study a game where individuals, embedded in a social network,[6] engage in two kinds of activities that are interdependent. Within each activity, there are local network externalities amongst the players: a player pays more attention to her friends' decisions than to others' choices. Importantly, we allow for substitution or complementarity across activities.

There is an important literature in applied mathematics on quadratic games that provides existence and uniqueness results, as well as compact closed-form solutions, for arbitrary deterministic games with n players, multiple (real-valued) actions for each player, and arbitrary quadratic payoff functions. The only conditions this literature imposes are concavity of the payoff function in own actions as well as a nonsingularity condition on a certain matrix. In this literature, the existence and uniqueness of solutions to general linear-quadratic games are very well understood.[7] The essential idea is that the first-order conditions form a linear system, and then one can apply standard linear algebra results on the existence and uniqueness of its solution. With one activity per agent, Ballester et al. (2006) (BCZ hereafter) offered an interpretation of the solution by viewing the best-response parameters (the slope of an agent's optimal action in every other's action) as a network. It turned out that the actions taken in equilibrium correspond to individuals' *centralities* in this network, in the sense of the Katz-Bonacich centrality.[8] However, the assumption of a one-dimensional choice for each agent in that paper is restrictive. One might wonder whether one can usefully interpret actions in terms of network position in the more general and realistic setting where agents take multiple actions. In the present paper, we answer this affirmatively by showing

[1] Zimmerman (2003); Calvó-Armengol et al. (2009); Sacerdote (2001, 2011, 2014)

[2] Gaviria and Raphael (2001); Kawaguchi (2004); Lundborg (2006); Norton et al. (1998); Fletcher (2010).

[3] Gaviria and Raphael (2001); Kawaguchi (2004); Clark and Loheac (2007); Keng and Huffman (2010).

[4] Ludwig et al. (2001); Kling et al. (2005); Bayer et al. (2009); Patacchini and Zenou (2012); Damm and Dustmann (2014); Lindquist and Zenou (2014).

[5] Ioannides and Loury (2004).

[6] The economics of networks is a growing field. See, in particular, the overviews by Jackson (2008); Jackson (2011); Ioannides (2012); Jackson (2014); Jackson and Zenou (2015); Jackson et al. (2017).

[7] A particularly concise and clear exposition of this literature is provided by Başar (2000) (Section 10.1, pp. 58-59).

[8] This centrality measure has been proposed by Katz (1953) and Bonacich (1987). The Katz-Bonacich centrality counts the number of all paths (not just shortest paths) emanating from a given node, weighted by a decay factor that decreases with the length of these paths. It was originally interpreted as an index of influence or power of the actors in a social network.

that the Nash-Bonacich connection discovered in BCZ aggregates nicely to multiple actions, and one can have a clean characterization of equilibria in the multiple-action framework.

We then derive some comparative statics results. Contrary to BCZ, we show that increasing network synergy (or the social multiplier) does *not* always increase effort in each activity and that denser networks do *not* always increase the efforts of the players. While with one activity, BCZ show that the influences are monotone, in this paper, both results strongly depend on whether the two activities are substitutes or complements.

In the spirit of BCZ, we also investigate the *key player* problem in this multiple-activity setting. Suppose that the government intends to remove one player to maximally reduce the total activity in the network. Which player should the government target? In the single-activity model, BCZ show that the key player is the one with the highest inter-centrality index. This index is a measure that takes into account both the direct reduction of activity, following the removal of the key player, and the indirect impact on the activities of the players directly connected to the key player. Compared to BCZ, with two activities, there are more possible ways of defining the key player. For example, the planner's objective can be to minimize one activity without paying attention to the other, minimize the sum of both activities, or any weighted combination of these two activities. In particular, if the planner wants to minimize one activity (for example crime), we show that the key player selection with two activities can be substantially different from the one in a single-activity case.

We then extend our model in different directions. First, we consider a model where players choose more than two activities; in fact, any finite number of activities. Second, we incorporate cross-network effects so that there are spillover effects within but also between activities. Third, we include heterogeneous substitution effects so that the degree of substitution or complementarity between activities is player-specific. We also look at heterogeneous network effects where spillover effects are heterogeneous and differ between activities. We also consider different networks for each activity. For example, there can be a criminal network which is different to the one for individuals studying together. Finally, we investigate a distinct but related model, the so-called local-average model, where agents lose utility by deviating from the social norms of their friends.

To the best of our knowledge, there are only two papers that have tackled the issue of multiple activities in an explicit *social network analysis*. From a theoretical perspective, we are only aware of the paper by Belhaj and Deroïan (2014), who characterize the equilibrium in a network game in which each player has only a fixed amount of resources to allocate between two activities. Similar to our model, there are peer effects within each activity but the total effort in both activities in their model is fixed and equal to one. In other words, the authors assume that the two activities are perfect substitutes. This implicitly means that they are dealing with only one choice of effort, since effort in one activity is equal to one minus the effort in the other activity. Because of this restriction, they focus on very different issues such as specialization and polarization of agents and multiple equilibria. In this regard, we allow activities to be substitutes or complements, and one activity needs not be tied perfectly to the other. Belhaj and Deroïan (2014) also discuss the impact of heterogeneous costs and large social multiplier effects on outcomes. These features are not captured in our paper since our focus is different. The second paper is that of Cohen-Cole et al. (2017). It is mainly an empirical paper where the authors study the identification of social interactions with multivariate activities.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the equilibrium outcomes and discusses some interesting special cases. Section 4 presents some comparative statics analyses. Section 5 deals with the key player policy when players can choose more than one activity. Section 6 extends our model in different directions. Section 7 discusses the implications of our model. Finally, Section 8 concludes. Appendix A provides some results on matrices and defines the Katz-Bonacich centrality. Appendix B provides some conditions that guarantee that the Nash equilibrium efforts are strictly positive. Appendix C derives some preliminary comparative-statics results. Appendix D derives the equilibrium utility of each agent while Appendix E provides some examples with specific networks to illustrate our main results. Appendix F characterizes the Nash equilibrium with two activities when the utility function incorporates both local-aggregate and local-average effects. Appendix G provides all the detailed derivations and proofs of the paper.

2 Model

Consider a social network $\mathbf{G}$ with n players, indexed by $i = 1, 2, \cdots, n$. Denote by $\mathbf{G} = (g_{ij})$ the adjacency matrix of the network. We assume $g_{ii} = 0$ – there is no self loop – and $g_{ij} = g_{ji}$, i.e., $\mathbf{G}$ is symmetric. Let $\mathcal{N} = \{1, 2, \cdots, n\}$ denote the set of players. Each player intends to engage in two kinds of activities, A and B, which are interdependent.

Utilities and decisions. Each player i chooses the levels of two activities, x_i^A for activity A and x_i^B for activity B, where x_i^A and x_i^B are real numbers. Let $\mathbf{x}_i = (x_i^A, x_i^B)$. For ease of exposition, we define $\mathbf{x}_{-i} := (x_1^A, x_1^B, ..., x_{i-1}^A, x_{i-1}^B, x_{i+1}^A, x_{i+1}^B, ..., x_n^A, x_n^B)$ as the decisions selected by players other than i. To explicitly model the interdependence between activities, player i's utility function is expressed as follows:

$$\begin{aligned} u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \quad = \quad & a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B \right\} \\ & + \delta \sum_{j=1}^{n} g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^{n} g_{ij} x_i^B x_j^B . \end{aligned} \tag{1}$$

This utility function (1) consists of two parts. The first part, $a_i^A x_i^A + a_i^B x_i^B - \left\{\frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B\right\}$, corresponds to the utility of providing efforts in the two activities, independently of the efforts of the other individuals in the network. Parameter a_i^l measures the intrinsic marginal utility (or ability) of activity $l = A, B$ for player i. The quadratic terms capture the decreasing marginal returns from each activity, and the cross-activity term $\beta x_i^A x_i^B$ depicts the interconnection between the efforts in the two activities. The parameter β, which is assumed to take values in $(-1, 1)$,[9] measures the substitutability or complementarity between the two activities. Observe that $\partial^2 u_i / \partial x_i^A \partial x_i^B = -\beta$. Therefore, when β is *positive*, the two activities are *substitutes*. When $\beta = 0$, the two activities are *independent*. When β is *negative*, the activities are *complements*.

The second part of (1), $\delta \sum_{j=1}^n g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^n g_{ij} x_i^B x_j^B$, captures the *network externalities*. The parameter $\delta \geq 0$ describes the intensity of the network effects or *network spillovers*. Indeed, when $\delta = 0$, each player's utility depends entirely on her own effort decisions. A higher δ indicates a higher dependence of a player's utility on others' decisions. We assume $\delta \geq 0$ to capture *strategic complementarities* in each activity, i.e.,

$$\frac{\partial^2 u_i}{\partial x_i^A \partial x_j^A} = \delta g_{ij} \geq 0, \quad \frac{\partial^2 u_i}{\partial x_i^B \partial x_j^B} = \delta g_{ij} \geq 0.$$

We will often offer illustrations of our results using the following examples. Activity A corresponds to crime as in the standard single-activity setup. If the two activities are substitutes, activity B corresponds to education; in contrast, if the two activities are complements, activity B corresponds to drug consumption.[10] This means that our utility function can be interpreted as:

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \underbrace{a_i^A x_i^A - \frac{1}{2}(x_i^A)^2 + \delta \sum_{j=1}^n g_{ij} x_i^A x_j^A}_{\text{net proceeds from crime}} + \underbrace{a_i^B x_i^B - \frac{1}{2}(x_i^B)^2 + \delta \sum_{j=1}^n g_{ij} x_i^B x_j^B}_{\text{net benefits from the other activity}} - \underbrace{\beta x_i^A x_i^B}_{\text{degree of interdependence}}.$$

The first part captures the benefits and costs from committing *crime*. In this context, a_i^A measures the individual crime productivity or ability of each player i while δ measures the degree of social interactions in crime, i.e. the intensity of peer effects in crime. It is indeed well-established that

[9] We take this interval because we need β to be, in absolute value, less than 1 (see Assumption 1). Otherwise, u_i would not be concave in $\mathbf{x}_i$.

[10] According to Goldstein (1985), drug use can positively affect criminal activity through three channels. The first is the "pharmacological" effect: drug use may increase aggression and therefore violent crime. The second is the "economic" effect: some users turn to crime to finance expenditures on drugs. The third is the "systemic" effect: violence occurs in the drug market because the participants cannot rely on contracts and courts to resolve disputes. There is a lot of empirical research that shows that a significant proportion of those apprehended for a range of criminal offences are frequent illicit drug users. In particular, research has shown a strong correlation between the level of drug use and level of criminal involvement—at both an aggregate and individual level (see e.g. Anglin and Speckart (1988); Nurco (1998); Makkai (2002); Bean (2014)). For example, Corman and Mocan (2000) find a positive relationship between drug use and robberies and burglaries.

delinquency is, to some extent, a group phenomenon, and that the sources of crime and delinquency are located in the intimate social networks of individuals (see e.g. Sutherland (1947); Sarnecki (2001); Warr (2002); Haynie (2001); Patacchini and Zenou (2008, 2012); Lindquist and Zenou (2014); Liu et al. (2012)) and peer effects are very strong in criminal decisions (Ludwig et al. (2001); Kling et al. (2005); Bayer et al. (2009); Damm and Dustmann (2014)).[11]

The second part captures the net benefits from the other activity. When $\beta \geq 0$, the two activities are substitutes, for example crime (activity A) and *education* (activity B). When $\beta \leq 0$, the two activities are complements, for example, crime and *drug consumption*. As a result, the second part captures the net benefits from either education or drug consumption. In particular, a_i^B measures either the individual ability in education ($\beta \geq 0$) or the individual marginal utility of consuming drugs ($\beta \leq 0$). In the empirical literature, a_i^A or a_i^B are usually measured by the observable characteristics of player i related to the activity. Finally, the last part indicates the degree of interaction between the two activities. If they are substitutes (resp. complements), then increasing effort in crime decreases (increases) the marginal utility of the other activity by β.

Two benchmark models. It is worthwhile discussing two special cases that correspond to well-known models, which are widely studied in the literature. In the first benchmark model, we consider the scenario when $\delta = 0$ so that the network plays no role. In this case, individual i's utility function reduces to:

$$u_i = a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B \right\}.$$

This is, for example, a model that is commonly used in the industrial organization literature to study production differentiation (Dixit (1979); Singh and Vives (1984)). In that case, the utility function is reduced to the case with n independent players, i.e., the utility is separable across different players.

In the second benchmark model, we consider the scenario when $\beta = 0$. In that case, player i's utility reduces to:

$$u_i = \left\{ a_i^A x_i^A - \frac{1}{2}(x_i^A)^2 + \delta \sum_{j=1}^{n} g_{ij} x_i^A x_j^A \right\} + \left\{ a_i^B x_i^B - \frac{1}{2}(x_i^B)^2 + \delta \sum_{j=1}^{n} g_{ij} x_i^B x_j^B \right\},$$

which is just the sum of the payoffs from exerting efforts in activities A and B. Notice that the two terms in the big brackets are totally separable. This model has been widely used in the literature

[11]For example, Bayer et al. (2009)) consider the influence that juvenile offenders serving time in the same correctional facility have on each other's subsequent criminal behavior. They find strong evidence of learning effects in criminal activities since exposure to peers with a history of committing a particular crime increases the probability that an individual who has already committed the same type of crime recidivates that crime.

on games on networks (Jackson and Zenou (2015)).

Let us now study the general case when $\delta \geq 0$, and $-1 < \beta < 1$. Let $\lambda_1(\mathbf{G})$ be the spectral radius of matrix $\mathbf{G}$.[12]

Assumption 1. $1 - |\beta| - \delta\lambda_1(\mathbf{G}) > 0$.

Assumption 1 guarantees the existence and uniqueness of the equilibrium. When $\beta = 0$, this condition is equivalent to $\delta < 1/\lambda_1(\mathbf{G})$, which is commonly assumed in the network literature with a single activity (see Ballester et al. (2006) and Zhou and Chen (2015)).

3 Nash equilibrium

3.1 Equilibrium characterization

Let

$$\mathbf{x}^A = \begin{bmatrix} x_1^A \\ \vdots \\ x_n^A \end{bmatrix}, \mathbf{x}^B = \begin{bmatrix} x_1^B \\ \vdots \\ x_n^B \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix}$$

Define $\mathbf{a}^A, \mathbf{a}^B$ and $\mathbf{A} = \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix}$ in a similar way. Also define $\mathbf{I}_n$ to be the $n \times n$ identity matrix and

$$\mathbf{M}^+ := [(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \sum_{k=0}^{\infty} \frac{(\delta\mathbf{G})^k}{(1+\beta)^{1+k}} \text{ and } \mathbf{M}^- := [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \sum_{k=0}^{\infty} \frac{(\delta\mathbf{G})^k}{(1-\beta)^{1+k}} \quad (2)$$

Theorem 1. *Suppose that Assumption 1 holds. Then, for any* $\mathbf{a}^A$ *and* $\mathbf{a}^B$*, there exists a unique Nash equilibrium given by:*

$$\begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{M}^+ + \mathbf{M}^-}{2} & \frac{\mathbf{M}^+ - \mathbf{M}^-}{2} \\ \frac{\mathbf{M}^+ - \mathbf{M}^-}{2} & \frac{\mathbf{M}^+ + \mathbf{M}^-}{2} \end{bmatrix} \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{M}^+(\mathbf{a}^A + \mathbf{a}^B) + \mathbf{M}^-(\mathbf{a}^A - \mathbf{a}^B) \\ \mathbf{M}^+(\mathbf{a}^A + \mathbf{a}^B) - \mathbf{M}^-(\mathbf{a}^A - \mathbf{a}^B) \end{bmatrix}.$$

Let us first comment on the conditions for the existence and uniqueness of the equilibrium derived in Theorem 1. Assumption 1 requires that the network effect is not too strong. A similar condition is commonly assumed in social network games with only one activity (see Ballester et al. (2006) and Zhou and Chen (2015)). In this paper, we generalize this condition when players choose two activities. Another technical issue is the possibility of corner solutions for the equilibrium efforts. In Proposition 5 in Appendix B, we give the conditions on parameters for which the

[12] It is also equal to its largest eigenvalue by the Perrron-Frobenius Theorem since $\mathbf{G}$ is a nonnegative symmetric matrix.

equilibrium efforts are always strictly positive. It turns out that, when $\beta \leq 0$ (activities are complements, or independent), Assumption 1 guarantees that $\mathbf{x}^A \succeq \mathbf{0}$ and $\mathbf{x}^B \succeq \mathbf{0}$. When $\beta > 0$ (activities are substitutes), we show that if either condition (14) (which imposes some restriction on the degree of substitution between two activities) or condition (15) (which limits the heterogeneity between two activities) holds, then $\mathbf{x}^A \succeq \mathbf{0}$ and $\mathbf{x}^B \succeq \mathbf{0}$.

From Theorem 1, we can determine the equilibrium utility of each agent. We perform such an analysis in Appendix D, Proposition 9. We obtain:

$$\begin{aligned} u_i^* &= \frac{1}{2}(x_i^{A*})^2 + \frac{1}{2}(x_i^{B*})^2 + \beta x_i^{A*} x_i^{B*} \\ &= \frac{1}{1+\beta}\left[b_i\left(\mathbf{G}, \frac{\delta}{1+\beta}, \overline{\mathbf{a}}\right)\right]^2 + \frac{1}{1-\beta}\left[b_i\left(\mathbf{G}, \frac{\delta}{1-\beta}, \widehat{\mathbf{a}}\right)\right]^2, \end{aligned} \tag{3}$$

where $\overline{\mathbf{a}} := \left(\mathbf{a}^A + \mathbf{a}^B\right)/2$ is the average ability and $\widehat{\mathbf{a}} := \left(\mathbf{a}^A - \mathbf{a}^B\right)/2$ is half of the difference in abilities. This shows that the equilibrium utilities can be expressed as the sum of the squares of two Katz-Bonacich centralities (which are formaly defined in Appendix A.2) as in the case with one action.

Theorem 1 provides a clean characterization of the equilibrium outcomes and has some intuitive interpretations. It characterizes the equilibrium of a quadratic game with two activities where the social network is explicitly modeled. We show that the *average action* across the two activities taken by each player is determined by a social interaction matrix multiplied by the *average ability* across the two activities while (half of) the *difference between the actions* is determined by another social-interaction matrix times (half of) the *difference in abilities* between the two actions. Moreover, Theorem 1 leads to the following equations:

$$\overline{\mathbf{x}} = \mathbf{M}^{+}\overline{\mathbf{a}} \text{ and } \widehat{\mathbf{x}} = \mathbf{M}^{-}\widehat{\mathbf{a}} \tag{4}$$

where $\overline{\mathbf{x}} := \left(\mathbf{x}^A + \mathbf{x}^B\right)/2$ is the average effort and $\widehat{\mathbf{x}} := \left(\mathbf{x}^A - \mathbf{x}^B\right)/2$ corresponds to half of the difference in efforts. Let us show how we obtain these two equations. Each individual i chooses x_i^A and x_i^B that maximizes (1). The first-order conditions are given by:

$$\begin{cases} x_i^A + \beta x_i^B = a_i^A + \delta \sum_{j=1}^n g_{ij} x_j^A \\ x_i^B + \beta x_i^A = a_i^B + \delta \sum_{j=1}^n g_{ij} x_j^B \end{cases}$$

Taking the sum of these two first-order conditions leads to:

$$(1+\beta)\frac{\left(x_i^A + x_i^B\right)}{2} = \frac{\left(a_i^A + a_i^B\right)}{2} + \delta \sum_{j=1}^n g_{ij} \frac{\left(x_j^A + x_j^B\right)}{2},$$

while taking the difference yields:

$$(1-\beta)\frac{\left(x_i^A - x_i^B\right)}{2} = \frac{(a_i^A - a_i^B)}{2} + \delta\sum_{j=1}^{n} g_{ij}\frac{\left(x_j^A - x_j^B\right)}{2}.$$

In matrix form, these two equations can be written as:

$$\frac{\mathbf{x}^A + \mathbf{x}^B}{2} = [(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}\frac{(\mathbf{a}^A + \mathbf{a}^B)}{2},$$

and

$$\frac{\mathbf{x}^A - \mathbf{x}^B}{2} = [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}\frac{(\mathbf{a}^A - \mathbf{a}^B)}{2},$$

which leads to (4) by using (2).

