

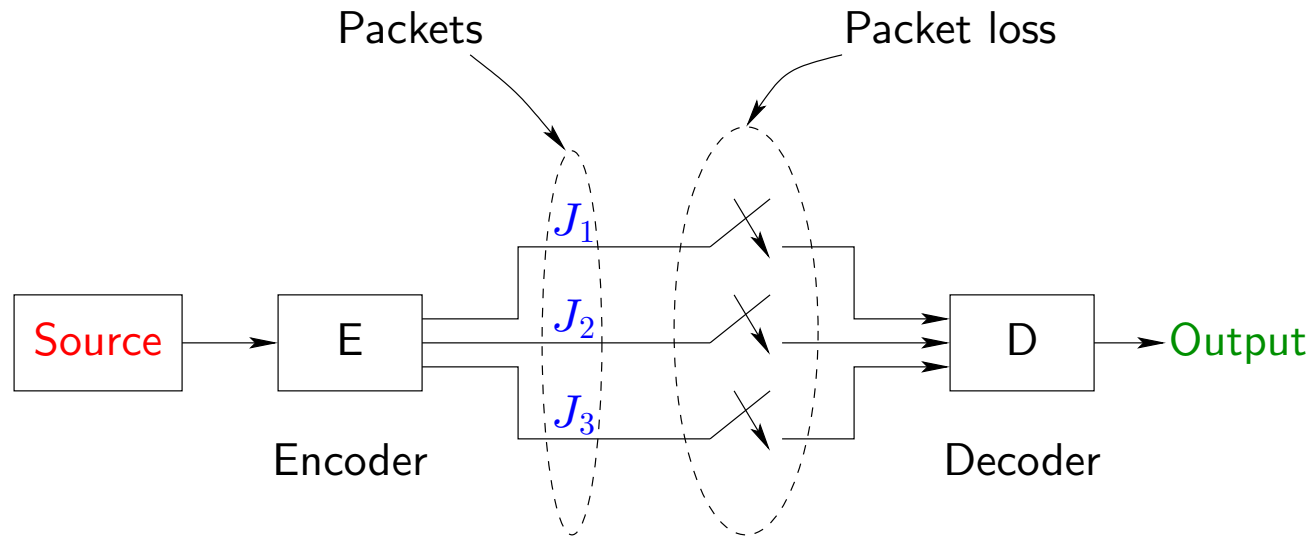
Multiple Description Coding with Many Channels

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Joint work with Gerhard Kramer (Bell Labs)
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Packet Networks



Model: Packets are either **lost completely** or received **error-free**.

How do we deal with packet losses?

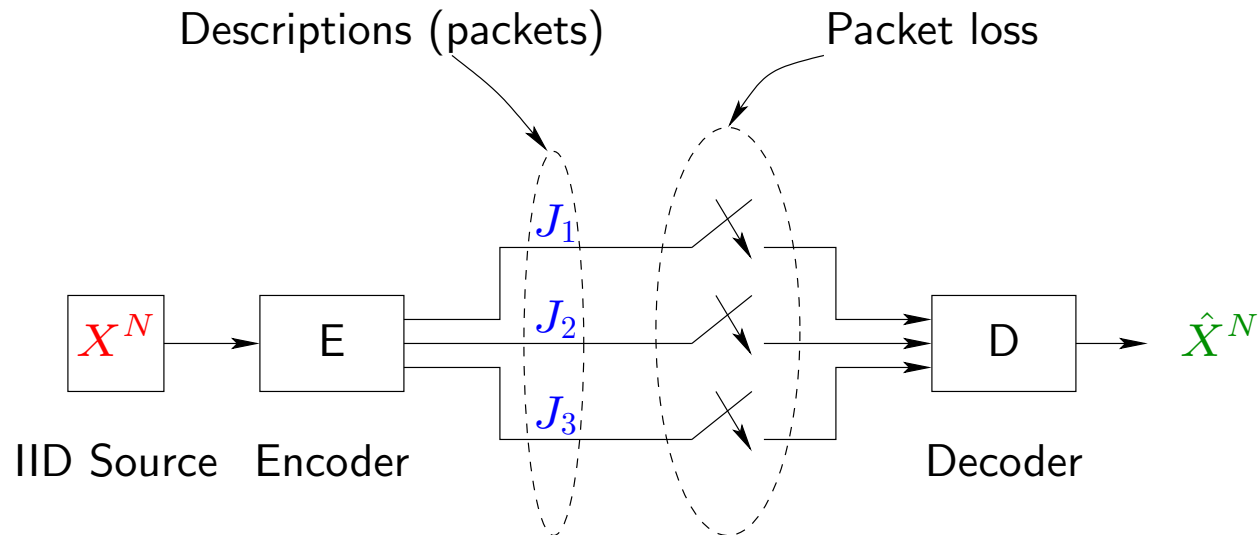
- Request a retransmission.
 - Good for loss-less transmission.
 - Not feasible for **real time** data such as **voice** and **video**.

