Multiple Invariance *ESPRIT*

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***Abstract*— *ESPRIT* is a recently developed technique for high-resolution signal parameter estimation. For the specific problem of direction-of-arrival (DOA) estimation, a decrease in computational complexity of orders of magnitude over other high-resolution methods is achieved by exploiting an *invariance structure* designed into the sensor array. Previously, *ESPRIT* took advantage of only *one* such invariance per dimension of the parameter vector. The ubiquitous uniform linear array is an example of a sensor array possessing many such invariances, and the question of which invariance to use naturally arises. In this paper, *ESPRIT* is extended to address the problem of exploiting all the invariances simultaneously. The nonlinear multiple invariance *ESPRIT* algorithm is derived and compared via simulation to various standard *ESPRIT* solutions possible for arrays with multiple invariances.**

1. Introduction

IN MANY SIGNAL PROCESSING applications, the objective is to estimate a set of unknown parameters upon which *deterministic* signals measured by an array of sensors depend. Direction-of-arrival estimation of narrowband sources and detection of exponentials in noise (*e.g.* linear system identification) are classic examples. These problems naturally possess multi-dimensional geometric characteristics that have only recently been recognized.

In many of the recently developed algorithms (*e.g.* Schmidt's MUSIC [1], Burg's MEM [2], and Capon's ML [3]) however, the multidimensional nature of the problem is retained only to obtain an estimate of the signal subspace. Once the data have been thus reduced, the multidimensional aspects are eschewed in favor of one-dimensional line searches for intersections of estimated signal subspaces and the set of all possible array responses (*i.e.*, the *array manifold*). Most of the computational effort involved in these algorithms is expended in performing these one-dimensional searches.

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A robust, computationally efficient technique known as *ESPRIT* [4] has recently been developed. It mitigates most of the computational complexity of previous high resolution techniques by exploiting a *displacement invariance* designed into the sensor array. *ESPRIT* also manifests lower sensitivity to array errors than previous high resolution methods, and, through use of a *total least-squares* (TLS) minimization criterion, yields apparently unbiased parameter estimates even at low SNRs.

In many situations, however, the sensor array possesses more than one displacement invariance. The ubiquitous uniform linear array is a prime example. For such arrays, the standard *ESPRIT* algorithm is applied by first selecting a pair of subarrays that satisfy one of the many inherent invariances. The question naturally arises as to whether or not there is an *optimal* choice of subarrays. More fundamentally, the issue of the proximity of any standard *ESPRIT* solution to the *optimal* solution that exploits all the invariances simultaneously is most certainly relevant. These are the questions to which this paper is addressed. In the following section, three equivalent formulations of the standard TLS *ESPRIT* solution are presented. Though certainly of pedagogical interest, the equivalence also demonstrates how TLS estimates of the array manifold vectors may be obtained, and provides a framework for extending *ESPRIT* to arrays with multiple invariances. The extension of the algorithm to such arrays is elucidated in Section 3, and some simulation results are given in Section 4.

2. TLS *ESPRIT*

In the standard TLS *ESPRIT* problem [4], the sensor array is assumed to be composed of two identical subarrays separated by a fixed displacement vector $\mathbf{\Delta}$. For such arrays, the manifold of steering vectors has the form $\mathbf{A}_e = [\mathbf{A}^T \ \mathbf{\Phi}^T \mathbf{A}^T]^T$, where $\mathbf{A}$ is $m \times d$ for m sensors (per subarray) and d sources, and where $\mathbf{\Phi}$ is a diagonal matrix. For the DOA estimation problem, the diagonal elements of $\mathbf{\Phi}$ are given by $\exp\{-j2\pi\Delta \sin\theta_i/c\}, i =$

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$1, \cdots, d$, where c is the wave propagation speed and θ_i is the direction-of-arrival (DOA) of the i^{th} source. For notational purposes, we partition the d "signal" eigenvectors $\mathbf{E}$ of the array covariance matrix in the same fashion as $\mathbf{A}_e$, and note that, in the absence of noise, a full rank $d \times d$ matrix $\mathbf{T}$ would exist such that

