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Group analysis, nonlinear self-adjointness, conservation laws, and soliton solutions for the mKdV systems

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Abstract. We study the symmetry groups, conservation laws, solitons, and singular solitary waves of some versions of systems of the modified KdV equations.

Keywords: systems of modified KdV equation, group analysis, conservation laws, exact solutions.

1 Introduction

One of the better known and studied partial differential equations (PDEs) is the Korteweg–de Vries (KdV) equation being, inter alia, a model for shallow water waves. Other versions of the KdV equation are the modified KdV (mKdV) equation, systems of KdV, and the complex KdV equations, amongst others. The literature abounds with these studies, see, e.g., [11–15] and [1]. Soliton studies of the equation and other equations may be found in [3] and [8]. *A number of authors have dealt with a* 4×4 *matrix spectral problem with three potentials. In some cases, a new hierarchy of nonlinear evolution* *equations was derived* (*see* [3] *and* [8]). *Some typical equations in the hierarchy are a new generalized Hirota–Satsuma coupled KdV system.* We study the invariances, conservation laws, solitons, and singular solitary waves in the study below. Two distinct versions of systems of the modified KdV equations are considered (see [6, 7] and references therein).

2 The equation I

The first version of the system of modified KdV equations we consider takes the form

$$\begin{aligned} &F_1 = u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3v_x w - 3w_x v = 0, \\ &F_2 = v_t + v_{xxx} - 3v_x u = 0, \qquad F_3 = w_t + w_{xxx} - 3w_x u = 0. \end{aligned} \tag{1}$$

2.1 Invariance and conservation

Here, we analyse system (1) using its invariance properties [2, 4, 5, 9, 10] and conservation laws [2]. The *formal* Lagrangian and *multiplier* approaches to the latter are used for the construction of conservation laws.

2.1.1 Adjoint system and conditions for self-adjointness

Considering system (1), we give the following formal Lagrangian:

$$\begin{aligned} L = \bar{u}\biggl(u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3v_x w - 3w_x v\biggr) \\ + \bar{v}(v_t + v_{xxx} - 3v_x u) + \bar{w}(w_t + w_{xxx} - 3w_x u), \end{aligned} \tag{2}$$

where $\bar{u}, \bar{v}$, and $\bar{w}$ are three new dependent variables.

The adjoint system is given by

$$F_1^* = \frac{\delta L}{\delta u} = 0, \qquad F_2^* = \frac{\delta L}{\delta v} = 0, \qquad F_3^* = \frac{\delta L}{\delta w} = 0,$$

here

$$\begin{aligned} \frac{\delta L}{\delta u} &= \frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} - D_x \frac{\partial L}{\partial u_x} + D_x D_x \frac{\partial L}{\partial u_{xx}} - D_x D_x D_x \frac{\partial L}{\partial u_{xxx}}, \\ \frac{\delta L}{\delta v} &= \frac{\partial L}{\partial v} - D_t \frac{\partial L}{\partial v_t} - D_x \frac{\partial L}{\partial v_x} + D_x D_x \frac{\partial L}{\partial v_{xx}} - D_x D_x D_x \frac{\partial L}{\partial v_{xxx}}, \\ \frac{\delta L}{\delta w} &= \frac{\partial L}{\partial w} - D_t \frac{\partial L}{\partial w_t} - D_x \frac{\partial L}{\partial w_x} + D_x D_x \frac{\partial L}{\partial w_{xx}} - D_x D_x D_x \frac{\partial L}{\partial w_{xxx}}. \end{aligned} \tag{3}$$

On the basis of Eq. (2), one can get the adjoint system

$$\begin{aligned} F_1^* &= -\bar{u}_t - 3u\bar{u}_x - 3\bar{v}v_x - 3\bar{w}w_x + \frac{1}{2}\bar{u}_{xxx} = 0, \\ F_2^* &= -\bar{v}_t + 3w\bar{u}_x + 3u_x\bar{v} + 3u\bar{v}_x - \bar{v}_{xxx} = 0, \\ F_3^* &= -\bar{w}_t + 3v\bar{u}_x + 3u_x\bar{w} + 3u\bar{w}_x - \bar{w}_{xxx} = 0. \end{aligned}$$

