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Research Article

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Multiple and sign-changing solutions for discrete Robin boundary value problem with parameter dependence

Abstract: In this paper, we study second-order nonlinear discrete Robin boundary value problem with parameter dependence. Applying invariant sets of descending flow and variational methods, we establish some new sufficient conditions on the existence of sign-changing solutions, positive solutions and negative solutions of the system when the parameter belongs to appropriate intervals. In addition, an example is given to illustrate our results.

1 Introduction

Throughout this paper, let **N**, **Z** and **R** denote the sets of all nature numbers, integers and real numbers, respectively. We consider the following second-order nonlinear difference equation with Robin boundary value problem (BVP for short)

$$\begin{cases} -\Delta^2 x(k-1) = \alpha f(k, x(k)), & k \in [1, T], \\ x(0) = \Delta x(T) = 0, \end{cases} \tag{1}$$

where $T \geq 2$ is a given integer, $[1, T] =: \{1, 2, \cdots, T\}$, parameter $\alpha > 0$, $f : [1, T] \times \mathbf{R} \to \mathbf{R}$ is continuous in the second variable, Δ denotes the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$, $\Delta^2 x(k) = \Delta(\Delta x(k))$.

Discrete nonlinear equations with parameter dependence play an important role in describing many physical problems, such as nonlinear elasticity theory or mechanics and engineering topics [1, 2]. In recent years, some authors also contributed to the study of (1) and obtained some interesting results. For example, when $\alpha = 1$, Jiang and Zhou [3] employed strongly monotone operator and critical point theory to establish the existence of nontrivial positive solutions. By virtue of variational methods and critical point theory, Guo and Song [4] investigated the existence of positive solutions. Zhang and Xu [5] obtained the existence and uniqueness theorems of nontrivial solutions. For results on nonlinear difference equations with other boundary value problems, we can see [6, 7] and references therein. With reference to the sign-changing solution, many scholars studied it for differential equations by a variety of methods and techniques, such

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as Leray-Schauder degree theory [8], fixed point index theory [9], topological degree theory [10], critical point theory and invariant sets of descending flow [11 – 13] etc. However, to the author's knowledge, there were few papers [14] that considered the sign-changing solutions for discrete boundary problems. One of the reasons is the fact that the traditional pasting techniques of differential equations are not applicable in the world of difference equations. The corresponding anchoring techniques are rather complicated, see the detailed discussion in [15].

View from above reasons and motivated by [14], the purpose of this paper is to apply invariant sets of descending flow and variational techniques to get some sufficient conditions for the existence of sign-changing solutions, negative solutions and positive solutions to (1).

In the following, we first consider the linear eigenvalue problem corresponding to (1)

$$\begin{cases} -\Delta^2 x(k-1) = \lambda x(k), & k \in [1, T], \\ x(0) = \Delta x(T) = 0, \end{cases} \tag{2}$$

Let λ_k be eigenvalues of (2) and $\{z_k\}_{k=1}^{T}$ be the corresponding eigenvectors of $\{\lambda_k\}_{k=1}^{T}$, then

$$\lambda_k = 4\sin^2\left(\frac{2k-1}{2T+1} \cdot \frac{\pi}{2}\right), \qquad z_k(j) = \sin\frac{(2k-1)\pi j}{2T+1}, \quad k, j = 1, 2, \ldots, T.$$

Obviously, $\lambda_1 = 4\sin^2\frac{\pi}{2(2T+1)} > 0$ and $z_1(j) = \sin\frac{j\pi}{2T+1} > 0$ for $j \in [1, T]$.

In this paper, we focus on the following assumptions:

$(J_1) f_0 = \max_{k\in[1,T]} \limsup_{u\to 0} \left|\frac{f(k, u)}{u}\right| < \lambda_1.$

$(J_2) \lim_{|u|\to\infty} \frac{f(k, u)}{u} = r$ for $k \in [1, T]$, where $r \in (0, +\infty)$ is a constant, or $r = +\infty$, $\nu > 2$ and $C > 0$ satisfy

$$|f(k, u)| \le C(1 + |u|^{\nu-1}), \qquad k \in [1, T], \quad u \in \mathbf{R}.$$

(J_3) Either $(i) \lim_{|u|\to\infty} [uf(k, u) - 2F(k, u)] = -\infty$, uniformly for $k \in [1, T]$,

or $\qquad (ii) \lim_{|u|\to\infty} [uf(k, u) - 2F(k, u)] = +\infty$, uniformly for $k \in [1, T]$,

where $F(k, u) = \int_0^u f(k, s)ds$.

