Multiple Positive Fixed Points of Nonlinear Operators on Ordered Banach Spaces

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1. Introduction. In this paper we consider the existence of multiple positive fixed points of completely continuous nonlinear operators defined on the cone K of an ordered Banach space E. Our main results give sufficient conditions for such an operator to have two, and in some cases three, positive fixed points.

The methods developed here improve a well-known multiple fixed point technique formulated by Krasnosel'skii and Stecenko [6] for certain boundary value problems and Hammerstein integral operators. Those authors considered an operator A bounded above and below by suitable order-preserving operators A_1 and A_2. They showed that if A_1 and A_2 have alternating sections of rapid and slow growth, it is sometimes possible to find disjoint order intervals (that is, sets of the form $\langle x, y\rangle = \{z \in K : x \leq z \leq y\}$) left invariant by A. If order intervals in K are bounded, the existence of a fixed point of A in each of the A-invariant order intervals then follows from Schauder's fixed point theorem. Using the fixed point index, Amann [1] has shown, under assumptions very similar to those in [6], that additional "intermediate" fixed points can sometimes be found outside the invariant order intervals. Results closely related to those of [6] and [1] can be found in [2], [3], [7], [8], and [9]. (Also, see [10].)

Here we investigate the existence of multiple positive fixed points of operators that need not satisfy the stringent monotonicity and growth assumptions imposed by the methods of [6]. In fact, the results proved here require no monotonicity assumptions whatever on the operator A. Hence they allow the determination of multiple fixed points that stem, in some sense, from a "change in direction" of A in the cone as well as in a change in the growth rate of A.

Consider, as an example, the nonlinear boundary value problem

$$x''(t) = -f(x(t)), \qquad 0 \leq t \leq 1, \tag{1.1}$$

$$x(0) = 0 = x'(1), \tag{1.2}$$

where f maps $\mathbf{R}$ continuously into $\mathbf{R}$. The solutions of (1.1)-(1.2) are the fixed points of the operator A defined on $C[0, 1]$, the space of real-valued, continuous functions on $[0, 1]$, by

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(1.3) $$Ax(t) = \int_0^1 G(t, s) f(x(s)) ds$$

where

(1.4) $$G(t, s) = \min\{t, s\}.$$

Our abstract results can be applied to the operator A to prove the following theorem.

Theorem 1.1 *Suppose there exist numbers a and d, with $0 < d < a$, satisfying the following properties:*

(i) $f(x) \geq 0, \quad 0 \leq x \leq 2a,$

(ii) $f(x) < 2d, \quad 0 \leq x \leq d;$

and

(iii) $f(x) \geq 4a, \quad a \leq x \leq 2a.$

Then the boundary value problem (1.1)-(1.2) has at least two nonnegative solutions. If, in addition, f is bounded above on $[0, \infty)$, then (1.1)-(1.2) has at least three nonnegative solutions.

The operator A defined by (1.3) is completely continuous and, by condition (i), maps a portion of the cone $C_+[0, 1]$ (the nonnegative functions in $C[0, 1]$) back into $C_+[0, 1]$. Clearly f (and hence A) may be badly nonmonotonic, and in general there do not exist two disjoint A-invariant order intervals. In fact, under the conditions of Theorem 1.1 the method of Krasnosel'skii and Stecenko would yield only one solution of (1.1)-(1.2).

In studying the existence of multiple positive fixed points of operators A that may be very nonmonotonic on a cone K, we have found it useful to consider (as an analog to order intervals) sets of the form

$$S(\alpha, a, b) = \{x \in K : \alpha(x) \geq a \quad \text{and} \quad \|x\| \leq b\},$$

where α is a concave positive functional defined on K. An important feature of our method is that the sets $S(\alpha, a, b)$ (and, in some cases, the domain of A) are not required to be left invariant by A. To a large extent, it is this feature that produces a generality and ease of application of our abstract results that is not possible with the approach of Krasnosel'skii and Stecenko.

In Section 2 we collect the definitions and notions basic to our work. The main results are stated and proved in Section 3, and in Section 4 we give a proof of Theorem 1.1 and provide additional applications of our abstract results. We conclude the paper by deriving a simplified version of the central result of Krasnosel'skii and Stecenko (Theorem 2 of [6]) for Hammerstein integral operators.

2. Definitions. Let E be a real Banach space. A closed, convex set $K \subset E$ is called a *(positive) cone* if the following conditions are satisfied:

(i) if $x \in K$, then $\lambda x \in K$ for $\lambda \geq 0$;

(ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A cone K induces a partial ordering $\leq$ in E by

$$x \leq y \text{ if and only if } y - x \in K.$$

A Banach space E with a partial ordering $\leq$ induced by a cone K is called an *ordered Banach space*. By a *completely continuous map* we mean a continuous function which takes bounded sets into relatively compact sets. In this paper we consider completely continuous maps which take some subset K_c, $0 < c \leq \infty$, of a cone K back into K, where $K_c = \{x \in K : \|x\| \leq c\}$, $0 < c < \infty$, and $K_\infty = K$.

Of particular importance to our applications is the ordered Banach space $C(\Omega)$, the continuous real-valued functions on the compact region Ω in $\mathbb{R}^n$. $C(\Omega)$ is endowed with the usual sup norm, and the cone of interest is $C_+(\Omega)$, the nonnegative functions in $C(\Omega)$.