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_X \\ \mathbf{E}_Y \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A\Phi} \end{bmatrix} \mathbf{T} . \tag{1}$$

The fundamental result of the TLS ***ESPRIT*** algorithm is that $\mathbf{\Phi}$ can be estimated as the eigenvalues of a matrix $\mathbf{\Psi}$ obtained from the following minimization problem:

Given subspace estimates $\mathbf{E}_X$ *and* $\mathbf{E}_Y$*, find a matrix* $\mathbf{F} \in \mathbb{C}^{2d \times d}$ *to minimize*

$$J = \|[\mathbf{E}_X \mid \mathbf{E}_Y]\,\mathbf{F}\|_F^2 \tag{2}$$

subject to

$$\mathbf{F}^*\mathbf{F} = \mathbf{I} , \tag{3}$$

where

$$\mathbf{F} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{F}_X \\ \mathbf{F}_Y \end{bmatrix} , \tag{4}$$

$$\mathbf{\Psi} = -\mathbf{F}_X[\mathbf{F}_Y]^{-1} . \tag{5}$$

The above minimization problem is not, however, the most common statement of the TLS linear parameter estimation problem. In [5], for example, the problem is posed in the following manner:

Given $\mathbf{C}, \mathbf{D} \in \mathbb{C}^{m \times d}$*,* $m > d$*, find* $\mathbf{R_C}$ *and* $\mathbf{R_D}$ *of minimum Frobenius*[2] *norm, and* $\mathbf{X}$ *such that*

$$[\mathbf{C} + \mathbf{R}_C]\,\mathbf{X} = \mathbf{D} + \mathbf{R}_D . \tag{6}$$

Note that in general $\mathbf{D}$ need not have the same number of columns as $\mathbf{C}$ though for the applications described herein, $\mathbf{C}$ and $\mathbf{D}$ have the same dimensions and $\mathbf{X}$ is square. The connection with the original formulation of TLS ***ESPRIT*** given above can be obtained by noting that $\mathbf{R}_C$ and $\mathbf{R}_D$ represent errors in the subspace estimates $\mathbf{E}_X = \mathbf{C}$ and $\mathbf{E}_Y = \mathbf{D}$ respectively, and that $\mathbf{X}$ is the operator $\mathbf{\Psi}$ whose eigenvalues are the parameters of interest. Thus, the TLS ***ESPRIT*** estimation problem can also be stated as follows:

Given subspace estimates $\mathbf{E}_X$ *and* $\mathbf{E}_Y$*, find an operator* $\mathbf{\Psi}$ *and residual matrices* $\mathbf{R}_{E_X}, \mathbf{R}_{E_Y}$ *to minimize*

$$J = \|[\mathbf{R}_{E_X} \mid \mathbf{R}_{E_Y}]\|_F^2 \tag{7}$$

[2] If knowledge of the covariance of the errors is available (in general in the form of a 4-tensor), a weighted Frobenius norm can be employed. Since for the problems considered herein it can be argued that the errors are independent and identically distributed (at least asymptotically), the *weighting matrix* or metric is proportional to the identity.

subject to

$$[\mathbf{E}_X + \mathbf{R}_{E_X}]\,\mathbf{\Psi} = \mathbf{E}_Y + \mathbf{R}_{E_Y} . \tag{8}$$