Our results read as follows:

Theorem 1.1. *Suppose (J_1), (J_2) hold and $r > \frac{\lambda_2}{\alpha}$. If $\alpha \in \left(\frac{1}{2}, +\infty\right)$ and r is not an eigenvalue of (2), then (1) has at least three nontrivial solutions, one sign-changing, one positive and one negative.*

Theorem 1.2. *Suppose $(J_1) - (J_3)$ hold and $r > \frac{\lambda_2}{\alpha}$. If r is an eigenvalue of (2), then (1) has at least three nontrivial solutions, one sign-changing, one positive and one negative.*

The remainder of this paper is organized as follows. After introducing some notations and preliminary results in Section 2, we complete the proof of Theorem 1.1 and give an example to illustrate our result in Section 3.

2 Variational structure and preliminary results

Given $m \ge 0$, let $G = \{x : [0, T+1] \to \mathbf{R} | x(0) = \Delta x(T) = 0\}$ be a T-dimensional Hilbert space which is equipped with the inner product

$$\langle x, y\rangle_m = \sum_{k=1}^{T}[\Delta x(k-1)\Delta y(k-1) + mx(k)y(k)], \qquad \text{for all } x, y \in G,$$

then the induced norm $\|\cdot\|_m$ is

$$\|x\|_m = \left(\sum_{k=1}^{T}[|\Delta x(k-1)|^2 + m|x(k)|^2]\right)^{\frac{1}{2}}, \qquad \text{for all } x \in G.$$

Let H be the T-dimensional Hilbert space equipped with the usual inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. It is easy to see that G is isomorphic to H, $\|\cdot\|_m$ and $\|\cdot\|$ are equivalent. Denote $x^+ = \max\{x, 0\}$, $x^- = \min\{x, 0\}$. Then for any $x \in H$, $\langle\cdot, \cdot\rangle_m \geq 0$.

Define functional $I : H \to \mathbf{R}$ as

$$I(x) = \frac{1}{2} \sum_{k=1}^{T} |\Delta x(k-1)|^2 - \alpha \sum_{k=1}^{T} F(k, x(k)). \tag{3}$$

For any $x = (x(1), x(2), \cdots, x(T))^\top \in H$, $I(x)$ can be rewritten as

$$I(x) = \frac{1}{2}(Ax, x) - \alpha \sum_{k=1}^{T} F(k, x(k)), \tag{4}$$

here α^τ is the transpose of the vector α on H, A is $T \times T$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

Remark 2.1. *In fact, many existing results are applicable. Namely, one can apply numerous results for the variational formulation* (4), *see* [16 − 18].

For $m \geq 0$, consider *BVP*

$$\begin{cases} -\Delta^2 x(k-1) + mx(k) = h(k), & k \in [1, T], \\ x(0) = \Delta x(T) = 0, \end{cases} \tag{5}$$

where $h : [1, T] \to \mathbf{R}$. It is not hard to see that (5) and the system of linear algebra equations $(A + mI)x = h$ are equivalent, then the unique solution of (5) can be expressed by

$$x = (A + mI)^{-1}h. \tag{6}$$

On the other side, we have

Lemma 2.2. *The unique solution of* (5) *is*

$$x(k) = \sum_{s=1}^{T} G_m(k, s)h(s), \quad k \in [0, T+1],$$

here $G_m(k, s)$ *can be written as*

$$G_m(k, s) = \begin{cases} \dfrac{(p^{T-s} - p^{s-T} - p^{T+1-s} + p^{s-T-1})(p^k - p^{-k})}{W}, & 0 \leq k \leq s \leq T+1, \\ \dfrac{(p^{T-k} - p^{k-T} - p^{T+1-k} + p^{k-T-1})(p^s - p^{-s})}{W}, & 0 \leq s \leq k \leq T+1, \end{cases}$$

$$W = (p^{T+1} - p^{-T-1} - p^T + p^{-T})(p^{-1} - p), p = \frac{2 + m + \sqrt{4m + m^2}}{2}.$$