We have been led by studies of various integral operators arising in applied problems to consider maps $A : K_c \to K$ satisfying the following property:

(2.1) A has a continuous extension $A_1 : K \to K$ such that range A_1 = range A and A_1 has no fixed points in $K \setminus K_c$.

In the following we give examples of maps A satisfying property (2.1). In these examples, and throughout the paper, K denotes a positive cone of some ordered Banach space.

Example 2.1. (a) Suppose $A : K_c \to K$ is completely continuous and $A(K_c) \subset K_c$. Define $A_1 : K \to K$ by

$$A_1x = \begin{cases} Ax & \text{if } x \in K_c \\ A\left(\dfrac{cx}{\|x\|}\right) & \text{if } x \in K \setminus K_c. \end{cases}$$

Then range A_1 = range A, A_1 is continuous, and all fixed points of A_1 must lie in K_c, so that condition (2.1) is satisfied.

(b) More generally, assume that $A : K_c \to K$ is completely continuous and that $Ax \in K_c$ for each $x \in K_c$ with $\|x\| = c$. Again define A_1 as in Example 2.1 (a). Clearly range A_1 = range A and A_1 is continuous. If $x \in K \setminus K_c$, then $A_1x = A\left(\dfrac{cx}{\|x\|}\right) \in K_c$, so that $A_1x \neq x$. Hence condition (2.1) is satisfied.

Example 2.2. Consider the nonlinear differential equation

$$\frac{d}{dt}\left[h(t, x)\frac{dx}{dt}\right] + f(t, x) = 0, \tag{2.2}$$

with boundary conditions:

$$x'(0) - ax(0) = 0, \; x'(1) = 0; \tag{2.3}$$

or

(2.4) $$x(0) = x'(1) = 0.$$

Such boundary value problems are used as mathematical models for a number of physical problems, including nonlinear heat conduction in one dimension, chemical reactions in adiabatic tubular reactors, and (with $h \equiv 1$) final value control problems. We consider functions h and f mapping $[0, 1] \times [0, c]$ $(c \leq \infty)$ continuously into $[0, \infty)$ such that h is bounded away from zero. Set $K = C_+[0, 1]$. Solutions to the boundary value problems (2.2)-(2.3) or (2.2)-(2.4) are fixed points of the completely continuous operator A on K_c defined by

(2.5)
$$Ax(t) = \kappa[ah(0, x(0))]^{-1} \int_0^1 f(s, x(s))ds + \int_0^t [h(u, x(u))]^{-1}\left[\int_u^1 f(s, x(s))ds\right]du,$$

where $\kappa = 1$ if (2.3) is assumed to hold and $\kappa = 0$ if (2.4) is assumed to hold. Suppose $f(t, c) = 0$, $0 \leq t \leq 1$, and define h_1 and f_1 by

$$h_1(t, x) = \begin{cases} h(t, x), & 0 \leq x \leq c, \\ h(t, c), & c < x, \end{cases}$$

and

$$f_1(t, x) = \begin{cases} f(t, x), & 0 \leq x \leq c, \\ 0, & c < x. \end{cases}$$

If A_1 is defined as in (2.5) with f and h replaced by f_1 and h_1, respectively, then A_1 is an extension of A which satisfies property (2.1). It is easy to see that A_1 is continuous and that range A_1 = range A. Let x be a fixed point of A_1 and note that $x(t) = A_1x(t)$ is a nondecreasing function of t. Assume that $x(1) \geq c$. Then there exists t_0, $0 < t_0 \leq 1$, such that $x(t_0) = c$ and $x(t) \geq c$ for $t_0 \leq t \leq 1$. Therefore for $u \geq t_0$, $\int_u^1 f_1(s, x(s))ds = 0$, so that

$$\begin{aligned} A_1x(1) &= \kappa[ah(0, x(0))]^{-1} \int_0^1 f_1(s, x(s))ds \\ &\quad + \int_0^{t_0} [h_1(u, x(u))]^{-1}\left[\int_u^1 f_1(s, x(s))ds\, du\right] \\ &= A_1x(t_0) = c. \end{aligned}$$

Hence $x \in K_c$ and A_1 has no fixed points in $K \setminus K_c$.

A notion central to our results is that of a *concave positive functional* on a cone K, that is, a continuous map $\alpha : K \to [0, \infty)$ satisfying

$$\alpha(\lambda x + (1 - \lambda)y) \geq \lambda\alpha(x) + (1 - \lambda)\alpha(y), \qquad 0 \leq \lambda \leq 1.$$

For example, if x_0 is an interior element of K, it is not difficult to show that the map $\alpha : K \to [0, \infty)$ defined by

$$\alpha(x) = \max \{t : tx_0 \leq x\}$$

is a concave positive functional on K. Also, consider the cone $K = C_+(\Omega)$, where Ω is a compact subset of $\mathbb{R}^n$. Let Ω_1 be a closed subset of Ω. Then the maps defined by

$$\alpha(x) = \min_{t \in \Omega_1} x(t)$$

and

$$\alpha(x) = \int_{\Omega_1} x(t)dt$$

are concave positive functionals on K.

If α is a concave positive functional on the cone K, a set of the form

$$S(\alpha, a, b) = \{x \in K : a \leq \alpha(x) \quad \text{and} \quad \|x\| \leq b\}$$

is closed, bounded, and convex in K. As will become evident, in proofs of existence of multiple fixed points of nonmonotonic operators on a cone, the sets $S(\alpha, a, b)$ often serve as suitable replacements for the order intervals commonly used in connection with monotonically increasing operators.