Alternate Approach:

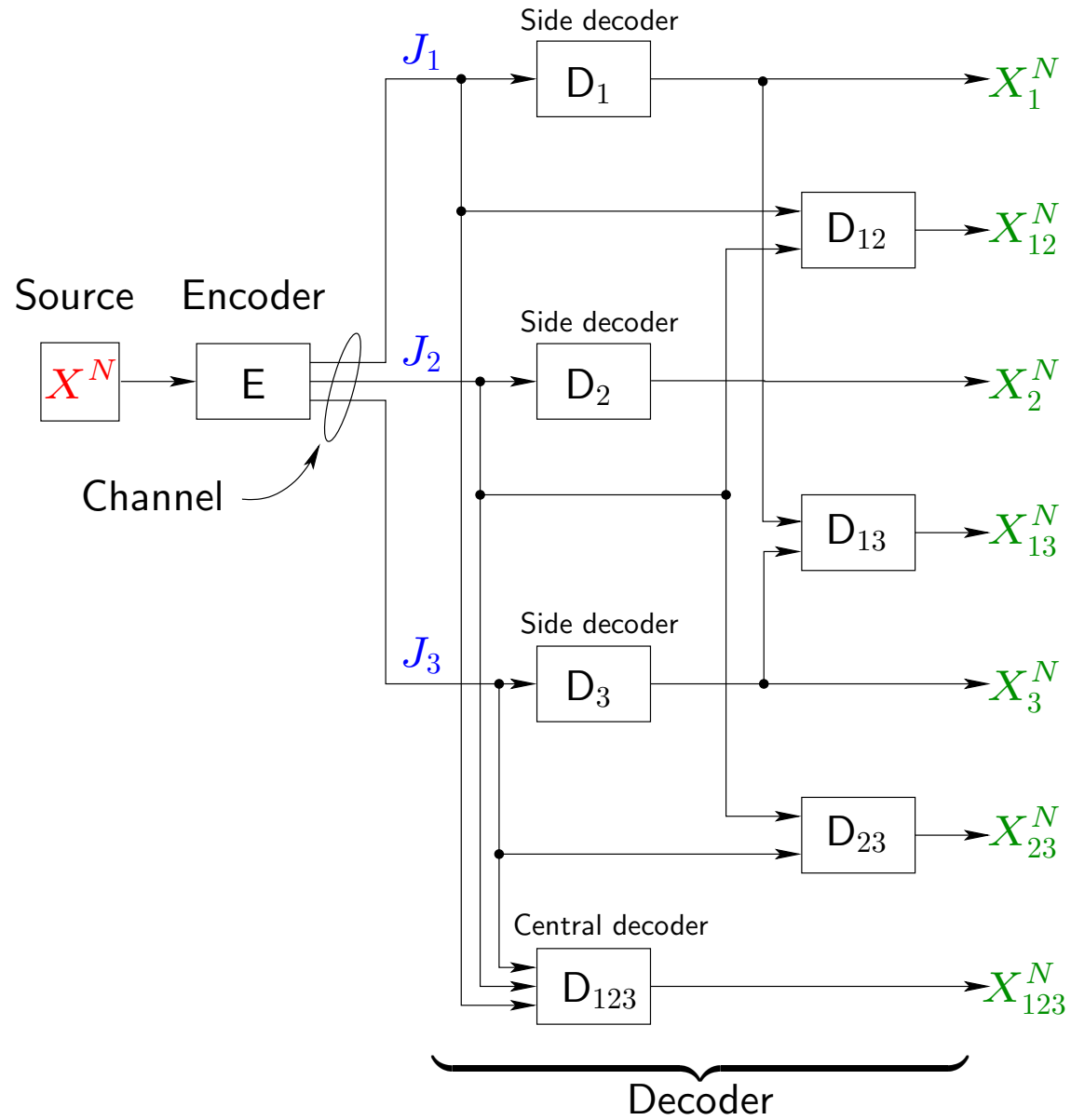
- Reconstruct using available packets.
 - Requires adding **redundancy** to packets (**coding**).
 - **Advantage**: Graceful degradation of output quality when packet losses increase.

