Classics in Mathematics

Charles B. Morrey, Jr.

Multiple Integrals in the Calculus of Variations

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Reprint of the 1966 Edition

Springer

Originally published as Vol. 130 of the series *Grundlehren der mathematischen Wissenschaften*

ISBN 978-3-540-69915-6 e-ISBN 978-3-540-69952-1

DOI 10.1007/978-3-540-69952-1

Classics in Mathematics ISSN 1431-0821

Library of Congress Control Number: 2008932928

Mathematics Subject Classification (2000): 49-xx, 58Exx, 35N15, 46E35, 46E39

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Die Grundlehren der mathematischen Wissenschaften

in Einzeldarstellungen
mit besonderer Berücksichtigung
der Anwendungsgebiete

Band 130

Multiple Integrals in the Calculus of Variations

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Springer-Verlag Berlin Heidelberg New York 1966

Library of Congress Catalog Card Number 66-24365

Printed in Germany

Title No. 5113

Preface

The principal theme of this book is "the existence and differentiability of the solutions of variational problems involving multiple integrals." We shall discuss the corresponding questions for single integrals only very briefly since these have been discussed adequately in every other book on the calculus of variations. Moreover, applications to engineering, physics, etc., are not discussed at all; however, we do discuss *mathematical* applications to such subjects as the theory of harmonic integrals and the so-called "$\bar{\partial}$-Neumann" problem (see Chapters 7 and 8). Since the plan of the book is described in Section 1.2 below we shall merely make a few observations here.

In order to study the questions mentioned above it is necessary to use some very elementary theorems about convex functions and operators on Banach and Hilbert spaces and some special function spaces, now known as "SOBOLEV spaces". However, most of the facts which we use concerning these spaces were known before the war when a different terminology was used (see CALKIN and MORREY [5]); but we have included some powerful new results due to CALDERON in our exposition in Chapter 3. The definitions of these spaces and some of the proofs have been made simpler by using the most elementary ideas of distribution theory; however, almost no other use has been made of that theory and no knowledge of that theory is required in order to read this book. Of course we have found it necessary to develop the theory of linear elliptic systems at some length in order to present our desired differentiability results. We found it particularly essential to consider "weak solutions" of such systems in which we were often forced to allow discontinuous coefficients; in this connection, we include an exposition of the DE GIORGI—NASH—MOSER results. And we include in Chapter 6 a proof of the analyticity of the solutions (on the interior and at the boundary) of the most general non-linear analytic elliptic system with general regular (as in AGMON, DOUGLIS, and NIRENBERG) boundary conditions. But we confine ourselves to functions which are analytic, of class C^∞, of class C_μ^n or C^n (see § 1.2), or in some Sobolev space H_p^m with m an integer ≥ 0 (except in Chapter 9). These latter spaces have been defined for all real m in a domain (or manifold) or on its boundary and have been used by many authors in their studies of linear systems. We have not included a study of these spaces since (i) this book is already sufficiently long, (ii) we took no part in this development, and (iii) these spaces are adequately discussed in other *books* (see A. FRIEDMAN [2], HORMANDER [1], LIONS [2]) as well as in many papers (see § 1.8 and papers by LIONS and MAGENES).

The research of the author which is reported on in this book has been partially supported for several years by the Office of Naval Research under contract Nonr 222(62) and was partially supported during the year 1961—62, while the author was in France, by the National Science Foundation under the grant G—19782.

Berkeley, August 1966

CHARLES B. MORREY, JR.

Contents

Chapter 1

Introduction

Chapter 2

Semi-classical results

Chapter 3

The spaces H_p^m and H_{p0}^m

Chapter 4

Existence theorems

Chapter 5

Differentiability of weak solutions

Chapter 6

Regularity theorems for the solutions of general elliptic systems and boundary value problems

Chapter 7

A variational method in the theory of harmonic integrals

Chapter 8

The $\bar{\partial}$-NEUMANN problem on strongly pseudo-convex manifolds

Chapter 9

Introduction to parametric Integrals; two dimensional problems

Chapter 10

The higher dimensional PLATEAU problems

Multiple Integrals in the Calculus of Variations

Chapter 1

Introduction

1.1. Introductory remarks

The principal theme of these lectures is "the existence and differentiability of the solutions of variational problems involving multiple integrals." I shall discuss the corresponding questions for single integrals only very briefly since these have been adequately discussed in every book on the calculus of variations (see, for instance, AKHIEZER [1], BLISS [1], BOLZA [1], CARATHEODORY [2], FUNK [1], PARS [1]. Moreover, I shall not discuss applications to engineering, physics, etc., at all, although I shall mention some *mathematical* applications.

In general, I shall consider integrals of the form

$$I(z,G) = \int_G f[x, z(x), \nabla z(x)]\,dx \tag{1.1.1}$$

where G is a domain,

$$x = (x^1, \ldots, x^\nu),\ z = (z^1, \ldots, z^N),\ dx = dx^1 \ldots dx^\nu, \tag{1.1.2}$$

$z(x)$ is a vector function, ∇z denotes its gradient which is the set of functions $\{z^i_{,\alpha}\}$, where $z^i_{,\alpha}$ denotes $\partial z^i/\partial x^\alpha$, and $f(x, z, p)$ ($p = \{p^i_\alpha\}$) is generally assumed continuous in all its arguments. The integrals

$$\int_a^b \sqrt{1 + (dz/dx)^2}\,dx \quad\text{and}\quad \iint_G \left[\left(\frac{\partial z}{\partial x^1}\right)^2 + \left(\frac{\partial z}{\partial x^2}\right)^2\right] dx^1\,dx^2$$

are familiar examples of integrals of the form (1.1.1.) in which $N = 1$ in both cases, $\nu = 1$ in the first case, $\nu = 2$ in the second case and the corresponding functions f are defined respectively by

$$f(x, z, p) = \sqrt{1 + p^2}, \qquad f(x, z, p) = p_1^2 + p_2^2$$

where we have omitted the superscripts on z and p since $N = 1$. The second integral is a special case of the *Dirichlet integral* which is defined in general by

$$D(z,G) = \int_G |\nabla z|^2\,dx,\ f(x, z, p) = |p|^2 = \sum_{i,\alpha} (p^i_\alpha)^2. \tag{1.1.3}$$

Another example is the *area integral*

$$A(z,G) = \iint_G \sqrt{\left[\frac{\partial(z^2, z^3)}{\partial(x^1, x^2)}\right]^2 + \left[\frac{\partial(z^3, z^1)}{\partial(x^1, x^2)}\right]^2 + \left[\frac{\partial(z^1, z^2)}{\partial(x^1, x^2)}\right]^2}\,dx^1\,dx^2 \tag{1 1.4}$$

which gives the area of the surface

$$z^i = z^i(x^1, x^2), \quad (x^1, x^2) \in G, \quad i = 1, 2, 3. \tag{1.1.5}$$

It is to be noticed that the area integral has the special property that it is invariant under diffeomorphisms (1—1 differentiable mappings, etc.) of the domain G onto other domains. This is the first example of an *integral in parametric form*. I shall discuss such integrals later (in Chapters 9 and 10).

I shall also discuss briefly integrals like that in (1.1.1) but involving derivatives of higher order. And, of course, the variational *method* has been used in problems which involve a "functional" not at all like the integral in (1.1.1); as for example in proving the Riemann mapping theorem where one minimizes $\sup |f(z)|$ among all schlicht functions $f(z)$ defined on the given simply connected region G for which $f(z_0) = 0$ and $f'(z_0) = 1$ at some given point z_0 in G.