A further reformulation of this constrained minimization problem provides the framework for generalizing TLS ***ESPRIT*** to arrays with multiple invariances. Starting with the TLS formulation in equations (7) and (8), define $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{E}_X + \mathbf{R}_{E_X}$. Substituting into equation (8) gives $\mathbf{E}_Y + \mathbf{R}_{E_Y} = \mathbf{B\Psi}$. Rearranging these equations gives

$$\mathbf{R}_{E_X} = \mathbf{E}_X - \mathbf{B} , \tag{9}$$

$$\mathbf{R}_{E_Y} = \mathbf{E}_Y - \mathbf{B\Psi} , \tag{10}$$

so the minimization becomes

$$\min_{\{\mathbf{B},\mathbf{\Psi}\}} J = \min_{\{\mathbf{B},\mathbf{\Psi}\}} \left\| \begin{bmatrix} \mathbf{E}_X \\ \mathbf{E}_Y \end{bmatrix} - \begin{bmatrix} \mathbf{B} \\ \mathbf{B\Psi} \end{bmatrix} \right\|_F^2 . \tag{11}$$

An important point to be made here is that the minimization of equation (11) is nothing more than a least-squares fit to the model of equation (1), with $\mathbf{B} = \mathbf{AT}$ and $\mathbf{\Psi} = \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}$. This formulation of the problem is much clearer than that given in equations (2) through (5) for at least two reasons. First, it is obvious from (11) why $\mathbf{\Phi}$ is given as the eigenvalues of the estimated operator $\mathbf{\Psi}$. Secondly, the question of how to naturally incorporate multiple displaced subarrays into the problem is easily answered, as will be explained in the following section. A proof formally establishing the equivalence of the minimization problems posed in equation (11) and in equations (2) through (5) can be found in the appendix.

3. Multiple Invariance *ESPRIT*

Using the alternative formulation of the minimization as given in equation (11), the standard ***ESPRIT*** solution can be easily extended to arrays with multiple invariances. Assuming the presence of p subarrays, each with its own displacement vector $\mathbf{\Delta}_i$, the multiple invariance ***ESPRIT*** problem can be stated as follows:

Given p *subspace estimates* $\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_{p-1} \in \mathbb{C}^{m \times d}$*, find an operator* $\mathbf{\Psi} \in \mathbb{C}^{d \times d}$ *and a subspace estimate* $\mathbf{B} \in \mathbb{C}^{m \times d}$ *to minimize*

$$J = \left\| \begin{bmatrix} \mathbf{E}_0 - \mathbf{B} \\ \mathbf{E}_1 - \mathbf{B\Psi} \\ \mathbf{E}_2 - \mathbf{B\Psi}^{\frac{\Delta_2}{\Delta_1}} \\ \vdots \\ \mathbf{E}_{p-1} - \mathbf{B\Psi}^{\frac{\Delta_{p-1}}{\Delta_1}} \end{bmatrix} \right\|_F^2 .$$

Note that the $\mathbf{\Delta}_i$*'s are collinear, though not necessarily of equal magnitude.*

The estimates of the parameters of interest are again given as the eigenvalues of the operator $\mathbf{\Psi}$. For the special case of a uniform linear array, $\mathbf{\Delta}_i = i\mathbf{\Delta}_1 \quad \forall\ i = 1,\ldots,p-1$, and the cost function becomes

$$J = \left\| \begin{bmatrix} \mathbf{E}_0 - \mathbf{B} \\ \mathbf{E}_1 - \mathbf{B\Psi} \\ \mathbf{E}_2 - \mathbf{B\Psi}^2 \\ \vdots \\ \mathbf{E}_{p-1} - \mathbf{B\Psi}^{p-1} \end{bmatrix} \right\|_F^2 .$$

Note that in this case p is the number of sensors and $m = 1$.