System (1) is said to be nonlinearly self-adjoint on the condition of the adjoint system (3) if it satisfies the following formulae:

$$\begin{aligned} F_1^*|_{\bar u=P(x,t,u,v,w),\,\bar v=Q(x,t,u,v,w),\,\bar w=R(x,t,u,v,w)} &= \lambda_{11}F_1+\lambda_{12}F_2+\lambda_{13}F_3=0,\\ F_2^*|_{\bar u=P(x,t,u,v,w),\,\bar v=Q(x,t,u,v,w),\,\bar w=R(x,t,u,v,w)} &= \lambda_{21}F_1+\lambda_{22}F_2+\lambda_{23}F_3=0,\\ F_3^*|_{\bar u=P(x,t,u,v,w),\,\bar v=Q(x,t,u,v,w),\,\bar w=R(x,t,u,v,w)} &= \lambda_{31}F_1+\lambda_{32}F_2+\lambda_{33}F_3=0 \end{aligned}$$

with $\bar u = P(x,t,u,v,w), \bar v = Q(x,t,u,v,w), \bar w = R(x,t,u,v,w)$ not equal to zero, and the coefficients λ_{ij} $(i,j=1,2,3)$ are to be fixed later.

From the coefficients of u_t, v_t, w_t one can get

$$\begin{aligned} &\lambda_{11}=-P_u, &&\lambda_{12}=-P_v, &&\lambda_{13}=-P_w,\\ &\lambda_{21}=-Q_u, &&\lambda_{22}=-Q_v, &&\lambda_{23}=-Q_w,\\ &\lambda_{31}=-R_u, &&\lambda_{32}=-R_v, &&\lambda_{33}=-R_w, \end{aligned}$$

and then putting these into the system, one can derive

$$\begin{aligned} &P=c_1tu-\frac{1}{3}c_1x+c_2u+c_3,\\ &Q=(c_1t+c_2)w, \qquad R=(c_1t+c_2)v. \end{aligned} \tag{4}$$

We have the following theorem.

Theorem 1. *System* (1) *is said to be nonlinear self-adjoint if* P, Q, *and* R *are given by* (4).

2.1.2 Conservation laws and Invariance analysis

It can be shown by a well established procedure that (1) admits the algebra of Lie point symmetries generated by

$$\begin{aligned} &X_1=\frac{\partial}{\partial x}, \qquad X_2=\frac{\partial}{\partial t},\\ &X_3=3t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-2u\frac{\partial}{\partial u}-4v\frac{\partial}{\partial v}, \qquad X_4=-v\frac{\partial}{\partial v}w\frac{\partial}{\partial w}. \end{aligned}$$

These may be associated with conservation laws of the system. In fact, in the case of a variational system with Noether symmetries, the Noether symmetries are associated with conservation laws via Noether's theorem. This procedure is not employed here.

A conservation law of (1) is the divergence equation $D_t\Phi^t + D_x\Phi^x = 0$ satisfied along the solutions of the equations in which $T = (\Phi^x, \Phi^t)$ is the conserved vector/flow.

In order to derive the conservation laws, we employ the following theorem.

Theorem 2. (*See* [5].) *For any Lie point, Lie–Bäcklund and nonlocal symmetry* $X = \xi^i\partial_{x^i} + \eta^\alpha\partial_{u^\alpha}$ *provides a conservation law for a system and its adjoint system. Then the* *conservation vector T is defined by the expression*

$$\Phi^i = \xi^i L + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right)\right] + D_j(W^\alpha)\left[\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) - D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) + \cdots\right],$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

Since system (1) is nonlinearly self-adjoint, we can get the conserved vectors using Theorem 2. Also, via the symmetry operator and the formal Lagragian, they become

$$\Phi^t = \xi^1 L + W^1\left(\frac{\partial L}{\partial u_t}\right) + W^2\left(\frac{\partial L}{\partial v_t}\right) + W^3\left(\frac{\partial L}{\partial w_t}\right),$$