We shall consider only problems in which the domain G is fixed; variations in G may be taken care of by transformations of coordinates. We shall usually consider problems involving fixed boundary values; we shall discuss other problems but will not derive the *transversality* conditions for such problems.

1.2. The plan of the book: notation

In this chapter we attempt to present an overall view of the principal theme of the book as stated at the beginning of the preceding section. However, we do not include a discussion of integrals in parametric form; these are discussed at some length in Chapters 9 and 10. The material in this book is not presented in its logical order. A possible logical order would be § 1.1—1.5, Chapter 2, Chapter 3, §§ 5.1—5.8, § 5.12, Chapter 6, §§ 1.6—1.9, §§ 4.1, 4.3, 4.4. Then the reader must skip back and forth as required among the material of § 1.10, 1.11, 4.2, 5.9, 5.10 and 5.11. Then the remainder of the book may be read substantially in order. Actually, Chapters 7 and 8 could be read immediately after § 5.8.

We begin by presenting background material including derivations, under restrictive hypotheses, of Euler's equations and the classical necessary conditions of Legendre and Weierstrass. Next, we include a brief and incomplete presentation of the classical so-called "sufficiency" conditions, including references to other works where a more complete presentation may be found.

The second half of this chapter presents a reasonably complete outline of the existence and differentiability theory for the solutions of variational problems. This begins with a brief discussion of the development of the direct methods and of the successively more general classes of "admissible" functions, culminating in the so-called "Sobolev spaces".

These are then defined and discussed briefly after which two theorems on lower-semicontinuity are presented. These are not the most general theorems possible but are selected for the simplicity of their proofs which, however, assume that the reader is willing to grant the truth of some well-known theorems on the Sobolev spaces. The relevant theorems about these spaces are proved in Chapter 3 and more general lower-semicontinuity and existence theorems are presented in Chapter 4.

In Section 1.10 the differentiability results are stated and some preliminary results are proved. In Section 1.11, an outline of the differentiability theory is presented. It is first shown that the solutions are "weak solutions" of the Euler equations. The theory of these non-linear equations is reduced to that of linear equations which, initially, may have discontinuous coefficients. The theory of these general linear equations is discussed in detail in Chapter 5. However, the higher order differentiability for the solutions of *systems* of Euler equations required the same methods as are used in studying systems of equations of higher order. Accordingly, we present in Chapter 6 many of the results in the two recent papers of AGMON, DOUGLIS, and NIRENBERG ([1], [2]) concerning the solutions and weak solutions of such systems. Both the L_p-estimates and the SCHAUDER-type estimates (concerning HÖLDER continuity) are presented. We have included sections in both Chapters 5 und 6 proving the analyticity, including analyticity at the boundary, of the solutions of both linear and non-linear analytic elliptic equations and systems; the most general "properly elliptic" systems with "complementing boundary conditions" (see § 6.1) are treated. The proof of analyticity in this generality is new. In Chapter 2 we present well-known facts about harmonic functions and generalized potentials and conclude with proofs of the CALDERON-ZYGMUND inequalities and of the maximum principle for the solutions of second order equations.

In Chapters 7 and 8, we present applications of the variational method to the HODGE theory of harmonic integrals and to the so-called $\bar{\partial}$-NEUMANN problem for exterior differential forms on strongly pseudo-convex complex analytic manifolds with boundary. In Chapter 9, we present a brief discussion of ν-dimensional parametric problems in general and then discuss the two dimensional Plateau problem in Euclidean space and on a Riemannian manifold. The chapter concludes with the author's simplified proof of the existence theorem of CESARI [4], DANSKIN, and SIGALOV [2] for the general two dimensional parametric problem and some incomplete results concerning the differentiability of the solutions of such problems. In Chapter 10, we present the author's simplification of the very important resent work of REIFENBERG [1], [2], and [3] concerning the higher dimensional PLATEAU problem and the author's extension of these results to varieties on a Riemannian manifold.

Notations. For the most part, we use standard notations. G and D will denote domains which are bounded unless otherwise specified. We denote the boundary of D by ∂D and its closure by $\bar{D}$. We shall often use the notation $D \subset \subset G$ to mean that $\bar{D}$ is compact and $\bar{D} \subset G$. $B(x_0, R)$ denotes the ball with center at x_0 and radius R. γ_ν and Γ_ν denote the ν-measure and $(\nu - 1)$-measure of $B(0,1)$ and $\partial B(0,1)$, respectively. We often denote $\partial B(0,1)$ by Σ. Most of the time (unless otherwise specified) we let R_q be q-dimensional number space with the usual metric and abbreviate $B(0,R)$ to B_R, denote by σ the $(\nu - 1)$-plane $x^\nu = 0$, and define

$$\begin{aligned} &R_\nu^+ = \{x \mid x^\nu > 0\}, && R_\nu^- = \{x \mid x^\nu < 0\} && \\ &G_R = B_R \cap R_\nu^+, && \Sigma_R = \partial B_R \cap R_\nu^+, && \sigma_R = B_R \cap \sigma \\ &G_R^- = B_R \cap R_\nu^-, && \Sigma_R^- = \partial B_R \cap R_\nu^-. && \end{aligned} \tag{1.2.1}$$

If S is a set in R_q, $|S|$ denotes its Lebesgue q-measure; if x is a point, $d(x, S)$ denotes the distance of x from S. We define

$$[a, b] = \{x \mid a^\alpha \le x^\alpha \le b^\alpha, \quad \alpha = 1, \ldots, \nu, \; x \in R_\nu\}.$$

In the case of boundary integrals, we often use dx'_α to denote $n_\alpha \, dS$ where dS is the surface area and n_α is the α-th component of the *exterior normal*. We say that a function $u \in C^n(G)$ iff (if and only if) u and its partial derivatives of order $\le n$ are continuous on G and $u \in C^n(\bar{G})$ iff $u \in C^n(G)$ and each of its derivatives of order $\le n$ can be extended to be continuous on $\bar{G}$. If $0 < \mu \le 1$, $u \in C^n_\mu(G)$ (or $C^n_\mu(\bar{G})$) $\Leftrightarrow$ (i.e. iff) $u \in C^n(G)$ (or $C^n(\bar{G})$) and all the derivatives of order $\le n$ satisfy a Hölder (Lipschitz if $\mu = 1$) condition on each compact subset of G (or on the whole of $\bar{G}$ as extended). If $u \in C^0_\mu(\bar{G})$, then $h_\mu(u, \bar{G}) = \sup |x_2 - x_1|^{-\mu} \cdot |u(x_2) - u(x_1)|$ for x_1 and $x_2 \in \bar{G}$ and $x_1 \ne x_2$. A domain G is said to be of class C^n (or C^n_μ, $0 < \mu \le 1$) iff G is bounded and each point P_0 of ∂G is in a neighborhood $\mathfrak{n}$ on $\bar{G}$ which can be mapped by a $1-1$ mapping of class C^n (or C^n_μ), together with its inverse, onto $G_R \cup \sigma_R$ for some R in such a way that P_0 corresponds to the origin and $\mathfrak{n} \cup \partial G$ corresponds to σ_R. If $u \in C^1(G)$, we denote its derivatives $\partial u / \partial x^\alpha$ by $u_{,\alpha}$. If $u \in C_2(G)$, then $\nabla^2 u$ denotes the tensor $u_{,\alpha\beta}$ where α and β run independently from to ν. Likewise $\nabla^3 u = \{u_{,\alpha\beta\gamma}\}$, etc., and $|\nabla^2 u|^2 = \sum_{\alpha,\beta} |u_{,\alpha\beta}|^2$, etc. If G is also of class C^1, then Green's theorem becomes (in our notations)

$$\int_G u_{,\alpha}(x) \, dx = \int_{\partial G} u \, n_\alpha \, dS = \int_{\partial G} u \, dx'_\alpha .$$