Proof. First study the homogeneous equation of (5)

$$\begin{cases} -\Delta^2 x(k-1) + mx(k) = 0, & k \in [1, T], \\ x(0) = \Delta x(T) = 0, \end{cases} \tag{7}$$

then the corresponding characteristic equation of (7) is $p^2 - (2+m)p + 1 = 0$. Consider $m > 0$ ($m = 0$ is trivial), then $(2 + m)^2 - 4 > 0$, which means we have

$$p_1 = \frac{2 + m + \sqrt{4m + m^2}}{2}, \qquad p_2 = \frac{2 + m - \sqrt{4m + m^2}}{2}.$$

Then two independent solutions of (7) can be expressed by $x_1(k) = p_1^k$ and $x_2(k) = p_2^k$. Therefore, the general solution of (5) is $x(k) = a_1(k)p_1^k + a_2(k)p_2^k$.

The next step is to determine coefficients $a_1(k)$ and $a_2(k)$. Using the method of variation of constant, we get the general solution of (5) as

$$x(k) = \left[a_1(0) + \sum_{s=1}^{k} \frac{p_2^s h(s)}{p_2 - p_1} \right] p_1^k + \left[a_2(0) - \sum_{s=1}^{k} \frac{p_1^s h(s)}{p_2 - p_1} \right] p_2^k.$$

From initial conditions, we find $a_1(0) = -a_2(0)$ and

$$a_1(0) = \sum_{s=1}^{T} \frac{(p_1^T p_2^s - p_1^s p_2^T - p_2^s p_1^{T+1} + p_1^s p_2^{T+1})}{(p_1^{T+1} - p_2^{T+1} - p_1^T + p_2^T)(p_2 - p_1)} h(s).$$

Write $W = (p_1^{T+1} - p_2^{T+1} - p_1^T + p_2^T)(p_2 - p_1)$, $p_1 = p_2^{-1} = p$, then

$$x(k) = \frac{1}{W} \Big[\sum_{s=1}^{T} (p^{T-s} - p^{s-T} - p^{T+1-s} + p^{s-T-1})(p^k - p^{-k})$$

$$+ \sum_{s=1}^{k} (P^{k-s} - P^{s-k})(p^{T+1} - p^{-T-1} - p^T + p^{-T}) \Big] h(s)$$

which means the proof of Lemma 2.2 is completed. □

Remark 2.3. *From Lemma 2.2, for $k, s \in [1, T]$, there holds $G_m(k, s) = G_m(s, k) > 0$. For any $x \in H$, $k \in [1, T]$, define $K_m, f_m, A_m : H \to H$ as follows*

$$(K_m x)(k) = \sum_{s=1}^{T} G_m(k, s)x(s), \qquad (f_m x)(k) = f(k, x(k)) + mx(k),$$

$$A_m = K_m f_m$$

where $A_m : H \to H$ is a completely continuous operator. Combining (6) with Lemma 2.2, we get $K_m = (A + mI)^{-1}$.

Remark 2.4. *According to Lemma 2.2, it is not difficult to see that $\{x(k)\}_{k=0}^{T+1}$ is a solution of (1) if and only if $x = \{x(k)\}_{k=1}^{T} \in H$ is a fixed point of A_m.*

Lemma 2.5. *The functional I defined by (3) is Fréchet differentiable on H and $I'(x)$ has the expression $I'(x) = x - K_m f_m x$ for $x \in H$.*