3. Main results. Most of the proofs in this section involve the fixed point index, the basic properties of which are listed in the following lemma. A proof of this lemma based on the Leray-Schauder degree theory can be found in [3].

Lemma 3.1. *Let Q be a retract of a Banach space E. For every open subset U of Q and every completely continuous map $A : \bar{U} \to Q$ which has no fixed points on ∂U = boundary of U, there exists an integer $i(A, U, Q)$ satisfying:*

(i) if $A : \bar{U} \to U$ is a constant map, then $i(A, U, Q) = 1$;

(ii) if U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\bar{U} \setminus (U_1 \cup U_2)$, then $i(A, U, Q) = i(A, U_1, Q) + i(A, U_2, Q)$, where $i(A, U_k, Q) = i(A|\bar{U}_k, U_k, Q)$, $k = 1, 2$;

(iii) if I is a compact interval in $\mathbb{R}$ and $h : I \times \bar{U} \to Q$ is a continuous map with relatively compact range such that $h(\lambda, x) \neq x$ for $(\lambda, x) \in I \times \partial U$, then $i(h(\lambda, \cdot), U, Q)$ is well-defined and independent of λ;

(iv) if $i(A, U, Q) \neq 0$, then A has at least one fixed point in U;

(v) if Q_1 is a retract of Q and $A(\bar{U}) \subset Q_1$, then $i(A, U, Q) = i(A, U \cap Q_1, Q_1)$, where $i(A, U \cap Q_1, Q_1) = i(A|\overline{U \cap Q_1}, U \cap Q_1, Q_1)$;

(vi) if V is open in U and A has no fixed points in $\bar{U} \setminus V$, then $i(A, U, Q) = i(A, V, Q)$.

Our first result gives sufficient conditions for an operator $A : K_c \to K$ to have at least one nonzero fixed point.

Theorem 3.2. *Suppose* $A : K_c \to K$ *is completely continuous and suppose there exist a concave positive functional* α *with* $\alpha(x) \leq \|x\|$ $(x \in K)$ *and numbers* $b > a > 0$ $(b \leq c)$ *satisfying the following conditions:*

(1) $\{x \in S(\alpha, a, b): \alpha(x) > a\} \neq \phi$, *and* $\alpha(Ax) > a$ *if* $x \in S(\alpha, a, b,)$;

(2) $Ax \in K_c$ *if* $x \in S(\alpha, a, c)$;

(3) $\alpha(Ax) > a$ *for all* $x \in S$ (α, a, c) *with* $\|Ax\| > b$.

Then A has a fixed point x in $S(\alpha, a, c)$.

Proof. Set $U = \{x \in S(\alpha, a, c) : \alpha(x) > a\}$. Then U is the interior of $S(\alpha, a, c)$ in K_c. Suppose that $x \in \partial U$ is a fixed point of A. Then $\alpha(x) = a$ and either $x \in S(\alpha, a, b)$ or $\|x\| > b$; but if $x \in S(\alpha, a, b)$, then $\alpha(x) = \alpha(Ax) > a$, and if $\|x\| > b$, then $\|Ax\| > b$ and $\alpha(x) = \alpha(Ax) > a$. Hence A has no fixed points in ∂U, and there exists an integer $i(A, U, K_c)$ satisfying properties (i)–(vi) of Lemma 3.1.

Choose $x_0 \in S(\alpha, a, b)$ such that $\alpha(x_0) > a$, and define the map $h : [0, 1] \times \bar{U} \to K_c$ by

$$h(t, x) = (1 - t)Ax + tx_0.$$

Clearly h is continuous and $h([0, 1] \times \bar{U})$ is relatively compact. Suppose there exists $(t, x) \in [0, 1] \times \partial U$ such that $h(t, x) = x$. Then $\alpha(x) = a$. If $\|Ax\| > b$, then by condition (3) $\alpha(Ax) > a$, so that

$$\begin{aligned}\alpha(x) &= \alpha(h(t, x)) = \alpha((1 - t)Ax + tx_0)\\ &\geq (1 - t)\alpha(Ax) + t\alpha(x_0) > a,\end{aligned}$$

a contradiction. On the other hand, if $\|Ax\| \leq b$, then

$$\|x\| = \|(1 - t)Ax + tx_0\| \leq (1 - t)\|Ax\| + t\|x_0\| \leq b,$$

so that $x \in S(\alpha, a, b)$. Hence, by condition (1), $\alpha(Ax) > a$ and again we arrive at the contradiction $\alpha(x) = \alpha((1 - t)Ax + tx_0) > a$. It follows that for each $(t, x) \in [0, 1] \times \partial U$, $h(t, x) \neq x$. Therefore by (i) and (iii) of Lemma 3.1, $i(A, U, K_c) = i(x_0, U, K_c) = 1$. Hence by (iv) of Lemma 3.1, A has a fixed point in U.

Remark 1. Condition (3) of Theorem 3.2 will be satisfied if either of the following conditions holds:

(i) $\alpha(Ax) \geq \dfrac{a}{b} \|Ax\|, \qquad x \in S(\alpha, a, c);$

(ii) $\|Ax\| - \alpha(Ax) \leq b - a, \qquad x \in S(\alpha, a, c).$

In applications of Theorem 3.2 and of the results which follow, it is often easier to establish the validity of (i) or (ii) than to establish the more general condition (3) directly.