Sometimes when we wish to consider u as a function of some single x^α, we write $x = (x^\alpha, x'_\alpha)$ and $u(x) = u(x^\alpha, x'_\alpha)$ where x'_α denotes the remaining x^β. One dimensional or $(\nu - 1)$-dimensional integrals are then indicated as might be expected. We often let α denote a "multi-index", i.e. a vector $(\alpha_1, \ldots, \alpha_\nu)$ in which each α_i is a non-negative integer. We then define

$$|\alpha| = \alpha_1 + \cdots + \alpha_\nu, \; D^\alpha u = \frac{\partial^{|\alpha|} u}{(\partial x^1)^{\alpha_1} \ldots (\partial x^\nu)^{\alpha_\nu}} \; (u \in C^{|\alpha|}(G))$$

$$\alpha! = (\alpha_1!) \ldots (\alpha_\nu!), \; C_\alpha = \frac{|\alpha|!}{\alpha!}, \xi^\alpha = (\xi^1)^{\alpha_1} \ldots (\xi^\nu)^{\alpha_\nu}.$$

Using this notation

$$|\nabla^m u|^2 = \sum_{|\alpha|=m} C_\alpha |D^\alpha u|^2.$$

We shall denote constants by C or Z with or without subscripts. These constants will, perhaps depend on other constants; in this case we may write $C = C(h, \mu)$ if C depends only on h and μ, for example. However, even though we may distinguish between different constants in some discussion by inserting subscripts, there is no guarantee that C_2, for example, will always denote the same constant. We sometimes denote the support of u by spt u. We denote by $C_c^\infty(G)$, $C_c^n(G)$, and $C_{\mu c}^n(G)$ the sets of functions in $C^\infty(G)$, $C^n(G)$, or $C_\mu^n(G)$, respectively, which have support in G (i.e. which vanish on and near ∂G). But it is handy to say that u has support in $G_R \cup \sigma_R \Leftrightarrow u$ vanishes on and near Σ_R (see 1.2.1); we allow $u(x)$ to be $\neq 0$ on σ_R.

1.3. Very brief historical remarks

Problems in the calculus of variations which involve only single integrals ($\nu = 1$) have been discussed at least since the time of the BERNOULLI's. Although there was some early consideration of double integrals, it was RIEMANN who aroused great interest in them by proving many interesting results in function theory by assuming DIRICHLET's *principle* which may be stated as follows: *There is a unique function which minimizes the* DIRICHLET *integral among all functions of class C^1 on a domain G and continuous on $\bar{G}$ which takes on given values on the boundary ∂G and, moreover, that function is harmonic on G.*

RIEMANN's work was criticized on the grounds that just because the integral was bounded below among the competing functions it didn't follow that the greatest lower bound was *taken on* in the class of competing functions. In fact an example was given of a (1-dimensional) integral of the type (1.1.1) for which there is no minimizing function and another was given of continuous boundary values on the unit circle such that $D(z, G) = +\infty$ for every z as above having those boundary values.

The first example is the integral (see COURANT [3])

$$(1.3.1) \qquad I(z, G) = \int_0^1 \left[1 + \left(\frac{dz}{dx}\right)^2\right]^{1/4} dx, \; G = (0,1),$$

the admissible functions z being those $\in C^1$ on $[0,1]$ with

$$z(0) = 0 \text{ and } z(1) = 1.$$

Obviously $I(z, G) > 1$ for every such z, $I(z, G)$ has no upper bound and if we define

$$z_r(x) = \begin{cases} 0 & , 0 \leq x \leq r \qquad , 0 < r < 1, \\ -1 + [1 + 3(x - r)^2/(1 - r)^2]^{1/2}, & r \leq x \leq 1 \end{cases}$$

we see that $I(z_r, G) \to 1$ as $r \to 1^-$.

The second example is the following (see COURANT [3]): It is now known that Dirichlet's principle holds for a circle and that each function harmonic on the unit circle has the form

$$(1.3.2) \qquad w(r, \Theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\Theta + b_n \sin n\Theta), \ (a_n, b_n \text{ const}),$$

in polar coordinates and that the Dirichlet integral is

$$(1.3.3) \qquad D(w, G) = \pi \sum_{n=1}^{\infty} n (a_n^2 + b_n^2)$$

provided this sum converges. But if we define

$$a_n = k^{-2} \text{ if } n = k! \quad , a_n = b_n = 0 \text{ otherwise},$$

we see that the series in (1.3.2) converges uniformly but that in (1.3.3) reduces to

$$\pi \sum_{k=1}^{\infty} \frac{k!}{k^4}$$

which diverges.

DIRICHLET's principle was established rigorously in certain important cases by HILBERT, LEBESGUE [2] and others shorly after 1900. That was the beginning of the so-called "direct methods" of the calculus of variations of which we shall say more later.

There was renewed interest in one dimensional problems with the advent of the MORSE theory of the critical points of functionals in which M. MORSE generalized his theory of critical points of functions defined on finite-dimensional manifolds [1] to certain functionals defined on infinite-dimensional spaces [2], [3]. He was able to obtain the MORSE inequalities between the numbers of possibly "unstable" (i.e. critical but not minimizing) geodesics (and unstable minmal surfaces) having various indices (see also MORSE and TOMPKINS, [1]—[4]). Except for the latter (which could be reduced to the case of curves), MORSE's theory was applied mainly to one-dimensional problems. However, within the last two years, SMALE and PALAIS and SMALE have found a modification of MORSE's theory which is applicable to a wide class of multiple integral problems.

Variational methods are beginning to be used in differential geometry. For example, the author and Eells (see MORREY and EELS, MORREY,

[11] and Chapter 7) developed the HODGE theory ([1], [2]) by variational methods (HODGE's original idea [1]). HÖRMANDER [2], KOHN [1], SPENCER (KOHN and SPENCER), and the author (MORREY [19], [20]) have applied variational techniques to the study of the $\bar{\partial}$-Neumann problem for exterior differential forms on complex analytic manifolds (see Chapter 8; the author encountered this problem in his work on the analytic embedding of real-analytic manifolds (MORREY [13]). Very recently, EELLS and SAMPSON have proved the existence of "harmonic" mappings (i.e. mappings which minimize an intrinsic Dirichlet integral) from one compact manifold into a manifold having negative curvature. Since the inf. of this integral is zero if the dimension of the compact manifold >2, they found it necessary to use a gradient line method which led to a non-linear system of parabolic equations which they then solved; the curvature restriction was essential in their work.