Proof. For any $x, y \in H$, using the mean value theorem, it follows

$$I(x + y) - I(x) = \frac{1}{2} \sum_{k=1}^{T} |\Delta y(k-1)|^2 + \sum_{k=1}^{T} [\Delta x(k-1)\Delta y(k-1) - \alpha f(k, x(k) + \theta(k)y(k))y(k)],$$

here $\theta(k) \in (0, 1)$, $k \in [1, T]$. As f is continuous in x, we find

$$I(x + y) - I(x) - \langle x, y \rangle_m + \sum_{k=1}^{T} (\alpha f(k, x(k)) + mx(k))y(k) = \|y\|_m o(1)$$

which leads to

$$\lim_{\|y\|_m \to 0} \frac{1}{\|y\|_m} \Big(\alpha \sum_{k=1}^{T} [f(k, y(k) - f(k, x(k) + \theta(k)y(k)))y(k)] + \frac{1}{2}\|y\|_m^2 - \frac{1}{2}m\|y\|^2 \Big) = 0,$$

thus I is Fréchet differentiable on H and

$$\langle I'(x), y \rangle_m = \langle x, y \rangle_m - \sum_{k=1}^{T} (\alpha f(k, x(k)) + mx(k))y(k). \tag{8}$$

On the other side, for all $x = \{x(k)\} \in H$, $z = \{z(k)\} \in H$, there holds

$$\sum_{k=1}^{T} \Delta^2 x(k-1) y(k) = \sum_{k=1}^{T} [\Delta x(k) y(k) - \Delta x(k-1) y(k)] = -\sum_{k=1}^{T} \Delta x(k-1) \Delta y(k-1).$$

Making use of the definition of inner product and Lemma 2.2, we get

$$\langle x - K_m f_m x, y \rangle_m = \langle x, y \rangle_m - \sum_{k=1}^{T} (\alpha f(k, x(k)) + mx(k)) y(k)$$

then $\langle I'(x), y \rangle_m = \langle x - K_m f_m x, y \rangle_m$ for all $x, y \in H$, i.e., $I'(x) = x - K_m f_m x$. This completes the proof of Lemma 2.5. □

Remark 2.6. *According to Lemma 2.5 and Remark 2.4, we find out that critical points of I defined on H are precisely solutions of* (1).

Definition 2.7 ([19]). *Let I be a C^1 functional defined on E. I is said to satisfy Palais-Smale condition ((PS)condition for short) if any sequence $\{u_n\} \subset E$ for which $I(u_n)$ is bounded and $I'(u_n) \to 0 (n \to \infty)$ possesses a convergent subsequence in E.*

Definition 2.8 ([20]). *Assume I be a C^1 functional defined on E. If any sequence $\{u_n\}$ such that $I(u_n)$ is bounded and $(1 + \|u_n\|_m) \|I'(u_n)\|_m \to 0$ as $n \to \infty$ has a convergent subsequence in E, then we say that I satisfies the Cerami condition ((C) condition for short).*

Lemma 2.9 ([21]). *Let H be a Hilbert space, there are two open convex subsets B_1 and B_2 on H with $A_m(\partial B_1) \subset B_1, A_m(\partial B_2) \subset B_2$ and $B_1 \cap B_2 \neq \varnothing$. If $I \in C^1(H, \mathbf{R})$ satisfies the (PS) condition and $I'(x) = x - A_m x$ for all $x \in H$. Assume there is a path $g : [0, 1] \to H$ such that*

$$g(0) \in B_1 \setminus B_2, \quad g(1) \in B_2 \setminus B_1,$$

and

$$\inf_{u \in \overline{B_1} \cap \overline{B_2}} I(x) > \sup_{\tau \in [0,1]} I(g(\tau)),$$

then I has at least four critical points, one in $H \setminus (\overline{B_1} \cup \overline{B_2})$, one in $B_1 \setminus \overline{B_2}$, one in $B_1 \cap B_2$, and one in $B_2 \setminus \overline{B_1}$.

Remark 2.10. *By Theorem 5.1 [20], we can replace (PS) condition by weaker (C) condition in Lemma 2.9.*

In this paper, we will analyse the properties of the flow, pay close attention to the direction and the destination to which the flow goes, and seek the limit along the flow. We are interested in those points in H across which the flow does not go to infinity and work for seeking such points in H. If we have such a point, then the flow curve crossing it goes ultimately to a critical point. It seems that one would obtain many critical points if he or she is given many such points. However, even if there may be many such points, we cannot get more than one critical point in general since the different flow curves may ultimately go to the same critical point. In order to get more critical points, we will define the concept of invariant set of descending flow and then we will divide the whole space H into several invariant subsets of descending flow. In this way, we can get more than one critical point.