Remark 2. Note that, in Theorem 3.2, the sets $S(\alpha, a, b)$ and $S(\alpha, a, c)$ (and in this theorem, the set K_c) are not required to be left invariant by A. In practice, it is usually very difficult to construct invariant concave sets in a cone other than order intervals containing zero and sets of the form K_r, and the requirement that other types of sets be left invariant by an operator severely restricts the applicability of a fixed point result for a cone.

In the following two theorems (Theorems 3.3 and 3.4) we place additional restrictions on the operator A of Theorem 3.2 and establish the existence of at least three fixed points of A. The use of the fixed point index in Theorems 3.3 and 3.4 is similar to the proof of Theorem 2 in [1]. However, in [1] it is assumed that the domain D and two disjoint, convex subsets of D are left invariant by A. In Theorem 3.3 we assume the invariance of the domain K_c and of a set $K_a \subset K_c$, and in Theorem 3.4 we assume the invariance of only the smaller set K_a.

Theorem 3.3. *Suppose $A : K_c \to K_c$ is completely continuous, and suppose there exist a concave positive functional α with $\alpha(x) \leq \|x\|$ ($x \in K$) and numbers a, b, and d, with $0 < d < a < b \leq c$, satisfying the following conditions:*
(1) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \phi$ and $\alpha(Ax) > a$ if $x \in S(\alpha, a, b)$;
(2) $\|Ax\| < d$ if $x \in K_d$;
(3) $\alpha(Ax) > a$ for all $x \in S(\alpha, a, c)$ with $\|Ax\| > b$.
Then A has at least three fixed points in K_c.

Proof. Let $U_1 = \{x \in K_c : \|x\| < d\}$ and $U_2 = \{x \in S(\alpha, a, c):\alpha(x) > a\}$. Then U_1 and U_2 are convex open sets in K_c and A has no fixed points on $\partial U_1 \cup \partial U_2 = \partial(U_1 \cup U_2)$. By (ii) of Lemma 3.1,

$$i(A, K_c, K_c) = i(A, U_1 \cup U_2, K_c) + i(A, K_c \setminus (\overline{U_1 \cup U_2}), K_c)$$

and

$$i(A, U_1 \cup U_2, K_c) = i(A, U_1, K_c) + i(A, U_2, K_c),$$

so that

$$i(A, K_c \setminus (\overline{U_1 \cup U_2}), K_c) = i(A, K_c, K_c) - \sum_{j=1}^{2} i(A, U_j, K_c).$$

Suppose V is a convex open subset of K_c such that $A : V \to V$ and A has no fixed points on ∂V. By property (v) of Lemma 3.1, $i(A, V, K_c) = i(A, V, \bar{V})$, since $\bar{V}$ is a retract of K_c. Fix x_0 in V and define $h : [0, 1] \times \bar{V} \to \bar{V}$ by

$$h(t, x) = (1 - t)Ax + tx_0.$$

Now if $h(t, x) = x$ for some $x \in \partial V$, then $t = 0$, since otherwise $h(t, x) \in V$. But then $Ax = x$ for some $x \in \partial V$, which is assumed not to be the case. Then by (i) and (iii) of Lemma 3.1, $i(A, V, \bar{V}) = i(x_0, V, \bar{V}) = 1$.

Now $A(U_1) \subset U_1$ and $A(K_c) \subset K_c$, so that $i(A, U_1, K_c) = 1 = i(A, K_c, K_c)$. (Note that A has no fixed points on the boundary of U_1 in K_c, and that the boundary of K_c in K_c is empty.) Also, it follows from the proof of Theorem 3.2 that $i(A, U_2, K_c) = 1$. Therefore $i(A, K_c \setminus (\overline{U_1 \cup U_2}), K_c) = 1 - 2 = -1$. By property (iv) of Lemma 3.1, A has a fixed point in $K_c \setminus (\overline{U_1 \cup U_2})$. By Schauder's theorem, A has a fixed point in U_1, and by Theorem 3.2, A has a fixed point in U_2. Therefore A has at least three fixed points in K_c.

In the following theorem we replace the assumption in Theorem 3.3 that $A(K_c) \subset K_c$ with the more general property (2.1). However, condition (3) of Theorem 3.3 must be modified somewhat.

Theorem 3.4. *Let $A : K_c \to K$ be a completely continuous operator satisfying property (2.1). Suppose there exist a concave positive functional α with $\alpha(x) \leq \|x\|$ $(x \in K)$ and numbers a, b, and d, with $0 < d < a < b \leq c$, satisfying the following conditions:*

(1) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \phi$ and $\alpha(Ax) > a$ if $x \in S(\alpha, a, b)$;

(2) $\|Ax\| < d$ if $x \in K_d$;

(3) $\alpha(Ax) > a$ if $x \in K_c$ and $\|Ax\| > b$.

Then A has at least three fixed points in K_c.

Proof. Let A_1 be the extension of A described in property (2.1) and choose $r \geq c$ such that $A_1(K_r) \subset K_r$. Note that conditions (1) and (2) of Theorem 3.3 hold for A_1. If $x \in S(\alpha, a, r)$ and $\|A_1 x\| > b$, then $A_1 x = Ay$ for some $y \in K_c$ and $\alpha(A_1 x) = \alpha(Ay) > a$ (since $\|Ay\| > b$). Hence condition (3) of Theorem 3.3 is satisfied for K_r and A_1, and A_1 has at least three fixed points in K_r. Since A_1 has no fixed points in $K \diagdown K_c$, these fixed points lie in K_c and therefore are fixed points of A.