Corollary 1. *Suppose that Assumption 1 holds. Then the unique Nash equilibrium can be written as:*

$$\mathbf{x}^A = \frac{1}{1+\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{(1+\beta)}, \overline{\mathbf{a}}\right) + \frac{1}{1-\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{(1-\beta)}, \widehat{\mathbf{a}}\right),$$

$$\mathbf{x}^B = \frac{1}{1+\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{(1+\beta)}, \overline{\mathbf{a}}\right) - \frac{1}{1-\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{(1-\beta)}, \widehat{\mathbf{a}}\right).$$

This corollary shows that, in addition to the expressions in Theorem 1, the equilibrium efforts can also be expressed as the sum of two Katz-Bonacich centralities, where the decay factors are $\delta/(1+\beta)$ and $\delta/(1-\beta)$ while the weights are $\overline{\mathbf{a}}$ and $\widehat{\mathbf{a}}$, the average and (half of) the difference in abilities, respectively. When $\beta > 0$, the two activities are substitutes, and $\frac{\delta}{(1+\beta)} < \frac{\delta}{(1-\beta)}$ while, when $\beta < 0$, they are complements and we have the opposite. Indeed, when activities are complements (such as crime and drug consumption), they reinforce each other so that exerting more effort in crime induces individuals to consume more drugs and both activities generate more spillovers to other individuals in the network. On the contrary, when activities are substitutes (crime and education), they oppose each other so that when individuals commit more crime, they influence other criminals to commit more crimes, which induce them and their friends to study less.

This means, in particular, that it is not necessary the most active individuals who are the most central in the network (as in Ballester et al. (2006)). Indeed, we have seen that, in the two-activity case, $\overline{\mathbf{x}} = \mathbf{M}^{+}\overline{\mathbf{a}}$ and $\widehat{\mathbf{x}} = \mathbf{M}^{-}\widehat{\mathbf{a}}$, which is similar to the quadratic game with one action where $\mathbf{x} = \mathbf{M}\,\mathbf{a}$ and $\mathbf{M} := [\mathbf{I}_n - \delta\mathbf{G}]^{-1}$. However, this also shows that one of the key result in quadratic games with one activity, namely the fact that the more central an agent is (in terms of Katz-Bonacich centrality), the higher is her effort level (see Ballester et al. (2006)), is only true in the case of complementary actions ($\beta < 0$) but not always true when actions are substitutes ($\beta > 0$), especially if β is large enough. In other words, if activities are substitutes (i.e. crime and education), then the most central player will have the highest effort in the network in one activity (say crime) but can have the lowest one in the other activity (education).

Let us illustrate this result by considering a star-shaped network where agent 1 is in the center and agents 2 and 3 are in the periphery so that the adjacency matrix is given by:

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In the one activity model (say crime), agent 1 will always exert the highest effort. Assume, for example, that $\delta = 0.1$ and that all agents have the same level of ex ante ability given by $\mathbf{a} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. The largest eigenvalue of $\mathbf{G}$ is $\sqrt{2}$, i.e. $\lambda_1(\mathbf{G}) = \sqrt{2}$. Then, since $1 - \delta\lambda_1(\mathbf{G}) > 0$, in the one-activity model, the equilibrium efforts are given by:

$$\mathbf{x} = \mathbf{M}\,\mathbf{a} = \begin{pmatrix} 3.67 \\ 3.37 \\ 3.37 \end{pmatrix}$$

Not surprisingly, agent 1 exerts the highest effort.

Consider now the two-activity model. Assume still that $\delta = 0.1$ and $\beta = \pm 0.4$ so that $1 - |\beta| - \delta\lambda_1(\mathbf{G}) > 0$. Assume that $a_i^A = 4$ and $a_i^B = 2$ for all i so that $\overline{\mathbf{a}} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ and $\widehat{\mathbf{a}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Consider first the case when the activities are *substitutes*, i.e. $\beta = 0.4$. We obtain:

$$\overline{\mathbf{x}} = \mathbf{M}^{+}\overline{\mathbf{a}} = \begin{pmatrix} 2.47 \\ 2.32 \\ 2.32 \end{pmatrix} \text{ and } \widehat{\mathbf{x}} = \mathbf{M}^{-}\widehat{\mathbf{a}} = \begin{pmatrix} 2.35 \\ 2.06 \\ 2.06 \end{pmatrix}$$

and thus

$$\mathbf{x}^A = \overline{\mathbf{x}} + \widehat{\mathbf{x}} = \begin{pmatrix} 4.82 \\ 4.38 \\ 4.38 \end{pmatrix} \text{ and } \mathbf{x}^B = \overline{\mathbf{x}} - \widehat{\mathbf{x}} = \begin{pmatrix} 0.12 \\ 0.26 \\ 0.26 \end{pmatrix}$$

We see that, in a star-shaped network, the central agent (agent 1) exerts the highest effort in activity A but the lowest effort in activity B, even though her average effort as well as her difference in efforts are the highest. This is true because β is high enough. For example, it is easily verified that for $\beta = 0.2$, the central agent exerts the highest effort in both activities. Indeed, we have:

$$\mathbf{x}^A = \begin{pmatrix} 4.57 \\ 4.20 \\ 4.20 \end{pmatrix} \text{ and } \mathbf{x}^B = \begin{pmatrix} 1.34 \\ 1.29 \\ 1.29 \end{pmatrix}$$

When the two activities are *complements*, i.e. $\beta = -0.4$, it is easily verified that:

$$\mathbf{x}^A = \begin{pmatrix} 6.56 \\ 5.88 \\ 5.88 \end{pmatrix} \text{ and } \mathbf{x}^B = \begin{pmatrix} 4.92 \\ 4.34 \\ 4.34 \end{pmatrix}$$

so that the central agent exerts the highest effort in both activities.

Using (4), let us explain these results. We assume that all agents have a higher productivity in exerting activity A than activity B. As a result, because of her position, the most central agent exerts the highest effort in activity A as in the single activity case. She generates a lot of synergies in activity A to her neighbors, who, in turn, have a strong impact on her. This is the network spillover effect. Then, when β is positive and large enough, the two activities are strong substitutes and, as a result, the most central agent, who exerts the highest effort in activity A, will sharply reduce her effort in activity B. When β is not too large, then the reduction in effort in activity B is not large enough to compensate her high effort in activity A due to her position in the network that generates strong network spillovers. When β is negative such that both activities are complements, then clearly being central leads to more effort in one activity, which, because of complementarity, leads to more effort in the other activity. This model thus highlights the fact that the position in the network (network spillover effects), the relative productivity of one activity over the other one and the degree of substitutability or complementarity between the two activities are crucial to understand the behavior of agents in a network. For example, if one wants to have an effective policy that reduces juvenile crime, then one needs not only to look at how central delinquents are in criminal networks but also at their involvement in other activities than crime.

More generally, we believe that the result obtained in (4) is a clean way to characterize the Nash equilibrium with multiple activities. In particular, it shows *that quadratic games with linear-best reply functions aggregate nicely to multiple activities* since it *explains why the average effort and the difference in efforts obey similar formulas to the one-activity case.*[13] It should be, however, clear that the quadratic games are actually the only ones that aggregate nicely to multiple activities. In Section 6, we show that this nice equilibrium characterization is usually true where we consider different extensions of the model. However, when the spillover parameter δ is different between the two activities (i.e. $\delta^A \neq \delta^B$) and/or the network itself is different between the two activities

[13]This is also true for the equilibrium utility (see (3)).

(i.e. $\mathbf{G}^A \neq \mathbf{G}^B$), we show in Section 6.4 that our model with linear-best reply functions *does not* aggregate nicely to multiple activities since efforts are not anymore equal to Katz-Bonacich centralities.

We also believe that the characterization result obtained in (4) could be very useful for potential econometric applications of this analysis. Indeed, if one would like to empirically test our theoretical model, then using (4), one would only need to simultaneously estimate the following system of equations for each $i = 1, ...n$:

$$\begin{cases} \overline{x}_i = \beta_1 \, \overline{a}_i + \delta_1 \sum_{j=1}^n g_{ij} \overline{x}_j \\ \widehat{x}_i = \beta_2 \, \widehat{a}_i + \delta_2 \sum_{j=1}^n g_{ij} \widehat{x}_j \end{cases}$$

where $\beta_1 := 1/(1+\beta)$, $\beta_2 := 1/(1-\beta)$, $\delta_1 := \delta/(1+\beta)$, $\delta_2 := \delta/(1-\beta)$. These equations are relatively simple to test since, in terms of data, they only involve the average and difference of both outcome efforts and abilities.

In terms of data, one may wonder what is the economic rationale behind caring about the total effort level (and also the difference) across the two activities. In particular, in terms of units what does it mean to add up two different activities? Interestingly, Cohen-Cole et al. (2017) have tested the two-activity model with the National Longitudinal Survey of Adolescent Health (Add Health), which collects national representative information on 7th-12th graders in both public and private schools in the United States. They consider two subtituable activities, namely screen activities (TV, video games, etc.) and education. The latter (academic performance) is the average grade (converted to a four point scale, with $A = 4$, $B = 3$, etc.) in English (or language arts), mathematics, history (or social studies) and science while the former is the logarithm of the total number of hours spent on watching TV/videos and playing video/computer games in a week. The authors use the logarithm to alleviate the problem of measurement errors when a student reports spending a lot of time on watching TV/videos and playing video/computer games. After taking the logarithm, the authors show that both variables (education and screen activities) have similar mean (2.87 and 2.84) and standard deviation (0.73 and 0.82) and thus comparable values. In that case, summing or differencing these two variables does not pose any problem.

3.2 Some interesting special cases

Ex ante homogeneous individuals. Let us study a special case when the only heterogeneity of the players only stems from their network position.

Corollary 2. *Suppose that Assumption 1 holds. Then, when $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, the unique Nash equilibrium is symmetric and given by:*

$$\mathbf{x}^A = \mathbf{x}^B = \mathbf{M}^{+}\mathbf{a} = \frac{1}{1+\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{1+\beta}, \mathbf{a}\right).$$

Indeed, when $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, the game becomes symmetric with respect to both activities and thus has a unique symmetric equilibrium where both efforts are strictly positive. In that case, although each player chooses the same amount of activities for A and B, this amount still differs across players depending on their network position, which is captured in Corollary 2 by the Katz-Bonacich centrality. Notice that, in equilibrium, the marginal utility vector and network parameter are both modulated by $1/(1+\beta)$, which reflects the interdependence between both activities.

No network effects. Let us return to the first benchmark model introduced above where $\delta = 0$. We have the following result:

Corollary 3. *Assume that $\delta = 0$ and that for all i, $a_i^A > \beta a_i^B$ and $a_i^B > \beta a_i^A$. Then, the unique Nash equilibrium is given by:*

$$\mathbf{x}^A = \frac{\mathbf{a}^A - \beta \mathbf{a}^B}{1-\beta^2} \text{ and } \mathbf{x}^B = \frac{\mathbf{a}^B - \beta \mathbf{a}^A}{1-\beta^2}.$$

Independent activities. Assume that $\beta = 0$ so that the two activities are independent. We have:

Corollary 4. *Assume that $\beta = 0$ and $\delta < 1/\lambda_1(\mathbf{G})$. Then, the unique Nash equilibrium is given by:*

$$\mathbf{x}^A = \mathbf{b}(\mathbf{G}, \delta, \mathbf{a}^A) \text{ and } \mathbf{x}^B = \mathbf{b}(\mathbf{G}, \delta, \mathbf{a}^B).$$

Furthermore, the equilibrium payoff of player i is equal to:

$$u_i(\mathbf{x}^A, \mathbf{x}^B) = \frac{1}{2}(x_i^{A*})^2 + \frac{1}{2}(x_i^{B*})^2 = \frac{1}{2}\left[b_i(\mathbf{G}, \delta, \mathbf{a}^A)\right]^2 + \frac{1}{2}\left[b_i(\mathbf{G}, \delta, \mathbf{a}^B)\right]^2.$$

Indeed, when $\beta = 0$, the equilibrium activity for A and for B is just given by the Katz-Bonacich centrality measure. This is the same formula as in the single-activity case (Ballester et al. 2006).

4 Comparative Statics

The clean characterization result derived in (4) helps us determine different comparative statics results. Indeed, in Theorem 1 or in equation (4), we showed that the equilibrium efforts are mainly determined by $\mathbf{M}^+$ and $\mathbf{M}^-$. In Appendix C, we derive some preliminary comparative-statics results on the matrices $\mathbf{M}^+$ and $\mathbf{M}^-$. In particular, in Propositions 6 and 8, we show how we can rank these different matrices, in particular, with respect to $\mathbf{M}$, depending on the values of β and $\mathbf{G}$ while, in Proposition 7, we show how δ and β affect these matrices. It is then relatively easy to derive the comparative statics results of the effect of productivities, network synergies, degree of interdependence between activities and network density on the equilibrium effort in each activity.

4.1 Effects of the intrinsic productivities $\mathbf{a}^A$ and $\mathbf{a}^B$

First, we study the impact of increasing the marginal utility (or productivity) for activity A. Let $\overline{\mathbf{M}} := (\mathbf{M}^+ + \mathbf{M}^-)/2$ and $\widehat{\mathbf{M}} := (\mathbf{M}^+ - \mathbf{M}^-)/2$.

Proposition 1. *Suppose that Assumption 1 holds. Then, for any network* $\mathbf{G}$*, we have:*

$$\begin{aligned} \frac{\partial \mathbf{x}^A}{\partial \mathbf{a}^A} &= \overline{\mathbf{M}} \succ \mathbf{0}, \\ \frac{\partial \mathbf{x}^B}{\partial \mathbf{a}^A} &= \widehat{\mathbf{M}} \begin{cases} \preceq \mathbf{0} & \text{if } \beta > 0 \\ = \mathbf{0} & \text{if } \beta = 0 \\ \succeq \mathbf{0} & \text{if } \beta < 0 \end{cases}. \end{aligned}$$

Indeed, when a player i's marginal benefit of activity A, say crime (i.e. her intrinsic productivity in crime), goes up, all other players in the network increase their effort in crime (activity A) because of spillover effects. This is due to strategic complementarities in efforts for all path-connected players in the network. On the other hand, because the players are also interconnected via the other activity B, their effort levels in activity B will also be affected. When the two activities are substitutes ($\beta > 0$) so that activity B is education, every individual chooses to study less (lower level of activity B). By contrast, when the two activities are complements ($\beta < 0$), say crime and drug consumption, the effort increase in crime leads to a boost in drug consumption for all individuals in the network.

4.2 Connection to supermodular games

The network game described above with two activities can easily be viewed as a supermodular game.

First, let us consider the simpler case when $\beta < 0$, i.e. the *two activities are complements.* In this case, we define the game Γ with strategy space $\mathbf{S}_i := \mathbf{R}^2$ for each player i and utility functions given by (1). We can verify that the game Γ is a supermodular game for each fixed set of parameters $\mathbf{t} := (\mathbf{a}^A, \mathbf{a}^B)$. Specifically, we can verify that for each i:

- The strategy space of player i $\mathbf{S}_i$ is a lattice.[14]

[14] Technically, the lattice $\mathbf{S}_i$ should be a complete lattice to guarantee existence of pure strategy Nash Equilibrium.

- The payoff function $\mathbf{u}_i$ is continuous and supermodular on $\mathbf{S}_i$ for every $\mathbf{x}_{-i}$.
- $\mathbf{u}_i$ satisfies increasing differences in $(\mathbf{S}_{-i}, \mathbf{S}_i)$.
- $\mathbf{u}_i$ is supermodular in $(\mathbf{x}_i, \mathbf{t})$ for every $\mathbf{x}_{-i}$.

Therefore, the standard monotonicity results (Milgrom and Roberts (1990), Vives (1990) and Milgrom and Shannon (1994)) in supermodular games give us the desired monotone results: Both $\mathbf{x}^A$ and $\mathbf{x}^B$ are monotone in $\mathbf{t}$, i.e., in $\mathbf{a}^A$ and $\mathbf{a}^B$. This is what was obtained in Proposition 1, i.e. $\frac{\partial \mathbf{x}^A}{\partial \mathbf{a}^A} \succ \mathbf{0}$, $\frac{\partial \mathbf{x}^B}{\partial \mathbf{a}^A} \succ \mathbf{0}$ and $\frac{\partial \mathbf{x}^A}{\partial \mathbf{a}^B} \succ \mathbf{0}$, $\frac{\partial \mathbf{x}^B}{\partial \mathbf{a}^B} \succ \mathbf{0}$.

Second, when $\beta > 0$ (the *two activities are substitutes*), the game Γ is no longer supermodular. This is because the inclusion of substitutable activities violates the second item above (the cross sign $\frac{\partial^2 u_i}{\partial x_i^A \partial x_i^B} = -\beta$ is negative). However, we can change the variables by setting

$$\tilde{\mathbf{x}}_i = (x_i^A, -x_i^B), \quad \tilde{\mathbf{t}} := (\mathbf{a}^A, -\mathbf{a}^B).$$

It is routine to check that the new game $\tilde{\Gamma}$, with the new strategy space and new parameters, will satisfy the four conditions above. Therefore, the Nash equilibrium is monotone in the new parameters. Returning to what we found in the second part of Proposition 1, we see that $\mathbf{x}^A$ is increasing in $\mathbf{a}^A$ but decreasing in $\mathbf{a}^B$. In contrast, $\mathbf{x}^B$ is increasing in $\mathbf{a}^B$ but decreasing in $\mathbf{a}^A$.

This sign-change trick is standard for transforming a linear Cournot game with two firms into a supermodular game (Vives 1990). In general, it will not work for arbitrary oligopoly models. Moreover, the game we study here is more complicated than the one used in standard oligopoly models. There are n players, each player has multi-dimensional strategy space, and the interactions between inter-player actions and intra-player actions can have different signs. Yet, we are able to establish the supermodularity using this kind of trick.