3 Proof of main result

Let convex cones $\Lambda = \{x \in H : x \geq 0\}$ and $-\Lambda = \{x \in H : x \leq 0\}$. The distance respecting to $\| \cdot \|_m$ in H is written by $dist_m$. For arbitrary $\varepsilon > 0$, we denote

$$B_\varepsilon^+ = \{x \in H : \text{dist}_m(x, \Lambda) < \varepsilon\}, \quad B_\varepsilon^- = \{x \in H : \text{dist}_m(x, -\Lambda) < \varepsilon\}.$$

Notice that B_ε^+ and B_ε^- are open convex subsets on H with $B_\varepsilon^+ \cap B_\varepsilon^- \neq \varnothing$ and $H \smallsetminus (B_\varepsilon^+ \cup B_\varepsilon^-)$ contains only sign-changing functions.

Lemma 3.1. *Suppose one of the following condition holds.*
(i) $r = +\infty$ or
(ii) $r < +\infty$ is not an eigenvalue of (2), here r is defined by (J_2).
Then the functional I defined by (3) satisfies (PS) condition for all $\alpha \in \left(\dfrac{1}{2}, +\infty \right)$.

Proof. (i) Assume $r = +\infty$. Let $\{x_n\} \subset H$ be a (PS) sequence. Since H is a finite dimensional space, we only need to show $\{x_n\}$ is bounded. If $r = +\infty$, choosing a constant $y > 0$, for all $(k, u) \in [1, T] \times \mathbf{R}$, we have $F(k, u) \geq \lambda_T u^2 - y$. Then

$$I(x_n) = \frac{1}{2}(Ax_n, x_n) - \alpha \sum_{k=1}^{T} F(k, x_n(k)) \leq \left(\frac{1}{2} - \alpha \right) \lambda_T \|x_n\|^2 + T\alpha y, \tag{9}$$

thus $\|x_n\|^2 \leq \dfrac{2I(x_n) - 2T\alpha y}{(1 - 2\alpha)\lambda_T}$ is bounded for $\alpha > \dfrac{1}{2}$.

(ii) suppose $r < +\infty$ is not an eigenvalue of (2). We are now ready to prove that $\{x_n\}$ is bounded. Arguing by contradiction, we suppose there is a subsequence of $\{x_n\}$ with $\rho_n = \|x_n\| \to +\infty$ as $n \to \infty$ and for each $k \in [1, T]$, either $\{x_n(k)\}$ is bounded or $x_n(k) \to +\infty$. Put $y_n = \dfrac{x_n}{\rho_n}$. Clearly, $\|y_n\| = 1$. Then there have a subsequence of $\{y_n\}$ and $y \in H$ satisfying that $y_n \to y$ as $n \to \infty$. Write $d_n = \left(\dfrac{f(1, x_n(1))}{x_n(1)} y_n(1), \ldots, \dfrac{f(T, x_n(T))}{x_n(T)} y_n(T) \right)^\top$.

Since $\lim\limits_{|u| \to +\infty} \dfrac{f(k, u)}{u} = r$ for all $k \in [1, T]$ and $I'(x_n) = x_n - K_m f_m x_n$, we get

$$\frac{I'(x_n)}{\rho_n} = y_n - \frac{1}{\rho_n} K_0 f_0 x_n = y_n - K_0 d_n \to y - K_0 r y.$$

For $\dfrac{I'(x_n)}{\rho_n} \to 0$ as $n \to \infty$, we have $y - K_0 r y \to 0$. In view of Lemma 2.5, we find that r is an eigenvalue of matrix A, which contradicts the assumption. So $\{x_n\}$ is bounded and the proof is finished. $\qquad \square$

Lemma 3.2. *I satisfies (C) condition under (J_3).*

Proof. First assume $(J_3)(i)$ is satisfied. There exists a constant $M_1 > 0$ such that $\{x_n\} \subset H$ with $I(x_n) \leq M_1$ and $(1 + \|x_n\|_m)\|I'(x_n)\|_m \leq M_1$, then

$$-3M_1 \leq \alpha \sum_{k=1}^{T} [x_n(k)f(k, x_n(k)) - 2F(k, x_n(k))]. \tag{10}$$