It is possible to obtain two fixed points of A even if A does not satisfy property (2.1). In this case condition (3) of Theorem 3.4 must be replaced by stronger conditions of the type in Remark 1.

Theorem 3.5. *Suppose $A : K_c \to K$ is completely continuous, and suppose there exist a concave positive functional α with $\alpha(x) \leq \|x\|$ $(x \in K)$ and numbers a and d, with $0 < d < a < c$, satisfying the following conditions:*

(1) $\{x \in S(\alpha, a, c) : \alpha(x) > a\} \neq \phi$ and $\alpha(Ax) > a$ if $x \in S(\alpha, a, c)$;

(2) $\|Ax\| < d$ if $x \in K_d$;

and either

(3) $\|Ax\| - \alpha(Ax) \leq c - a$ for each $x \in K_c$ such that $\|Ax\| > c$;

or

(4) $\alpha(Ax) > \dfrac{a}{c}\|Ax\|$ for each $x \in K_c$ such that $\|Ax\| > c$.

Then A has at least two fixed points in K_c.

Proof. The existence of a fixed point x_1 in K_d follows from Schauder's fixed point theorem. To prove the existence of a second fixed point in K_c, define the auxiliary operator $B : K_c \to K_c$ by

$$Bx = \begin{cases} Ax & \text{if } \|Ax\| \leq c \\ \dfrac{cAx}{\|Ax\|} & \text{if } \|Ax\| > c. \end{cases}$$

Clearly B is completely continuous and $\|Bx\| < d$ if $x \in K_d$. Suppose $x \in S(\alpha, a, c)$. If $\|Ax\| \leq c$, then $\alpha(Bx) = \alpha(Ax) > a$. If $\|Ax\| > c$, then $\alpha(Bx) \geq \frac{c}{\|Ax\|}\alpha(Ax)$, for

$$\alpha(Bx) = \alpha\left(\frac{cAx}{\|Ax\|}\right) = \alpha\left(\frac{c}{\|Ax\|}Ax + \left(1 - \frac{c}{\|Ax\|}\right)0\right)$$

$$\geq \frac{c}{\|Ax\|}\alpha(Ax) + \left(1 - \frac{c}{\|Ax\|}\right)\alpha(0) = \frac{c}{\|Ax\|}\alpha(Ax).$$

Therefore if $\|Ax\| > c$ and condition (3) holds, then

$$\begin{aligned}\|Bx\| - \alpha(Bx) &\leq c[1 - \alpha(Ax)/\|Ax\|] \\ &= c\|Ax\|^{-1}[\|Ax\| - \alpha(Ax)] \\ &\leq c\|Ax\|^{-1}(c - a) < c - a,\end{aligned}$$

so that $\alpha(Bx) > \|Bx\| + a - c = a$. If $\|Ax\| > c$ and condition (4) holds, then

$$\alpha(Bx) \geq c\alpha(Ax)\|Ax\|^{-1} > c\|Ax\|^{-1}[ac^{-1}\|Ax\|] = a.$$

The hypotheses of Theorem 3.3 are now satisfied (with $b = c$) for the operator B. By the proof of Theorem 3.3, B has a fixed point $x_2 \in K_c \setminus (K_d \cup S(\alpha, a, c))$. Therefore $\alpha(x_2) < a$. If $\|Ax_2\| > c$ and condition (3) holds, then

$$\begin{aligned}a > \alpha(x_2) &= \alpha(Bx_2) \geq c\|Ax_2\|^{-1}\alpha(Ax_2) \\ &\geq c\|Ax_2\|^{-1}[\|Ax_2\| + a - c] = c - c\|Ax_2\|^{-1}(c - a) \\ &\geq c - (c - a) = a,\end{aligned}$$

a contradiction. Finally, if $\|Ax_2\| > c$ and condition (4) holds, then

$$\begin{aligned}a > \alpha(x_2) &= \alpha(Bx_2) \geq c\|Ax_2\|^{-1}\alpha(Ax_2) \\ &> c\|Ax_2\|^{-1}[ac^{-1}\|Ax_2\|] = a,\end{aligned}$$

a contradiction. Therefore $\|Ax_2\| \leq c$, and $Ax_2 = Bx_2 = x_2$.

4. Applications. Consider the Hammerstein integral operator defined on $C(\Omega)$ by

$$Ax(t) = \int_\Omega G(t, s)f(s, x(s))ds, \qquad t \in \Omega. \tag{4.1}$$

Here Ω is a compact region in $\mathbb{R}^n$, $f\colon \Omega \times [0, c] \to [0, \infty)$ is continuous, and $G\colon \Omega \times \Omega \to [0, \infty)$ is such that A is completely continuous on $K_c = [C_+(\Omega)]_c$. (For example, if G is continuous, then A is completely continuous.) Let Ω_1 be a closed subset of Ω of positive Lebesgue measure, and assume that

$$\|G\| \equiv \sup_{t \in \Omega} \int_\Omega G(t, s)ds < \infty,$$

$$\epsilon \equiv \inf_{t \in \Omega_1} \int_{\Omega_1} G(t, s)ds > 0,$$

and

$$\delta \equiv \sup_{\substack{u \in \Omega_1, \\ t \in \Omega}} \int_{\Omega} |G(t, s) - G(u, s)|ds > 0.$$

Theorem 3.3 can be reduced to the following result for Hammerstein integral operators.