1.4. The Euler equations

After a number of special problems had been solved, EULER deduced in 1744 the first general necessary condition, now known as EULER's equation, which must be satisfied by a minimizing or maximizing arc. His derivation, given for the case $N = \nu = 1$, proceeds as follows: Suppose that the function z is of class C^1 on $[a, b]$ $(= G)$ minimizes (for example) the integral $I(z, G)$ among all similar functions having the same values at a and b. Then, if ζ is *any* function of class C^1 on $[a, b]$ which vanishes at a and b, the function $z + \lambda\zeta$ is, for every λ, of class C^1 on $[a, b]$ and has the same values as z at a and b. Thus, if we define

$$\varphi(\lambda) = I(z + \lambda\zeta, G) = \int_a^b f[x, z(x) + \lambda\zeta(x), z'(x) + \lambda\zeta'(x)]\, dx \tag{1.4.1}$$

φ must take on its minimum for $\lambda = 0$. If we assume that f is of class C^1 in its arguments, we find by differentiating (1.4.1) and setting $\lambda = 0$ that

$$\int_a^b \{\zeta'(x)\cdot f_p[x, z(x), z'(x)] + \zeta(x) f_z[x, z(x), z'(x)]\}\, dx = 0 \qquad \left(f_p = \frac{\partial f}{\partial p}, \text{ etc.}\right). \tag{1.4.2}$$

The integral in (1.4.2) is called the *first variation* of the integral I; it is supposed to vanish for every ζ of class C^1 on $[a, b]$ which vanishes at a and b. *If we now assume that f and z are of class C^2 on* $[a, b]$ (EULER had no compunctions about this) we can integrate (1.4.2) by parts to obtain

$$\int_a^b \zeta(x)\cdot\left\{f_z - \frac{d}{dx} f_p\right\} dx = 0, \quad f_p = f_p[x, z(x), \nabla z(x)], \text{ etc.} \tag{1.4.3}$$

Since (1.4.3) holds for all ζ as above, it follows that the equation

$$\frac{d}{dx} f_p = f_z \tag{1.4.4}$$

must hold. This is *Euler's equation* for the integral I in this simple case. If we write out (1.4.4) in full, we obtain

$$f_{pp} \cdot z'' + f_{pz} z' + f_{px} = f_z \tag{1.4.5}$$

which shows that Euler's equation is non-linear and of the second order. It is, however, linear in z''; equations which are linear in the derivatives of highest order are frequently called *quasi-linear*. The equation evidently becomes singular whenever $f_{pp} = 0$. Hence *regular* variational problems are those for which f_{pp} never vanishes; in that case, it is assumed that $f_{pp} > 0$ which turns out to make minimum problems more natural than maximum problems.

It is clear that this derivation generalizes to the most general integral (1.1.1) provided that f and the minimizing (or maximizing, etc.) function z is of class C^2 on the closed domain G which has a sufficiently smooth boundary. Then, if z minimizes I among all (vector) functions of class C^1 with the same boundary values and ζ is any such vector which vanishes on the boundary or G, it follows that $z + \lambda\zeta$ is a "competing" or "admissible" function for each λ so that if φ is defined by

$$\varphi(\lambda) = I(z + \lambda\zeta, G) \tag{1.4.6}$$

then $\varphi'(0) = 0$. This leads to the condition that

$$\int_G \sum_{i=1}^{N} \left\{ \sum_{\alpha=1}^{\nu} \zeta^i_{,\alpha} f_{p^i_\alpha} + \zeta^i f_{z^i} \right\} dx = 0 \tag{1.4.7}$$

for all ζ as indicated. The integral in (1.4.7) is the *first variation* of the general integral (1.1.1). Integrating (1.4.7) by parts leads to

$$\int_G \sum_{i=1}^{N} \zeta^i \cdot \left\{ f_{z^i} - \sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^\alpha} f_{p^i} \right\} dx = 0.$$

Since this is zero for all vectors ζ, it follows that

$$\sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^\alpha} f_{p^i_\alpha} = f_{z^i}, \quad i = 1, \ldots, N \tag{1.4.8}$$

which is a quasi-linear system of partial differential equations of the second order. In the case $N = 1$, it reduces to

$$\sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^\alpha} f_{p_\alpha} = f_z, \quad \text{or}$$

$$\sum_{\alpha,\beta=1}^{\nu} f_{p_\alpha p_\beta} z_{,\alpha\beta} + \sum_{\alpha=1}^{\nu} (f_{p_\alpha z} z_{,\alpha} + f_{p_\alpha x^\alpha}) = f_z. \tag{1.4.9}$$

The equation (1.4.9) is evidently singular whenever the quadratic form

$$\sum_{\alpha,\beta} f_{p_\alpha p_\beta}(x, z, p)\,\lambda_\alpha \lambda_\beta{}^* \tag{1.4.10}$$

in λ is degenerate.

We notice from (1.4.5) that if $N = \nu = 1$ and f depends only on p and the problem is regular, then Euler's equation reduces to

$$z'' = 0.$$

In general, if f depends only on $p (= p^i_\alpha)$, Euler's equation has the form

$$\sum_{j,\alpha,\beta} f_{p^i_\alpha p^j_\beta} z^j_{,\alpha\beta} = 0, \quad i = 1, \ldots, N$$

and every linear vector function is a solution. In particular, if $N = 1$ and $f = |p|^2$, Euler's equation is just Laplaces equation

$$\Delta z \equiv \sum_\alpha z_{,\alpha\alpha} = 0.$$

In case $f = (1 + |p|^2)^{1/4}$ as in the first example in § 1.3, we see that

$$4 f_{pp} = (2 - p^2)(1 + p^2)^{-7/4}$$

which is not always > 0. On the other hand $f_{pp} > 0$ if $|p| < \sqrt{2}$ so classical results which we shall discuss later (see § 1.6) show that the linear function $z(x) = x$ minimizes the integral among all arcs having $|z'(x)| \leq \sqrt{2}$.

We now revert to equation (1.4.9). If we take, for instance, $N = 1$, $\nu = 2$, $f = p_1^2 - p_2^2$, then (1.4.9) becomes

$$z_{,11} - z_{,22} \left(\equiv \frac{\partial^2 z}{\partial (x^1)^2} - \frac{\partial^2 z}{\partial (x^2)^2} \right) = 0$$

which is of *hyperbolic type*. Moreover, the integral (1.1.1) with this f obviously has no minimum or maximum, whatever boundary values are given for z. Anyhow, it is well known that boundary value problems are not natural for equations of hyperbolic type. If $\nu > 2$ a greater variety of types may occur, depending on the signature of the quadratic form (1.4.10). A similar objection occurs in all cases except those in which the form (1.4.10) is *positive definite* or negative definite; we shall restrict ourselves to the case where it is positive definite. In this case Euler's equation is of *elliptic type*. The choice of this condition on f is re-enforced by analogy with the case $\nu = 1$; in that case $f_{pp} \geq 0$ *implies the convexity* (see § 1.8) *of f as a function of p* for each (x, z) and the non-negative definiteness of the form (1.4.10) is equivalent to the convexity of f as a function of $p_1, \ldots, p_\nu$ for each set $(x^1, \ldots, x^\nu, z)$. Our choice is re-enforced further by the classical derivation given in the next section.

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* Greek indices are summed from 1 to ν and Latin indices are summed from 1 to N. Hereafter we shall usually employ the summation convention in which repeated indices are summed and summation signs omitted.