The minimization of J over $\{\mathbf{\Psi} \in \mathbb{C}^{d\times d}, \mathbf{B} \in \mathbb{C}^{m\times d}\}$ is a nonlinear, computationally complex problem. A somewhat simpler expression for J can be obtained by solving for $\mathbf{B}$ explicitly in terms of $\mathbf{\Psi}$. Defining

$$\mathbf{E} \stackrel{\text{def}}{=} [\mathbf{E}_0 \,|\, \mathbf{E}_1 \,|\, \cdots \,|\, \mathbf{E}_{p-1}],$$

$$\bar{\mathbf{\Psi}} \stackrel{\text{def}}{=} \left[\mathbf{I} \,|\, \mathbf{\Psi} \,|\, \mathbf{\Psi}^{\frac{\Delta_2}{\Delta_1}} \,|\, \cdots \,|\, \mathbf{\Psi}^{\frac{\Delta_{p-1}}{\Delta_1}}\right]$$

and using standard properties of the *trace* operator, the cost function becomes

$$J = Tr\left\{\left[\mathbf{E} - \mathbf{B}\bar{\mathbf{\Psi}}\right]^* \left[\mathbf{E} - \mathbf{B}\bar{\mathbf{\Psi}}\right]\right\} .$$

Solving $\partial_{\mathbf{B}} J = 0$ yields

$$\mathbf{B} = \mathbf{E}\bar{\mathbf{\Psi}}^* \left[\bar{\mathbf{\Psi}}\bar{\mathbf{\Psi}}^*\right]^{-1}, \qquad (12)$$

which, when substituted for $\mathbf{B}$ in the expression of (3) for the cost function J, gives

$$J = Tr\left\{\mathbf{E}^*\mathbf{E}(\mathbf{I} - \bar{\mathbf{\Psi}}^*[\bar{\mathbf{\Psi}}\bar{\mathbf{\Psi}}^*]^{-1}\bar{\mathbf{\Psi}})\right\} , \qquad (13)$$

which must now be minimized over $\mathbf{\Psi}$. For the specific case of a uniform linear array, the partial derivative of J with respect to $\mathbf{\Psi}$ can be succinctly written as

$$\partial_{\mathbf{\Psi}} J = \bar{\mathbf{\Psi}}\mathbf{C}\tilde{\mathbf{\Psi}}^*, \qquad (14)$$

where

$$\tilde{\mathbf{\Psi}} \stackrel{\text{def}}{=} \left[\mathbf{I} \,|\, \mathbf{\Psi}^* \,|\, \mathbf{\Psi}^{*2} \,|\, \cdots \,|\, \mathbf{\Psi}^{*(p-1)}\right] ,$$

and where $\mathbf{C}$ is an upper-left triangular block Hankel matrix whose first block row is given by $[\mathbf{C}_1 \,|\, \mathbf{C}_2 \,|\, \cdots \,|\, \mathbf{C}_p]$, with $\mathbf{C}_i \stackrel{\text{def}}{=} \mathbf{B}^*\mathbf{B}\mathbf{\Psi}^i - \mathbf{E}_i^*\mathbf{B}$. Provided that Δ_i/Δ_1 is rational, the partial derivative $\partial_{\mathbf{\Psi}} J$ can also be obtained for the more general multiple invariance case, though the resulting expressions are quite cumbersome.

Since a closed form expression for $\mathbf{\Psi}$ is not possible, some type of search technique is required to achieve $\partial_{\mathbf{\Psi}} J = 0$. Fortunately, the original single invariance TLS ***ESPRIT*** algorithm can be applied to produce an excellent initial estimate as a starting point for a simple gradient search. The multiple invariance ***ESPRIT*** algorithm is thus implemented as follows:

1. Using the TLS ***ESPRIT*** algorithm, obtain an initial estimate of the operator $\hat{\mathbf{\Psi}}(0)$. Obtain $\hat{\mathbf{B}}(0)$ using (12). Repeat steps 2 and 3 until convergence ($\|\partial_{\mathbf{\Psi}} J\| \approx 0$).

2. Update $\hat{\mathbf{\Psi}}(k+1) = \hat{\mathbf{\Psi}}(k) + \alpha\partial_{\mathbf{\Psi}} J(k)$

3. Set $\hat{\mathbf{B}}(k+1) = \mathbf{E}\bar{\mathbf{\Psi}}(k)^* \left[\bar{\mathbf{\Psi}}(k)\bar{\mathbf{\Psi}}^*(k)\right]^{-1}$, and evaluate $\|\partial_{\mathbf{\Psi}} J(k+1)\|$.