Theorem 4.1. *Suppose there exist positive numbers a, b, and d, with $0 < d < a < b \leq c$, satisfying the following conditions:*
(1) $f(t, x) > a\epsilon^{-1}$ if $t \in \Omega_1$, $a \leq x \leq b$;
(2) $f(t, x) < d\|G\|^{-1}$ if $t \in \Omega$, $0 \leq x \leq d$;
(3) $f(t, x) \leq \delta^{-1}(b - a)$ if $t \in \Omega$, $0 \leq x \leq c$;
(4) $f(t, x) \leq c\|G\|^{-1}$ if $t \in \Omega$, $0 \leq x \leq c$.
Then the Hammerstein operator (4.1) has at least three fixed points in $K_c = [C_+(\Omega)]_c$.

Proof. Condition (2) implies that $\|Ax\| < d$ if $\|x\| \leq d$, and condition (4) insures that A maps K_c into K_c. Define α on K by $\alpha(x) = \min_{t \in \Omega_1} x(t)$. Obviously, $\alpha(x) \leq \|x\|$ and there exists $x \in S(\alpha, a, b)$ with $\alpha(x) > a$. If $x \in S(\alpha, a, b)$, then

$$\begin{aligned}\alpha(Ax) &= \min_{t \in \Omega_1} \int_{\Omega} G(t, s)f(s, x(s))ds \\ &\geq \min_{t \in \Omega_1} \int_{\Omega_1} G(t, s)f(s, x(s))ds \\ &> \min_{t \in \Omega_1} \int_{\Omega_1} G(t, s)\, a\epsilon^{-1}ds = a.\end{aligned}$$

Finally, if $x \in \dot{S}(\alpha, a, c)$, then for $t \in \Omega$ and $u \in \Omega_1$,

$$\begin{aligned}&\int_{\Omega} G(t, s)f(s, x(s))ds - \int_{\Omega} G(u, s)f(s, x(s))ds \\ &\leq \int_{\Omega} |G(t, s) - G(u, s)|f(s, x(s))ds \\ &\leq \int_{\Omega} |G(t, s) - G(u, s)|\delta^{-1}(b - a)ds \leq b - a.\end{aligned}$$

Thus $\|Ax\| - \alpha(Ax) \leq b - a$, and if $\|Ax\| > b$, then $\alpha(Ax) > a$. The theorem now follows from Theorem 3.3.

Proof of Theorem 1.1. It is easy to show that the solutions of (1.1)-(1.2) correspond to fixed points of the operator A defined by (1.3)-(1.4), and that A maps K_c into K, where c is any positive number such that $f(x) \geq 0$ for $x \in [0, c]$.

Also, an easy application of the Arzela-Ascoli theorem shows that A is completely continuous on K_c.

First we assume that conditions (i), (ii), and (iii) of Theorem 1.1 hold, and we apply Theorem 3.5 to establish the existence of at least two fixed points of A. If $x \in K_d$, then

$$\|Ax\| = Ax(1) = \int_0^1 G(1, s)f(x(s))ds < 2d \int_0^1 sds = d,$$

and condition (2) of Theorem 3.5 is satisfied. Let $\alpha(x) = \min_{1/2 \leq t \leq 1} x(t)$. Clearly $\{x \in S(\alpha, a, 2a) : \alpha(x) > a\} \neq \phi$, and if $x \in S(\alpha, a, 2a)$, then

$$\begin{aligned}
\alpha(Ax) &= \min_{1/2 \leq t \leq 1} \int_0^1 G(t, s)f(x(s))ds \\
&= \int_0^1 G\left(\frac{1}{2}, s\right)f(x(s))ds \\
&> \int_{1/2}^1 G\left(\frac{1}{2}, s\right)f(x(s))ds \\
&\geq \int_{1/2}^1 \left(\frac{1}{2}\right)(4a)ds = a.
\end{aligned}$$

Hence condition (1) of Theorem 3.5 is satisfied. Furthermore, if f is nonnegative on any interval $[0, c]$, and if $x \in K_c$ and $\|Ax\| > 2a$, then

$$\begin{aligned}
\alpha(Ax) &= \int_0^1 G\left(\frac{1}{2}, s\right)f(x(s))ds \\
&= \int_0^{1/2} sf(x(s))ds + \int_{1/2}^1 [f(x(s))/2]ds \\
&> \left[\int_0^{1/2} sf(x(s))ds + \int_{1/2}^1 sf(x(s))ds\right]\Big/2 \\
&= \left[\int_0^1 sf(x(s))ds\right]\Big/2 \\
&= \left[\int_0^1 G(1, s)f(x(s))ds\right]\Big/2 \\
&= Ax(1)/2 = \|Ax\|/2 = (c/2c)\|Ax\|.
\end{aligned}$$

If we let $c = 2a$, then condition (4) of Theorem 3.5 holds, and A has at least two fixed points in K_c.

We next assume that f is bounded above on $[0, \infty)$ and show that A has at least three fixed points in K. If $f(x) > 0$ for all $x > 2a$, then there exists $r > 2a$ such that A maps K_r into K_r, and the existence of three fixed points of A in K will follow from Theorem 3.3 (with $b = 2a$ and $c = r$), together with the preceding proof of the existence of two fixed points. If $f(r) = 0$ for some smallest r in $[2a, \infty)$, then the existence of three fixed points of A in K_r will follow from Theorem 3.4 and Example 2.2, since (1.1)−(1.2) can be written as (2.2)-(2.4) with $h(t, x) \equiv 1$ and $A|K_r$ can be written in the form

$$Ax(t) = \int_0^t \left[\int_u^1 f(x(s))ds \right] du.$$

Thus $A|K_r$ has an extension $A_1 : K \to K$ such that range A_1 = range $A|K_r$ and A_1 has no fixed points in K_r.