1.5. Other classical necessary conditions

Suppose that f is of class C^2 in its arguments, that $N = 1$, that z is of class C^1 on the closure of G, and that z minimizes $I(z, G)$ among all functions Z of the same class which coincide with z on the boundary and are sufficiently close to z in the C^1 norm, i.e.

$$|Z(x) - z(x)| \leq \delta, \quad |\nabla Z(x) - \nabla z(x)| \leq \delta, \quad x \text{ on } G.$$

In classical terminology, we say that z furnishes a *weak relative minimum* to $I(z, G)$. We shall show that this implies the non-negative definiteness of the form (1.4.10) when $z = z(x)$ and $p = \nabla z(x)$. We note that our hypotheses imply that for each ζ of the type above, vanishing on the boundary, the function $\varphi(\lambda)$, defined by (1.4.6) is of class C^2 for $|\lambda| \leq \lambda_0 (> 0)$ and has a relative minimum at $\lambda = 0$. This implies that

$$(1.5.1) \quad \begin{aligned} &\varphi''(0) = \int_G \Big[\sum_{\alpha,\beta} a^{\alpha\beta}(x)\, \zeta_{,\alpha} \zeta_{,\beta} + 2 \sum_\alpha b^\alpha(x)\, \zeta \zeta_{,\alpha} + c(x)\, \zeta^2\Big]\, dx \geq 0 \\ &a^{\alpha\beta}(x) = f_{p_\alpha p_\beta}[x, z(x), \nabla z(x)], \quad b^\alpha = f_{p_\alpha z}, \quad c = f_{zz}, \end{aligned}$$

for all ζ as described. The integral in (1.5.1) is called the *second variation* of the integral (1.1.1). By approximations, it follows that (1.5.1) holds for all LIPSCHITZ functions ζ which vanish on the boundary. Now let us select a point $x_0 \in G$ and a unit vector λ, and let us choose new coordinates y related to x by a transformation

$$(1.5.2) \quad y^\gamma = \sum_\alpha d_\alpha^\gamma (x^\alpha - x_0^\alpha), \quad x^\alpha - x_0^\alpha = \sum_\gamma d_\alpha^\gamma y^\gamma, \quad \lambda_\alpha = d_\alpha^1$$

where d is a constant orthogonal matrix so that λ is the unit vector in the y^1 direction, and define

$$(1.5.3) \quad \begin{aligned} \omega(y) &= \zeta[x(y)], \quad 'a^{\gamma\delta}(y) = a^{\alpha\beta}[x(y)]\, d_\alpha^\gamma d_\beta^\delta, \\ 'b^\gamma(y) &= b^\alpha[x(y)]\, d_\alpha^\gamma, \quad 'c(y) = c[x(y)]. \end{aligned}$$

Then if G' denotes the image of G,

$$\varphi''(0) = \int_{G'} \Big[\sum_{\gamma,\delta} 'a^{\gamma\delta}(y)\, \omega_{,\gamma}\, \omega_{,\delta} + 2'b^\gamma\, \omega \cdot \omega_{,\gamma} + 'c\,\omega^2\Big]\, dy \geq 0.$$

Now, choose $0 < h < H$ so small that the support of $\omega \subset G'$, where

$$(1.5.4) \quad \omega(y^1, \ldots, y^\nu) = \begin{cases} (h - |y^1|)(1 - r/H), & \text{if } |y^1| \leq h, \ 0 \leq r \leq H \\ 0 & \text{, otherwise} \end{cases}$$

$$r^2 = (y^2)^2 + \cdots + (y^\nu)^2.$$

Then if we divide $\varphi''(0)$ by the measure of the support of ω and then let h and $H \to 0$ so that $h/H \to 0$, we conclude that

$$'a^{11}(0) = a^{\alpha\beta}(x_0)\, d_\alpha^1 d_\beta^1 = a^{\alpha\beta}(x_0)\, \lambda_\alpha \lambda_\beta \geq 0$$

which is the stated result. This is called the *Legendre condition.*

If we repeat this derivation for the case of the general integral (1.1.1), assuming $f \in C^2$ of course, we obtain

$$\varphi''(0) = \int_G \Big\{\sum_{i,j}\sum_{\alpha,\beta} a_{ij}^{\alpha\beta}\zeta^i_{,\alpha}\zeta^j_{\beta} + 2\sum_\alpha b_{ij}^\alpha \zeta^i\zeta^j_{,\alpha} + c_{ij}\zeta^i\zeta^j\Big\}\,dx \geq 0 \tag{1.5.1'}$$

$$a_{ij}^{\alpha\beta}(x) = f_{p^i_\alpha p^j_\beta}[x, z(x), \nabla z(x)],\quad b_{ij}^\alpha = f_{z^i p^j_\alpha},\quad c_{ij} = f_{z^i z^j}.$$

Making the change of variables (1.5.2) and (1.5.3) and setting

$$\omega^i(y^1, \ldots, y^\nu) = \xi^i\omega(y^1, \ldots, y^\nu),\quad (i = 1, \ldots, N)$$

where ξ is an arbitrary constant vector and ω is defined by (1.5.4), and letting h and $H \to 0$ as above, we obtain

$$\sum_{i,j,\alpha,\beta} f_{p^i_\alpha p^j_\beta}[x_0, z(x_0), \nabla z(x_0)]\lambda_\alpha\lambda_\beta\xi^i\xi^j \geq 0 \text{ for all } \lambda, \xi, \tag{1.5.5}$$

which is known as the *Legendre-Hadamard condition* (HADAMARD [1]). In this case, we say that the integral (1.1.1) or the integrand f is *regular* if the inequality holds in (1.5.5) for all $\lambda \neq 0$ and $\xi \neq 0$. It turns out that the system (1.4.8) of Euler's equations is *strongly elliptic* in the sense defined by NIRENBERG [2].

Let us suppose, now, that $f \in C^2$ everywhere and that $z \in C^1$ on $\bar{G}$ and minimizes $I(z, G)$, as given by (1.1.1), among all such functions with the same boundary values. A simple approximation argument shows that z minimizes I among all LIPSCHITZ functions with the same boundary values. Let us choose $x_0 \in G$ and a unit vector λ, and let us introduce the y coordinates as in (1.5.2) and let us define (using part of the notation of (1.5.4)

$$\zeta^i_h(x) = \xi^i\omega_h[y(x)]$$

$$\omega_h(x) = \begin{cases} (y^1 + h)\,\varphi(rh^{-1/2}) , & -h \leq y^1 \leq 0 ,\quad 0 \leq r \leq h^{1/2}, \\ h^{1/2}(h^{1/2} - y^1)\cdot\varphi(rh^{-1/2}), & 0 \leq y^1 \leq h^{1/2},\quad 0 \leq r \leq h^{1/2}, \\ 0 , & \text{otherwise} \end{cases} \tag{1.5.6}$$

where $\varphi \in C^\infty$ on $[0,1]$ with $\varphi(0) = 1$ and $\varphi(\varrho) = 0$ for ϱ near 1. Since the first variation vanishes, we have

$$\int_G \big[f(x, z + \zeta_h, \nabla z + \nabla\zeta_h) - f(x, z, \nabla z) - \zeta^i_h f_{z^i} - \zeta^i_{h,\alpha} f_{p^i_\alpha}\big]\,dx \geq 0,$$

$$f_{z^i} = f_{z^i}(x, z, \nabla z),\quad f_{p^i_\alpha} = f_{p^i_\alpha}(x, z, \nabla z). \tag{1.5.7}$$

We notice first that the integrand is $O(h)$ (since ζ_h and $\nabla\zeta_h$ are both small) for $x \in R^2_h$ where $0 \leq y^1 \leq h^{1/2}$, $0 \leq r \leq h^{1/2}$. By setting $y^1 = h\eta^1$, $r = h^{1/2}\varrho$ in $R^1_h(-h \leq y^1 \leq 0,\ 0 \leq r \leq h^{1/2})$, dividing by $h^{(\nu+1)/2}$ and letting $h \to 0$, we obtain

$$\int_{\varrho\leq 1} \Big\{[f(x_0, z_0, p^i_{0\alpha} + \lambda_\alpha\xi^i\varphi(\varrho)] - f(x_0, z_0, p_0) - \\ - \sum_{i,\alpha}\lambda_\alpha\,\xi^i\,\varphi(\varrho) f_{p^i_\alpha}(x_0, z_0, p_0)\Big\}\,d\eta'_1 \geq 0.$$