$$\begin{aligned}\Phi^x = {}& \xi^2 L + W^1\left(\frac{\partial L}{\partial u_x} + D_x^2\frac{\partial L}{\partial u_{xxx}}\right) + W^2\left(\frac{\partial L}{\partial v_x} + D_x^2\frac{\partial L}{\partial v_{xxx}}\right)\\ &+ W^3\left(\frac{\partial L}{\partial w_x} + D_x^2\frac{\partial L}{\partial w_{xxx}}\right) + D_x\big(W^1\big)\left(-D_x\frac{\partial L}{\partial u_{xxx}}\right)\\ &+ D_x\big(W^2\big)\left(-D_x\frac{\partial L}{\partial v_{xxx}}\right) + D_x\big(W^3\big)\left(-D_x\frac{\partial L}{\partial w_{xxx}}\right)\\ &+ D_x^2\big(W^1\big)\left(\frac{\partial L}{\partial u_{xxx}}\right) + D_x^2\big(W^2\big)\left(\frac{\partial L}{\partial v_{xxx}}\right) + D_x^2\big(W^3\big)\left(\frac{\partial L}{\partial w_{xxx}}\right),\end{aligned}$$

where $W^1 = \eta^1 - \xi^1 u_t - \xi^2 u_x$, $W^2 = \eta^2 - \xi^1 v_t - \xi^2 v_x$, $W^3 = \eta^3 - \xi^1 w_t - \xi^2 w_x$.

For system (1), we have the following set of conserved vectors:

$$\begin{aligned}T_1 &= \left[\frac{1}{2}\big(3u^2 - wv_t + v(-6w + w_t) - u_{xx} - 2w_x v_{xx} + 2v_x w_{xx}\big),\right.\\ &\qquad \left. u + \frac{1}{2}(wv_x - vw_x)\right],\\ T_2 &= \left[-\frac{1}{2}xu^2 + tu^3 - \frac{u_x}{6} + \frac{1}{4}t{u_x}^2 + tv_x w_x + \frac{1}{6}xu_{xx} - \frac{1}{2}tuu_{xx}\right.\\ &\qquad \left. - twv_{xx} + v(xw - tw_{xx}),\ \frac{1}{6}\big(-2xu + 3tu^2 - 6tvw\big)\right],\\ T_3 &= \left[u^3 + \frac{{u_x}^2}{4} + v_x w_x - \frac{1}{2}uu_{xx} - wv_{xx} - vw_{xx},\ \frac{1}{2}\big(u^2 - 2vw\big)\right].\end{aligned} \tag{5}$$

One can show that these correspond, respectively, with the multipliers

$$Q_1 = (1, -w_x, v_x), \qquad Q_2 = \left(-\frac{1}{3}x + tu, -tw, -tv\right), \qquad Q_3 = (u, -w, -v).$$

2.2 Soliton solutions

This section will retrieve solitary wave, shock waves, and singular solitary wave solutions to the model given by system (1). The method of undetermined coefficients will be employed to obtain these solutions. The derivation of results is distributed in the following few subsections.

2.2.1 Solitary waves

To study solitary waves of system (1), we first assume a solution of the form

$$u(x,t) = A_1 \operatorname{sech}^{p_1} \tau, \tag{6}$$
$$v(x,t) = A_2 \operatorname{sech}^{p_2} \tau, \tag{7}$$
$$w(x,t) = A_3 \operatorname{sech}^{p_3} \tau, \tag{8}$$

where τ is given by

$$\tau = B(x - ct) \tag{9}$$

with B representing the inverse width, while c is the corresponding speed of such nonlinear wave. The parameter A_j for $j = 1, 2, 3$ are the amplitudes of the three solitary wave components. After substituting the trial functions (6)–(8) into system (1), one get, in a simplified form,

$$p_1\big(2c + p_1^2 B^2\big) A_1 \operatorname{sech}^{p_1}\tau - 6p_1 A_1^2 \operatorname{sech}^{2p_1}\tau - p_1(1+p_1)(2+p_1)A_1 B^2 \\ \times \operatorname{sech}^{2+p_1}\tau + 6(p_2+p_3)A_2A_3 \operatorname{sech}^{p_2+p_3}\tau = 0, \tag{10}$$
$$p_2^2 B^2 \operatorname{sech}^{p_2}\tau - 3A_1 \operatorname{sech}^{p_1+p_2}\tau - (1+p_2)(2+p_2)B^2 \operatorname{sech}^{2+p_2}\tau = 0, \tag{11}$$