Observe that showing that our network game with two activities can be viewed as a supermodular game helps us derive comparative statics results but does not help us in characterizing the Nash equilibrium as in Theorem 1 or in equation (4). Observe also that the precise quantitative calculations of the linear algebra help us understand the way the comparative statics results work. The fact that our game is supermodular just tells us that, for example, increasing $\mathbf{a}^A$ always increases $\mathbf{x}^A$. The exact calculation of these comparative statics results given in Proposition 1 shows the exact role of the social-interaction matrices $\mathbf{M}^+$ and $\mathbf{M}^-$ in this derivation. For example, it shows that $\frac{\partial \mathbf{x}^A}{\partial \mathbf{a}^A} = \overline{\mathbf{M}}$, which means that when the vector of abilities $\mathbf{a}^A$ increases by z percent, then the vector of efforts in activity A increases by $z\overline{\mathbf{M}}$ percent.

As we have shown the existence and uniqueness of the equilibrium in Theorem 1, all of the properties of supermodular games apply to the unique equilibrium characterized in Theorem 1.

4.3 Effects of the social multiplier δ

Next, we study the impact of increasing δ, the intensity of network effects or social multiplier, on equilibrium efforts.

Proposition 2. *Suppose that Assumption 1 holds. Then,*

$$\begin{aligned}\frac{\partial \mathbf{x}^A}{\partial \delta} &= \mathbf{G}[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\overline{\mathbf{a}} + \mathbf{G}[(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\widehat{\mathbf{a}},\\ \frac{\partial \mathbf{x}^B}{\partial \delta} &= \mathbf{G}[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\overline{\mathbf{a}} - \mathbf{G}[(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\widehat{\mathbf{a}}.\end{aligned}$$

When activities are ex ante homogeneous, i.e. $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$*, we obtain:*

$$\frac{\partial \mathbf{x}^A}{\partial \delta}|_{\mathbf{a}^A=\mathbf{a}^B=\mathbf{a}} = \frac{\partial \mathbf{x}^B}{\partial \delta}|_{\mathbf{a}^A=\mathbf{a}^B=\mathbf{a}} = \mathbf{G}[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\mathbf{a} \succeq \mathbf{0}.$$

When δ increases, it affects the equilibrium efforts in activities A and B in two different ways (see equation (4)), which is given by: $\overline{\mathbf{x}} = \mathbf{M}^{+}\overline{\mathbf{a}}$ and $\widehat{\mathbf{x}} = \mathbf{M}^{-}\widehat{\mathbf{a}}$. First, it impacts the mean activity as $\mathbf{M}^{+}$ is increasing in δ. Second, it also affects the difference between two activities through $\mathbf{M}^{-}$, which is increasing in δ. As the result, the net effect of δ on equilibrium efforts is ambiguous, depending on which effect dominates the other one. However, when we shut down the second channel by imposing that $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, both efforts go up by Corollary 2. In Appendix D, we will further investigate the general case when $\mathbf{a}^A \neq \mathbf{a}^B$ using specific networks to illustrate the non-monotone relationship between δ and equilibrium efforts.

4.4 Effects of the degree of interdependence between activities β

Let us now examine the influence of the interdependence between the two activities on equilibrium efforts.

Proposition 3. *Suppose that Assumption 1 holds. Then,*

$$\begin{aligned}\frac{\partial \mathbf{x}^A}{\partial \beta} &= -[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\overline{\mathbf{a}} + [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\widehat{\mathbf{a}},\\ \frac{\partial \mathbf{x}^B}{\partial \beta} &= -[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\overline{\mathbf{a}} - [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\widehat{\mathbf{a}}.\end{aligned}$$

When $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$*, we obtain:*

$$\frac{\partial \mathbf{x}^A}{\partial \beta}|_{\mathbf{a}^A=\mathbf{a}^B=\mathbf{a}} = \frac{\partial \mathbf{x}^B}{\partial \beta}|_{\mathbf{a}^A=\mathbf{a}^B=\mathbf{a}} = -[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2}\mathbf{a} \preceq \mathbf{0}.$$

When β increases, we have again the same two opposite forces that make the impact of β on $\mathbf{x}^A$ and $\mathbf{x}^B$ ambiguous. Indeed, when $\beta > 0$, increasing β makes the substitutable activities more substitutable. The difference in the two efforts becomes larger, but the sum becomes lower. The results clearly depend on whether the initial advantage in productivity is greater or lower in activity A compared to activity B. Interestingly, when $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, so that there is no ability difference between the agents and thus the efforts in both actions are the same, the effect of β on equilibrium efforts is negative.

If we now consider the complementary case when $\beta < 0$, then a higher β makes the activities more complementary. Similar comparative statics can be obtained. In particular, when $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, increasing β makes the activities even more complementary. In this case, $\frac{\partial \mathbf{x}^A}{\partial |\beta|}|_{\mathbf{a}^A=\mathbf{a}^B=\mathbf{a}} \succeq \mathbf{0}$, i.e., efforts are enhanced by a higher degree of complementarity.

4.5 Effect of network density G

Finally, we can consider the situation in which the network gets *denser*, for example, by increasing or adding links while keeping the number of individuals n constant. This is captured by $\mathbf{G}' \succeq \mathbf{G}$, where network $\mathbf{G}'$ is denser that network $\mathbf{G}$.

Proposition 4. *Suppose that Assumption 1 holds for both* $\mathbf{G}$ *and* $\mathbf{G}'$ *and that* $\mathbf{G}' \succeq \mathbf{G}$*. Then,*

$$\mathbf{x}^A(\mathbf{G}') + \mathbf{x}^B(\mathbf{G}') \succeq \mathbf{x}^A(\mathbf{G}) + \mathbf{x}^B(\mathbf{G}).$$

- *If, in addition,* $\mathbf{a}^A \geq \mathbf{a}^B$,

$$\mathbf{x}^A(\mathbf{G}') - \mathbf{x}^B(\mathbf{G}') \succeq \mathbf{x}^A(\mathbf{G}) - \mathbf{x}^B(\mathbf{G}).$$

- *When* $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$*, then,*

$$\mathbf{x}^A(\mathbf{G}') = \mathbf{x}^B(\mathbf{G}') \succeq \mathbf{x}^A(\mathbf{G}) = \mathbf{x}^B(\mathbf{G}).$$

If $\mathbf{a}^A \geq \mathbf{a}^B$ (productivity advantage for activity A), then in equilibrium both the average and difference in activities increase. As a result, we always have $\mathbf{x}^A(\mathbf{G}') \succeq \mathbf{x}^A(\mathbf{G})$. This is because there are more spillovers in network $\mathbf{G}'$, which favors activity A because of its initial productivity advantage. In other words, if individuals are "better" in committing crime than in studying, they will exert relatively more crime effort than education effort in a denser network. For crime and drug consumption, the same logic applies. If an individual has a productivity advantage in crime, then she will commit relatively more crime than consuming drugs in denser networks. However, it is not always true that $\mathbf{x}^B(\mathbf{G}') \succeq \mathbf{x}^B(\mathbf{G})$ (see Table 3 in Appendix E). Furthermore, if the two activities are ex ante symmetric, then we obtain the monotonicity result that both activities increase in denser networks.

In Appendix E, we illustrate these different comparative-statics results with the help of some specific networks. In particular, contrary to Ballester et al. (2006), we give some intuition on the result that, with two activities, denser networks do not necessarily lead to higher aggregate activity.

5 Key player policy

In this section, we investigate the key-player policy (Lindquist and Zenou (2014), Liu et al. (2012)). Suppose that the planner wants to remove the individual who maximally reduces total criminal activities. Then, which player should the planner target? The individual to be removed has been called the *key player* (Zenou (2016)) and an explicit formula has been given by Ballester et al. (2006) in a model where criminals can only choose one activity: effort in crime. We would like now to see if the key player will be the same if individuals exert efforts in *two* activities: crime and education (for substitutable activities) or crime and consuming drugs (for complementary activities). In other words, we would like to examine if our framework of multiple activities affects the key player selection.

5.1 Characterization

Mathematically, in the single-activity setting, the key player program is formulated as follows:

$$\max_i \left\{ \sum_{k=1}^{n} b_i(\mathbf{G}, \delta, \mathbf{a}) - \sum_{k \neq i} b_k(\mathbf{G}_{-i}, \delta, \mathbf{a}_{-i}) \right\}. \tag{5}$$

Here $\mathbf{G}_{-i}$ is the resulting network when player i is removed. The first term $\sum_{k=1}^{n} b_i(\mathbf{G}, \delta, \mathbf{a})$ is the sum of total efforts in the original network $\mathbf{G}$, while the second term $\sum_{k \neq i} b_k(\mathbf{G}_{-i}, \delta, \mathbf{a}_{-i})$ is the resulting equilibrium total effort when i is removed. Observe that in the single-activity case, the equilibrium effort of each player i is equal to her Katz-Bonacich centrality in the network and that is why total activities are equal to the sum of the Katz-Bonacich centralities in (5). The solution to this problem is characterized by the following lemma.

Lemma 1. *(Ballester et al. (2006)) Assume that $\delta < 1/\lambda_1(\mathbf{G})$. In the single-activity framework, the following identity holds:*

$$\left\{ \sum_{k=1}^{n} b_k(\mathbf{G}, \delta, \mathbf{a}) - \sum_{k \neq i} b_k(\mathbf{G}_{-i}, \delta, \mathbf{a}_{-i}) \right\} = \frac{b_i(\mathbf{G}, \delta, \mathbf{1}_n) b_i(\mathbf{G}, \delta, \mathbf{a})}{m_{ii}(\mathbf{G}, \delta)} := \bar{c}_i(\mathbf{G}, \delta, \mathbf{a}),$$

so that the key player is the individual who has the highest inter-centrality measure $\bar{c}_i(\mathbf{G}, \delta, \mathbf{a})$ *in the network.*

With two activities, there are multiple ways of defining the key player, depending on the objective function. First, suppose that the social planner cares about the reduction of the sum of both activities. Recall that

$$\mathbf{x}^A + \mathbf{x}^B = \mathbf{M}^+(\mathbf{a}^A + \mathbf{a}^B) = \mathbf{b}(\mathbf{G}, \frac{\delta}{1+\beta}, \frac{(\mathbf{a}^A + \mathbf{a}^B)}{1+\beta})$$

by Theorem 1, where $\mathbf{M}^+ = [(1+\beta)\mathbf{I} - \delta\mathbf{G}]^{-1} = \frac{1}{1+\beta}[\mathbf{I} - \frac{\delta}{(1+\beta)}\mathbf{G}]^{-1}$. In this case, by directly using Lemma 1, the key player is determined by:

$$c_i^{1,1}(\mathbf{G}, \delta, \mathbf{a}^A, \mathbf{a}^B) := \frac{1}{1+\beta}\bar{c}_i\left(\mathbf{G}, \frac{\delta}{(1+\beta)}, (\mathbf{a}^A + \mathbf{a}^B)\right).$$

Here the superscripts $(1,1)$ indicate that the objective function puts equal weights on both activities.

Alternatively, the social planner may only be concerned about activity A. Recall that

$$\mathbf{x}^A(\mathbf{G}) = \mathbf{M}^+\frac{(\mathbf{a}^A + \mathbf{a}^B)}{2} + \mathbf{M}^-\frac{(\mathbf{a}^A - \mathbf{a}^B)}{2}.$$

With this alternative objective (*minimizing total crime*), the key player index in this scenario, again following directly from Lemma 1, is:

$$c_i^{1,0}(\mathbf{G}, \delta, \mathbf{a}^A, \mathbf{a}^B) := \frac{1}{1+\beta}\bar{c}_i\left(\mathbf{G}, \frac{\delta}{(1+\beta)}, \overline{\mathbf{a}}\right) + \frac{1}{1-\beta}\bar{c}_i\left(\mathbf{G}, \frac{\delta}{(1-\beta)}, \widehat{\mathbf{a}}\right). \tag{6}$$

Equation (6) shows that the inter-centrality measure of player i with two actions is a combination of the intercentralities of player 1 with one action with different weights and decay factors.

In general, the social planner may care about the weighted sum $\lambda\sum_{i=1}^n x_i^A + \mu\sum_{i=1}^n x_i^B$ for any parameter pair (λ, μ). This induces a new key player index.

Theorem 2. *Suppose that Assumption 1 holds and that the social planner cares about the maximal reduction of the weighted sum of* $\lambda\sum x_i^A + \mu\sum x_i^B$ *for* $(\lambda, \mu) \in \mathbf{R}^2$. *The solution to the key player problem is characterized by the following generalized inter-centrality measure with multiple activities:*

$$c_i^{\lambda,\mu}(\mathbf{G}, \delta, \mathbf{a}^A, \mathbf{a}^B) = \mu\, c_i^{1,1}(\mathbf{G}, \delta, \mathbf{a}^A, \mathbf{a}^B) + (\lambda - \mu)c_i^{1,0}(\mathbf{G}, \delta, \mathbf{a}^A, \mathbf{a}^B).$$

When players are ex ante homogeneous, i.e., $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, *then*

$$c_i^{\lambda,\mu}(\mathbf{G}, \delta, \mathbf{a}, \mathbf{a}) = \frac{(\lambda+\mu)}{1+\beta}\bar{c}_i(\mathbf{G}, \frac{\delta}{(1+\beta)}, \mathbf{a}).$$

Theorem 2 shows that a key-player policy based on one activity (crime) can be misleading. If the objective of the planner is to reduce total crime, then the person who, once removed, reduces total crime the most, can differ between the one-activity and the two-activity case. This could be a problem since the planner could target the wrong person in the network.

Indeed, consider, first, crime and education and assume that the planner wants to reduce crime. When she removes a criminal from the network, there are now two effects. On the one hand, there are less spillovers in terms of crime so that the neighbors of the removed person exert less crime, which, in turn, leads to the fact that their neighbors reduce crime, and so on. On the other hand, by reducing crime, the planner induces the remaining criminals to increase their education effort, which, because of spillovers in education, induces their neighbors to also study more, and so forth. As a result, there is a virtuous effect of removing a criminal since it induces the remaining criminals to commit less crimes and to focus more on education.

Consider now crime and drug consumption. When a criminal is removed from the network, the remaining criminals reduce their crime effort and, because of complementarity between activities, also reduce their drug consumption. There is again a virtuous effect of removing a criminal in a network, which is not taken into account in the single activity case. This is why the key player in the single activity case can be different from the one in the two-activity case, as we show now by means of examples.

5.2 Examples

5.2.1 Determining the key player when the planner believes that players exert effort in only one activity (crime)

Theorem 2 characterizes the key player policy with multiple activities. Notably, even if the social planner cares about only one activity, the very existence of the other activity can substantially affect the key player policy. We now provide a concrete example to illustrate this point. Consider the network $\mathbf{G}$ in Figure 1 with $n = 11$ players. This is the network that was considered by Ballester et al. (2006) to illustrate their formula of the key player.

In this network, player 1 bridges together two fully intra-connected groups with five players each. Removing player 1 disrupts the network, whereby removing 2 decreases maximally the total number of network links. In the case of a single activity, the highest value of δ that is compatible with the intercentrality measure is

$$\delta < \frac{1}{\lambda_1(\mathbf{G})} = \frac{1}{4.404} = 0.227.$$

Assume that players are ex ante identical (i.e. $a_1 = ... = a_{11} = 1$). The intercentrality measure is

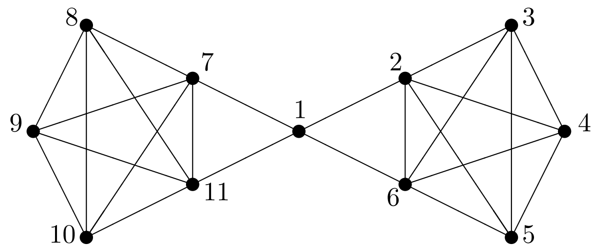

Figure 1: A bridge network.

then given by:

$$\bar{c}_i(\mathbf{G}, \delta, \mathbf{1}_n) = \frac{[b_i(\mathbf{G}, \delta, \mathbf{1}_n)]^2}{m_{ii}(\mathbf{G}, \delta)},$$

where $m_{ii}(\mathbf{G}, \delta)$ is the diagonal element of matrix $\mathbf{M}(\mathbf{G}, \delta)$ (defined in (12) in Appendix A.2). Note that we have only considered one activity so far. Table 1 gives the Bonacich and inter-centrality measures for two values of δ. A star identifies the highest value in each column.

δ	0.18		0.2	
Player Type	b_i	$\bar{c}_i$	b_i	$\bar{c}_i$
1	4.77	17.03	8.33	41.67*
2	5.23*	17.62*	9.17*	40.33
3	4.51	14.07	7.78	32.67

Table 1: Centrality measures for the key player policy.

Player 2 has the highest number of direct links and a wide span of indirect links through her link with player 1. As a result, player 2 has the highest Katz-Bonacich centrality and thus commits the highest level of crime. When δ is low ($\delta = 0.18$), player 2 is also the key player. When δ is higher ($\delta = 0.20$), player 1 becomes the key player. Now, indirect effects matter more and removing player 1 has the highest joint direct and indirect effects on aggregate outcomes.

Consider now our model with two activities and ex ante identical players ($a_1^A = ... = a_{11}^A = 1$ and $a_1^B = ... = a_{11}^B = 1$). The condition of existence and uniqueness of equilibrium (Assumption 1) is given by:

$$\delta < \frac{1 - |\beta|}{\lambda_1(\mathbf{G})} = 0.227 \left(1 - |\beta|\right).$$

We have seen in (6) that, if the planner wants to determine the key player considering only one activity (say activity A), then the intercentrality is given by

$$\frac{1}{1+\beta} \bar{c}_i \left(\mathbf{G}, \frac{\delta}{1+\beta}, \mathbf{1}_n \right) = \frac{1}{1+\beta} \frac{[b_i(\mathbf{G}, \delta / \left(1+\beta\right), \mathbf{1}_n)]^2}{m_{ii}(\mathbf{G}, \delta / \left(1+\beta\right))}.$$

5.2.2 Determining the key player when the planner believes that players exert effort in two substitutable activities (crime and education)

Let us take $\beta = 1/9 > 0$ so that activities A and B are substitutes (crime and education). As in Table 1, assume that $\delta = 0.2$.[15] In that case, if the planner wants to minimize crime (activity A), the key player is the one with the higher intercentrality $\bar{c}_i\left(\mathbf{G}, \frac{\delta}{1+\beta}, \mathbf{1}_n\right)$. Since $\beta = 1/9$ and $\delta = 0.2$, intercentrality is $\bar{c}_i\left(\mathbf{G}, \frac{\delta}{1+\beta}, \mathbf{1}_n\right) = \bar{c}_i\left(\mathbf{G}, 0.18, \mathbf{1}_n\right)$. We see from Table 1 that the key player is now player 2! In other words, if the planner ignores the fact that individuals exert efforts in *two substitutable activities*, she will wrongly think that the key player is player 1 while player 2 is in fact the key player. Indeed, when δ is relatively high, in the single activity case, the planner wants to remove player 1 because she is the one that generates the most crime spillovers (which intensity is measured by δ) due to her bridge position in the network, even though player 1 has less direct links than 2. When we consider both crime and education, player 2 becomes the key player because not only she generates crime spillovers but also generates important education spillovers due to her position in the network. Indeed, the intensity of the spillovers of both crime and education is now measured by $\delta/(1+\beta) < \delta$ when $\beta > 0$, so that the bridge position of player 1 becomes less important while the higher direct impact on neighbors of player 2 becomes more important.