We claim $\{x_n\}$ is bounded. Actually, if $\{x_n\}$ is unbounded, it possesses a subsequence of $\{x_n\}$ and some $k_0 \in [1, T]$ satisfying $|x_n(k_0)| \to +\infty$ as $n \to \infty$. According to $(J_3)(i)$, we get

$$x_n(k_0)f(k_0, x_n(k_0)) - 2F(k_0, x_n(k_0)) \to -\infty \quad as \quad n \to \infty,$$

and there is a constant $M_2 > 0$ such that $uf(k, u) - 2F(k, u) \leq M_2$ for any $k \in [1, T]$ and $u \in \mathbf{R}$. Therefore,

$$\sum_{k=1}^{T} [x_n(k)f(k, x_n(k)) - 2F(k, x_n(k))] \to -\infty$$

which contradicts (10). So our claim is proved and I satisfies the (C) condition.

Finally, assume $(J_3)(ii)$ hold. In a similar way as above, we find that I satisfies (C) condition. Then Lemma 3.2 is verified. $\qquad \square$

Lemma 3.3. *If (J_1) and (J_2) hold, there exist $m \geq 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, we have:*

(i) *if $x \in B_\varepsilon^-$ is a nontrivial critical point of I and $A_m(\partial B_\varepsilon^-) \subset B_\varepsilon^-$, then x is a negative solution of (1);*

(ii) *if $x \in B_\varepsilon^+$ is a nontrivial critical point of I and $A_m(\partial B_\varepsilon^+) \subset B_\varepsilon^+$, then x is a positive solution of (1).*

Proof. (i) According to (J_1) and (J_2), for all $u \neq 0$ and $k \in [1, T]$, there exists $m \geq 0$ such that

$$u(\alpha f(k, u) + mu) > 0. \tag{11}$$

Let $y = A_m(x)$ and $x^+ = \max\{x, 0\}$, $x^- = \min\{x, 0\}$ for $x \in H$. Since

$$\|x\|_m^2 = \sum_{k=1}^T [|\Delta x(k-1)|^2 + m|x(k)|^2] = x^\tau A x + m\|x\|^2 = \lambda\|x\|^2 + m\|x\|^2 = (\lambda + m)\|x\|^2,$$

it follows $\sqrt{\lambda_1 + m}\|x\| \leq \|x\|_m \leq \sqrt{\lambda_T + m}\|x\|$ and

$$\|x^+\| = \inf_{z \in -\Lambda} \|x - z\| \leq \frac{1}{\sqrt{m + \lambda_1}} \inf_{z \in -\Lambda} \|x - z\|_m = \frac{1}{\sqrt{m + \lambda_1}} dist_m(x, -\Lambda). \tag{12}$$

By (J_1) and (J_2), there exist constants $\tau > 0$, $C > 0$ and $\nu > 2$ such that

$$|\alpha f(k, u) + mu| \leq (m + \lambda_1 - \tau)|u| + \alpha C|u|^{\nu-1}, \quad \forall (k, u) \in [1, T] \times \mathbf{R}. \tag{13}$$

Choosing a constant $D > 0$, since $x \in H$, we have

$$|x|_\nu := (\sum_{k=1}^T |x(k)|^\nu)^{\frac{1}{\nu}} \leq D \min\{\|x\|, \|x\|_m\}, \quad \forall x \in H. \tag{14}$$

It is obvious that $|x|_2 = \|x\|$. Moreover, $y^+ = y - y^-$ and $y^- \in -\Lambda$ imply $dist_m(y, -\Lambda) \leq \|y - y^-\|_m = \|y^+\|_m$. Making use of (12), (15) and (14), we get

$$dist_m(y, -\Lambda)\|y^+\|_m \leq (\frac{m + \lambda_1 - \tau}{m + \lambda_1} dist_m(x, -\Lambda) + C_1(dist_m(x, -\Lambda))^{\nu-1})\|y^+\|_m,$$

here $C_1 = \dfrac{\alpha C D^\nu}{\sqrt{(m + \lambda_1)^{\nu-1}}}$. Hence

$$dist_m(y, -\Lambda) \leq \frac{m + \lambda_1 - \tau}{m + \lambda_1} dist_m(x, -\Lambda) + C_1(dist_m(x, -\Lambda))^{\nu-1}.$$