and

$$p_3^2 B^2 \operatorname{sech}^{p_3}\tau - 3A_1 \operatorname{sech}^{p_1+p_3}\tau - (1+p_3)(2+p_3)B^2 \operatorname{sech}^{2+p_3}\tau = 0 \tag{12}$$

for each equation in (1), respectively. The subtle balance between nonlinearity and dispersion allows one to equate

$$2 + p_1 = 2p_1, \tag{13}$$
$$2 + p_1 = p_2 + p_3, \tag{14}$$
$$p_1 + p_2 = 2 + p_2, \tag{15}$$
$$p_1 + p_3 = 2 + p_3 \tag{16}$$

from which one can obtain $p_1 = 2$, $p_2 = 2$, and $p_3 = 2$. Substituting this values of p_i into (10)–(12) leads to

$$\big(c + 2B^2\big)A_1 \operatorname{sech}^2\tau + 3\big(2A_2A_3 - A_1^2 - 2A_1B^2\big)\operatorname{sech}^4\tau = 0, \tag{17}$$
$$-4B^2 \operatorname{sech}^2\tau + 3\big(A_1 + 4B^2\big)\operatorname{sech}^4\tau = 0.$$

After adding both equations, one get from the coefficients of the linearly independent functions $\operatorname{sech}^j \tau$ for $j = 2, 4$

$$c = \frac{2(2 + A_1)B^2}{A_1} \tag{18}$$

and the relation

$$4B^2 + \left(1 - A_1 - 2B^2\right)A_1 + 2A_2A_3 = 0. \tag{19}$$

Also, from Eq. (17), setting the coefficients of the linearly independent functions to zero, one get the relations

$$B^2 = -\frac{c}{2} \tag{20}$$

and

$$B^2 = \frac{A_1^2 - 2A_2A_3}{2A_1}. \tag{21}$$

Equating the values of B^2 from (20) and (21), we also get

$$c = \frac{2A_2A_3 - A_1^2}{A_1}. \tag{22}$$

Thus, the solitary waves solution of system (1) takes the form

$$u(x,t) = A_1 \operatorname{sech}^2\left[B(x - ct)\right],$$
$$v(x,t) = A_2 \operatorname{sech}^2\left[B(x - ct)\right],$$
$$w(x,t) = A_3 \operatorname{sech}^2\left[B(x - ct)\right],$$

where the speed of the waves is given by (18) or (22), the amplitudes are related in (19), and the soliton width is provided in (20) or (21) provided

$$A_1 \neq 0.$$

2.2.2 Conserved quantites

From the densities given by system (5) the conserved quantities are as follows:

$$I_1 = \int_{-\infty}^{\infty} \left[2u + (wv_x - vw_x)\right] \mathrm{d}x = \frac{4A_1}{B},$$

$$I_2 = \int_{-\infty}^{\infty} \left(2xu - 3tu^2 + 6tvw\right) \mathrm{d}x = \frac{4t}{B}\left(2A_2A_3 - A_1^2\right).$$

For I_2 to be a conserved quantity, it is necessary to have $\mathrm{d}I_2/\mathrm{d}t = 0$. Thus, this conserved quantity exists provided

$$\frac{A_1^2}{A_2A_3} = 2, \qquad I_3 = \int_{-\infty}^{\infty} \left(u^2 - 2vw\right) \mathrm{d}x = \frac{4}{3B}\left(A_1^2 - 2A_2A_3\right),$$

where the densities are computed using the solitary wave solutions.