5.2.3 Determining the key player when the planner believes that players exert effort in two complementary activities (crime and drug consumption)

Let us now take: $\beta = -0.1 < 0$ so that activities A and B are complements (crime and drug consumption). As in Table 1, assume that $\delta = 0.18$.[16] In that case, if the planner wants to minimize crime (activity A), then the key player is the one with the highest intercentrality $\bar{c}_i\left(\mathbf{G}, \frac{\delta}{1+\beta}, \mathbf{1}_n\right)$. Since $\beta = -0.1$ and $\delta = 0.18$, this intercentrality is equal to:

$$\bar{c}_i\left(\mathbf{G}, \frac{\delta}{1+\beta}, \mathbf{1}_n\right) = \bar{c}_i\left(\mathbf{G}, 0.2, \mathbf{1}_n\right).$$

We see from Table 1 that, in this case, the key player is now player 1! In other words, if the planner ignores the fact that the individuals in the network exert efforts in *two complementary activities*,

[15] The condition of existence and uniqueness of equilibrium is satisfied since

$$\delta < \frac{1-|\beta|}{\lambda_1(\mathbf{G})} = 0.227\,(1-|\beta|) \Leftrightarrow 0.2 < 0.227\,(0.89) = 0.202.$$

[16] The condition of existence and uniqueness of equilibrium is satisfied since

$$\delta < \frac{1-|\beta|}{\lambda_1(\mathbf{G})} = 0.227\,(1-|\beta|) \Leftrightarrow 0.18 < 0.227\,(0.9) = 0.204.$$

she will wrongly think that the key player is player 2 while it is player 1. The explanation of this result is similar to the one above. Indeed, when we consider both crime and drug consumption, player 1 becomes the key player because the intensity of the spillovers of both crime and education is now measured by $\delta/(1+\beta) > \delta$ since $\beta < 0$. As a result, the bridge position of player 1 becomes more important while the higher number of direct links of player 2 becomes less crucial.

The above examples make a sharp argument that multiple activities matter for the criminal policies aiming at reducing crime.

6 Extensions

In this section, we consider different extensions of the model. In particular, we would like to investigate whether the neat characterization results obtained in Theorem 1 and in equation (4), which show that quadratic games with linear-best-response functions aggregate nicely to two activities, are still true when the model is extended.

6.1 More than two activities

So far, we only have considered two activities. In the real world, players exert efforts in more than two activities. Let us thus consider the case when there are more than two ($l \geq 2$) activities. As a result, the utility function (1) should be modified and written as:

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{t=1}^{l} a_i^t x_i^t - \frac{1}{2}\sum_{t=1}^{l}(x_i^t)^2 - \frac{1}{2}\beta \sum_{t=1}^{l}\sum_{s\neq t} x_i^s x_i^t + \delta \sum_{t=1}^{l}\left(\sum_{j=1}^{n} g_{ij} x_i^t x_j^t\right). \tag{7}$$

where subscripts i, j refer to the players while superscripts s and t refer to activities. As in the two activity cases, in (7), each player can choose among l different activities and network externalities only affect players within the same activity. In this framework, the parameter β lies in the interval $(-1/(l-1), 1)$.[17] In this new setup, we need a condition that generalizes assumption 1 for the case of l activities. We have:[18]

Assumption 2. $(1+(l-1)\beta) - \delta\lambda_1(\mathbf{G}) > 0$ *and* $1 - \beta - \delta\lambda_1(\mathbf{G}) > 0$.

[17] The payoff u_i in equation (7) is concave in $\mathbf{x}_i$ only if β lies in that interval.

[18] When β is positive, $(1-\beta) - \delta\lambda_1(\mathbf{G}) > 0$ implies that $(1+(l-1)\beta) - \delta\lambda_1(\mathbf{G}) > 0$ and so the constraint in this case reduces to $\beta + \delta\lambda_1(\mathbf{G}) < 1$. When β is negative, $(1+(l-1)\beta) - \delta\lambda_1(\mathbf{G}) > 0$ implies $(1-\beta) - \delta\lambda_1(\mathbf{G}) > 0$. Thus, the constraint in this case reduces to $(1-l)\beta + \delta\lambda_1(\mathbf{G}) < 1$.

Let

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & \beta & \cdots & \beta \\ \beta & 1 & \cdots & \beta \\ \vdots & \ddots & \ddots & \vdots \\ \beta & \cdots & \beta & 1 \end{bmatrix}, \tag{8}$$

$$\mathbf{x}^t = \begin{bmatrix} \mathbf{x}_1^t \\ \vdots \\ \mathbf{x}_n^t \end{bmatrix}, \ \mathbf{a}^t = \begin{bmatrix} \mathbf{a}_1^t \\ \vdots \\ \mathbf{a}_n^t \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^l \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \mathbf{a}^1 \\ \vdots \\ \mathbf{a}^l \end{bmatrix}.$$

We have the following result.

Theorem 3. *Suppose that Assumption 2 holds. When players exert l activities, there exists a unique Nash equilibrium given by*

$$\mathbf{X} = [\boldsymbol{\Psi} \otimes \mathbf{I}_n - \delta \mathbf{I}_l \otimes \mathbf{G}]^{-1}\mathbf{A} = \begin{bmatrix} \mathbf{W} & \boldsymbol{\Phi} & \cdots & \boldsymbol{\Phi} \\ \boldsymbol{\Phi} & \mathbf{W} & \cdots & \boldsymbol{\Phi} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{\Phi} & \cdots & \boldsymbol{\Phi} & \mathbf{W} \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 \\ \vdots \\ \mathbf{a}^l \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{W} &= \frac{[(1+(l-1)\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1} + (l-1)[(1-\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1}}{l}, \\ \boldsymbol{\Phi} &= \frac{[(1+(l-1)\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1} - [(1-\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1}}{l}. \end{aligned}$$

The equilibrium profile for activity t can be written as:

$$\mathbf{x}^t = \frac{1}{(1+(l-1)\beta)}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{(1+(l-1)\beta)}, \frac{\sum_{k=1}^{l} \mathbf{a}^k}{l}\right) + \frac{1}{1-\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{(1-\beta)}, \mathbf{a}^t - \frac{\sum_{k=1}^{l} \mathbf{a}^k}{l}\right).$$

Theorem 3 generalizes Theorem 1 and Corollary 1, which show that quadratic games with linear-best-response functions aggregate nicely to *multiple* activities.

Given the characterization of the Nash Equilibrium, we can easily derive the key player analysis in a similar way as in Section 5. For example, if the social planner only cares about activity t (crime) and thus wants to minimize total crime, the corresponding key player intercentrality measure is given by:

$$c_i^{(0,\cdots,0,1,0,\cdots,0)}(\mathbf{G},\delta,\mathbf{A}) = \frac{1}{(1+(l-1)\beta)}\bar{c}_i\left(\mathbf{G},\frac{\delta}{((1+(l-1)\beta))},\frac{\sum_{k=1}^{l}\mathbf{a}^k}{l}\right) + \frac{1}{1-\beta}\bar{c}_i\left(\mathbf{G},\frac{\delta}{(1-\beta)},\mathbf{a}^t-\frac{\sum_{k=1}^{l}\mathbf{a}^k}{l}\right).$$

As in Section 5, we can also consider other alternative objectives (such as a weighted sum of these aggregate activities).

6.2 Cross-network effects

So far we have not allowed for cross network effects, i.e., network externalities between different activities. Indeed, it is possible that the crime effort exerted by a person affects the best response of her direct friends in terms of education or drug consumption. If we go back to the two activities case, then the utility function (1) should now be written as:

$$u_i(\mathbf{x}_i,\mathbf{x}_{-i}) = a_i^A x_i^A + a_i^B x_i^B - \left\{\frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B\right\} + \delta\sum_{j=1}^{n} g_{ij}x_i^A x_j^A + \delta\sum_{j=1}^{n} g_{ij}x_i^B x_j^B + \mu\sum_{j=1}^{n} g_{ij}x_i^A x_j^B + \mu\sum_{j=1}^{n} g_{ij}x_i^B x_j^A. \tag{9}$$

where we have added two *cross-externality terms*: $\mu\sum_{j=1}^{n} g_{ij}x_i^A x_j^B + \mu\sum_{j=1}^{n} g_{ij}x_i^B x_j^A$, where μ captures the intensity of the *between-activity network effects*. Observe that, as before, $\delta \geq 0$ captures the intensity of the *within-activity network effects*. We need the following condition:

Assumption 3. $\max(\frac{|\delta+\mu|}{1+\beta},\frac{|\delta-\mu|}{1-\beta})\lambda_1(\mathbf{G}) < 1$.

We have the following result:

Theorem 4. *Suppose that Assumption 3 holds. Then, there exists a unique equilibrium characterized by*

$$\begin{bmatrix}\mathbf{x}^A\\ \mathbf{x}^B\end{bmatrix} = \begin{bmatrix}\frac{\tilde{\mathbf{M}}^+ + \tilde{\mathbf{M}}^-}{2} & \frac{\tilde{\mathbf{M}}^+ - \tilde{\mathbf{M}}^-}{2}\\ \frac{\tilde{\mathbf{M}}^+ - \tilde{\mathbf{M}}^-}{2} & \frac{\tilde{\mathbf{M}}^+ + \tilde{\mathbf{M}}^-}{2}\end{bmatrix}\begin{bmatrix}\mathbf{a}^A\\ \mathbf{a}^B\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\tilde{\mathbf{M}}^+(\mathbf{a}^A+\mathbf{a}^B) + \tilde{\mathbf{M}}^-(\mathbf{a}^A-\mathbf{a}^B)\\ \tilde{\mathbf{M}}^+(\mathbf{a}^A+\mathbf{a}^B) - \tilde{\mathbf{M}}^-(\mathbf{a}^A-\mathbf{a}^B)\end{bmatrix}$$

where matrices $\tilde{\mathbf{M}}^+$ *and* $\tilde{\mathbf{M}}^-$ *are givem by:*

$$\tilde{\mathbf{M}}^+ = [(1+\beta)\mathbf{I}_n - (\delta+\mu)\mathbf{G}]^{-1} = \sum_{k\geq 0}\frac{(\delta+\mu)^k\mathbf{G}^k}{(1+\beta)^{1+k}},$$

$$\tilde{\mathbf{M}}^- = [(1-\beta)\mathbf{I}_n - (\delta-\mu)\mathbf{G}]^{-1} = \sum_{k\geq 0}\frac{(\delta-\mu)^k\mathbf{G}^k}{(1-\beta)^{1+k}}.$$

Even when there are cross-network effects, we can still characterize the Nash equilibrium with two activities in a neat way and show results that are similar to that of Theorem 1 and Corollary 1.

More generally, this analysis is interesting since it allows for cross-network effects, which means that agents receive network spillovers from both activities. To better understand this result, let us go back to Section 3.1 where we characterize the equilibrium and show, in particular, that the most central agents in a network do not always exert the highest effort in both activities. As an example, we considered a star-shaped network where agent 1 is the star and agents 2 and 3 are in the periphery. We showed that, when $\delta = 0.1$, $\beta = 0.4$ (the two activities are substitutes), $\overline{\mathbf{a}} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ and $\widehat{\mathbf{a}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then

$$\mathbf{x}^A = \begin{pmatrix} 4.82 \\ 4.38 \\ 4.38 \end{pmatrix} \quad \text{and} \quad \mathbf{x}^B = \begin{pmatrix} 0.12 \\ 0.26 \\ 0.26 \end{pmatrix}$$

which means that the star (agent 1) exerts the highest effort in activity A but the lowest effort in activity B. Let us now allow for cross-network effects. Assume the same parameter values and $\mu = 0.05$. Using Theorem 4, we obtain:

$$\mathbf{x}^A = \begin{pmatrix} 4.64 \\ 4.26 \\ 4.26 \end{pmatrix} \quad \text{and} \quad \mathbf{x}^B = \begin{pmatrix} 0.69 \\ 0.60 \\ 0.60 \end{pmatrix}$$

We now obtain the opposite result, which shows that agent 1, who is the most central agent in the network, exerts the highest effort in both activities A and B. This is because her position in the network generates now a lot of network spillovers both in activities A and B, which means that we put more weight on the network position and less weight on the degree of substitution between the two activities. As a result, even if both activities are substitutes and β is quite high, more central agents tend to exert high efforts in both activities because of network spillovers in both activities. Consider, for example, crime and education. When agent 1's neighbors exert high effort in education, it directly positively affects agent 1's effort in education but now also in crime (cross-network effects). Similarly, when agent 1's neighbors exert high effort in crime, it positively affects agent 1's effort in crime but also in education. Because these effects reinforce each other, agent 1 ends up exerting high efforts in both activities.

6.3 Heterogeneous substitutions

So far, we have assumed that β, which captures the degree of interdependence between activities, was the same for all players in the network. In this section, we relax this assumption by assuming that it is player-specific. For that, we write the utility function (1) as follows:

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta_i x_i^A x_i^B \right\} + \delta \sum_{j=1}^{n} g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^{n} g_{ij} x_i^B x_j^B.$$

We need again to have the new following condition:

Assumption 4. $\max_i \left\{ \frac{1}{1-|\beta_i|} \right\} \delta \lambda_1(\mathbf{G}) < 1.$

Let $\mathbf{\Lambda}^\beta = diag(\beta_1, \cdots, \beta_n)$, which is diagonal matrix with β_i on its (i,i) entry.

Theorem 5. *Suppose that Assumption 4 holds. Then, there exists a unique equilibrium characterized by:*

$$\begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} = \begin{bmatrix} \frac{\hat{\mathbf{M}}^+ + \hat{\mathbf{M}}^-}{2} & \frac{\hat{\mathbf{M}}^+ - \hat{\mathbf{M}}^-}{2} \\ \frac{\hat{\mathbf{M}}^+ - \hat{\mathbf{M}}^-}{2} & \frac{\hat{\mathbf{M}}^+ + \hat{\mathbf{M}}^-}{2} \end{bmatrix} \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix},$$

where matrices $\hat{\mathbf{M}}^+$ *and* $\hat{\mathbf{M}}^-$ *are given by:*

$$\hat{\mathbf{M}}^+ = [\mathbf{I}_n + \mathbf{\Lambda}^\beta - \delta \mathbf{G}]^{-1}, \ \hat{\mathbf{M}}^- = [\mathbf{I}_n - \mathbf{\Lambda}^\beta - \delta \mathbf{G}]^{-1}.$$

We show here again that quadratic games with linear-best-response functions aggregate nicely to two activities, even where there are heterogenous substitutions between the two activities.

6.4 Heterogeneous network effects

Finally, we consider a last extension where we assume that the intensity of the network externalities is heterogeneous and differs between activities. This means that the intensity of social interactions in crime (activity A) between players is different to ones in, for example, education (activity B). We also assume that each activity corresponds to a different network so that players who are direct friends in a criminal network (activity A) may not be direct friends in the education network (activity B). As a result, we need to have *two* network matrices to describe the network interactions between the players. Specifically, let $\mathbf{G}^A = (g_{ij}^A)$ be the adjacency matrix for activity A and $\mathbf{G}^B = (g_{ij}^B)$ be the adjacency matrix for activity B. Assume that both matrices $\mathbf{G}^A$ and $\mathbf{G}^B$ are symmetric, i.e. the two networks are undirected. The utility function (1) can now be written as:

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B \right\} + \delta^A \sum_{j=1}^{n} g_{ij}^A x_i^A x_j^A + \delta^B \sum_{j=1}^{n} g_{ij}^B x_i^B x_j^B.$$

We have the following condition:

Assumption 5. $\max(\delta^A \lambda_1(\mathbf{G}^A), \delta^B \lambda_1(\mathbf{G}^B)) < 1 - |\beta|$.

In this setting, we can still characterize the unique equilibrium as follows.

Theorem 6. *Suppose that Assumption 5 holds. Then, with heterogeneous network effects, there exists a unique Nash equilibrium such that*

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n - \delta^A \mathbf{G}^A & \beta \mathbf{I}_n \\ \beta \mathbf{I}_n & \mathbf{I}_n - \delta^B \mathbf{G}^B \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{bmatrix} \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix},$$

where, by using the block matrix inverse formula,

$$\begin{aligned}
\mathbf{Z}_1 &= [\mathbf{I}_n - \delta^A \mathbf{G}^A - \beta^2 [\mathbf{I}_n - \delta^B \mathbf{G}^B]^{-1}]^{-1}, \\
\mathbf{Z}_2 &= -\beta [\mathbf{I}_n - \delta^A \mathbf{G}^A]^{-1} [\mathbf{I}_n - \delta^B \mathbf{G}^B - \beta^2 [\mathbf{I}_n - \delta^A \mathbf{G}^A]^{-1}]^{-1}, \\
\mathbf{Z}_3 &= -\beta [\mathbf{I}_n - \delta^B \mathbf{G}^B]^{-1} [\mathbf{I}_n - \delta^A \mathbf{G}^A - \beta^2 [\mathbf{I}_n - \delta^B \mathbf{G}^B]^{-1}]^{-1}, \\
\mathbf{Z}_4 &= [\mathbf{I}_n - \delta^B \mathbf{G}^B - \beta^2 [\mathbf{I}_n - \delta^A \mathbf{G}^A]^{-1}]^{-1}.
\end{aligned}$$

It is interesting to observe here that the clean characterization results obtained in Theorem 1 and in (4) are not anymore true. In other words, quadratic games with linear best-reply functions do *not* aggregate nicely to two activities when the spillover parameter δ is different between the two activities (i.e. $\delta^A \neq \delta^B$) and/or the network itself is different between the two activities (i.e. $\mathbf{G}^A \neq \mathbf{G}^B$).

The result obtained in Theorem 6 is, however, important because it characterizes the Nash equilibrium in efforts in two activities when agents are embedded in two different networks corresponding to the two activities and where spillovers are different between networks. The fact that individuals belong to different networks corresponding to different activities is relatively well documented. For example, using data from 75 rural villages in Karnataka, India, Banerjee et al. (2013) gathered social network data on thirteen dimensions, including which friends or relatives visit one's home, which friends or relatives the individual visits, with whom the individual goes to pray (at a temple, church, or mosque), from whom the individual would borrow money, to whom the individual would lend money, from whom they obtain advice, and to whom they give advice, etc. For each of these activities, Banerjee et al. (2013) define a network and show that they are quite different. For example, the networks of individuals who go together to pray and the one in which individuals obtain advice from are quite different, the latter being sparser than the latter. Using Theorem 6, we could easily use these network data to estimate the network spillover effects δ^A and δ^B and the degree of substitution β between these activities in each of these networks.

6.5 Alternative model

So far, we have developed a model where the utility function was given by (1). This has been referred to as the *local aggregate* model in the literature (Liu et al. (2014); Topa and Zenou (2015)) since what affects the utility of each player i is the *sum of efforts* (in each activity) of players connected to i in the network. For the one-activity case, there is an alternative model, first introduced by Patacchini and Zenou (2012), which is referred to as the *local average* model, and where players bear a cost from deviating to the *social norm* of the reference group (i.e., the average effort of the peers). Let $\mathbf{G}^* = \left(g_{ij}^*\right)$ denote the *row-normalized* adjacency matrix of $\mathbf{G}$, where $g_{ij}^* = g_{ij}/g_i$ and $g_i = \sum_{j=1}^n g_{ij}$ is the degree of player i. By construction, we have $0 \leq g_{ij}^* \leq 1$ and $\sum_{j=1}^n g_{ij}^* = 1$. In the case of two activities, the utility function in the *local average* model can be written as:

$$\begin{aligned} u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \quad &= \quad a_i^A x_i^A + a_i^B x_i^B - \left\{\frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B\right\} \\ &\quad -\frac{\gamma}{2}\left(x_i^A - \sum_{j=1}^n g_{ij}^* x_j^A\right)^2 - \frac{\gamma}{2}\left(x_i^B - \sum_{j=1}^n g_{ij}^* x_j^B\right)^2 . \end{aligned} \tag{10}$$

where γ is the *social-conformity* parameter. The condition is now given by:[19]

Assumption 6. $1 - |\beta| > 0$.