We may now choose a sequence $\{\varphi_n\}$ so $\varphi_n(\varrho) \to 1$ boundedly. This leads to

$$(1.5.8) \qquad f(x_0, z_0, p^i_{\alpha 0} + \lambda_\alpha \xi^i) - f(x_0, z_0, p_0) - \lambda_\alpha \xi^i f_{p^i_\alpha}(x_0, z_0, p_0) \geq 0$$

which is the *Weierstrass condition* (see GRAVES). In case $N = 1$, (1.5.8) yields the following more familiar form of this condition:

$$E(x, z, \nabla z, P) = f(x, z, P) - f(x, z, \nabla z) - (P_\alpha - z_{,\alpha}) f_{p_\alpha}(x, z, \nabla z) \geq 0$$
(1.5.9)

for all P and all x. The function $E(x, z, p, P)$ here defined is known as the *Weierstrass E-function.*

HESTENES and MACSHANE studied these general integrals in cases where $\nu = 2$. HESTENES and E. HÖLDER studied the second variation of these integrals. DEDECKER studied the first variation of very general problems on manifolds.

1.6. Classical sufficient conditions

A detailed account of classical and recent work in this field is given in the recent book by FUNK, pp. 410—433) where other references are given. I shall give only a brief introduction to this subject.

It is clear that the positiveness of the second variation along a function z guarantees that z furnishes a relative minimum to $I(Z, G)$ among all $Z(= z$ on $\partial G)$ in any finite dimensional space. However, if $N = 1$, a great deal more can be concluded, namely that z furnishes a *strong relative minimum* to I, i.e. minimizes $I(Z, G)$ among all $Z \in C^1(\bar{G})$ with $Z = z$ on ∂G for which $|Z(x) - z(x)| < \delta$ for some $\delta > 0$ regardless of the values of the derivatives. WEIERSTRASS was the first to prove such a theorem but his proof was greatly simplified by the use of HILBERT's invariant integral. Of course, the original proof was for the case $N = \nu = 1$; we present briefly an extension to the case $N = 1$, ν arbitrary.

Suppose G is of class C^2_μ, $z \in C^2_\mu(\bar{G})$, and f and f_p are of class C^3_μ in their arguments, $0 < \mu < 1$ (see § 1.2), and suppose that the second variation, as defined in (1.5.1), > 0 for each $\zeta \in C^1_c(\bar{G})$ (compact support). By a straightforward approximation, it follows that the second variation is defined for all $\zeta \in H^1_{20}(G)$ (see § 1.8). If we call the integral (1.5.1) $I_2(z; \zeta; G)$ we see from the theorems of § 1.8 below that I_2 is lower-semicontinuous with respect to weak convergence in $H^1_{20}(G)$. Moreover, from the assumed positive definiteness of the form (1.4.10), it follows from the continuity of the $a^{\alpha\beta}(x)$ (they $\in C^1_\mu(\bar{G})$ in fact) that there exist $m_1 > 0$ and M_1 such that

$$(1.6.1) \qquad a^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \geq m_1|\lambda|^2, \quad \sum_{\alpha\beta}[a^{\alpha\beta}(x)]^2 \leq M_1^2.$$

Then, from the SCHWARZ and CAUCHY inequalities, we conclude that there is a K such that

$$(1.6.2) \qquad I_2(z;\zeta;G) \geq \frac{m_1}{2}\int_G |\nabla\zeta|^2\,dx - K\int_G \zeta^2\,dx.$$

Since weak convergence in $H^1_{20}(G)$ implies strong convergence in $L_2(G)$ (RELLICH's theorem, Theorem 3.4.4), it follows that there is a ζ_0 in $H^1_{20}(G)$ (actually $C^2_\mu(\bar{G})$) which minimizes I_2 among all $\zeta \in H^1_{20}(G)$ for which $\int_G \zeta^2\,dx = 1$. Since we have assumed $I_2 > 0$ for every $\zeta \neq 0$, it follows that

$$(1.6.3) \qquad I_2(z;\zeta;G) \geq \lambda_1 \int_G \zeta^2\,dx,\ \lambda_1 > 0.$$

From the theory of §§ 5.2—5.6, it follows that there is a unique solution ζ of *Jacobi's equation*

$$(1.6.4) \qquad L\zeta \equiv \frac{\partial}{\partial x^\alpha}(a^{\alpha\beta}\zeta_{,\alpha}) + (b^\alpha_{,\alpha} - c)\zeta = 0$$

with given smooth boundary values. It is to be noted that JACOBI's equation is just the Euler equation (z fixed) corresponding to I_2. It is also the equation of variation of the Euler equation for the original I, i.e.

$$(1.6.5) \qquad L\zeta = \frac{\partial}{\partial\varrho}\left\{\frac{\partial}{\partial x^\alpha} f_{p_\alpha}[x, z+\varrho\zeta, \nabla z + \varrho\nabla\zeta] - f_z[\text{same}]\right\}_{\varrho=0}.$$

It follows from Theorems 6.8.5 and 6.8.6 that there is a unique solution of the Euler equation for all sufficiently near (in $C^2_\mu(\partial G)$) boundary values, in particular for the boundary values $z + \varrho$, and that $z = z(\varrho)$ satisfies an ordinary differential equation

$$(1.6.6) \qquad \frac{dz}{d\varrho} = F(z)$$

in the Banach space ($C^2_\mu(\bar{G})$), where $F(z)$ denotes the solution ζ of Jacobi's equation (1.6.4) with $z = z(\varrho)$ for which $\zeta = 1$ on ∂G. We shall show below that this solution ζ cannot vanish on $\bar{G}$ for ϱ sufficiently small; it is sufficient to do this for $\varrho = 0$, when $z(\varrho) =$ our solution z, on account of the continuity.

So, let ζ_1 be this solution. If $\zeta_1(x) < 0$ anywhere, then the set where this holds is an open set D and $\zeta_1 = 0$ on $\partial D(\subset G)$. Since ζ_1 is a solution on D, $I(\zeta_1, D) \leq 0$ since ζ_1 is minimizing on D. (D may not be smooth, but see Chapters 3—5). But if we set $\zeta = \zeta_1$ on D and $\zeta = 0$, otherwise, $\zeta \in H^1_{20}(G)$ so (1.6.3) holds and we must have $\zeta \equiv 0$. Hence $\zeta_1(x) \geq 0$ everywhere. Now, suppose $\zeta_1(x_0) = 0$. From Theorem 6.8.7, it follows that we may choose R so small that $B(x_0, R) \subset G$ and there is a non-vanishing solution ω of (1.6.4) on $B(x_0, R)$. Letting $\zeta_1 = \omega v$, we see that v satisfies the equation

$$a^{\alpha\beta} v_{,\alpha\beta} + (a^{\alpha\beta}_{,\beta} + 2\omega^{-1} a^{\alpha\beta} \omega_{,\beta}) v_{,\alpha} = 0$$

and $v(x_0) = 0$. But from the maximum principle as proved by *E.* HOPF [1] (see § 2.8), it follows that v cannot have a minimum interior to G. Accordingly $\zeta_1 \neq 0$ anywhere in $\bar{G}$.