The second approach is called **Multiple Description Coding**.

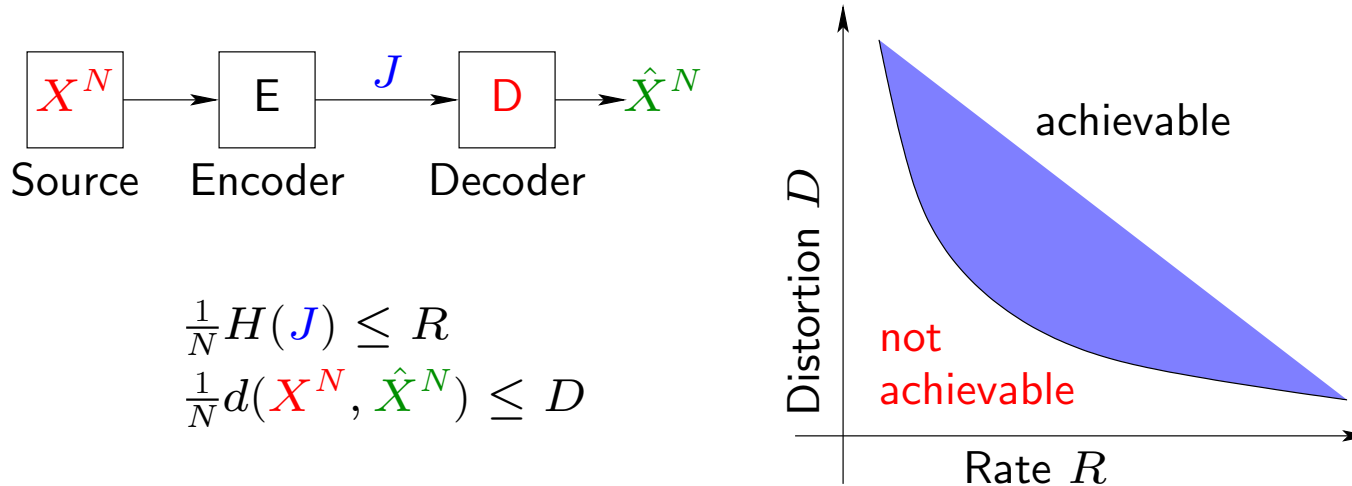
Example: Coding with 3 Descriptions



- The source is IID vector X^N (length N).
- The encoder produces L “descriptions” J_1, \dots, J_L of X^N .
- The decoder produces an output \hat{X}^N from the **available** descriptions.



Review of Rate-Distortion Theory



Theorem 1. [Shannon] *The RD region is the convex set $R \geq R(D)$ where*

$$R(D) = \min_{\hat{X}} I(X; \hat{X}) \quad \text{s.t.} \quad \mathbb{E}d(X, \hat{X}) \leq D$$

minimized over all \hat{X} jointly distributed with X .

Gaussian Source $X \sim N(0, 1)$: $R(D) = \frac{1}{2} \log\left(\frac{1}{D}\right)$

Multiple Description (MD) coding

Source: Length N vector X^N of i.i.d. random variables.

Encoder: $X^N \rightarrow \{J_1, \dots, J_L\}$ which are the L “descriptions” of X^N at rates R_1, \dots, R_L per source symbol.

Descriptions: $J_l = f_l(X^N)$, $H(J_l) \leq NR_l$, $l = 1, \dots, L$.

Decoder: Consists of $2^L - 1$ decoders: one for each non-empty subset of the available descriptions.

Decoder Outputs: $X_S^N = g_S(\{J_l : l \in \mathcal{S}\})$ where $\mathcal{S} \subseteq \{1, \dots, L\}$, $\mathcal{S} \neq \emptyset$.