Define

$$\mathbf{M}^{+*} := \left[(1+\gamma+\beta)\,\mathbf{I}_n - \gamma\mathbf{G}^*\right]^{-1} \text{ and } \mathbf{M}^{-*} := \left[(1+\gamma-\beta)\,\mathbf{I}_n - \gamma\mathbf{G}^*\right]^{-1}.$$

We have the following theorem which proof is omitted since it is very similar to that of Theorem 1.

Theorem 7. *Suppose that Assumption 6 holds and the utility function of each player i is given by (10). Then, for any $\mathbf{a}^A$ and $\mathbf{a}^B$, there exists a unique Nash equilibrium given by:*

$$\begin{bmatrix}\mathbf{x}^A \\ \mathbf{x}^B\end{bmatrix} = \begin{bmatrix}\frac{\mathbf{M}^{+*}+\mathbf{M}^{-*}}{2} & \frac{\mathbf{M}^{+*}-\mathbf{M}^{-*}}{2} \\ \frac{\mathbf{M}^{+*}-\mathbf{M}^{-*}}{2} & \frac{\mathbf{M}^{+*}+\mathbf{M}^{-*}}{2}\end{bmatrix}\begin{bmatrix}\mathbf{a}^A \\ \mathbf{a}^B\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\mathbf{M}^{+*}(\mathbf{a}^A+\mathbf{a}^B) + \mathbf{M}^{-*}(\mathbf{a}^A-\mathbf{a}^B) \\ \mathbf{M}^{+*}(\mathbf{a}^A+\mathbf{a}^B) - \mathbf{M}^{-*}(\mathbf{a}^A-\mathbf{a}^B)\end{bmatrix}.$$

[19] Observe that the largest eigenvalue of $\mathbf{G}^*$ is 1, i.e. $\lambda_1(\mathbf{G}) = 1$.

We see that Theorems 1 and 7 are very similar but still work differently. The main difference is in the definition of the matrices $\mathbf{M}^{+*}$ and $\mathbf{M}^{-*}$ as compared to $\mathbf{M}^{+}$ and $\mathbf{M}^{-}$. First, the term in brackets in front of $\mathbf{I}_n$ incorporates an additional term γ, which captures the *taste for conformity* of agents. Second, and more importantly, in the local-aggregate model, the network is described by the adjacency matrix $\mathbf{G}$ whereas, in the local-average model, it is the row-normalized adjacency matrix $\mathbf{G}^*$ that accounts for the network. This implies that the position in the network is crucial for determining the individual equilibrium efforts in each activity in the local-aggregate model while it is not important in the local-average model. In fact, if there is not heterogeneity ex ante (i.e. $a_i^A = a_i^B = a$, $\forall i$), the position in the network does not matter at all in the local-average model since all agents will exert the same effort in both activities while it is the main determinant of effort in the local-aggregate model. Indeed, with ex ante identical ability, in the local-aggregate model, $\mathbf{x}^A = \mathbf{x}^B = \mathbf{M}^{+}\overline{\mathbf{a}}$ while, in the local-average model, $\mathbf{x}^A = \mathbf{x}^B = \mathbf{M}^{+*}\overline{\mathbf{a}}$.

To illustrate this point, consider a star-shaped network where agent 1 is in the center and agents 2 and 3 are in the periphery. As in Section 3.1, assume that $\delta = 0.1$, $\beta = \pm 0.4$ and that all agents have the same level of ex ante ability, given by $\mathbf{a}^T = \begin{pmatrix} 3 & 3 & 3 \end{pmatrix}$. Consider the *local-aggregate* model. When the activities are *substitutes* ($\beta = 0.4$) and when they are complements ($\beta = -0.4$), the equilibrium efforts are respectively equal to:

$$\mathbf{x}^A(sub) = \mathbf{x}^B(sub) = \begin{pmatrix} 2.475 \\ 2.321 \\ 2.321 \end{pmatrix} \text{ and } \mathbf{x}^A(comp) = \mathbf{x}^B(comp) = \begin{pmatrix} 7.059 \\ 6.177 \\ 6.177 \end{pmatrix}$$

Not surprisingly, agent 1 located in the center of the network, exerts the highest effort in both activities.

Consider now the *local-average* model and assume that $\gamma = 0.1$. When the activities are *substitutes* ($\beta = 0.4$) and when they are complements ($\beta = -0.4$), the equilibrium efforts are respectively given by:

$$\mathbf{x}^A(sub) = \mathbf{x}^B(sub) = \begin{pmatrix} 2.143 \\ 2.143 \\ 2.143 \end{pmatrix} \text{ and } \mathbf{x}^A(comp) = \mathbf{x}^B(comp) = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

Here, the position in the network does not matter since all agents exert the same effort in both activities because they all conform to the same norm, which is the average effort in the network.

Finally, following Liu et al. (2014), we can incorporate both local-aggregate and local-average effects in the same utility function. This is referred to as the *hybrid model.* Appendix F characterizes the Nash equilibrium for the hybrid model and shows that, again, the linear best-reply functions aggregate nicely to multiple activities since the average activity $\overline{\mathbf{x}}$ and the difference in activities $\widehat{\mathbf{x}}$

can be expressed as direct functions of the average productivity $\overline{\mathbf{a}}$ and the difference in productivities $\widehat{\mathbf{a}}$ multiplied by some $\mathbf{M}$ matrix, which captures the network structure.

We believe that the main interesting aspect of the local-average model with two activities is that, in case of substitutable activities, there is a conflict between conforming to two *different opposite norms*. Consider crime and education. The utility function (7) then assumes that, for individual i, it is costly to deviate both from her friends' *average criminal effort* and her friends' *average education effort*. Theorem 7 gives the solution to this problem when activities can be either substitutes or complements. This solution again depends on the difference and the sum of the productivities in different activities. If agent i has some initial advantage in crime (higher ability), then she will be induced to follow more the social norm imposed by her friends in crime than in education because the former has higher returns than the latter. This will imply that conformity in crime will be higher than in education.

7 Policy implications

We have seen in Section 5 that a key-player policy based on one activity (crime) can be misleading. If the objective of the planner is to reduce total crime, then the person who, once removed, reduces total crime the most, can differ between the one-activity and the two-activity case. This could be a problem since the planner could target the wrong person in the network.

Furthermore, as stated above, if we consider the extension made in Section 6.4 with different networks for different activities, then the key-player policy will be even more interesting in both the local-aggregate and the local-average models. Indeed, if we consider policies that affect the social norm of education, our model helps us understand how they could spill over into the criminal network. In particular, consider the "No Excuses policy" implemented in the United States (Angrist et al. (2010), Angrist et al. (2012)), which is a highly standardized and widely replicated *charter model* that features a long school day, an extended school year, selective teacher hiring and strict behavior norms, and emphasizes traditional reading and math skills. The main objective is to change the *social norms* of disadvantaged kids by being very strict on discipline. This is a typical policy that is in accordance with the *local-average model* since, in the latter, only a change in social norms can affect the behavior of students.[20] As a result, a policy whose aim is to change the social norm of students in terms of education (for example, a norm that induce children to think that working and studying is a cool thing) can be an effective way to change the educational behavior of students. Angrist et al. (2012) focus on special needs students that may be underserved. Their

[20]In some inner cities in the United States, it has been shown that African Americans residing in poor areas may be ambivalent about learning standard English and performing well at school because this may be regarded as 'acting white', a negative social norm for these students (Fordham and Ogbu (1986); Wilson (1987); Fryer and Torelli (2010); Battu and Zenou (2010)).

results show average achievement gains of 0.36 standard deviations in math and 0.12 standard deviations in reading for each year spent at a charter school. Our model predicts even more impact since it will not only change the social norm in education but also that in crime. As a result, we believe that, by incorporating more than one activity into agents' decisions, we can better understand the larger impact of a policy on the two activities.

8 Conclusion

People interact on many dimensions: they commit crime, drink and smoke together, they also study and work together, share information about jobs, receive suggestions from and give advice to others, etc. In this paper, we theoretically investigate these issues by developing a network model where players choose more than one activity. Activities can be substitutes or complements and generate spillover effects to other players in the network.

Our key theoretical insights are as follows: (*i*) we provide a full characterization of the Nash equilibrium of the game for any network structure and for any number of activities, (*ii*) we show that the quadratic games with linear best-reply functions aggregate nicely to multiple activities, (*iii*) we provide comparative statics results showing how own productivity affects equilibrium efforts and how network density impacts on equilibrium outcomes, (*iv*) we determine who the key-player is when individuals exert more than one activity.

This implies that, compared to the BCZ one-activity model, we have new empirical implications that suggest the following empirical tests: (*i*) Does own productivity in one activity affect positively (negatively) the effort of the other activity when the two activities are complements (substitutes)? (*ii*) If there is an initial productivity advantage in one activity, then do both the average and the difference of efforts between these activities increase when the network becomes denser? (*iii*) Does the identity of the key player change when we switch from one to two activities?

We believe that these are important questions that should be empirically investigated. We leave them for future research.

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Appendix

In this Appendix, we first provide some notations, derive some matrix operations, derive some comparative-statics results, calculate the equilibrium utility of all agents, provide some examples with specific networks and then give the proofs of our main results.

A Matrix analysis and Katz-Bonacich centrality

A.1 Matrix analysis

Let us now introduce some notations on matrices and vectors. $\mathbf{I}_k$ is the $k \times k$ identity matrix, $\mathbf{J}_{pq}$ is the $p \times q$ matrix with only 1s, and $\mathbf{1}_n = \mathbf{J}_{n1}$ is a column vector with 1s, i.e.

$$\mathbf{I}_k = \begin{bmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{bmatrix}_{k\times k}, \quad \mathbf{J}_{pq} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{p\times q}, \ \mathbf{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}.$$

In addition, $\mathbf{0}$ is the zero matrix with suitable dimensions. Given two matrices $\mathbf{H}$ and $\mathbf{D}$, we say that $\mathbf{H} \preceq (\succeq)\mathbf{D}$ if component-wise $h_{ij} \leq (\geq) d_{ij}$ for all i, j, where $\{h_{11}, ..., h_{mn}\}$'s are the components of the matrix $\mathbf{H}$ and $\{d_{11}, ..., d_{mn}\}$'s are $\mathbf{D}$'s components. We call a matrix $\mathbf{H}$ a positive matrix if $\mathbf{H} \succeq \mathbf{0}$. A square symmetric matrix $\mathbf{H}$ is called positive definite if all of its eigenvalues (which are real numbers) are strictly positive. The transpose of a matrix $\mathbf{H}$ is denoted by $\mathbf{H}'$. For two matrices $\mathbf{H}_1$ and $\mathbf{H}_2$, their Kronecker product is defined as follows:

$$\mathbf{H}_1 \otimes \mathbf{H}_2 = \begin{bmatrix} h_{11}\mathbf{H}_2 & \cdots & h_{1t}\mathbf{H}_2 \\ \vdots & \ddots & \vdots \\ h_{s1}\mathbf{H}_2 & \cdots & h_{st}\mathbf{H}_2 \end{bmatrix}.$$

Moreover, $\otimes$ is a bi-linear operator, and it satisfies the following properties. For any matrices $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$, and $\mathbf{H}_4$,

$$(\mathbf{H}_1 \otimes \mathbf{H}_2)(\mathbf{H}_3 \otimes \mathbf{H}_4) = (\mathbf{H}_1\mathbf{H}_3) \otimes (\mathbf{H}_2\mathbf{H}_4), \quad (\mathbf{H}_1 \otimes \mathbf{H}_2)^{-1} = \mathbf{H}_1^{-1} \otimes \mathbf{H}_2^{-1}, \quad (\mathbf{H}_1 \otimes \mathbf{H}_2)' = (\mathbf{H}_1' \otimes \mathbf{H}_2').$$

Let $\mathcal{M}_n$ denote all $n \times n$ square real matrices. Let $\rho(\mathbf{H})$ be the spectral radius of the $n \times n$ matrix $\mathbf{H}$, i.e., $\rho(\mathbf{H}) := \max\{|\lambda|, \lambda \text{ is an eigenvalue of } \mathbf{H}\}$. For any fixed vector $\mathbf{w} \in \mathbf{R}^n$, the L_2 vector norm on $\mathbf{R}^n$ is the defined as follows: $||\mathbf{w}||_2 := \sqrt{\sum |w_i|^2}$. The L_2 vector norm on $\mathbf{R}^n$

induces a norm on matrix $\mathcal{M}_n$ as follows:

$$|||\mathbf{H}|||_{2\to 2} := \max_{||\mathbf{w}||_2=1} ||\mathbf{H}\mathbf{w}||_2 = \max_{\mathbf{w}\neq \mathbf{0}} \frac{||\mathbf{H}\mathbf{w}||_2}{||\mathbf{w}||_2}.$$

It is straightforward to show that:

- For any $\mathbf{H} \in \mathcal{M}_n$, $\rho(\mathbf{H}) \leq |||\mathbf{H}|||_{2\to 2}$;
- Given $\mathbf{H}, \mathbf{Z}$ in $\mathcal{M}_n$, we have $|||\mathbf{H}\mathbf{Z}|||_{2\to 2} \leq |||\mathbf{H}|||_{2\to 2}|||\mathbf{Z}|||_{2\to 2}$.
- $|||\mathbf{H}|||_{2\to 2} = \sqrt{\rho(\mathbf{H}'\mathbf{H})}$. Moreover if $\mathbf{H}$ is symmetry, then $|||\mathbf{H}|||_{2\to 2} = \rho(\mathbf{H})$.

The proofs of these properties can be found in a standard matrix analysis textbook.

Lemma 2. *If both* $\mathbf{H}$ *and* $\mathbf{Z}$ *are both symmetric* $n \times n$ *matrices, then* $\rho(\mathbf{H}\mathbf{Z}) \leq \rho(\mathbf{H})\rho(\mathbf{Z})$.[21]

Proof of Lemma 2. $\rho(\mathbf{H}\mathbf{Z}) \leq |||\mathbf{H}\mathbf{Z}|||_{2\to 2} \leq |||\mathbf{H}|||_{2\to 2}|||\mathbf{Z}|||_{2\to 2} = \rho(\mathbf{H})\rho(\mathbf{Z})$. The first inequality is because the spectral radius of a matrix is always no greater than any matrix norm. The second inequality follows from the sub-multiplicativity of the matrix norm. The last step follows from the fact that $\mathbf{H}$ is symmetric, and hence $|||\mathbf{H}|||_{2\to 2} = \rho(\mathbf{H})$. Similarly, we obtain $|||\mathbf{Z}|||_{2\to 2} = \rho(\mathbf{Z})$. □

Next, we state the *block matrix inversion formula*, whose proof is standard and thus is omitted.

Lemma 3. *For matrices* $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3,$ *and* $\mathbf{H}_4$*,*

$$\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_3 & \mathbf{H}_4 \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{H}_1 - \mathbf{H}_2\mathbf{H}_4^{-1}\mathbf{H}_3)^{-1} & -\mathbf{H}_1^{-1}\mathbf{H}_2(\mathbf{H}_4 - \mathbf{H}_3\mathbf{H}_1^{-1}\mathbf{H}_2)^{-1} \\ -\mathbf{H}_4^{-1}\mathbf{H}_3(\mathbf{H}_1 - \mathbf{H}_2\mathbf{H}_4^{-1}\mathbf{H}_3)^{-1} & (\mathbf{H}_4 - \mathbf{H}_3\mathbf{H}_1^{-1}\mathbf{H}_2)^{-1} \end{bmatrix}.$$

The following result is frequently used for proofs of the uniqueness and existence of Nash Equilibrium in various scenario, for convenience we present it as a Lemma.

Lemma 4. *Suppose* $\mathbf{z} \in \mathbf{R}^n$ *is a vector and* $\mathbf{Q} \in \mathcal{M}_n$ *is a square matrix. If the spectral radius of* $\mathbf{Q}$ *is less than 1, i.e.,* $\rho(\mathbf{Q}) < 1$*, the following mapping:*

$$\Xi(\mathbf{z}) = \mathbf{z}_0 + \mathbf{Q}\mathbf{z}$$

has a unique fixed point $\mathbf{z}^* = (\mathbf{I}_n - \mathbf{Q})^{-1}\mathbf{z}_0 = \ \mathbf{z}_0 + \mathbf{Q}\ \mathbf{z}_0 + \mathbf{Q}^2\ \mathbf{z}_0 + \cdots.$

[21] This inequality does not hold in general if we remove the symmetric matrix assumption. To see this, suppose $\mathbf{H} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{Z} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. Clearly $\rho(\mathbf{H}) = 0, \rho(\mathbf{Z}) = 1$. However, $\mathbf{H}\mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ and $\rho(\mathbf{H}\mathbf{Z}) = 3 > 0 = \rho(\mathbf{H})\rho(\mathbf{Z})$.

Proof of Lemma 4. Since $\rho(\mathbf{Q}) < 1$, all the eigenvalues of $\mathbf{Q}$ (which could be complex numbers) lie in the interior of the unit disk. Therefore, $Det(\mathbf{I}_n - \mathbf{Q}) \neq 0$, and so $(\mathbf{I}_n - \mathbf{Q})$ is invertible. Moreover, the following holds

$$(\mathbf{I}_n - \mathbf{Q})^{-1} = \mathbf{I}_n + \mathbf{Q} + \mathbf{Q}^2 + \cdots .$$

The infinite series on the right hand side converges as we have assumed $\rho(\mathbf{Q}) < 1$.

Suppose $\mathbf{z}^*$ is a fixed point of Ξ. Then $\mathbf{z}^* = \Xi(\mathbf{z})^* = \mathbf{z}_0 + \mathbf{Q}\mathbf{z}^*$, and equivalently $(\mathbf{I}_n - \mathbf{Q})\mathbf{z}^* = \mathbf{z}_0$. Therefore,

$$\mathbf{z}^* = (\mathbf{I}_n - \mathbf{Q})^{-1}\mathbf{z}_0 = \ \mathbf{z}_0 + \mathbf{Q}\ \mathbf{z}_0 + \mathbf{Q}^2\ \mathbf{z}_0 + \cdots .$$

The uniqueness also follows by the construction of of the $\mathbf{z}^*$. $\square$

A.2 Katz-Bonacich centrality

Let us define the Katz-Bonacich centrality.