Let $C_1(dist_m(x, -\Lambda))^{\nu-2} = \dfrac{\tau}{2(m + \lambda_1)}$, there holds

$$dist_m(A_m(x), -\Lambda) \leq \frac{2(m + \lambda_1) - \tau}{2(m + \lambda_1)} dist_m(x, -\Lambda). \tag{15}$$

Since $\dfrac{2(m + \lambda_1) - \tau}{2(m + \lambda_1)} < 1$, we obtain

$$A_m(x) \in B_\varepsilon^-, \quad \forall u \in B_\varepsilon^-.$$

If $x \in B_\varepsilon^-$ is a nontrivial critical point of I, it is clear that $I'(x) = x - A_m x = 0$. It follows from (15) that $x \in -\Lambda \setminus \{0\}$. Combining (11) and remark 2.3, we have $x(k) < 0$. Consequently, x is a negative solution of (1).

(ii) can be discussed similarly, we only need to change y^+ to y^- to prove (ii). For simplicity, we omit its proof. □

Lemma 3.4. *Suppose z_1, z_2 be eigenvectors corresponding to eigenvalues λ_1, λ_2 of (2) and $u \in H_2 = \text{span}\{z_1, z_2\}$. If $r > \dfrac{\lambda_2}{\alpha}$, then $I(x) \to -\infty$ as $\|x\|_m \to +\infty$.*

Proof. (1) If $r = +\infty$. From (9), for any $x \in H$, we have $I(x) \to -\infty$ as $\|x\|_m \to +\infty$.

(2) Assume $r \in \left(\dfrac{\lambda_2}{\alpha}, +\infty\right)$. For $u \in H_2$, $x = \varepsilon_1 z_1 + \varepsilon_2 z_2$. In general, we can suppose $(z_1, z_2) = 0$. Thus $\|x\|^2 = \langle x, x \rangle = \langle \varepsilon_1 z_1 + \varepsilon_2 z_2, \varepsilon_1 z_1 + \varepsilon_2 z_2 \rangle = \varepsilon_1^2 \|z_1\|^2 + \varepsilon_2^2 \|z_2\|^2$ and there exists ε satisfying $0 < \varepsilon <$

$\min \left\{ r - \dfrac{\lambda_1}{\alpha}, r - \dfrac{\lambda_2}{\alpha} \right\}$. From $\lim\limits_{|x| \to +\infty} \dfrac{f(k,x)}{x} = r$, for any $k \in [1, T]$ and $x \in \mathbf{R}$, there exists a constant $\zeta > 0$ such that

$$F(k, x) \geq \frac{r - \varepsilon}{2} x^2 - \zeta.$$

Then for $x \in H_2$, it follows

$$I(x) \leq \frac{1}{2}(\lambda_1 - \alpha r + \alpha\varepsilon)\varepsilon_1^2 \|z_1\|^2 + \frac{1}{2}(\lambda_2 - \alpha r + \alpha\varepsilon)\varepsilon_2^2 \|z_2\|^2 + T\zeta\alpha.$$

That is, $I(x) \to -\infty$ as $\|x\|_m \to +\infty$ for $\lambda_1 - \alpha r + \alpha\varepsilon < 0$ and $\lambda_2 - \alpha r + \alpha\varepsilon < 0$. $\qquad\square$

Now we are in the position to prove Theorem 1.1 by using Lemma 2.9.

Proof of Theorem 1.1. From (15), we get $\alpha F(k, x) + \dfrac{m}{2}|x|^2 \leq (m + \lambda_1 - \tau)\dfrac{1}{2}|x|^2 + \dfrac{\alpha C}{\nu}|x|^\nu$, which together with (14) gives that

$$I(x) \geq \frac{\tau}{2(m + \lambda_1)}\|x\|_m^2 - \frac{\alpha C D^\nu}{\nu}\|x\|_m^\nu.$$