Therefore it is possible to embed our solution z in a *field of extremals*. That is, there is a 1 parameter family $Z(x, \varrho)$ of solutions of Euler's equation where $Z(x, 0) = z(x)$, our given solution. $Z(x, \varrho) \in C^1(\bar{G} \times [-\varrho_0, \varrho_0])$ and $\in C^2_\mu(\bar{G})$ as a function of x for each ϱ with $z_\varrho > 0$. Consequently there are functions $P_\alpha(x, z)$ on the set Γ, where

$$\Gamma: x \in \bar{G},\ Z(x, -\varrho_0) \leq z \leq Z(x, \varrho_0), \tag{1.6.7}$$

which act as *slope-functions* for the field, i.e.

$$Z_{,\alpha}(x, \varrho) = P_\alpha[x, Z(x, \varrho)]. \tag{1.6.8}$$

By virtue of the facts that $Z(x, \varrho)$ satisfies Euler's equation for each ϱ, that (1.6.8) holds, and that if $(x, z) \in \Gamma$, then $z = Z(x, \varrho)$ for a unique ϱ on $[-\varrho_0, \varrho_0]$, we find that

$$\begin{gathered} f_z - f_{p_\alpha p_\beta}(P_{\beta x^\alpha} + P_{\beta z} P_\alpha) - f_{p_\alpha z} P_\alpha - f_{p_\alpha x^\alpha} = 0 \\ f_z = f_z[x, z, P(x, z)], \text{ etc., } (x, z) \in \Gamma. \end{gathered} \tag{1.6.9}$$

Let us define

$$\begin{gathered} I^*(z, G) = \int_G f^*(x, z, \nabla z)\, dx, \\ f^*(x, z, p) = f_{p_\alpha}[x, z, P(x, z)] \cdot [p_\alpha - P_\alpha(x, z)] + f[x, z, P(x, z)]. \end{gathered} \tag{1.6.10}$$

We observe that

$$\begin{gathered} f^*_z = [p_\alpha - P_\alpha(x, z)] \cdot \{f_{p_\alpha z} + f_{p_\alpha p_\beta} P_{\beta z}\} - P_{\alpha z} f_{p_\alpha} \\ + f_z + f_{p_\alpha} P_{\alpha z}; \quad f^*_{p_\alpha} = f_{p_\alpha}[x, z, P(x, z)]. \end{gathered} \tag{1.6.11}$$

Thus, if $z \in C^1(\bar{G})$ and $(x, z(x)) \in \Gamma$ for $x \in \bar{G}$, we see that

$$f^*_z[x, z(x), \nabla z(x)] - \frac{\partial}{\partial x^\alpha} f^*_{p_\alpha}[x, z(x), \nabla z(x)] = 0. \tag{1.6.12}$$

Accordingly *the integral $I^*(z, G)$ has the same value for all such z which have the same boundary values.* Moreover, if $z(x) \equiv Z(x, \varrho)$ for some ϱ, then

$$\begin{gathered} \nabla z(x) = P[x, z(x)], \quad f^*[x, z(x), \nabla z(x)] = f[x, z(x), \nabla z(x)], \\ I^*(z, G) = I(z, G). \end{gathered} \tag{1.6.13}$$

This integral $I^*(z, G)$ is known as *Hilbert's invariant integral*. Therefore, if $z \in C^1(\bar{G})$ and $(x, z(x)) \in \Gamma$ for all $x \in \bar{G}$, and $z(x) = z_0(x)$ on ∂G, then

$$\begin{aligned} I(z, G) - I(z_0, G) &= I(z, G) - I^*(z_0, G) = I(z, G) - I^*(z, G) \\ &= \int_G E[x, z(x), P\{x, z(x)\}, \nabla z(x)]\, dx \end{aligned} \tag{1.6.14}$$

where $E(x, z, P, p)$ is the Weierstrass E-function defined in (1.5.9). *Thus z_0 minimizes $I(z, G)$ among all such z, and hence furnishes a strong relative minimum to I, provided that*

$$E[x, z, P(x, z), p] \geq 0, \ (x, z) \in \Gamma, p \text{ arbitrary.} \tag{1.6.15}$$

This same proof shows that if (1.6.15) *holds for all (x, z, p) in some domain R, where all the (x, z) involved $\in \Gamma$, then $I(z_0, G) \leq I(z, G)$ for all z for which $[x, z(x), \nabla z(x)] \in R$ for all $x \in \bar{G}$.*

In the cases $\nu = 1$, the *Jacobi condition* is frequently stated in terms of "conjugate points". A corresponding condition for $\nu > 1$ is that the Jacobi equation has no non-zero solution which vanishes on the boundary ∂D of any sub-domain $D \subset G$; D may coincide with G or may not be smooth, in which case we say u vanishes on $\partial D \Leftrightarrow u \in H_{20}^1(D)$. The most interesting condition is that there exist a non-vanishing solution ω on $\bar{G}$; we have seen above that this is implied by the positivity of the second variation. If we then set $\zeta = \omega u$, where $u = 0$ on ∂G, then the reader may easily verify that

$$I_2(z; \zeta; G) = \int_G [\omega^2 a^{\alpha\beta} u_{,\alpha} u_{,\beta} - u^2 \omega L \omega] \, dx > 0$$

for all $u \in H_{20}^1(G)$, since $L\omega = 0$.

In cases where $\nu > 1$ and $N > 1$ it is still true (if we continue to assume the same differentiability for G, f, and z) that (1.6.2) holds with I_2 defined by (1.5.1′) even in the general regular case where (1.5.5) holds with the inequality for $\lambda \neq 0$ and $\xi \neq 0$. This is proved in § 5.2. So it is still true that if $I_2 > 0$ for all ζ, the Euler equations have a unique solution for sufficiently nearby boundary values. It is more difficult (but possible) to show that there is an N-parameter field of extremals and then it turns out that such a field does not lead so easily to an invariant integral. By allowing slope functions $P_\alpha^i(x, z)$ which are not "integrable" (i.e. there may not be z's such that $z_{,\alpha}^i = P_\alpha^i(x, z)$), Weyl [1] (see also De Donder) developed a comparatively simple field theory and showed the existence of his types of fields under certain conditions. His theory is succesful if f is convex in all the p_α^i. To treat more general cases, Caratheodory [1], Boerner, and Le Page have introduced more general field theories. The latter two noticed that exterior differential forms were a natural tool to use in forming the analog of Hilbert's invariant integral. However, the sufficient conditions developed so far are rather far from the necessary conditions and many questions remain to be answered.

1.7. The direct methods

The necessary and sufficient conditions which we have just discussed have presupposed the existence and differentiability of an extremal. In the cases $\nu = 1$, this was often proved using the existence theorems for ordinary differential equations. However, until recently, corresponding theorems for partial differential equations were not available so the direct methods were developed to handle this problem and to obtain results in the large for one dimensional problems.

As has already been said, HILBERT [1] and LEBESGUE [2] had solved the Dirichlet problem by essentially direct methods. These methods were exploited and popularized by TONELLI in a series of papers and a book ([1], [2], [3], [4], [5], [7], [8]), and have been and still are being used by many others. The idea of the direct methods is to show (i) that the integral to be minimized is lower-semicontinuous with respect to some kind of convergence, (ii) that it is bounded below in the class of "admissible functions," and (iii) that there is a *minimizing sequence* (i.e. a sequence $\{z_n\}$ of admissible functions for which $I(z_n, G)$ tends to its infimum in the class) which converges in the sense required to some admissible function.