Definition 1. *Assume $0 \leq \delta < 1/\lambda_1(\mathbf{G})$. Then, for any vector $\mathbf{a} = (a_1, \cdots, a_n)' \in \mathbf{R}^n$, the Katz-Bonacich centrality vector with weight $\mathbf{a}$ is defined as:*

$$\mathbf{b}(\mathbf{G}, \delta, \mathbf{a}) := \mathbf{M}(\mathbf{G}, \delta)\mathbf{a}, \tag{11}$$

where

$$\mathbf{M}(\mathbf{G}, \delta) = [\mathbf{I} - \delta\mathbf{G}]^{-1} = \mathbf{I} + \sum_{k \geq 1} \delta^k \mathbf{G}^k. \tag{12}$$

Let $b_i(\mathbf{G}, \delta, \mathbf{a})$ be the ith entry of $\mathbf{b}(\mathbf{G}, \delta, \mathbf{a})$. Let $m_{ij}(\mathbf{G}, \delta)$ be the ij entry of $\mathbf{M}(\mathbf{G}, \delta)$. Then,

$$b_i(\mathbf{G}, \delta, \mathbf{a}) = \sum_j m_{ij}(\mathbf{G}, \delta) a_j.$$

B Conditions under which Nash equilibrium efforts are positive

In Theorem 1, we show that

$$\mathbf{x}^A = \frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\mathbf{a}^A + \frac{\mathbf{M}^+ - \mathbf{M}^-}{2}\mathbf{a}^B \text{ and } \mathbf{x}^B = \frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\mathbf{a}^B + \frac{\mathbf{M}^+ - \mathbf{M}^-}{2}\mathbf{a}^A. \tag{13}$$

Clearly, Assumption 1 guarantees that both $\mathbf{M}^+$ and $\mathbf{M}^-$ are well-defined. Moreover, they are both positive matrices. When $\beta \leq 0$ (activities are complements, or independent), we have $\mathbf{M}^+ - \mathbf{M}^- \succeq$

$\mathbf{0}$. As a result, this automatically shows that $\mathbf{x}^A \succeq \mathbf{0}$ and $\mathbf{x}^B \succeq \mathbf{0}$.[22]

Let us now focus on the substitute case ($\beta > 0$). We need further conditions on the parameters to guarantee the nonnegativity of effort levels. Intuitively, from equation (13), we observe that when β is close to zero, $\frac{\mathbf{M}^+ - \mathbf{M}^-}{2}$ is close to the zero matrix. Hence, the second term in $\mathbf{x}^A$, $\frac{\mathbf{M}^+ - \mathbf{M}^-}{2}\mathbf{a}^B$, is dominated by the first term, $\frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\mathbf{a}^A$. Therefore, the summation of these two terms, $\mathbf{x}^A$, is nonnegative. More precisely, we rewrite $\mathbf{x}^A$ in equation (13) as follows:

$$\begin{aligned}
\mathbf{x}^A &= \frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\mathbf{a}^A + \frac{\mathbf{M}^+ - \mathbf{M}^-}{2}\mathbf{a}^B \\
&= \frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\left\{\mathbf{a}^A + \left(\frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\right)^{-1}\frac{\mathbf{M}^+ - \mathbf{M}^-}{2}\mathbf{a}^B\right\} \\
&= \frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\left\{\mathbf{a}^A - \beta[\mathbf{I} - \delta\mathbf{G}]^{-1}\mathbf{a}^B\right\},
\end{aligned}$$

where the second equality follows from the identity:

$$\left(\frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\right)^{-1}\left(\frac{\mathbf{M}^+ - \mathbf{M}^-}{2}\right) = -\beta[\mathbf{I} - \delta\mathbf{G}]^{-1}.$$

Given the positiveness of the matrix $\left(\frac{\mathbf{M}^+ + \mathbf{M}^-}{2}\right)$, $\mathbf{x}^A \succeq \mathbf{0}$ as long as $\mathbf{a}^A \succeq \beta[\mathbf{I} - \delta\mathbf{G}]^{-1}\mathbf{a}^B = \beta\, \mathbf{b}(\mathbf{G}, \delta, \mathbf{a}^B)$. The latter is true when $\beta \le \min_i(a_i^A / b_i(\mathbf{G}, \delta, \mathbf{a}^B))$. By the same logic, $\mathbf{x}^B \succeq \mathbf{0}$ when $\beta \le \min_i(a_i^B / b_i(\mathbf{G}, \delta, \mathbf{a}^A))$. In total, when

$$\beta \le \min\left\{\min_i\left(\frac{a_i^A}{b_i(\mathbf{G}, \delta, \mathbf{a}^B)}\right), \min_i\left(\frac{a_i^B}{b_i(\mathbf{G}, \delta, \mathbf{a}^A)}\right)\right\},$$

then $\mathbf{x}^A \succeq \mathbf{0}$ and $\mathbf{x}^B \succeq \mathbf{0}$.

Alternatively, we could impose condition on the heterogeneity between $\mathbf{a}^A$ and $\mathbf{a}^B$. Intuitively, when $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, we have $\mathbf{x}^A = \mathbf{x}^B = \mathbf{M}^+\mathbf{a} \succ \mathbf{0}$. By continuity, when the difference between $\mathbf{a}^A$ and $\mathbf{a}^B$ is small, both $\mathbf{x}^A$ and $\mathbf{x}^B$ are still nonnegative. More precisely, we rewrite $\mathbf{x}^A$ in equation

[22] As the game is supermodular when $\beta < 0$ (see discussion in section 4.2), we could also obtain the non-negativity of efforts from the perspective of the *Best Response Tatonnement*, which converges monotonically to the unique Nash Equilibrium, starting from any initial strategy profile.

(13) as follows:

$$\begin{aligned}
\mathbf{x}^A &= \mathbf{M}^+\frac{\mathbf{a}^A+\mathbf{a}^B}{2}+\mathbf{M}^-\frac{\mathbf{a}^A-\mathbf{a}^B}{2}\\
&= \mathbf{M}^+\left\{\frac{\mathbf{a}^A+\mathbf{a}^B}{2}+(\mathbf{M}^+)^{-1}\mathbf{M}^-\frac{\mathbf{a}^A-\mathbf{a}^B}{2}\right\}\\
&= \mathbf{M}^+\left\{\frac{\mathbf{a}^A+\mathbf{a}^B}{2}+(\mathbf{I}+2\beta\mathbf{M}^-)\frac{\mathbf{a}^A-\mathbf{a}^B}{2}\right\}\\
&= \mathbf{M}^+\left\{\mathbf{a}^A+\beta\mathbf{M}^{-1}(\mathbf{a}^A-\mathbf{a}^B)\right\},
\end{aligned}$$

where the second step follows from the identity: $(\mathbf{M}^+)^{-1}\mathbf{M}^- = \mathbf{I}+2\beta\mathbf{M}^-$. Since $\mathbf{M}^+$ is positive, $\mathbf{x}^A \succeq \mathbf{0}$ as long as

$$\mathbf{a}^A \succeq -\beta\mathbf{M}^{-1}(\mathbf{a}^A-\mathbf{a}^B) = \frac{\beta}{1-\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{1-\beta}, (\mathbf{a}^B-\mathbf{a}^A)\right).$$

By the same logic, $\mathbf{x}^B \succeq \mathbf{0}$ when $\mathbf{a}^B \succeq \frac{\beta}{1-\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{1-\beta}, (\mathbf{a}^A-\mathbf{a}^B)\right)$. In total, when

$$\mathbf{a}^A \succeq \frac{\beta}{1-\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{1-\beta}, (\mathbf{a}^B-\mathbf{a}^A)\right) \succeq -\mathbf{a}^B,$$

we have $\mathbf{x}^A \succeq \mathbf{0}$ and $\mathbf{x}^B \succeq \mathbf{0}$.

The following proposition summarizes our results:

Proposition 5. *Suppose that Assumption 1 holds.*

(i) When $\beta \le 0$ (activities are complement or independent), then $\mathbf{x}^A \succeq \mathbf{0}$ and $\mathbf{x}^B \succeq \mathbf{0}$.

(ii) When $\beta > 0$ (activities are substitutes), if either

$$\beta \le \min\left\{\min_i\left(\frac{a_i^A}{b_i(\mathbf{G},\delta,\mathbf{a}^B)}\right), \min_i\left(\frac{a_i^B}{b_i(\mathbf{G},\delta,\mathbf{a}^A)}\right)\right\} \tag{14}$$

or

$$\mathbf{a}^A \succeq \frac{\beta}{1-\beta}\mathbf{b}\left(\mathbf{G}, \frac{\delta}{1-\beta}, (\mathbf{a}^B-\mathbf{a}^A)\right) \succeq -\mathbf{a}^B \tag{15}$$

holds, then $\mathbf{x}^A \succeq \mathbf{0}$ and $\mathbf{x}^B \succeq \mathbf{0}$.

As discussed above, observe that condition (14) imposes some restriction on the degree of substitution between two activities, while condition (15) limits the heterogeneity between two activities. Observe also that these two conditions are only sufficient but not necessary for equilibrium efforts to be nonnegative.

C Some comparative-statics results on the matrices $\mathbf{M}^+$ and $\mathbf{M}^-$

Before deriving the general comparative-statics results in Section 4, we need to obtain some results on the matrices $\mathbf{M}^+$ and $\mathbf{M}^-$, which are more precisely defined in equation (23) in Appendix G that we report here for the ease of exposition:

$$\mathbf{M}^+(\mathbf{G}, \delta, \beta) = [(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \frac{1}{1+\beta}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{(1+\beta)}\right) = \sum_{k=0}^{\infty} \frac{(\delta\mathbf{G})^k}{(1+\beta)^{1+k}}.$$

$$\mathbf{M}^-(\mathbf{G}, \delta, \beta) = [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \frac{1}{1-\beta}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{(1-\beta)}\right) = \sum_{k=0}^{\infty} \frac{(\delta\mathbf{G})^k}{(1-\beta)^{1+k}}.$$

First, let us compare these two matrices with $\mathbf{M}(\mathbf{G}, \delta)$, which is defined in (12), i.e.

$$\mathbf{M}(\mathbf{G}, \delta) = [\mathbf{I} - \delta\mathbf{G}]^{-1} = \mathbf{I} + \sum_{k\geq 1} \delta^k \mathbf{G}^k.$$

Proposition 6. *Suppose that Assumption 1 holds.*

- *When $\beta \geq 0$, i.e. the activities A and B are substitutes,* $\mathbf{M}^-(\mathbf{G}, \delta, \beta) \succeq \mathbf{M}(\mathbf{G}, \delta) \succeq \mathbf{M}^+(\mathbf{G}, \delta, \beta)$;
- *When $\beta \leq 0$, i.e. the activities A and B are complements,* $\mathbf{M}^-(\mathbf{G}, \delta, \beta) \preceq \mathbf{M}(\mathbf{G}, \delta) \preceq \mathbf{M}^+(\mathbf{G}, \delta, \beta)$.
- *When $\beta = 0$, i.e. the activities A and B are independent,* $\mathbf{M}^-(\mathbf{G}, \delta, 0) = \mathbf{M}^+(\mathbf{G}, \delta, 0) = \mathbf{M}(\mathbf{G}, \delta) = \sum_{k=0}^{\infty} \delta^k \mathbf{G}^k$.

The results in Proposition 6 follow directly from the infinite series formula defined in each of the matrices above. For example, when the two activities are substitutes, i.e. $\beta \geq 0$, then $1 - \beta \leq 1 \leq 1 + \beta$ and we obtain the first result. Similar logic applies to other cases when $\beta < 0$ (complementary activities) and $\beta = 0$ (independent activities).

Proposition 7. *Suppose that Assumption 1 holds. Then,* $\mathbf{M}^+$ *is increasing in δ and decreasing in β while* $\mathbf{M}^-$ *is increasing in δ and increasing in β. Mathematically,*

$$\frac{\partial \mathbf{M}^+(\mathbf{G}, \delta, \beta)}{\partial \delta} = \mathbf{G}[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2} \succeq \mathbf{0}, \quad \frac{\partial \mathbf{M}^+(\mathbf{G}, \delta, \beta)}{\partial \beta} = -[(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2} \preceq \mathbf{0},$$

$$\frac{\partial \mathbf{M}^-(\mathbf{G}, \delta, \beta)}{\partial \delta} = \mathbf{G}[(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2} \succeq \mathbf{0}, \quad \frac{\partial \mathbf{M}^-(\mathbf{G}, \delta, \beta)}{\partial \beta} = [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-2} \succeq \mathbf{0}.$$

Next, we examine the effect of increased *network density*. We say $\mathbf{G}' \succeq \mathbf{G}$ if there are more links in $\mathbf{G}'$ than in $\mathbf{G}$.

Proposition 8. *Suppose that Assumption 1 holds for both* $\mathbf{G}$ *and* $\mathbf{G}'$ *and that* $\mathbf{G}' \succeq \mathbf{G}$*, then* $\mathbf{M}^+(\mathbf{G}', \delta, \beta) \succeq \mathbf{M}^+(\mathbf{G}, \delta, \beta)$ *and* $\mathbf{M}^-(\mathbf{G}', \delta, \beta) \succeq \mathbf{M}^-(\mathbf{G}, \delta, \beta)$*.*

As the network grows (in terms of links), the network effects become stronger and this explains the results of Proposition 8.

D Equilibrium utility

Using the equilibrium efforts given in Theorem 1, let us derive the equilibrium payoffs.

Proposition 9. *Suppose that Assumption 1 holds. Then, the equilibrium payoff for player* i *is given by:*

$$\begin{aligned} u_i^* &= \frac{1}{2}(x_i^{A*})^2 + \frac{1}{2}(x_i^{B*})^2 + \beta x_i^{A*} x_i^{B*} \\ &= \frac{1}{1+\beta}\left[b_i\left(\mathbf{G}, \frac{\delta}{1+\beta}, \frac{\mathbf{a}^A + \mathbf{a}^B}{2}\right)\right]^2 + \frac{1}{1-\beta}\left[b_i\left(\mathbf{G}, \frac{\delta}{1-\beta}, \frac{\mathbf{a}^A - \mathbf{a}^B}{2}\right)\right]^2. \end{aligned}$$

Proof of Proposition 9. In equilibrium, player i chooses $\mathbf{x}_i$ to maximize

$$\max_{\mathbf{x}_i} \begin{bmatrix} x_i^A & x_i^B \end{bmatrix} \begin{bmatrix} a_i^A + \delta \sum_{j=1}^n g_{ij} x_j^A \\ a_i^B + \delta \sum_{j=1}^n g_{ij} x_j^B \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x_i^A & x_i^B \end{bmatrix} \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} x_i^A \\ x_i^B \end{bmatrix}.$$

Plugging the first-order conditions (20) into the above equation yields:

$$\begin{aligned} u_i^* &= \begin{bmatrix} x_i^{A*} & x_i^{B*} \end{bmatrix} \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} x_i^{A*} \\ x_i^{B*} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x_i^{A*} & x_i^{B*} \end{bmatrix} \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} x_i^{A*} \\ x_i^{B*} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} x_i^{A*} & x_i^{B*} \end{bmatrix} \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} x_i^{A*} \\ x_i^{B*} \end{bmatrix} \\ &= \frac{1}{2}(x_i^{A*})^2 + \frac{1}{2}(x_i^{B*})^2 + \beta x_i^{A*} x_i^{B*}, \end{aligned}$$

where x_i^{A*} and x_i^{B*} are the equilibrium activities of player i.

Using the characterizations in Theorem 1 and Corollary 1

$$\begin{aligned} x_i^{A*} &= \frac{1}{1+\beta} b_i(\mathbf{G}, \frac{\delta}{1+\beta}, \frac{\mathbf{a}^A + \mathbf{a}^B}{2}) + \frac{1}{1-\beta} b_i(\mathbf{G}, \frac{\delta}{1-\beta}, \frac{\mathbf{a}^A - \mathbf{a}^B}{2}), \\ x_i^{B*} &= \frac{1}{1+\beta} b_i(\mathbf{G}, \frac{\delta}{1+\beta}, \frac{\mathbf{a}^A + \mathbf{a}^B}{2}) - \frac{1}{1-\beta} b_i(\mathbf{G}, \frac{\delta}{1-\beta}, \frac{\mathbf{a}^A - \mathbf{a}^B}{2}), \end{aligned}$$

we can further simplify the expression:

$$\begin{aligned} u_i^* &= \frac{1}{2}(x_i^{A*})^2 + \frac{1}{2}(x_i^{B*})^2 + \beta x_i^{A*} x_i^{B*} \\ &= (1+\beta)\left(\frac{x_i^{A*} + x_i^{B*}}{2}\right)^2 + (1-\beta)\left(\frac{x_i^{A*} - x_i^{B*}}{2}\right)^2 \\ &= \frac{1}{1+\beta} b_i^2(\mathbf{G}, \frac{\delta}{1+\beta}, \frac{\mathbf{a}^A + \mathbf{a}^B}{2}) + \frac{1}{1-\beta} b_i^2(\mathbf{G}, \frac{\delta}{1-\beta}, \frac{\mathbf{a}^A - \mathbf{a}^B}{2}). \end{aligned}$$

This completes the proof. □

We see that, contrary to the single activity case (Corollary 4), the cross-activity term $\beta x_i^A x_i^B$ in the cost function directly affects the trade-off between the two activities. Note that the two activities can be either substitutes or complements, and the interdependence between activities creates a driving force beyond the network effects generated by other players' activities. In equilibrium, a player's payoff incorporates both of these factors and can be therefore expressed as a sum of the squared of two Katz-Bonacich centralities.

E Some examples

Let us illustrate our main results with some specific networks.

E.1 Regular networks

We first consider the family of *regular* graphs, which are networks for which players have the same number of links. We say that $\mathbf{G}$ is regular with degree d, if each node has exactly d neighbors, i.e., $\mathbf{G}\mathbf{1}_n = d\mathbf{1}_n$. Figure 2 provides an example of a regular graph with degree 2, which is symmetric. It is a circle network and is denoted by $\mathbf{O}_4$.

For a regular network of degree d, when $\mathbf{a}^A = \mathbf{a}^B = a\mathbf{1}_n$, it is easily verified that the unique Nash equilibrium is given by:

$$\mathbf{x}^A = \mathbf{x}^B = \frac{a}{1+\beta-d\delta}\mathbf{1}_n. \tag{16}$$

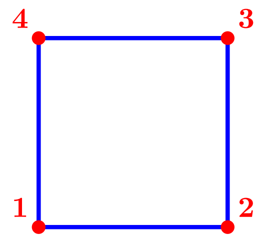

Figure 2: A regular graph (circle) of degree 2

Assume that $\mathbf{a}^A = a^A \mathbf{1}_n$, $\mathbf{a}^B = a^B \mathbf{1}_n$, which means that everybody in the same activity has the same ex ante productivity but productivities differ between activities. For a regular network of degree d, it is easily verified that the equilibrium outcomes are given by:

$$\mathbf{x}^A = \frac{1}{2}\left\{\frac{a^A + a^B}{1+\beta-d\delta}\mathbf{1}_n + \frac{a^A - a^B}{1-\beta-d\delta}\mathbf{1}_n\right\},$$

$$\mathbf{x}^B = \frac{1}{2}\left\{\frac{a^A + a^B}{1+\beta-d\delta}\mathbf{1}_n - \frac{a^A - a^B}{1-\beta-d\delta}\mathbf{1}_n\right\}.$$

We see here clearly how the equilibrium efforts exerted by the individuals strongly depend on the sum and average of initial productivities a^A and a^B, on the network spillover intensity δ and on the network structure, captured here by the degree d of each individual. For example, we see the ambiguous effect of a denser network (captured here by a higher degree d) or more spillover intensity (higher δ) in equilibrium efforts. If $a^A > a^B$, then a denser network or more spillover intensity increases x^A but has (generally) an ambiguous effect on x^B.