4. Simulation Results

To investigate the performance improvement of the iterated ***ESPRIT*** algorithm over the single invariance ***ESPRIT*** algorithm, simulations were conducted using a 12 element uniform linear array (ULA). The array was split into varying numbers of subarrays, in each case using every element once and only once. Though there are a wide variety of such arrays, they can be grouped into sets of arrays that have the same $\mathbf{\Delta}$ for rational comparison of the results between estimates from different subarray structures.

Maintaining $\Delta = 1$ (in normalized units of the ULA element separation), there are 6 ways of decomposing the 12 element ULA into subarrays. If elements in a subarray are indicated by $\{\}$, corresponding elements in the various subarray sets are assumed to be identical and separated by $\mathbf{\Delta}$. Thus, the decomposition into 2 subarrays is indicated by $\{1,3,5,7,9,11\}$,$\{2,4,6,8,10,12\}$ where the elements of the ULA have been labeled from 1 to 12. Elements 1 and 2 are therefore identical and separated by $\mathbf{\Delta}$ as are 3 and 4, 5 and 6, *etc.*. For this decomposition there is only one invariance. The decomposition into three subarrays, with two invariances, is given by $\{1,4,7,10\}$,$\{2,5,8,11\}$,$\{3,6,9,12\}$, where now elements 1, 2, and 3 are assumed to be identical and form a three element uniform linear subarray with element separation $\mathbf{\Delta}$. The decomposition into 4, 6 and finally 12 subarrays should now be easily seen. For the latter case, each subarray is composed of a single element of the underlying ULA, and for this case iterated ***ESPRIT*** is exploiting the entire invariance structure of the array.

For each of the decompositions, the single invariance ***ESPRIT*** results were generated using maximally overlapping subarrays with $\Delta = 1$. For the 3 subarray decomposition, this corresponds to combining subarrays 1 and 2 into a single subarray and subarrays 2 and 3 into the second subarray. Four elements are shared in common. The ***ESPRIT*** estimates were also used as initial conditions for the iterations. Due to the manifestly unbiased nature of the ***ESPRIT*** estimates, the iteration converged quite rapidly. With convergence criterion $\|\partial_{\mathbf{\Psi}} J(k+1)\|_F < 10^{-4}$, the average number of iterations for 2,3, and 5 invariances was 7, and for 11 invarances, 21 iterations were required on average.

The results shown in Figures 1 through 5 are for a scenario in which two sources are present at *electrical angles* ($\phi = 2\pi\Delta\sin\theta/\lambda$) of 0.17π and 0.22π radians. The SNR per element was assumed to be 0dB for both sources and the sources were uncorrelated. The number of snapshots in each trial was 100, and 1000 Monte Carlo trials were run. The quantities shown in Figures 1 through 2 are the magnitude of the mean of the error in the complex pole location over the 1000 trials as a function of the number of invariances and its standard deviation. From Figure 3 it can be seen that the bias in the mean is negligible in both ***ESPRIT*** and iterated ***ESPRIT***. Since iterated ***ESPRIT*** reduces to ***ESPRIT*** when there is only one invariance to exploit, the results are identical. As the number of invariances increases to the maximum of 11, a gradual improvement of up to 30% is achieved. It should be noted however, that for the 12 element subarray, there is a significant latitude in the choice of Δ for ***ESPRIT*** and simulations indicate that minimum variance estimates are obtained for Δ's near half the maximum possible (again using all the elements once and only once), in this case $\Delta = 3$.