Next, we consider the boundary value problem

$$\beta x''(t) - x'(t) + f(x(t)) = 0, \qquad 0 \le t \le 1, \beta > 0, \tag{4.2}$$

$$\beta x'(0) - x(0) = 0, x'(1) = 0, \tag{4.3}$$

where f maps $[0, \infty)$ continuously into $(-\infty, \infty)$. This boundary value problem arises in the theory of adiabatic tubular chemical reactors and has been studied extensively for the case in which f is the Arrhenius reaction rate

$$f(x) = p(q - x) \exp(-k/(1 + x)). \tag{4.4}$$

(See, *e.g.*, [4], [5], [10].)

The Green's function for (4.2)-(4.3) is given by

$$G(t, s) = \begin{cases} \exp \dfrac{t - s}{\beta}, & 0 \le t \le s \le 1 \\ 1, & 0 \le s \le t \le 1, \end{cases}$$

and solutions of (4.2)-(4.3) can be identified with fixed points of the completely continuous operator $A : C_+[0, 1] \to C[0, 1]$ defined by

$$Ax(t) = \int_0^1 G(t, s) f(x(s)) ds.$$

Theorem 4.2. *Suppose there exist positive numbers a and d with* $0 < d < a$ *such that*

(1) $f(x) \ge 0$ *if* $0 \le x \le ae^{1/\beta}$;

(2) $f(x) < d$ *if* $0 \le x \le d$;

and

(3) $f(x) > a\,(\beta - \beta e^{-1/\beta})^{-1}$ *if* $a \le x \le ae^{1/\beta}$.

Then (4.1)-(4.2) has at least two nonnegative solutions. If, in addition, $f(x)$ *is bounded above on* $[0, \infty)$, *or if there exists* $c \ge ae^{1/\beta}$ *such that* $0 \le f(x) \le c$ *for* $0 \le x \le c$, *then (4.1)-(4.2) has at least three nonnegative solutions.*

Proof. Let $K = C_+[0, 1]$, and let $\alpha(x) = \min_{0 \le t \le 1} x(t)$, $x \in K$. Note that $G(t, s)$ is an increasing function of t for fixed s. If $x \in K$ and $\|x\| \le d$, then

$$\|Ax\| = Ax(1) = \int_0^1 G(1, s)f(x(s))ds$$
$$< d \int_0^1 G(1, s)ds = d.$$

Further, if $x \in S(\alpha, a, ae^{1/\beta})$ then

$$\alpha(Ax) = Ax(0) = \int_0^1 e^{-s/\beta}f(x(s))ds$$
$$> a(\beta - \beta e^{-1/\beta})^{-1} \int_0^1 e^{-s/\beta}ds = a.$$

Finally, if $x \in K$, $\|Ax\| > 0$, and $f(x(s)) \geq 0$ for $0 \leq s \leq 1$, then

$$\alpha(Ax) = Ax(0) = \int_0^1 e^{-s/\beta}f(x(s))ds$$
$$> e^{-1/\beta} \int_0^1 f(x(s))ds = e^{-1/\beta}Ax(1)$$
$$= e^{-1/\beta}\|Ax\| = a(ae^{1/\beta})^{-1}\|Ax\|.$$

Thus the existence of at least two nonnegative solutions follows from Theorem 3.5 (with $c = ae^{1/\beta}$).

Now if $0 \leq f(x) \leq c$ for $0 \leq x \leq c$, where $c \geq ae^{1/\beta}$, then

$$\|Ax\| = Ax(1) = \int_0^1 f(x(s))ds \leq c, \; x \in K_c.$$

Hence, if we set $b = ae^{1/\beta}$, the existence of three nonnegative solutions follows from Theorem 3.3.

If f is bounded above and is nonnegative, there exists $c \geq ae^{1/\beta}$ such that $0 \leq f(x) \leq c$ for $0 \leq x \leq c$, and again there exist at least three nonnegative solutions. If f is bounded above and $f(x) < 0$ for some $x \in [0, \infty)$, then there exists $c > ae^{1/\beta}$ such that $f(c) = 0$ and $f(x) \geq 0$ for $0 \leq x \leq c$. Define f_1: $[0, \infty) \to [0, \infty)$ by

$$f_1(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq c \\ 0 & \text{if } x > c. \end{cases}$$

Then the operator

$$Ax(t) = \int_0^1 G(t, s)f_1(x(s))ds \tag{4.5}$$

has at least three fixed points in K. Since (4.2) with f replaced by f_1 can be written in the form

$$\frac{d}{dt}\left(\beta e^{-t/\beta}\frac{dx}{dt}\right) + e^{-t/\beta}f_1(x(t)) = 0,$$

(4.2)-(4.3) is a special case of the boundary value problem (2.2)-(2.3) in Example 2.2. Thus all solutions of (4.2)-(4.3) (again, with f replaced by f_1) and all fixed points of the operator (4.5) must have norm no greater than c. Hence these three solutions are solutions of the original boundary value problem (4.2)-(4.3).