In the last example ($L = 3$):

$$\mathcal{S} = \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \text{ or } \{1, 2, 3\}$$

Problem Statement

- **Problem:** What is the Rate-Distortion (RD) region?
- The **rates** (L parameters) are

$$R_1, \dots, R_L.$$

- The **distortions** ($(2^L - 1)$ parameters) are

$$D_{\mathcal{S}} = \frac{1}{N} \mathbb{E} d(\mathbf{X}^N, \mathbf{X}_{\mathcal{S}}^N), \quad \mathcal{S} \subseteq \{1, \dots, L\}, \mathcal{S} \neq \emptyset$$

- The RD region is $(L + 2^L - 1)$ -dimensional.
- **Remark:** For $L = 1$, it is Shannon's RD region.

$L = 2$ case

The RD region is the set of possible rates and distortions as $N \rightarrow \infty$:

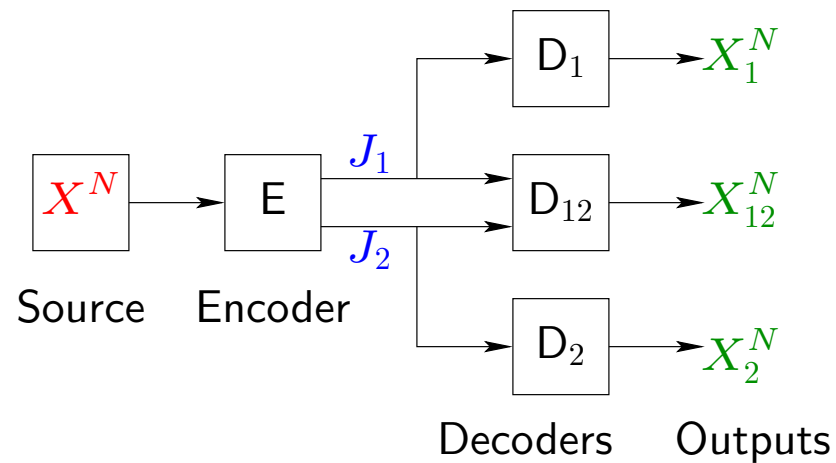
$$R_1 = \frac{1}{N}H(J_1)$$

$$D_1 = \frac{1}{N}Ed(\mathbf{X}^N, \mathbf{X}_1^N)$$

$$R_2 = \frac{1}{N}H(J_2)$$

$$D_2 = \frac{1}{N}Ed(\mathbf{X}^N, \mathbf{X}_2^N)$$

$$D_{12} = \frac{1}{N}Ed(\mathbf{X}^N, \mathbf{X}_{12}^N)$$



Review of Past Research

El Gamal and Cover (1982) found an achievable region for $L = 2$:

$$R_1 \geq I(\mathbf{X}; \mathbf{X}_1)$$

$$R_2 \geq I(\mathbf{X}; \mathbf{X}_2)$$

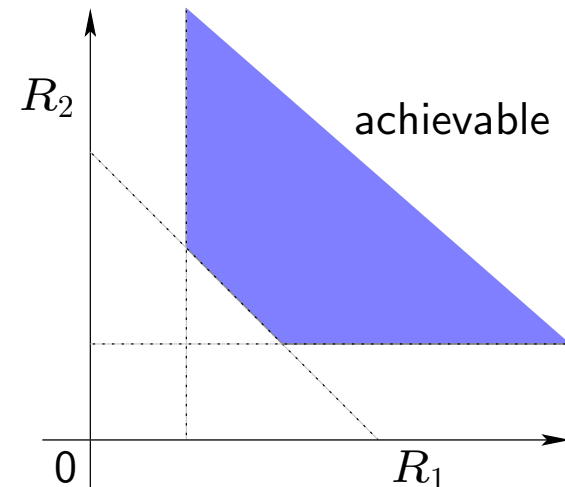
$$R_1 + R_2 \geq I(\mathbf{X}; \mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_{12}) + I(\mathbf{X}_1; \mathbf{X}_2)$$

$$D_S \geq \text{Ed}(\mathbf{X}, \mathbf{X}_S), \quad S = 1, 2, 12$$

where $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_{12}$ are any r.v.'s jointly distributed with the source \mathbf{X} .

Remark 1: The convex hull of this region is achievable by time-sharing.

Remark 2: Gives the RD region for the **Gaussian source**.



Rate region for fixed $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_{12}$ such that $D_S \geq \text{Ed}(\mathbf{X}, \mathbf{X}_S)$.