2.2.3 Shock waves

We assume shock wave solutions of the form

$$u(x,t) = A_1 \tanh^{p_1} \tau, \tag{23}$$
$$v(x,t) = A_2 \tanh^{p_2} \tau, \tag{24}$$
$$w(x,t) = A_3 \tanh^{p_3} \tau, \tag{25}$$

where τ is defined as in (9). In this case, the parameters A_j for $j = 1, 2, 3$ and B are free parameters. Then, by substituting ansatz (23)–(25) into system (1) and simplifying, one get

$$\begin{aligned} &6p_1 A_1^2 \tanh^{2p_1-1} \tau - 6p_1 A_1^2 \tanh^{2p_1+1} \tau \\ &\quad - 6(p_2+p_3) A_2 A_3 \tanh^{p_2+p_3-1} \tau + 6(p_2+p_3) A_2 A_3 \tanh^{p_2+p_3+1} \tau \\ &\quad + p_1(p_1+1)(p_1+2) A_1 B^2 \tanh^{p_1+3} \tau - p_1(p_1-2)(p_1-1) A_1 B^2 \tanh^{p_1-3} \tau \\ &\quad + p_1\bigl[-2\bigl(c-B^2\bigr) + 3p_1(p_1-1)B^2\bigr] A_1 \tanh^{p_1-1} \tau \\ &\quad - p_1\bigl[-2\bigl(c-B^2\bigr) + 3p_1(p_1+1)B^2\bigr] A_1 \tanh^{p_1+1} \tau = 0, \end{aligned} \tag{26}$$

$$\begin{aligned} &\bigl[2+3p_2(1+p_2)\bigr] B^2 \tanh^{p_2+1} \tau - \bigl[2+3p_2(p_2-1)\bigr] B^2 \tanh^{p_2-1} \tau \\ &\quad - (p_2+1)(p_2+2) B^2 \tanh^{p_2+3} \tau + (p_2-1)(p_2-2) B^2 \tanh^{p_2-3} \tau \\ &\quad + 3A_1 \tanh^{p_1+p_2+1} \tau - 3A_1 \tanh^{p_1+p_2-1} \tau = 0, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\bigl[2+3p_3(1+p_3)\bigr] B^2 \tanh^{p_3+1} \tau - \bigl[2+3p_3(p_3-1)\bigr] B^2 \tanh^{p_3-1} \tau \\ &\quad - (p_3+1)(p_3+2) B^2 \tanh^{p_3+3} \tau + (p_3-1)(p_3-2) B^2 \tanh^{p_3-3} \tau \\ &\quad + 3A_1 \tanh^{p_1+p_3+1} \tau - 3A_1 \tanh^{p_1+p_3-1} \tau = 0. \end{aligned} \tag{28}$$

As for the case in the previous section, balancing principle leads to $p_1 = 2$, $p_2 = 2$, and $p_3 = 2$. This values of p_i reduce (26)–(28) into

$$\begin{aligned} &-\bigl(c-4B^2\bigr) A_1 \tanh \tau + \bigl[\bigl(c-10B^2\bigr) A_1 + 3A_1^2 - 6A_2 A_3\bigr] \tanh^3 \tau \\ &\quad + 3\bigl[2A_1 B^2 + 2A_2 A_3 - A_1^2\bigr] \tanh^5 \tau = 0, \\ &-8B^2 \tanh \tau + \bigl(20B^2 - 3A_1\bigr) \tanh^3 \tau + 3\bigl(A_1 - 4B^2\bigr) \tanh^5 \tau = 0. \end{aligned} \tag{29}$$

In this case, adding both equations and setting the coefficients of the resulting linearly independent functions $\tanh^j \tau$ for $j = 1, 3, 5$ allow one to obtain two different expressions for the speed:

$$c = \frac{4(A_1-2)B^2}{A_1}, \tag{30}$$

$$c = 3 + 10B^2 - 3A_1 + \frac{6A_2 A_3 - 20B^2}{A_1} \tag{31}$$

and the identity

$$2A_1B^2 + 2A_2A_3 + A_1 - 4B^2 - 3A_1^2 = 0. \tag{32}$$

After coupling the (30) and (31), it follows that

$$A_1 = \frac{1}{2}\bigl(1 + 2B^2 + \sqrt{1 + 4(B^2 - 3)B^2 + 8A_2A_3}\,\bigr). \tag{33}$$

Notice also that by setting to zero the coefficients of the linearly independent functions in equation (29) we obtain

$$c = 4B^2 \tag{34}$$

and

$$B = \frac{A_1^2 - 2A_2A_3}{2A_1}, \tag{35}$$

whenever

$$\bigl(A_1^2 - 2A_2A_3\bigr)A_1 > 0. \tag{36}$$

Then we have that the shock waves solution of system (1) is in the form

$$u(x,t) = A_1 \tanh^2\bigl[B(x - ct)\bigr],$$
$$v(x,t) = A_2 \tanh^2\bigl[B(x - ct)\bigr],$$
$$w(x,t) = A_3 \tanh^2\bigl[B(x - ct)\bigr],$$

where A_1 is provided in (33), while its relation with the amplitudes A_2 and A_3 is given in identity (32). Two expressions for the soliton speed (30) and (31) allow to obtain an expression for A_1, while the speed provided in (34) leads to the formula of the inverse soliton width showed in (35) subject to the constraint (36).