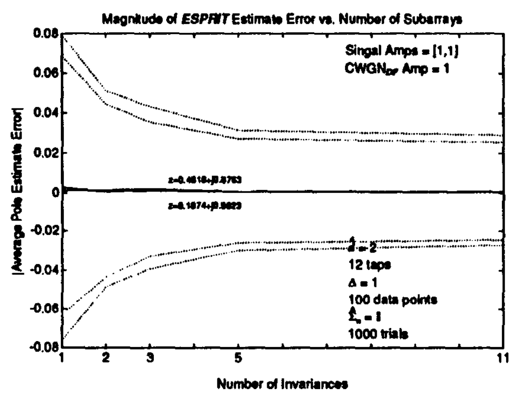

Figure 1: Bias and Standard Deviation of ***ESPRIT*** Estimates versus Number of Invariances

In Figures 4 and 5, standard and iterated ***ESPRIT*** are compared with root-MUSIC, an *analytic* version of the well-known MUSIC algorithm applicable to uniform linear arrays and constant amplitude plane waves. The standard ***ESPRIT*** results are obtained from maximally overlapping 11-element subarrays, while iterated ***ESPRIT*** is exploiting all 11 invariances present. The bias in the root-MUSIC estimates is easily seen. As discussion of root-MUSIC and the inherent biases in the algorithm can be found in [4].

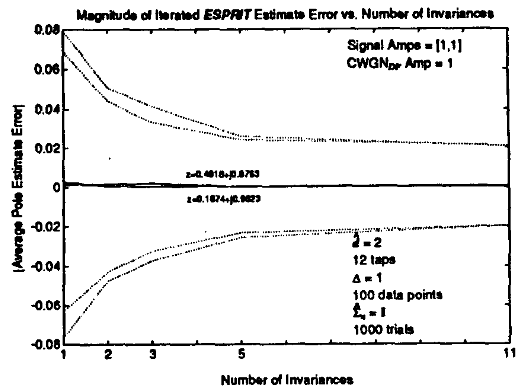

Figure 2: Bias and Standard Deviation of Iterated ***ESPRIT*** Estimates versus Number of Invariances

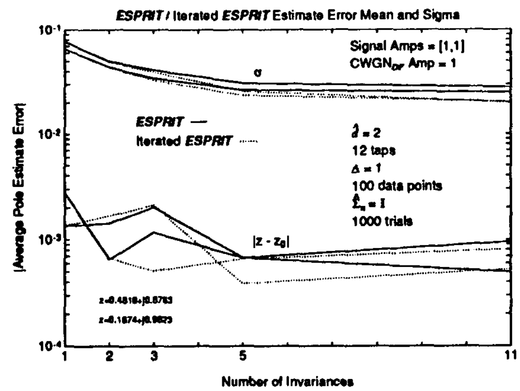

Figure 3: Comparison of ***ESPRIT*** and Iterated ***ESPRIT*** Estimate Bias and Standard Deviation

5. Concluding Remarks

In this paper, an extension of the TLS ***ESPRIT*** algorithm to arrays with multiple one-dimensional invariances has been presented. It was shown how a simple reformulation of the TLS minimization problem leads to a framework into which a given multiple invariance structure can be easily incorporated. One drawback of using the algorithm to exploit such invariances is that a *closed-form* solution to the problem can not be obtained. However, using the standard TLS ***ESPRIT*** estimate as a starting value, a simple gradient search technique converges rapidly to the desired solution. To date, simulations performed for a variety of scenarios indicate that exploiting the full invariance structure of an array offers some improvement in performance (reduction in estimate error variance) over the standard single invariance ***ESPRIT*** solution. The performance of the standard ***ES-***

