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Multiple Contrast Tests in the Presence of Heteroscedasticity

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1. Situation

For $j=1,\dots,n_i$ and $i=1,\dots,k$, let X_{ij} denote the j^{th} observation under the i^{th} treatment. Suppose the X_{ij} to be independently normal with means μ_i and possibly unequal variances σ_i^2. We are interested in the vector of ratio contrasts $\gamma = (\gamma_1,\dots,\gamma_q)^T$ where for $1 \le l \le q$

$$\gamma_l = \frac{\sum_{i=1}^{k} c_{li}\mu_i}{\sum_{i=1}^{k} d_{li}\mu_i}.$$

The vectors $c_l = (c_{l1},\dots,c_{lk})^T$ and $d_l = (d_{l1},\dots,d_{lk})^T$ consist of real constants. The hypothesis to test is

$$H_0: \ \gamma_l = \theta_l \quad \forall l = 1,\dots,q$$

against several alternatives due to given testing problems and for given relative thresholds $\theta_1,\dots,\theta_q$.

Problem: Existing approaches are not robust!

2. Procedure

Test statistic for $1 \le l \le q$:

$$T_l = \frac{\sum_{i=1}^{k}(c_{li} - \theta_l d_{li})\overline{X}_i}{\sqrt{\sum_{i=1}^{k}(c_{li} - \theta_l d_{li})^2 \hat{\sigma}_i^2 / n_i}}$$

→ Separate sample variances

Degrees of freedom for $1 \le l \le q$:

$$df_l = \frac{\left(\sum_{i=1}^{k}(c_{li} - \theta_l d_{li})^2 \hat{\sigma}_i^2 / n_i\right)^2}{\sum_{i=1}^{k} \frac{\left((c_{li} - \theta_l d_{li})^2 \hat{\sigma}_i^2 / n_i\right)^2}{n_i - 1}}$$

→ Due to Satterthwaite (1946)

Correlation matrix $R = (\rho_{lm})$ with $(1 \le l \ne m \le q)$

$$\rho_{lm} = \frac{\sum_{i=1}^{k}(c_{li} - \theta_l d_{li})(c_{mi} - \theta_m d_{mi})\hat{\sigma}_i^2 / n_i}{\sqrt{\sum_{i=1}^{k}(c_{li} - \theta_l d_{li})^2 \hat{\sigma}_i^2 / n_i}\sqrt{\sum_{i=1}^{k}(c_{mi} - \theta_m d_{mi})^2 \hat{\sigma}_i^2 / n_i}}$$

→ Plug-in of the sample variances

So, each test statistic T_l $(1 \le l \le q)$ is compared with a **separate** – „its own" – **quantile** coming from a q-variate t-distribution with adjusted degrees of freedom and a correlation matrix for which a variance plug-in is used. This procedure is referred to as **PI**.

3. Competing approaches

HOM: Multiple contrast test (MCT) for homogeneous variances and originally for difference contrasts; pooled sample variance; correlations without variance estimator; common degree of freedom
→ same quantile for all contrasts

GH: Originally for all-pair comparisons as a difference contrast (Games and Howell, 1976); separate sample variances; correlations without variance estimator; separate degrees of freedom due to Satterthwaite (1946)
→ separate quantiles for the contrasts

HTL: Originally for unbalanced settings (Hochberg and Tamhane, 1987) or for special contrasts only (Tamhane and Logan, 2004); separate sample variances; average of correlations and degrees of freedom as in **PI**
→ same quantile for all contrasts

4. Simulation studies

- Different settings and allocations, numbers of treatments, contrasts
- Global and local α level focused (weak and strong control of FWER)
- 100000 simulation runs in statistic software R (mvtnorm)

Example: Dunnett contrast, one-sided, $\theta_i = 1.25$ $(1 \le i \le q)$, $\alpha = 0.05$

Setting	HOM	GH	PI	HTL
n=(10,10,10), σ=(10,10,50)	0.0471	0.0556	0.0496	0.0506
n=(4,13,13), σ=(10,10,50)	0.0055	0.0654	0.0527	0.0585
n=(13,13,4), σ=(10,10,50)	0.1970	0.0545	0.0514	0.0770
n=(10,10,10), σ=(30,30,30)	0.0492	0.0487	0.0485	0.0487
n=(10,10,10,10,10), σ=(10,10,10,10,50)	0.0634	0.0562	0.0498	0.0496
n=(6,11,11,11,11), σ=(10,10,10,10,50)	0.0338	0.0619	0.0500	0.0485
n=(11,11,11,11,6), σ=(10,10,10,10,50)	0.1462	0.0573	0.0520	0.0593
n=(10,10,10,10,10), σ=(30,30,30,30,30)	0.0497	0.0500	0.0493	0.0493

Results:

- **HOM** does not control the α level
- **PI** almost exact, tightest α ranges
- **GH** and **HTL** often too both liberal and conservative, respectively
- **GH** differs more dependent on the special contrasts
- **HTL** differs more dependent on settings, widest α ranges

5. Simultaneous confidence intervals

Lower bounds of approximate $(1-\alpha)100\%$ SCI of **PI** for $\gamma = (\gamma_1,\dots,\gamma_q)^T$:

$$\hat{\theta}_l^{(l)} = \frac{-B_l - \sqrt{B_l^2 - 4A_l C_l}}{2A_l}$$

$$A_l = \left(\sum_{i=1}^{k} d_{li}\overline{X}_i\right)^2 - t_{q,1-\alpha}^2(df_l, R)\sum_{i=1}^{k} d_{li}^2\hat{\sigma}_i^2 / n_i, \quad C_l = \left(\sum_{i=1}^{k} c_{li}\overline{X}_i\right)^2 - t_{q,1-\alpha}^2(df_l, R)\sum_{i=1}^{k} c_{li}^2\hat{\sigma}_i^2 / n_i,$$

$$B_l = -2\left(\left(\sum_{i=1}^{k} c_{li}\overline{X}_i\right)\left(\sum_{i=1}^{k} d_{li}\overline{X}_i\right) - t_{q,1-\alpha}^2(df_l, R)\sum_{i=1}^{k} c_{li}d_{li}\hat{\sigma}_i^2 / n_i\right), \quad A_l > 0, \quad 1 \le l \le q$$

6. Conclusions

- Adjusted degrees of freedom not sufficient to handle heteroscedasticity, plug-in **variance estimators necessary**
- New approach (**PI**) keeps FWER best for all contrasts and settings
- Taking **averages** of correlations and degrees of freedom **too rough**
- Also other approaches studied which take the minimum (conservative) or maximum (liberal) of the Welch-adjusted degrees of freedom, respectively
- Same theory also considered for MCT of DIFFERENCES in means with similar results
- R code available from first author

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