For the nonlinearity (4.4), the boundary value problem (4.2)-(4.3) has a unique solution provided $k \leq 4 + 4/q$ (see [5]). If $k > 4 + 4q$, then $f(x)/x$ is increasing in an interval $[r_1, r_2]$ and is decreasing elsewhere in $[0, \infty)$, where

$$r_1 = (2k + 2q)^{-1}\{kq - 2q - [kq(kq - 4q - 4)]^{1/2}\},$$

and

$$r_2 = (2k + 2q)^{-1}\{kq - 2q + [kq(kq - 4q - 4)]^{1/2}\}.$$

If p in (4.4) is chosen so that $f(r_1)/r_1 < 1$ and $f(r_2)/r_2 > 1$, then (4.2)-(4.3)-(4.4) will have at least three solutions for a range of values of β. Choose $d = r_1$, $a \in (r_1, r_2]$ with $f(a)/a > 1$, and $b > r_2$ such that $f(a) = f(b)$. Then Theorem 4.2 will apply to give three solutions provided β satisfies the inequalities

$$ae^{1/\beta} \leq b$$

and

$$\beta - \beta e^{-1/\beta} > a/f(a).$$

We conclude by demonstrating how the notion of a concave positive functional can be used to obtain an improved version of the central result of Krasnosel'skii and Stecenko [6, Theorem 2] for the Hammerstein integral operator (4.1). We have eliminated superfluous hypotheses and present a simpler formulation and simpler proof of essentially the same result. In the following, Ω, Ω_1, f, G, $\|G\|$, and ϵ are the same as in the first paragraph of Section 4; the quantity δ defined in that paragraph is not used here.

Theorem 4.3. *Suppose there exist numbers a and b, with $0 < a < b$, such that*

(1) $a/\epsilon \leq f(s, x)$ if $s \in \Omega_1$,

and

(2) $f(s, x) \leq b/\|G\|$ if $s \in \Omega$ and $0 \leq x \leq b$.

Then there is a nonnegative function $x \in C(\Omega)$ such that $x(t) \leq b$ if $t \in \Omega$, $x(t) \geq a$ if $t \in \Omega_1$, and $Ax = x$.

Proof. Define $\alpha : C(\Omega) \to \mathbb{R}$ by $\alpha(x) = \min_{t \in \Omega_1} x(t)$. If $x \in S(\alpha, a, b)$, then for each $t \in \Omega_1$,

$$\begin{aligned} Ax(t) &= \int_\Omega G(t, s)f(s, x(s))ds \\ &\geq \int_{\Omega_1} G(t, s)f(s, x(s))ds \end{aligned}$$

$$\geq (a/\epsilon) \int_{\Omega_1} G(t, s)ds \geq a.$$

Thus $\alpha(Ax) \geq a$. Furthermore, for each $t \in \Omega$,

$$0 \leq Ax(t) = \int_\Omega G(t, s)f(s, x(s))ds$$

$$\leq \int_\Omega G(t, s)(b/\|G\|)ds \leq b.$$

Therefore A leaves $S(\alpha, a, b)$ invariant, and the theorem now follows from Schauder's fixed point theorem.

Remark 3. In Theorem 2 of [6], it is assumed that there exist functions $f_1(s, x)$ and $f_2(s, x)$, nondecreasing in x, such that $0 \leq f_1(s, x) \leq f(s, x) \leq f_2(s, x)$ for each $s \in \Omega$ and $x \in [0, b]$. Conditions (1) and (2) take the form

(1)′ $a/\epsilon \leq f_1(s, a)$ for each $s \in \Omega$,

and

(2)′ $f_2(s, b) \leq b/\|G\|$ for each $s \in \Omega$.

Note that condition (1) of Theorem 4.3 is required to hold only for $s \in \Omega_1$.

Remark 4. As with Theorem 2 of [6], the preceding result may be expanded into a multiple fixed point result. However, as indicated earlier with regard to Theorem 2 of [6], the applications of such a fixed point result would be very limited. We have presented Theorem 4.3 only to demonstrate the relative simplicity of our methods compared with those of [6].

Remark 5. If strict inequality were assumed to hold in condition (1) of Theorem 4.3, then Theorem 4.3 would be a special case of Theorem 3.2.

References

1. H. Amann, *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Functional Analysis **11** (1972), 346–384.
2. H. Amann, *Existence of multiple solutions for nonlinear elliptic boundary value problems*, Indiana Univ. Math. J. **21** (1972), 925–935.
3. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review **18** (1976), 620–709.
4. N. R. Amundson & D. Luss, *Qualitative and quantitative observations on the tubular reactor*, Canadian J. Chem. Eng. **46** (1968), 424–433.
5. D. Cohen, *Multiple stable solutions of nonlinear boundary value problems arising in chemical reactor theory*, SIAM J. Appl. Math. **20** (1971), 1–13.
6. M. A. Krasnosel'skii & V. Ja. Stecenko, *Some nonlinear problems with many solutions*, Amer. Math. Soc. Transl., Ser. 2, **54** (1966), 29–48.
7. T. Laetsch, *Existence and bounds for multiple solutions of nonlinear equations*, SIAM J. Appl. Math. **18** (1970), 389–400.
8. S. V. Parter, *Solutions of a differential equation arising in chemical reactor processes*, SIAM J. Appl. Math. **26** (1974), 687–716.
9. M. Steuerwalt, *Stability of solutions of mildly nonlinear elliptic problems*, Indiana Univ. Math. J. **25** (1976), 1083–1103.

10. L. R. Williams & R. W. Leggett, *Multiple fixed point theorems for problems in chemical reactor theory*, Journal Math. Anal. Appl. **69** (1979).

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