2.3 Singular solitary waves (type I)

To obtain the first type of singular solitary waves, we adopt the solutions of the form

$$u(x,t) = A_1 \operatorname{csch}^{p_1}\tau, \tag{37}$$
$$v(x,t) = A_2 \operatorname{csch}^{p_2}\tau, \tag{38}$$
$$w(x,t) = A_3 \operatorname{csch}^{p_3}\tau \tag{39}$$

with τ as defined in (9). In order to calculate the unknown parameters p_i for $i = 1, 2, 3$, we make a substitution of (37)–(39) into system (1) and simplify, it takes the form

$$p_1\bigl(2c + p_1^2B^2\bigr)A_1 \operatorname{csch}^{p_1}\tau - 6p_1A_1^2 \operatorname{csch}^{2p_1}\tau + p_1(1 + p_1)(2 + p_1)A_1B^2$$
$$\times \operatorname{csch}^{2+p_1}\tau + 6(p_2 + p_3)A_2A_3 \operatorname{csch}^{p_2+p_3}\tau = 0, \tag{40}$$

$$p_2^2B^2 \operatorname{csch}^{p_2}\tau - 3A_1 \operatorname{csch}^{p_1+p_2}\tau + (1 + p_2)(2 + p_2)B^2 \operatorname{csch}^{p_2+2}\tau = 0, \tag{41}$$

and

$$p_3^2 B^2 \operatorname{csch}^{p_3} \tau - 3A_1 \operatorname{csch}^{p_1+p_3} \tau + (1+p_3)(2+p_3) B^2 \operatorname{csch}^{p_3+2} \tau = 0. \tag{42}$$

In this case, we have

$$2 + p_1 = 2p_1,$$
$$2 + p_1 = p_2 + p_3,$$
$$2 + p_2 = p_1 + p_2,$$

which yields $p_1 = p_2 = p_3 = 2$. This values of p_i reduce (40)–(42) into

$$(c + 2B^2) A_1 \operatorname{csch}^2 \tau + 3(2A_1 B^2 + 2A_2 A_3 - A_1^2) \operatorname{csch}^4 \tau = 0 \tag{43}$$

and

$$-4B^2 \operatorname{csch}^2 \tau - 3(A_1 - 4B^2) \operatorname{csch}^4 \tau = 0.$$

By adding the last two equations one get, after equating to zero the coefficients of the linearly independent functions $\operatorname{csch}^j \tau$ for $j = 2, 4$,

$$c = \frac{2(2 - A_1)B^2}{A_1} \tag{44}$$

and the identity

$$4B^2 - (1 + A_1 - 2B^2)A_1 + 2A_2 A_3 = 0. \tag{45}$$

Notice also thet from Eq. (43) it is possible to retrieve (20)–(22). Finally, we have that the type-I singular solitary waves solution of system (1) are in the form

$$u(x,t) = A_1 \operatorname{csch}^2 [B(x - ct)],$$
$$v(x,t) = A_2 \operatorname{csch}^2 [B(x - ct)],$$
$$w(x,t) = A_3 \operatorname{csch}^2 [B(x - ct)],$$

where the corresponding speed is given in (44), and relation between soliton width, amplitudes A_2 and A_3 is provided in (45). Also, another possible expression for the speed is given in (18) along with two possible expressions for the solitary wave width (20) and (21) depending on the situation at hand.