- **Ozarow (1980)** computed an **outer bound** on the RD region for $L = 2$ for the Gaussian source. The bound **meets the inner bound** by El Gamal and Cover.
- **Zhang and Berger (1987)** provided a stronger achievable result than El Gamal and Cover for $L = 2$. For the **binary symmetric source** with Hamming distortion measure, their result provides a **strict improvement**.
- **Wolf, Wyner and Ziv (1980), Witsenhausen and Wyner (1981), Zhang and Berger (1983)** provided some results for the binary symmetric source.

An Achievable Region for $L > 2$

Theorem 2. *The RD region contains the rates and distortions satisfying*

$$\begin{aligned} \sum_{l \in \mathcal{S}} R_l &\geq (|\mathcal{S}| - 1)I(\mathbf{X}; \mathbf{X}_\emptyset) - H(\mathbf{X}_\mathcal{U} : \mathcal{U} \in 2^{\mathcal{S}} | \mathbf{X}) \\ &\quad + \sum_{\mathcal{T} \subseteq \mathcal{S}} H(\mathbf{X}_\mathcal{T} | \mathbf{X}_\mathcal{U} : \mathcal{U} \in 2^{\mathcal{T}} - \mathcal{T}) \\ D_{\mathcal{S}} &\geq \text{Ed}_{\mathcal{S}}(\mathbf{X}, \mathbf{X}_{\mathcal{S}}) \end{aligned}$$

for every $\emptyset \neq \mathcal{S} \subseteq \mathcal{L} = \{1, \dots, L\}$ and some joint distribution between outputs $\{\mathbf{X}_{\mathcal{S}}\}$ and the source \mathbf{X} .

Remark: This result generalizes the results of **El Gamal and Cover**, and of **Zhang and Berger**.

Gaussian Source: Outer Bound on the RD Region

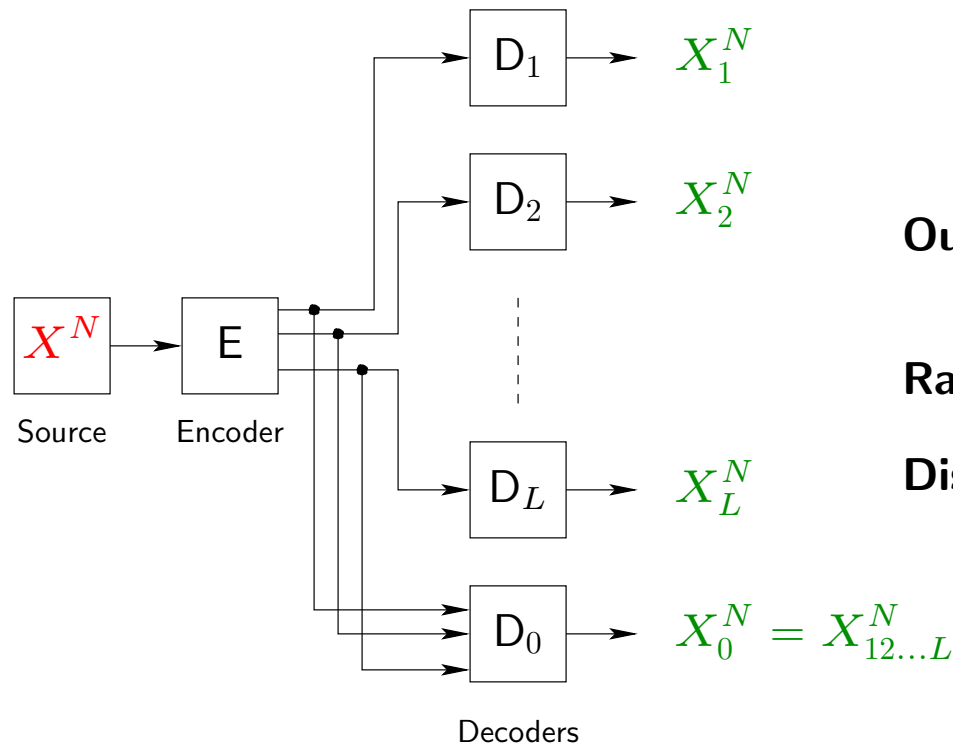
- **Gaussian source:** $X \sim N(0, 1)$.
- **Squared-error distortion:** $d(x, y) = |x - y|^2$.
- An **outer bound** on the RD region:

Theorem 3. *The RD region is contained in*

$$\exp\left(-2 \sum_{k \in \mathcal{K}} R_k\right) \leq \min_{\{\mathcal{K}_m\}_{m=1}^M} \inf_{\lambda \geq 0} \left(D_{\mathcal{K}} \frac{\prod_{m=1}^M (D_{\mathcal{K}_m} + \lambda)}{(D_{\mathcal{K}} + \lambda)(1 + \lambda)^{M-1}} \right), \quad \forall \mathcal{K} \in 2^{\mathcal{L}}$$

minimized over all partitions $\{\mathcal{K}_m\}$ of \mathcal{K} .