Let us now illustrate Propositions 2 and 3, which showed that the impact of δ and β on the equilibrium efforts were ambiguous.

Impact of network spillovers and degree of interdependence between activities.

Start with homogeneous players so that $a^A = a^b = 3$. When the parameters are set to: $\delta = 0.2, \beta = 0.2$ for the circle network of Figure 2 ($d = 2$), we have $x_i^A = x_i^B = 3.75$, When we increase δ from 0.20 to 0.21, both activities increases from 3.75 to 3.85. Similarly if we increase β from 0.20 to 0.25, both activities decrease from 3.75 to 3.53. Indeed, when $\delta > 0$ and the two activities are substitutes ($\beta = 0.2 > 0$), increasing δ always increases the effort in both activities because network effects reinforce each other between the two activities. The effect of β is less obvious. When β increases, the cost of substituting the two activities increases and efforts in both activities decrease.

Consider now some inter-group heterogeneity so that $a^A = 4, a^B = 2$. Using the same parameter values for the circle network ($\delta = 0.2$, $\beta = 0.2$, $d = 2$), we obtain: $x_i^A = 6.25 > x_i^B = 1.25$. This is because activity A has a higher marginal utility than activity B (productivity advantage). When δ increases from 0.20 to 0.21, effort in activity A increases from 6.25 to 6.48, but effort in activity B decreases from 1.25 to 1.21. This is because the two activities are substitutes (crime and education). So, if individuals have higher returns from crime than from education, they will exert more crime effort than education effort. Because of higher spillover effects (δ increases), their friends will also increase their criminal activities, which will feed back to the initial individuals. As a result, criminal activities will increase and education activities will be reduced. When we increase β from 0.20 to 0.25, activity B drops from 1.25 to 0.57, but activity A increases from 6.25 to 6.39. Indeed, when β increases, crime and education become more substitutable and thus, because of the initial productivity advantage in crime, crime effort increases, which reduces even more education effort. We summarize the results in the following Table 2.

$(a^a, a^b) \setminus (\delta, \beta)$	(0.2, 0.2)	(0.21, 0.2)	(0.2, 0.25)
$(3, 3)$	(3.75, 3.75)	(3.85, 3.85)	(3.53, 3.53)
$(4, 2)$	(6.25, 1.25)	(6.48, 1.21)	(6.39, 0.57)

Table 2: Values of (x_i^A, x_i^B) for different parameter values in the circle network of Figure 2.

Impact of network intensity. We can also compare the circle network $\mathbf{O}_4$ in Figure 2 and the corresponding *complete network* $\mathbf{K}_4$ where we add links 13 and 24 to network $\mathbf{O}_4$. Clearly, $\mathbf{K}_4$ is *denser* than $\mathbf{O}_4$. As above, we set $\delta = 0.2, \beta = 0.2$.

Assume ex ante homogeneity $a^A = a^B = 4$. Given these parameter choices, the equilibrium efforts in activities A and B are equal to $(5, 5)$ for every node (player) in the circle network $\mathbf{O}_4$ and $(6.67, 6.67)$ for every node in the complete network $\mathbf{K}_4$. We observe that both activities increase in the denser network under the symmetry assumption.

Suppose instead that $a^A = 5, a^B = 3$. The equilibrium efforts are now given by $(7.50, 2.50)$ for the circle network $\mathbf{O}_4$ and by $(11.67, 1.67)$ for the complete network $\mathbf{K}_4$. The difference between two activities in the denser network $\mathbf{K}_4$ is larger than in the circle network because, compared to the symmetric case, activity A goes up and activity B goes down as $1.67 < 2.5$. The fact that activity B is lower in a denser network is a novel feature that arises only in games with multiple activities. By contrast, in the single-activity framework, the equilibrium activity always increases when network $\mathbf{G}$ becomes denser (Ballester et al. (2006)). The interdependence between multiple activities is the main driving force behind this type of non-monotonicity results. When $\beta = 0$, it never happens. All of these results are consistent with Proposition 4, and are summarized in Table 3.

network	$\mathbf{O}_4$		$\mathbf{K}_4$	
(a^A, a^b)	x_i^A	x_i^B	x_i^A	x_i^B
$(4,4)$	5.00	5.00	6.67	6.67
$(5,3)$	7.50	2.50	11.67	1.67

Table 3: Equilibrium activities for two different networks.

E.2 Bipartite networks

Consider now bipartite graphs (denoted by $\mathbf{K}_{pq}$). In a bipartite graph $\mathbf{K}_{pq}$, there are two disjoint groups P and Q such that any node in P is connected to any node in Q. Let $p = |P|, q = |Q|$. Thus, the network size is equal to: $n = p + q$. Figure 3 provides an example of a bipartite networks for the cases of $p = 2$ and $q = 3$.

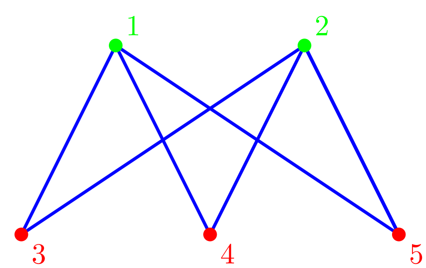

Figure 3: A bipartite graph for $\mathbf{K}_{2,3}$.

It can be shown that, when $\mathbf{a}^A = \mathbf{a}^B = \mathbf{a}$, the unique Nash equilibrium is given by:

$$\mathbf{x}^A = \mathbf{x}^B = \begin{bmatrix} \frac{1}{1+\beta}[\mathbf{I}_p + \frac{\delta^2 q}{(1+\beta)^2 - \delta^2 qp}\mathbf{J}_{pp}] & \frac{\delta}{(1+\beta)^2-\delta^2 pq}\mathbf{J}_{pq} \\ \frac{\delta}{(1+\beta)^2-\delta^2 pq}\mathbf{J}_{qp} & \frac{1}{1+\beta}[\mathbf{I}_q + \frac{\delta^2 p}{(1+\beta)^2-\delta^2 qp}\mathbf{J}_{qq}] \end{bmatrix} \mathbf{a}.$$

where $\mathbf{J}_{pq}$ is the $p \times q$ matrix of 1s. Furthermore, if we assume that $a_i^A = a_i^B = a$ for all i, we obtain:

$$\mathbf{x}^A = \mathbf{x}^B = \begin{bmatrix} \frac{a(1+\beta+\delta q)}{(1+\beta)^2 - pq\delta^2}\mathbf{1}_p \\ \frac{a(1+\beta+\delta p)}{(1+\beta)^2-pq\delta^2}\mathbf{1}_q \end{bmatrix} = \begin{bmatrix} x^p \mathbf{1}_p \\ x^q \mathbf{1}_q \end{bmatrix},$$

where

$$x^p = \frac{a(1+\beta+\delta q)}{(1+\beta)^2 - pq\delta^2}, \quad \text{and} \quad x^q = \frac{a(1+\beta+\delta p)}{(1+\beta)^2 - pq\delta^2}.$$

Here, x^p and x^q are the activity levels of the players in groups P and Q, respectively. It is easy to see that x^p, x^q are monotonically increasing in p, q, δ, and are decreasing in β. Moreover, we find that

$$x^p > x^q \text{ if and only if } q > p.$$

That is, the players in the smaller group put more effort than those in the larger group when players are homogeneous ex ante.

F The hybrid model with multiple activities

Liu et al. (2014) have developed the *hybrid model*, which incorporates both the local-aggregate model of Section 2 and the local-average model of Section 6.5. Let us extend the hybrid model when agents exert two activities. The utility function can now be written as:

$$\begin{aligned} u_i(\mathbf{x}_i, \mathbf{x}_{-i}) &= a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B \right\} \\ &\quad + \delta \sum_{j=1}^{n} g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^{n} g_{ij} x_i^B x_j^B - \frac{\gamma}{2} \left(x_i^A - \sum_{j=1}^{n} g_{ij}^* x_j^A \right)^2 - \frac{\gamma}{2} \left(x_i^B - \sum_{j=1}^{n} g_{ij}^* x_j^B \right)^2 \end{aligned} \tag{17}$$

Assumption 7. $1 - |\beta| + \gamma - \lambda_1(\delta \mathbf{G} + \gamma \mathbf{G}^*) > 0$.

Denote $\mathbf{M}^{++} := [(1+\gamma+\beta)\,\mathbf{I}_n - \delta\mathbf{G} - \gamma\mathbf{G}^*]^{-1}$ and $\mathbf{M}^{--} := [(1+\gamma+\beta)\,\mathbf{I}_n - \delta\mathbf{G} - \gamma\mathbf{G}^*]^{-1}$.

Theorem 8. *Suppose that Assumption 7 holds and the utility function of each player i is given by (17). Then, for any $\mathbf{a}^A$ and $\mathbf{a}^B$, there exists a unique Nash equilibrium given by: $\overline{\mathbf{x}} = \mathbf{M}^{++}\overline{\mathbf{a}}$ and $\widehat{\mathbf{x}} = \mathbf{M}^{--}\widehat{\mathbf{a}}$ or, equivalently, by:*

$$\begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{++}\overline{\mathbf{a}} + \mathbf{M}^{--}\widehat{\mathbf{a}} \\ \mathbf{M}^{++}\overline{\mathbf{a}} - \mathbf{M}^{--}\widehat{\mathbf{a}} \end{bmatrix}$$

Proof of Theorem 8. The first-order conditions are given by:

$$\begin{aligned} (1+\gamma)\, x_i^A &= a_i^A - \beta x_i^B + \delta \sum_{j=1}^{n} g_{ij} x_j^A + \gamma \sum_{j=1}^{n} g_{ij}^* x_j^A \\ (1+\gamma)\, x_i^B &= a_i^B - \beta x_i^A + \delta \sum_{j=1}^{n} g_{ij} x_j^B + \gamma \sum_{j=1}^{n} g_{ij}^* x_j^B \end{aligned}$$

If we sum these two equations, we obtain:

$$(1+\gamma+\beta)\left(x_i^A+x_i^B\right)=a_i^A+a_i^B+\delta\sum_{j=1}^{n} g_{ij}\left(x_j^A+x_j^B\right)+\gamma\sum_{j=1}^{n} g_{ij}^*\left(x_j^A+x_j^B\right)$$

or, in matrix form,

$$\overline{\mathbf{x}}=\mathbf{M}^{++}\overline{\mathbf{a}} \tag{18}$$

where $\overline{\mathbf{x}}:=\left(\mathbf{x}^A+\mathbf{x}^B\right)/2$, $\overline{\mathbf{a}}:=\left(\mathbf{a}^A+\mathbf{a}^B\right)/2$ and $\mathbf{M}^{++}:=\left[(1+\gamma+\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*\right]^{-1}$. If we now subtract these two equations, we obtain:

$$(1+\gamma-\beta)\left(x_i^A-x_i^B\right)=a_i^A-a_i^B+\delta\sum_{j=1}^{n} g_{ij}\left(x_j^A-x_j^B\right)+\gamma\sum_{j=1}^{n} g_{ij}^*\left(x_j^A-x_j^B\right)$$

or, in matrix form,

$$\widehat{\mathbf{x}}=\mathbf{M}^{--}\widehat{\mathbf{a}} \tag{19}$$

where $\widehat{\mathbf{x}}:=\left(\mathbf{x}^A-\mathbf{x}^B\right)/2$, $\widehat{\mathbf{a}}:=\left(\mathbf{a}^A-\mathbf{a}^B\right)/2$ and $\mathbf{M}^{--}:=\left[(1+\gamma-\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*\right]^{-1}$.

Thus, using (18) and (19), we obtain:

$$\begin{bmatrix}\mathbf{x}^A\\ \mathbf{x}^B\end{bmatrix}=\begin{bmatrix}\mathbf{M}^{++}\overline{\mathbf{a}}+\mathbf{M}^{--}\widehat{\mathbf{a}}\\ \mathbf{M}^{++}\overline{\mathbf{a}}-\mathbf{M}^{--}\widehat{\mathbf{a}}\end{bmatrix}$$

which is equivalent to

$$\mathbf{x}^A=\left[(1+\gamma+\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*\right]^{-1}\frac{\left(\mathbf{a}^A+\mathbf{a}^B\right)}{2}+\left[(1+\gamma-\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*\right]^{-1}\frac{\left(\mathbf{a}^A-\mathbf{a}^B\right)}{2}$$

$$\mathbf{x}^B=\left[(1+\gamma+\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*\right]^{-1}\frac{\left(\mathbf{a}^A+\mathbf{a}^B\right)}{2}-\left[(1+\gamma-\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*\right]^{-1}\frac{\left(\mathbf{a}^A-\mathbf{a}^B\right)}{2}$$

We need to show that $(1+\gamma+\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*$ and $(1+\gamma-\beta)\,\mathbf{I}_n-\delta\mathbf{G}-\gamma\mathbf{G}^*$ are invertible. A condition for this to be true is Assumption 7. □

G Proofs

Proof of Theorem 1: The first-order conditions are given by:

$$\begin{bmatrix}1 & \beta\\ \beta & 1\end{bmatrix}\begin{bmatrix}x_i^A\\ x_i^B\end{bmatrix}=\begin{bmatrix}a_i^A+\delta\sum_{j=1}^{n} g_{ij}x_j^A\\ a_i^B+\delta\sum_{j=1}^{n} g_{ij}x_j^B\end{bmatrix}. \tag{20}$$

Taking the sum of first-order conditions in (20) yields:

$$(1+\beta)\frac{x_i^A + x_i^B}{2} = \frac{(a_i^A + a_i^B)}{2} + \delta \sum_{j=1}^{n} g_{ij} \frac{x_j^A + x_j^B}{2},$$

which can be expressed in matrix form:

$$(1+\beta)\frac{\mathbf{x}^A + \mathbf{x}^B}{2} = \frac{(\mathbf{a}^A + \mathbf{a}^B)}{2} + \delta \mathbf{G} \frac{\mathbf{x}^A + \mathbf{x}^B}{2}.$$

Therefore, we obtain

$$\frac{\mathbf{x}^A + \mathbf{x}^B}{2} = [(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} \frac{(\mathbf{a}^A + \mathbf{a}^B)}{2}. \tag{21}$$

On the other hand, taking the difference in (20) yields:

$$\begin{aligned} & (1-\beta)\frac{x_i^A - x_i^B}{2} = \frac{(a_i^A - a_i^B)}{2} + \delta \sum_{j=1}^{n} g_{ij} \frac{x_j^A - x_j^B}{2} \\ \Longrightarrow \qquad & (1-\beta)\frac{\mathbf{x}^A - \mathbf{x}^B}{2} = \frac{(\mathbf{a}^A - \mathbf{a}^B)}{2} + \delta \mathbf{G} \frac{\mathbf{x}^A - \mathbf{x}^B}{2}. \end{aligned}$$

Therefore,

$$\frac{\mathbf{x}^A - \mathbf{x}^B}{2} = [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} \frac{(\mathbf{a}^A - \mathbf{a}^B)}{2}. \tag{22}$$

Combing equations (21) and (22), we obtain the theorem. $\square$

Proof of Corollary 1. We have

$$\begin{cases} \mathbf{M}^{+}(\mathbf{G}, \delta, \beta) = [(1+\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \frac{1}{1+\beta}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{(1+\beta)}\right) = \sum_{k=0}^{\infty} \frac{(\delta\mathbf{G})^k}{(1+\beta)^{1+k}}. \\ \mathbf{M}^{-}(\mathbf{G}, \delta, \beta) = [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \frac{1}{1-\beta}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{(1-\beta)}\right) = \sum_{k=0}^{\infty} \frac{(\delta\mathbf{G})^k}{(1-\beta)^{1+k}}. \end{cases} \tag{23}$$

Both series expansions in (23) converge as $\frac{\delta\lambda_1(\mathbf{G})}{1\pm\beta} < 1$ by Assumption 1. As a result, we can rewrite the equilibrium equations of Theorem 1 as follows:

$$\begin{cases} \mathbf{x}^A = \frac{1}{1+\beta}\mathbf{b}(\mathbf{G}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A+\mathbf{a}^B)}{2}) + \frac{1}{1-\beta}\mathbf{b}(\mathbf{G}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A-\mathbf{a}^B)}{2}) \\ \mathbf{x}^B = \frac{1}{1+\beta}\mathbf{b}(\mathbf{G}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A+\mathbf{a}^B)}{2}) - \frac{1}{1-\beta}\mathbf{b}(\mathbf{G}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A-\mathbf{a}^B)}{2}) \end{cases}.$$

Equivalently, we have, for each player i:

$$\begin{cases} \mathbf{x}_i^A = \frac{1}{1+\beta}b_i(\mathbf{G}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A+\mathbf{a}^B)}{2}) + \frac{1}{1-\beta}b_i(\mathbf{G}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A-\mathbf{a}^B)}{2}) \\ \mathbf{x}_i^B = \frac{1}{1+\beta}b_i(\mathbf{G}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A+\mathbf{a}^B)}{2}) - \frac{1}{1-\beta}b_i(\mathbf{G}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A-\mathbf{a}^B)}{2}) \end{cases}.$$

This proves the result. $\square$

Proofs of Corollaries 2, 3 and 4. These are direct consequences of Theorem 1. Therefore, we omit the details. $\square$

Proof of Proposition 6. From (23), we immediately see that if $\beta \geq 0$,

$$\sum_{k=0}^{\infty} \frac{(\delta \mathbf{G})^k}{(1-\beta)^{1+k}} \succeq \sum_{k=0}^{\infty} \frac{(\delta \mathbf{G})^k}{(1+0)^{1+k}} \succeq \sum_{k=0}^{\infty} \frac{(\delta \mathbf{G})^k}{(1+\beta)^{1+k}}, \text{ as } 1+\beta \geq 1+0 \geq 1-\beta.$$

Therefore, $\mathbf{M}^-(\mathbf{G}, \delta, \beta) \succeq \mathbf{M}(\mathbf{G}, \delta) \succeq \mathbf{M}^+(\mathbf{G}, \delta, \beta)$. The other cases can be shown similarly. $\square$

Proof of Proposition 7. By definition,

$$\mathbf{M}^+(\mathbf{G}, \delta, \beta)[(1+\beta)\mathbf{I}_n - \delta \mathbf{G}] = \mathbf{I}_n.$$

Taking the derivative with respect to δ yields

$$\frac{\partial \mathbf{M}^+(\mathbf{G}, \delta, \beta)}{\partial \delta}[(1+\beta)\mathbf{I}_n - \delta \mathbf{G}] - \mathbf{M}^+(\mathbf{G}, \delta, \beta)\mathbf{G} = \mathbf{0}.$$

Therefore,

$$\frac{\partial \mathbf{M}^+(\mathbf{G}, \delta, \beta)}{\partial \delta} = \mathbf{M}^+(\mathbf{G}, \delta, \beta)\mathbf{G}[(1+\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1} = \mathbf{G}[(1+\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-2}.$$

The last equality uses the fact that the matrices $\mathbf{G}$ and $\mathbf{M}^+(\mathbf{G}, \delta, \beta) = [(1+\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1}$ are commutative. Hence, the order of multiplication is irrelevant. The other results can be shown similarly. $\square$