2.4 Singular solitary waves (type II)

For the case of type-II singular solitary waves, we adopt a solution of the form

$$u(x,t) = A_1 \coth^{p_1} \tau, \tag{46}$$
$$v(x,t) = A_2 \coth^{p_2} \tau, \tag{47}$$
$$w(x,t) = A_3 \coth^{p_3} \tau \tag{48}$$

with τ as defined in (9). The substitution of ansatz (46)–(48) into system (1) leads to

$$\begin{aligned}
&p_1\bigl[-2\bigl(c-B^2\bigr)+3p_1(p_1+1)B^2\bigr]A_1\coth^{p_1+1}\tau\\
&\quad-p_1\bigl[-2\bigl(c-B^2\bigr)+3p_1(p_1-1)B^2\bigr]A_1\coth^{p_1-1}\tau\\
&\quad-p_1(p_1+1)(p_1+2)A_1B^2\coth^{p_1+3}\tau+p_1(p_1-1)(p_1-2)A_1B^2\coth^{p_1-3}\tau\\
&\quad-6(p_2+p_3)A_2A_3\coth^{p_2+p_3+1}\tau+6(p_2+p_3)A_2A_3\coth^{p_2+p_3-1}\tau\\
&\quad+6p_1A_1^2\coth^{2p_1+1}\tau-6p_1A_1^2\coth^{2p_1-1}\tau=0,
\end{aligned}\tag{49}$$

$$\begin{aligned}
&3A_1\coth^{p_1+p_2+1}\tau-3A_1\coth^{p_1+p_2-1}\tau\\
&\quad-(p_2+1)(p_2+2)B^2\coth^{p_2+3}\tau+(p_2-1)(p_2-2)B^2\coth^{p_2-3}\tau\\
&\quad+\bigl(2+3p_2(p_2+1)\bigr)B^2\coth^{p_2+1}\tau-\bigl(2+3p_2(p_2-1)\bigr)B^2\coth^{p_2-1}\tau=0,
\end{aligned}\tag{50}$$

and

$$\begin{aligned}
&3A_1\coth^{p_1+p_3+1}\tau-3A_1\coth^{p_1+p_3-1}\tau\\
&\quad-(p_3+1)(p_3+2)B^2\coth^{p_3+3}\tau+(p_3-1)(p_3-2)B^2\coth^{p_3-3}\tau\\
&\quad+\bigl(2+3p_3(p_3+1)\bigr)B^2\coth^{p_3+1}\tau-\bigl(2+3p_3(p_3-1)\bigr)B^2\coth^{p_3-1}\tau=0.
\end{aligned}\tag{51}$$

From the balancing principle one get the parameter values $p_1=p_2=p_3=2$. Then (49)–(51) reduce to

$$\begin{aligned}
&\bigl(c-4B^2\bigr)A_1\coth\tau+\bigl[6A_2A_3-\bigl(c-10B^2\bigr)A_1-3A_1^2\bigr]\coth^3\tau\\
&\quad-3\bigl(2A_1B^2+2A_2A_3-A_1^2\bigr)\coth^5\tau=0
\end{aligned}\tag{52}$$

and

$$-8B^2\coth\tau+\bigl(20B^2-3A_1\bigr)\coth^3\tau+\bigl(3A_1-12B^2\bigr)\coth^5\tau=0.$$

In this case, one get two different expressions for the speed

$$c=\frac{4(A_1+2)B^2}{A_1},\tag{53}$$

$$c=10B^2-3-3A_1+\frac{20B^2+6A_2A_3}{A_1}\tag{54}$$

and the identity

$$A_1\bigl(1+A_1-2B^2\bigr)-4B^2-2A_2A_3=0.\tag{55}$$

From the two expressions of the speed (53) and (54) it is possible to obtain

$$A_1=\frac{1}{2}\bigl(2B^2-1+\sqrt{1+4(B^2+3)B^2+8A_2A_3}\,\bigr).\tag{56}$$

Additionally, from Eq. (52), setting the coefficients of the linearly independent functions, one get the relations

$$B^2 = \frac{c}{4} \tag{57}$$

and

$$B^2 = \frac{(c + 3A_1)A_1 - 6A_2A_3}{10A_1}. \tag{58}$$

Equating both expressions of B^2 leads to another expression for the speed

$$c = \frac{2(A_1^2 - 2A_2A_3)}{A_1}. \tag{59}$$

Therefore, the type-II singular solitary waves solution of system (1) is

$$\begin{aligned}
u(x,t) &= A_1 \coth^2\big[B(x - ct)\big],\\
v(x,t) &= A_2 \coth^2\big[B(x - ct)\big],\\
w(x,t) &= A_3 \coth^2\big[B(x - ct)\big],
\end{aligned}$$

where the speed of the waves is given by (53) or (54), the amplitude A_1 is given in (56), and its relation with A_2, A_3, and the soliton width is provided in (55). The soliton width may be retrieve either from (57) or (58), and another expression for the speed given in (59) comes from this two expressions of B.