Tonelli applied these methods to many single integral problems and some double integral problems. In doing this he found it expedient to use uniform convergence (at least on interior domains) and to allow absolutely continuous functions (satisfying the given boundary conditions) as admissible for one dimensional problems; and he defined what he called absolutely continuous functions of two variables ([6]) to handle certain double integral problems (see the next section). In the double integral problems ($N = 1$, $\nu = 2$), he found it expedient to require that $f(x, z, p)$ satisfy conditions such as

$$m|p|^k - K \leq f(x, z, p), \quad k \geq 2, \quad m > 0, \text{ where} \tag{1.7.1}$$

$$f(x, z, p) \geq 0 \quad \text{and} \quad f(x, z, 0) = 0 \quad \text{if} \quad k = 2, \tag{1.7.2}$$

in order to obtain equicontinuous minimizing sequences. However, Tonelli was not able to get a general theorem to cover the case where f satisfies (1.7.1) with $1 < k < 2$. Moreover, if one considers integrals in which $\nu > 2$, one soon finds that one would have to require $k > \nu$ in order to ensure that the functions in any minimizing sequence would be equicontinuous, at least on interior domains (see Theorem 3.5.2). To see this, one needs only to notice that the functions

$$\log\log(1 + 1/|x|), \quad 1/|x|^h, \quad 0 < |x| < 1$$

are limits of C^1 functions z_n in which

$$\int |\nabla z_n|^\nu \, dx \quad \text{and} \quad \int |\nabla z_n|^k \, dx \quad \text{for} \quad k < \nu/(h+1)$$

are uniformly bounded over the unit ball G. In the "borderline case" $k = \nu$, it is possible, in case f satisfies the supplementary condition (1.7.2), to replace an arbitrary minimizing sequence of continuous functions by a minimizing sequence each member of which is *monotone in the* *sense of Lebesgue* (i.e. takes on its max. and min. values on the boundary of each compact sub-domain); the new sequence is equicontinuous on interior domains (see ε 4.3).

However, even this Lebesgue smoothing process does not work in general for $1 < k < \nu$. In order to get a more complete existence theory, the writer and CALKIN (MORREY [5], [6], [7]) found it expedient to allow as admissible, functions which are still more general than Tonelli's functions and to allow correspondingly more general types of convergence. The new spaces of functions can now be identified with the Banach spaces $H_p^1(G)$ (see the next section) (or the Sobolev spaces $W_p^1(G)$) which have been and still are being used by many writers in many different connections (see § 1.8). In this way, the writer was able to obtain very general existence theorems. Unfortunately the solution shown to exist was known only to be in one of these general spaces and hence wasn't even known to be continuous, let alone of class C^2! So these existence theorems in themselves have only minor interest. However, at the same time,[1] the writer was able to show in the case $\nu = 2$ (N arbitrary) that these very general solutions were, in fact, of class C^2 after all provided that f satisfied the conditions (1.10.8) below with $k = \nu = 2$. A greatly simplified presentation of this old work is to be found in the author's paper [15]; recent developments have permitted further simplifications and extensions which we shall discuss later.

In general, it is still not known that the solutions are continuous if f satisfies (1.7.1) with $1 < k < \nu$. In the case $\nu = 2$, $N = 1$ TONELLI [8] showed that the solutions in this case are continuous if f is such that there is a unique minimizing function in the small. More recently, SIGALOV ([1], [3]) showed that the solution surfaces always posess conformal maps (possibly with vertical segments) in the case $\nu = 2$, $N = 1$. In the case $\nu = 2$, $N = 1$, $f = f(p)$ it was proved a long time ago by A. HAAR (see also RADO [2]) that there is a unique minimizing function z which is defined on a strictly convex domain G and which satisfies a Lipschitz condition with constant L, provided that any linear function which coincides with the given boundary values at three different points on the boundary has slope $\leq L$. This result has recently been generalized (not quite completely) by GILBARG and STAMPACCHIA [3]. The author has completed the extension of HAAR's results and has extended those

[1] This work was completed during the year 1937—38 and the author lectured on it in the seminar of Marston Morse during the spring of 1938 and also reported on this work in an invited lecture to the American Mathematical Society at its meeting in Pasadena, California, on December 2, 1939 [6]. The necessary theorems about the H_p^1 spaces (called $\mathfrak{P}_p$ at that time) (see § 1.8) were published in the papers referred to above by CALKIN and the author [5]. The remainder of the work was first published in a paper (MORREY (7]) which was released in December 1943; the manuscript had been approved for publication in 1939.

of STAMPACCHIA and GILBARG. These results are presented and proved in § 4.2. The results include GILBARG's existence theorem for equations of the form (1.10.13) with $N = 1$, $B_i = 0$, $A^\alpha = A^\alpha(p)$. The advantages of these theorems are that one can restrict one's self to LIPSCHITZ functions z and no assumption has to be made about how $f(x, z, p)$ behaves as $|p| \to \infty$. The convexity assumption for all (x, z, p) is suggested by the conditions in §§ 1.4—1.6.

In the existence theorems mentioned above, the author considered integrals of the form (1.1.1) in which ν and N are arbitrary but in which f is convex in all the p^i_α; with this convexity assumption, no difficulties were introduced in the proofs by allowing $N > 1$. The results have been extended and the old proofs greatly simplified by SERRIN ([1], [2]); we shall present (in § 1.8) a simple lower-semicontinuity proof based on some of his ideas and on some ideas of TONELLI. However, for $N > 1$, the proper condition would be to assume that (1.5.8) and/or (1.5.5) held for all (x, z, p, λ, ξ). The author has studied these general integrals ([9]) and found that if f satisfies the conditions

$$m V^k - K \leq f(x, z, p) \leq M V^k, \quad |f_p(x, z, p)| \leq M V^{k-1}, \quad k > 1$$
$$|f_x|, |f_z| \leq M V^k, \quad m > 0, \quad V = (1 + |p|^2)^{1/2}$$

then a necessary and sufficient condition that $I(z, G)$ be lower semicontinuous on the space $H^1_k(G)$ with respect to uniform convergence is that f be quasi-convex as a function of p. A function $f(p)$, $p = \{p^i_\alpha\}$ is quasi-convex if and only if it is continuous and

$$\int_G f[p_0 + \nabla \zeta(x)]\, dx \geq f(p_0) \cdot |G|, \quad \zeta \in C^1_c(G);$$

that is, linear vectors give the absolute minimum to $I(z, G)$ among all z with such boundary values (note that linear functions always satisfy EULER's equation if $f \in C^2$). A *necessary* condition for quasi-convexity is just (1.5.5). The author showed that (1.5.5) is *sufficient* for quasi-convexity if $f(p)$ is of one of the two following forms

$$(1)\ f(p) = a^{\alpha\beta}_{ij} p^i p^j \qquad (a^{\alpha\beta}_{ij}) \text{ const.}$$
$$(2)\ f(p) = F(D_1, \ldots, D_{\nu+1}), \quad N = \nu + 1$$

where F is homogeneous of degree 1 in the D_i and D_i is the determinant of the submatrix obtained by omitting the i-th column of the p^i_α matrix; or if for each p, there exist alternating constant tensors A^α_i, $A^{\alpha\beta}_{ij}$, ..., such that

$$f(p + \pi) \geq f(p) + A^\alpha_i \pi^i_\alpha + A^{\alpha\beta}_{ij} \pi^i_\alpha \pi^j_\beta + \cdots + A^{\alpha_1 \ldots \alpha_\mu}_{i_1 \ldots i_\mu} \pi^{i_1}_{\alpha_1} \ldots \pi^{i_\mu}_{\alpha_\mu}$$

for all π. Under some additional conditions the integral is lower-semicontinuous with respect to weak convergence in $H^1_k(G)$. We discuss these integrals in § 4.4. Recently Norman MEYERS has extended the author's