Proof of Proposition 8. Since $\mathbf{G}' \succeq \mathbf{G}$, for all $k \geq 1$ $(\mathbf{G}')^k \succeq (\mathbf{G})^k$. Therefore, we obtain

$$\mathbf{M}^+(\mathbf{G}', \delta, \beta) = \sum_{k=0}^{\infty} \frac{(\delta \mathbf{G}')^k}{(1+\beta)^{1+k}} \succeq \sum_{k=0}^{\infty} \frac{(\delta \mathbf{G})^k}{(1+\beta)^{1+k}} = \mathbf{M}^+(\mathbf{G}, \delta, \beta).$$

Similarly we can show $\mathbf{M}^-(\mathbf{G}', \delta, \beta) \succeq \mathbf{M}^-(\mathbf{G}, \delta, \beta)$. $\square$

Proofs of Propositions 1 − 4. These follow directly from Theorem 1 and the comparative statics results in Proposition 6, Proposition 7, and Proposition 8, therefore the detailed proofs are omitted. $\square$

Proof of Lemma 1. Please see the original proof in Ballester et al. (2006), or an alternative derivation in Zhou and Chen (2015). $\square$

Proof of Theorem 2. First we consider the case with $\lambda = \mu = 1$. By Theorem 1, in equilibrium

$$\mathbf{x}^A + \mathbf{x}^B = \mathbf{M}^{+}(\mathbf{a}^A + \mathbf{a}^B) = \mathbf{b}(\mathbf{G}, \frac{\delta}{1+\beta}, \frac{(\mathbf{a}^A + \mathbf{a}^B)}{1+\beta}).$$

Therefore, the sum of both activities is

$$\sum_{k=1}^{n} b_k(\mathbf{G}, \frac{\delta}{1+\beta}, \frac{(\mathbf{a}^A + \mathbf{a}^B)}{1+\beta}). \tag{24}$$

When player i is removed from the network, the resulting network is just $\mathbf{G}_{-i}$, and marginal utilities vectors are $\mathbf{a}^A_{-i}$, and $\mathbf{a}^B_{-i}$. Therefore, the sum of both activities in equilibrium, by analogy to (24), is

$$\sum_{k \neq i} b_k(\mathbf{G}_{-i}, \frac{\delta}{1+\beta}, \frac{(\mathbf{a}^A + \mathbf{a}^B)_{-i}}{1+\beta}). \tag{25}$$

The difference between (24) and (25), by Lemma 1, is

$$\frac{1}{1+\beta}\bar{c}_i(\mathbf{G}, \frac{\delta}{(1+\beta)}, (\mathbf{a}^A + \mathbf{a}^B)) := c_i^{1,1}(\mathbf{G}, \delta, \mathbf{a}^A, \mathbf{a}^B).$$

Now we solve the case with $\lambda = 1, \mu = 0$. By Theorem 1, we obtain

$$\mathbf{x}^A = \frac{1}{1+\beta}\mathbf{b}(\mathbf{G}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A + \mathbf{a}^B)}{2}) + \frac{1}{1-\beta}\mathbf{b}(\mathbf{G}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A - \mathbf{a}^B)}{2}).$$

Therefore, the sum of activity A is

$$\frac{1}{1+\beta}\sum_{k=1}^{n} b_k(\mathbf{G}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A + \mathbf{a}^B)}{2}) + \frac{1}{1-\beta}\sum_{k=1}^{n} b_k(\mathbf{G}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A - \mathbf{a}^B)}{2}).$$

When i is removed from the network $\mathbf{G}$, the sum of activity A now becomes

$$\frac{1}{1+\beta}\sum_{k \neq i} b_k(\mathbf{G}_{-i}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A + \mathbf{a}^B)_{-i}}{2}) + \frac{1}{1-\beta}\sum_{k \neq i} b_k(\mathbf{G}_{-i}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A - \mathbf{a}^B)_{-i}}{2}).$$

Taking the difference yields

$$\frac{1}{1+\beta}\bar{c}_i(\mathbf{G}, \frac{\delta}{(1+\beta)}, \frac{(\mathbf{a}^A + \mathbf{a}^B)}{2}) + \frac{1}{1-\beta}\bar{c}_i(\mathbf{G}, \frac{\delta}{(1-\beta)}, \frac{(\mathbf{a}^A - \mathbf{a}^B)}{2}) := c_i^{1,0}(\mathbf{G}, \delta, \mathbf{a}^A, \mathbf{a}^B).$$

Since the equilibrium activities are linear in $\mathbf{a}^A, \mathbf{a}^B$, and the objective is also linear, the generalized inter-centrality measure with multiple activities under weights (λ,μ) is

$$c_i^{\lambda,\mu}(\mathbf{G},\delta,\mathbf{a}^A,\mathbf{a}^B)=\mu c_i^{1,1}(\mathbf{G},\delta,\mathbf{a}^A,\mathbf{a}^B)+(\lambda-\mu)c_i^{1,0}(\mathbf{G},\delta,\mathbf{a}^A,\mathbf{a}^B).$$

This completes the proof. □

Proof of Theorem 3. The first-order conditions are

$$a_i^t-(x_i^t+\beta\sum_{j\neq i}x_j^t)+\delta\sum g_{ij}x_j^t=0,\quad t=1,2,\cdots l; i=1,2,\cdots,n.$$

In matrix form, we have

$$(\mathbf{\Psi}\otimes\mathbf{I}_n)\mathbf{X}=\mathbf{A}+\delta(\mathbf{I}_l\otimes\mathbf{G})\mathbf{X},$$

and therefore

$$\mathbf{X}=[\mathbf{\Psi}\otimes\mathbf{I}_n-\delta\mathbf{I}_l\otimes\mathbf{G}]^{-1}\mathbf{A}.$$

More precisely, we can conjecture that

$$[\mathbf{\Psi}\otimes\mathbf{I}_n-\delta\mathbf{I}_l\otimes\mathbf{G}]^{-1}=\begin{bmatrix}\mathbf{I}_n-\delta\mathbf{G} & \beta\mathbf{I}_n & \cdots & \beta\mathbf{I}_n\\ \beta\mathbf{I}_n & \mathbf{I}_n-\delta\mathbf{G} & \cdots & \beta\mathbf{I}_n\\ \vdots & \ddots & \ddots & \vdots\\ \beta\mathbf{I}_n & \cdots & \beta\mathbf{I}_n & \mathbf{I}_n-\delta\mathbf{G}\end{bmatrix}^{-1}=\begin{bmatrix}\mathbf{W} & \mathbf{\Phi} & \cdots & \mathbf{\Phi}\\ \mathbf{\Phi} & \mathbf{W} & \cdots & \mathbf{\Phi}\\ \vdots & \ddots & \ddots & \vdots\\ \mathbf{\Phi} & \cdots & \mathbf{\Phi} & \mathbf{W}\end{bmatrix}.$$

The matrices $\mathbf{W}$ and $\mathbf{\Phi}$ must satisfy the following conditions:

$$(\mathbf{I}_n-\delta\mathbf{G})\mathbf{W}+\beta(l-1)\mathbf{\Phi}=\mathbf{I}_n,\quad(\mathbf{I}_n-\delta\mathbf{G})\mathbf{\Phi}+\beta(\mathbf{W}+(l-2)\mathbf{\Phi})=\mathbf{0}.$$

Solving these equations together, we obtain

$$\begin{aligned}\mathbf{W}&=\frac{[(1+(l-1)\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}+(l-1)[(1-\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1},}{l}\\ \mathbf{\Phi}&=\frac{[(1+(l-1)\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}-[(1-\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}}{l}.\end{aligned}$$

This gives the equilibrium activities.

For uniqueness, we can write down the best response mapping, in multiple activities, as follows:

$$\begin{aligned}\mathbf{BR}(\mathbf{X}):=&\quad(\mathbf{\Psi}\otimes\mathbf{I}_n)^{-1}\mathbf{A}+\delta(\mathbf{\Psi}\otimes\mathbf{I}_n)^{-1}(\mathbf{I}_l\otimes\mathbf{G})\mathbf{X}\\ =&\quad(\mathbf{\Psi}^{-1}\otimes\mathbf{I}_n)\mathbf{A}+\delta(\mathbf{\Psi}^{-1}\otimes\mathbf{G})\mathbf{X}.\end{aligned}$$

The spectral radius of $\delta\mathbf{G}$ is $\delta\lambda_1(\mathbf{G})$, and the spectral radius of $\mathbf{\Psi}^{-1}$ is $\max(\frac{1}{1-\beta},\frac{1}{1+(l-1)\beta})$ as Ψ^{-1}

has only two distinct eigenvalues, $\frac{1}{1-\beta}$ and $\frac{1}{1+(l-1)\beta}$. Therefore the spectral radius of the Kronecker product $\delta(\mathbf{\Psi}^{-1} \otimes \mathbf{G})$ equals $\delta\lambda_1(\mathbf{G}) \max(\frac{1}{1-\beta}, \frac{1}{1+(l-1)\beta})$, which is strictly less than 1 by Assumption 2[23]. The uniqueness result follows from Lemma 4 in Appendix A. □

Proof of Theorem 4. The first-order conditions from the players' best responses can be written as:

$$\begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} = \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix} + \begin{bmatrix} \delta\mathbf{G} & \mu\mathbf{G} \\ \mu\mathbf{G} & \delta\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix}.$$

Taking the sum yields

$$(1+\beta)(\mathbf{x}^A + \mathbf{x}^B) = (\mathbf{a}^A + \mathbf{a}^B) + (\delta+\mu)\mathbf{G}(\mathbf{x}^A + \mathbf{x}^B),$$

or equivalently

$$(\mathbf{x}^A + \mathbf{x}^B) = [(1+\beta)\mathbf{I}_n - (\delta+\mu)\mathbf{G}]^{-1}(\mathbf{a}^A + \mathbf{a}^B).$$

Similarly, the substraction leads to the variation component:

$$(\mathbf{x}^A - \mathbf{x}^B) = [(1-\beta)\mathbf{I}_n - (\delta-\mu)\mathbf{G}]^{-1}(\mathbf{a}^A - \mathbf{a}^B).$$

Combining these two equations yields the equilibrium efforts $\mathbf{x}^A$ and $\mathbf{x}^B$, thus the existence is proved by construction. For uniqueness, we can write down the best response mapping as follows:

$$\mathbf{BR}(\mathbf{X}) := \begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}^{-1} \begin{bmatrix} \delta\mathbf{G} & \mu\mathbf{G} \\ \mu\mathbf{G} & \delta\mathbf{G} \end{bmatrix}}_{:\mathbf{\Omega}} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix}.$$

Note that

$$\begin{aligned} \mathbf{\Omega} &= \left\{ \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \otimes \mathbf{I}_n \right\}^{-1} \left\{ \begin{bmatrix} \delta & \mu \\ \mu & \delta \end{bmatrix} \otimes \mathbf{G} \right\} = \left\{ \frac{1}{1-\beta^2} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \otimes \mathbf{I}_n \right\} \left\{ \begin{bmatrix} \delta & \mu \\ \mu & \delta \end{bmatrix} \otimes \mathbf{G} \right\} \\ &= \left\{ \frac{1}{1-\beta^2} \begin{bmatrix} \delta-\beta\mu & \mu-\beta\delta \\ \mu-\beta\delta & \delta-\beta\mu \end{bmatrix} \otimes \mathbf{G} \right\}. \end{aligned}$$

The eigenvalues of $\frac{1}{1-\beta^2} \begin{bmatrix} \delta-\beta\mu & \mu-\beta\delta \\ \mu-\beta\delta & \delta-\beta\mu \end{bmatrix}$ are $\frac{\delta+\mu}{1+\beta}$ and $\frac{\delta-\mu}{1-\beta}$. Thus, its spectral radius is $\max(\frac{|\delta+\mu|}{1+\beta}, \frac{|\delta-\mu|}{1-\beta})$. Moreover, the spectral radius of $\mathbf{G}$ is $\lambda_1(\mathbf{G})$. Therefore, the spectral radius of $\mathbf{\Omega}$ is $\max(\frac{|\delta+\mu|}{1+\beta}, \frac{|\delta-\mu|}{1-\beta})\lambda_1(\mathbf{G})$, which is strictly less than 1 by Assumption 3. The rest just follows from Lemma 4 in Appendix A.
□

[23] The spectral radius of the Kronecker product of two matrices equals the product of the two spectral radius.

Proof of Theorem 5. To characterize the equilibrium, the first-order conditions can be written as:

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{\Lambda}^\beta \\ \mathbf{\Lambda}^\beta & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} = \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix} + \begin{bmatrix} \delta\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix}.$$

Here $\mathbf{\Lambda}^\beta = diag(\beta_1, \cdots, \beta_n)$ is a diagonal matrix with β_i on its (i,i) entry. Taking the sum yields:

$$(\mathbf{I}_n + \mathbf{\Lambda}^\beta)(\mathbf{x}^A + \mathbf{x}^B) = (\mathbf{a}^A + \mathbf{a}^B) + (\delta + \mu)\mathbf{G}(\mathbf{x}^A + \mathbf{x}^B),$$

or equivalently

$$(\mathbf{x}^A + \mathbf{x}^B) = [\mathbf{I}_n + \mathbf{\Lambda}^\beta - \delta\mathbf{G}]^{-1}(\mathbf{a}^A + \mathbf{a}^B).$$

Similarly, the variation term is

$$(\mathbf{x}^A - \mathbf{x}^B) = [\mathbf{I}_n - \mathbf{\Lambda}^\beta - \delta\mathbf{G}]^{-1}(\mathbf{a}^A - \mathbf{a}^B).$$

The existence is thus proved by construction. For uniqueness, we can write down the best response mapping as follows

$$\mathbf{BR}(\mathbf{X}) := \begin{bmatrix} \mathbf{I}_n & \mathbf{\Lambda}^\beta \\ \mathbf{\Lambda}^\beta & \mathbf{I}_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}^A \\ \mathbf{a}^B \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{I}_n & \mathbf{\Lambda}^\beta \\ \mathbf{\Lambda}^\beta & \mathbf{I}_n \end{bmatrix}^{-1} \begin{bmatrix} \delta\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{G} \end{bmatrix}}_{:\mathbf{\Sigma}} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix}.$$

Note that the eigenvalues of $\begin{bmatrix} \mathbf{I}_n & \mathbf{\Lambda}^\beta \\ \mathbf{\Lambda}^\beta & \mathbf{I}_n \end{bmatrix}$ are $1 \pm \beta_i, i = 1, 2, \cdots, n$. Therefore, its spectral radius of $\begin{bmatrix} \mathbf{I}_n & \mathbf{\Lambda}^\beta \\ \mathbf{\Lambda}^\beta & \mathbf{I}_n \end{bmatrix}^{-1}$ is $\max_i(\frac{1}{1\pm\beta_i}) = \max_i \left\{\frac{1}{1-|\beta_i|}\right\}$.

Recall that the spectral radius of $\begin{bmatrix} \delta\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{G} \end{bmatrix}$ is $\delta\lambda_1(\mathbf{G})$. Thus, the spectral radius of $\mathbf{\Sigma}$ is

$$\rho(\mathbf{\Sigma}) \leq \rho\left(\begin{bmatrix} \mathbf{I}_n & \mathbf{\Lambda}^\beta \\ \mathbf{\Lambda}^\beta & \mathbf{I}_n \end{bmatrix}^{-1}\right) \rho\left(\begin{bmatrix} \delta\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{G} \end{bmatrix}\right) = \max_i \left\{\frac{1}{1-|\beta_i|}\right\} \delta\lambda_1(\mathbf{G}) < 1,$$

where the first inequality follows from Lemma 2 as both $\begin{bmatrix} \mathbf{I}_n & \mathbf{\Lambda}^\beta \\ \mathbf{\Lambda}^\beta & \mathbf{I}_n \end{bmatrix}^{-1}$ and $\begin{bmatrix} \delta\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{G} \end{bmatrix}$ are symmetric matrices, and the last inequality is by Assumption 4. The uniqueness result follows from Lemma 4 in Appendix A. □

Proof of Theorem 6. Following similar steps, we obtain the first-order conditions in matrix form as follows:

$$\begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}^{A*} \\ \mathbf{x}^{B*} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{A} \\ \mathbf{a}^{B} \end{bmatrix} + \begin{bmatrix} \delta^A\mathbf{G}^A & \mathbf{0} \\ \mathbf{0} & \delta^B\mathbf{G}^B \end{bmatrix} \begin{bmatrix} \mathbf{x}^{A} \\ \mathbf{x}^{B} \end{bmatrix}.$$

Therefore, we obtain

$$\mathbf{X}^* = \begin{bmatrix} \mathbf{x}^{A*} \\ \mathbf{x}^{B*} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n - \delta^A\mathbf{G}^A & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n - \delta^B\mathbf{G}^B \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}^{A} \\ \mathbf{a}^{B} \end{bmatrix}.$$

The inverse matrix above can be computed using Lemma 3 about Block Matrix Inversion Formula.

To show the uniqueness, we need to show the best response mapping defined below

$$\mathbf{BR}(\mathbf{X}) := \begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}^{A} \\ \mathbf{a}^{B} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}^{-1} \begin{bmatrix} \delta^A\mathbf{G}^A & \mathbf{0} \\ \mathbf{0} & \delta^B\mathbf{G}^B \end{bmatrix}}_{:\boldsymbol{\Upsilon}} \begin{bmatrix} \mathbf{x}^{A} \\ \mathbf{x}^{B} \end{bmatrix}$$

has a unique fixed point. Notice that $\begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}^{-1}$ has only two distinct eigenvalues, $1/(1+\beta)$ and $1/(1-\beta)$. Therefore, its spectral radius is $1/(1-|\beta|)$.

The eigenvalues of $\begin{bmatrix} \delta^A\mathbf{G}^A & \mathbf{0} \\ \mathbf{0} & \delta^B\mathbf{G}^B \end{bmatrix}$ are $\delta^A\lambda_i(\mathbf{G}^A), i = 1, \cdots, n$, and $\delta^B\lambda_i(\mathbf{G}^B), i = 1, \cdots, n$, where $\{\lambda_i(\mathbf{G}^t), i = 1, \cdots, n\}$ are the eigenvalues of $\mathbf{G}^t$ for $t = A, B$. By Perron-Frobenius Theorem, the spectral radius of a nonnegative matrix equals its largest eigenvalue. As a consequence, its spectral radius equals $\max(\delta^A\lambda_1(\mathbf{G}^A), \delta^B\lambda_1(\mathbf{G}^B))$.

Moreover, both $\begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}^{-1}$ and $\begin{bmatrix} \delta^A\mathbf{G}^A & \mathbf{0} \\ \mathbf{0} & \delta^B\mathbf{G}^B \end{bmatrix}$ are symmetric as $\mathbf{G}^A$ and $\mathbf{G}^B$ are symmetric. Applying Lemma 2 , we obtain that

$$\rho(\boldsymbol{\Upsilon}) \leq \rho\left(\begin{bmatrix} \mathbf{I}_n & \beta\mathbf{I}_n \\ \beta\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}^{-1}\right) \rho\left(\begin{bmatrix} \delta^A\mathbf{G}^A & \mathbf{0} \\ \mathbf{0} & \delta^B\mathbf{G}^B \end{bmatrix}\right) = \frac{1}{1-|\beta|} \max(\delta^A\lambda_1(\mathbf{G}^A), \delta^B\lambda_1(\mathbf{G}^B)) < 1$$

by Assumption 5. The uniqueness just follows from Lemma 4 in Appendix A. □