It follows from (12) that $\|x^\pm\| \leq \dfrac{1}{\sqrt{m + \lambda_1}} dist_m(x, \mp\Lambda) \leq \dfrac{1}{\sqrt{m + \lambda_1}}\varepsilon_0$ for any $x \in \overline{B_\varepsilon^+} \cap \overline{B_\varepsilon^-}$. Then there is $c_0 > -\infty$ such that $\inf\limits_{x \in \overline{B_\varepsilon^+} \cap \overline{B_\varepsilon^-}} I(x) = c_0$. Moreover, in view of Lemma 3.4, we can choose $R > 2\varepsilon_0$ such that $I(x) < c_0 - 1$ for all $x \in H_2$ and $\|x\|_m = R$. To apply Lemma 2.9, we define a path $g : [0, 1] \to H_2$ as

$$g(s) = R\frac{z_1 \cos(\pi s) + z_2 \sin(\pi s)}{\|z_1 \cos(\pi s) + z_2 \sin(\pi s)\|_m}.$$

By direct computation, we get

$$g(0) = R\frac{z_1}{\|z_1\|_m} \in B_\varepsilon^+ \setminus B_\varepsilon^-, \qquad g(1) = -R\frac{z_1}{\|z_1\|_m} \in B_\varepsilon^- \setminus B_\varepsilon^+,$$

$$\inf_{x \in \overline{B_\varepsilon^+} \cap \overline{B_\varepsilon^-}} I(x) > \sup_{\tau \in [0,1]} I(g(\tau)).$$

Combining Lemmas 3.1, 3.3 and 2.9, we find there is a critical point in $H \setminus (\overline{B_\varepsilon^+} \cup \overline{B_\varepsilon^-})$ corresponding to a sign-changing solution of (1). Moreover, we also have a critical point in $B_\varepsilon^+ \setminus \overline{B_\varepsilon^-}$ $(B_\varepsilon^- \setminus \overline{B_\varepsilon^+})$ corresponding to a positive solution(a negative solution) of (1). The proof is completed. $\qquad\square$

Remark 3.5. *By Lemma 3.2 and Remark 2.10, we find the proof of Theorem 1.2 is analogous to the proof of Theorem 1.1 and we therefore omit it.*

Finally, we exhibit an example to illustrate Theorem 1.1.

Example 3.6. *Consider* (1) *with* $\alpha = 2$ *and* $f(k, x) = \dfrac{|x| - n}{|x| + 1} mx$. *Here* $m > 2\sin^2\dfrac{3\pi}{2(2T + 1)}$, $0 < n < \dfrac{4\sin^2\dfrac{\pi}{2(2T + 1)}}{m}$. *Then*

$$F(k, u) = \begin{cases} \dfrac{mx^2}{2} - m(n + 1)(x - ln(1 + x)), & x \geq 0, \\ \dfrac{mx^2}{2} + m(n + 1)(x + ln(1 - x)), & x < 0. \end{cases}$$

By direct computation, we have $\lim\limits_{|x| \to \infty} [2F(k, x) - xf(k, x)] = -\infty$ *uniformly for* $k \in [1, T]$ *and* $\lambda_1 = 4\sin^2\dfrac{\pi}{2(2T + 1)}$, $\lambda_2 = 4\sin^2\dfrac{3\pi}{2(2T + 1)}$. *In addition,* $f_0 = \max\limits_{k \in [1, T]} \limsup\limits_{x \to 0}\left|\dfrac{f(k, x)}{x}\right| = mn < \lambda_1$ *and* $\lim\limits_{|x| \to \infty} \dfrac{f(k, x)}{x} = r = m > \dfrac{\lambda_2}{2}$. *Then* (1) *satisfies conditions of Theorem 1.1, thus it has at least a positive solution, a negative solution and a sign-changing solution.*

For the case $T = 2$, here $m > 2\sin^2\frac{3\pi}{10} \approx 1.28$, $0 < n < \dfrac{4\sin^2\frac{\pi}{10}}{m} \approx \dfrac{0.36}{m}$, thus we can choose $m = 1.5$ and $n = 0.1$. After not very complicated calculation, we find $(0, 0.161, 0.298, 0.298)$, $(0, 8.73, -5.77, -5.77)$, $(0, -0.1616, -0.2975, -0.2975)$ and $(0, -8.73, 5.77, 5.77)$ are nontrivial approximate solutions of (1).

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