3 The equation II

Another version of the mKdV system is given by

$$\begin{aligned}
&u_t - \frac{1}{2}u_{xxx} + 3u^2u_x - \frac{3}{2}(vw)_{x,x} - 3(uvw)_x = 0,\\
&v_t + v_{xxx} + 3(vu_x)_x + 3v^2w_x - 6uvu_x - 3u^2v_x = 0,\\
&w_t + w_{xxx} + 3(wu_x)_x + 3w^2v_x - 6uwu_x - 3u^2w_x = 0.
\end{aligned}$$

3.1 Adjoint system and conditions for self-adjointness

Here, we get the following formal Lagrangian:

$$\begin{aligned}
L = \bar{u}\bigg(&u_t - \frac{1}{2}u_{xxx} + 3u^2u_x - \frac{3}{2}(vw)_{x,x} - 3(uvw)_x\bigg)\\
&+ \bar{v}\big(v_t + v_{xxx} + 3(vu_x)_x + 3v^2w_x - 6uvu_x - 3u^2v_x\big)\\
&+ \bar{w}\big(w_t + w_{xxx} + 3(wu_x)_x + 3w^2v_x - 6uwu_x - 3u^2w_x\big),
\end{aligned}$$

where $\bar{u}, \bar{v}$ and $\bar{w}$ are three new dependent variables. As previous step, one can derive the adjoint system

$$\begin{aligned}
F_1^* &= -\bar{u}_t + \frac{1}{2}\bar{u}_{xxx} - 3u^2\bar{u}_x + 3\bar{u}_x vw + 3\bar{v}_{xx}v + 3\bar{v}_x v_x - 3v^2\bar{v}_x \\
&\quad - 6vv_x\bar{v} + 6uv\bar{v}_x + 3w\bar{w}_{xx} + 3\bar{w}_x w_x + 6uw\bar{w}_x = 0, \\
F_2^* &= -\bar{v}_t - \bar{v}_{xxx} - \frac{3}{2}\bar{u}_{xx}w + 3\bar{u}u_x w - 3u_x\bar{v}_x + 6\bar{v}vu_x + 3u^2\bar{v}_x \\
&\quad - 6ww_x\bar{w} - 3w^2\bar{w}_x = 0, \\
F_3^* &= -\bar{w}_t - \bar{w}_{xxx} - \frac{3}{2}\bar{u}_{xx}v + 3u\bar{u}_x v - 3\bar{w}_x u_x + 6w\bar{w}v_x + 3u^2\bar{w}_x = 0.
\end{aligned}$$

Also, we get the conditions of nonlinear self-adjointness to be

$$P = c_1, \qquad Q = 0, \qquad R = 0.$$

3.2 Lie symmetry analysis and conservation laws

The generators of the algebra of Lie point symmetries are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = -3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}, \\
X_4 &= -3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2w\frac{\partial}{\partial w},
\end{aligned}$$

and the system admits the conserved vector

$$\begin{aligned}
T = \biggl[&\frac{1}{4}\bigl(4ku^3 + 3u^4 - 18u^2vw + 3v^2w^2 + {u_x}^2 - 6kwv_x \\
&- 2v_xw_x - 2ku_{xx} - 2u\bigl(3wv_x + 3v(2kw + w_x) + u_{xx}\bigr) \\
&+ 2wv_{xx} + 2v(6wu_x - 3kw_x + w_{xx})\bigr),\ \frac{1}{2}\bigl(2ku + u^2 + vw\bigr)\biggr]
\end{aligned}$$

with associated multiplier

$$Q = \left(u + k,\ \frac{1}{2}w,\ \frac{1}{2}v\right),$$

where k is a constant.

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