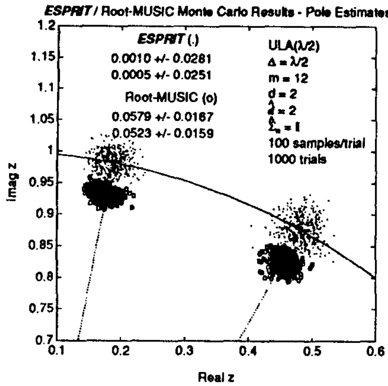

Figure 4: ***ESPRIT*** and Root-MUSIC Estimated Pole Locations

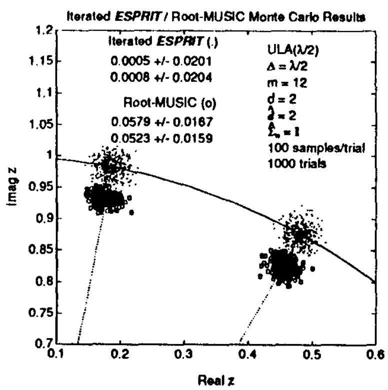

Figure 5: Iterated ***ESPRIT*** and Root-MUSIC Estimated Pole Locations

PRIT algorithm has proved to be remarkably similar to that of the multiple invariance version however.

Appendix

Equivalence of Two Forms of the TLS *ESPRIT* Problem

Here the equivalence between the operator Ψ obtained from equations (2) through (5) and that obtained from equation (11) is established. Using standard properties of the trace operator, the minimization of equation (2) can be rewritten as

$$\min_{\mathbf{F}} J = \min_{\mathbf{F}} Tr\{\mathbf{E}_{XY}^* \mathbf{E}_{XY} P_F\} ,$$

where

$$\mathbf{E}_{XY} \stackrel{\text{def}}{=} [\mathbf{E}_X \mid \mathbf{E}_Y] ,$$

and where $P_F = \mathbf{F}[\mathbf{F}^*\mathbf{F}]^{-1}\mathbf{F}^* = \mathbf{F}\mathbf{F}^*$ is the projection onto the (full rank) columns of $\mathbf{F}$. It is shown in [4] that the TLS ***ESPRIT*** estimate of Ψ for this formulation of the problem is given by $\Psi = -\mathbf{E}_{12}\mathbf{E}_{22}^{-1}$, where

$$\mathbf{E}_{XY}^* \mathbf{E}_{XY} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \mathbf{\Lambda} \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix}^*$$

is the eigendecomposition of $\mathbf{E}_{XY}^* \mathbf{E}_{XY}$.

Returning to the expression of equation (13), we see that the minimization problem of (11) may be rewritten as

$$\min_{\Psi} J = \min_{\Psi} Tr\left\{\mathbf{E}_{XY}^* \mathbf{E}_{XY} P_{\underline{\Psi}^*}^{\perp}\right\} .$$

The two minimization problems are thus manifestly equivalent[3] since both are minimizations of the same functional form over rank d projection matrices in $\mathbb{C}^{2d\times 2d}$. To see that the operator Ψ obtained from (13) is identical to that obtained from the original formulation, note that by the orthogonality of the eigenvectors of $\mathbf{E}_{XY}^* \mathbf{E}_{XY}$, $P_{\underline{\Psi}^*}^{\perp} = P_{\mathbf{E}_2}$ implies $P_{\underline{\Psi}^*} = P_{\mathbf{E}_1}$. Therefore there exists a non-singular matrix $\mathbf{T}$ such that

$$\begin{bmatrix} \mathbf{I} \\ \Psi^* \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11} \\ \mathbf{E}_{21} \end{bmatrix} \mathbf{T} . \tag{15}$$

The estimate of Ψ is thus given by $\mathbf{E}_{11}^{-*}\mathbf{E}_{21}^*$, and the orthogonality relation

$$\mathbf{E}_{11}^*\mathbf{E}_{12} + \mathbf{E}_{21}^*\mathbf{E}_{22} = 0 \tag{16}$$

now easily establishes the equivalence of the two estimates of Ψ as was to be shown.

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[3] There is a subtle difference in the constraints however. Mathematically speaking, the set of matrices of the form $\underline{\Psi}^*$ is a subset of all full-rank matrices in $\mathbb{C}^{2d\times d}$ since full-rank matrices whose upper $d \times d$ block is not invertible can not be converted into the form of $\underline{\Psi}^*$. For the problems considered herein, this set has zero probability of occurring and need not be considered further.