Special Case: L Channels and $L + 1$ Decoders



Outputs: $X_1^N, X_2^N, \dots, X_L^N$
and $X_0^N = X_{12\dots L}^N$

Rates: R_1, R_2, \dots, R_L

Distortions: D_1, D_2, \dots, D_L
and D_0

Keep only **side** and **central** decoders. Ignore all other decoders.

Inner and Outer Bounds on the RD region

- **Inner Bound:** Computable from our **achievable region** (Theorem 2).
- **Outer Bound:** Computable from Theorem 3 for the **Gaussian source**
- **Tightness of Bounds:** The inner and outer bounds meet for over some range of rates and distortions for the Gaussian source.

Example: 3-Channel 4-Decoder Problem

Take $L = 3$, $D_1 = D_2 = D_3 = 1/2$ and $D_0 = 1/16$.

Outer Bound:

$$R_l \geq 0.5, \quad l = 1, 2, 3$$

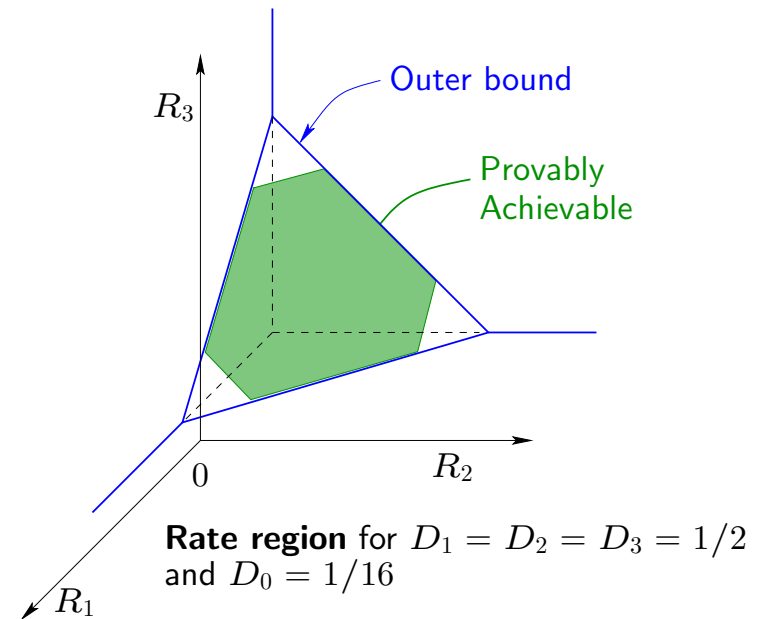
$$R_1 + R_2 + R_3 \geq 2.1755$$

Achievable Rates:

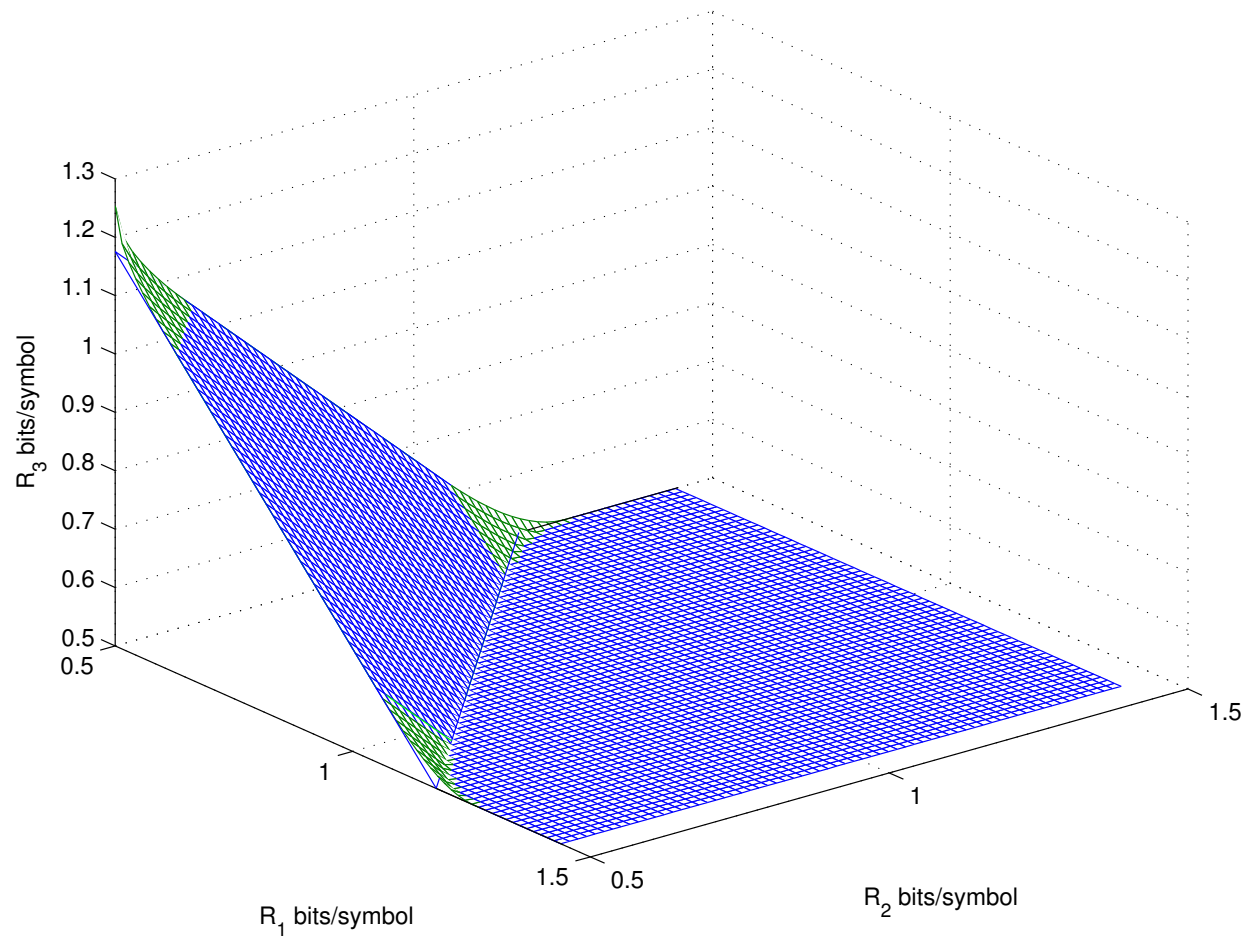
$$R_l \geq 0.5, \quad l = 1, 2, 3$$

$$R_1 + R_2 + R_3 = 2.1755$$

$$R_l + R_m \geq 1.1258, \quad l < m$$



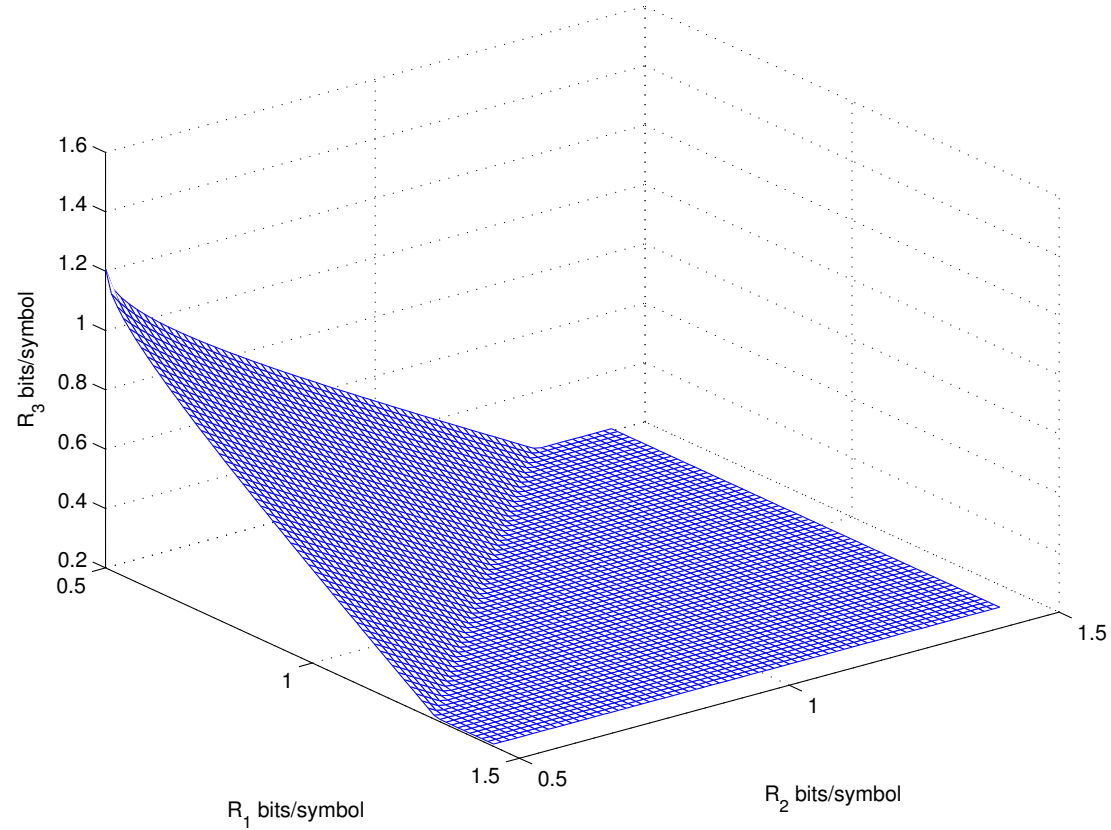
Remark: Excess rate = $2.1755 - R(D_0) = 0.1755$ bits.



Blue Region: Inner and outer bounds meet on a hexagon.

Green Region: Inner bound (does not meet outer bound).

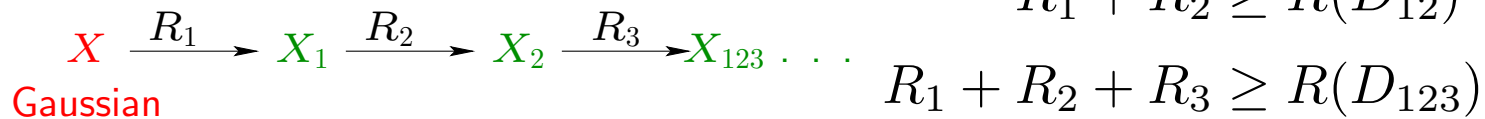
Another Example: $L = 3$, $D_1 = D_2 = 1/2$, $D_3 = 3/4$, and $D_0 = 1/16$:



The RD Region

Gaussian Sources are Successively Refinable

Chains:

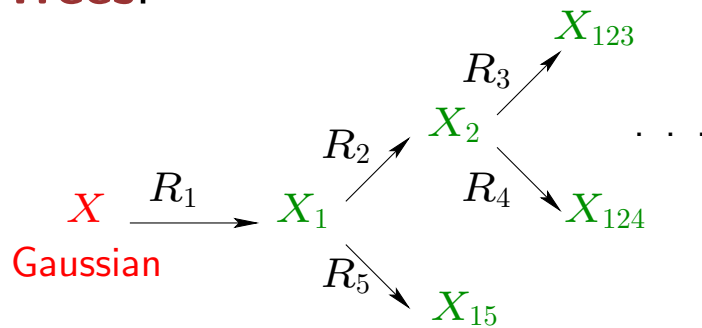


$$R_1 \geq R(D_1)$$

$$R_1 + R_2 \geq R(D_{12})$$

$$R_1 + R_2 + R_3 \geq R(D_{123})$$

Trees:



$$R_1 \geq R(D_1)$$

$$R_1 + R_2 \geq R(D_{12})$$

$$R_1 + R_2 + R_3 \geq R(D_{123})$$

$$R_1 + R_2 + R_4 \geq R(D_{124})$$

$$R_1 + R_5 \geq R(D_{15})$$

Summary

- **Multiple Description Coding:**
 - Motivation: coding for packet networks.
- **Results:**
 - An achievable region
 - An outer bound for the Gaussian source.
- **Still unsolved:**
 - The Gaussian problem for $L \geq 3$
 - Non-Gaussian sources