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Multiple Radial Solutions for a Class of Elliptic Systems with Singular Nonlinearities (*).

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Summary. – *We study radial solutions $u = (u_1, u_2)$ in an exterior domain of $\boldsymbol{R}^N$ $(N \geq 3)$ of the elliptic system $-\Delta u + V'(u) = 0$, where V is a positive and singular potential. We look for solutions which satisfy Dirichlet boundary conditions and vanish at infinity. We prove existence of infinitely many radial solutions, which can be topologically classified by their winding numbers around the singularity of V. Furthermore, we study some qualitative properties of such solutions.*

1. – Introduction and statement of the results.

In the present paper, we aim to prove the existence of infinitely many radial solutions $u = (u_1, u_2)$ of the following problem:

$$\text{(P)} \qquad \begin{cases} -\Delta u + V'(u) = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ \lim\limits_{|x| \to \infty} u(x) = 0\,, \end{cases}$$

where $\Omega = \boldsymbol{R}^N \setminus \overline{B}_1(0)$, $N \geq 3$ and V is a C^1 real map defined in an open subset of $\boldsymbol{R}^2$. By $V' = (V_{\xi_1}, V_{\xi_2})$ we denote the gradient of V.

By some Pohozaev-type arguments (cf. [9, 1]), it is easy to see that

$$\text{(1.1)} \qquad -\Delta u + g'(u) = 0 \quad \text{in } \boldsymbol{R}^N \quad (N \geq 3)$$

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with $g \in C^1(\boldsymbol{R}^k, \boldsymbol{R})$, $g \geq 0$, $k \geq 1$, does not admit nonconstant solutions with finite energy (for $k = 1$, this result is also known as Derrick Theorem). These nonexistence results are based on scaling arguments thus, in order to regain solutions for (1.1), it is convenient to replace $\boldsymbol{R}^N$ by a domain which is not invariant under scalings. On the other hand, even if this is the case (1.1) may admit no nontrivial solutions at all if g is a smooth function, as can be easily checked.

In view of such results, we set problem (P) in the exterior domain of a ball, and we consider a singular potential, namely we assume

(V1) there exists $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2) \neq (0, 0)$ such that $V \in C^1(\boldsymbol{R}^2 \setminus \{\bar{\xi}\}, \boldsymbol{R})$;

(V2) $V(\xi) \geq V(0)$ for any $\xi \in \boldsymbol{R}^2 \setminus \{\bar{\xi}\}$.

As concernes the behaviour of V around the singularity, we assume a Strong Force type condition (see [10]):

(V3) there exist m, $\delta > 0$ such that $V(\xi) \geq m|\xi - \bar{\xi}|^{-2}$ for any $0 < |\xi - \bar{\xi}| < \delta$.

Let us introduce some notation. Let $\mathcal{D}^{1,2}(\Omega, \boldsymbol{R}^2)$ be, as usual, the closure of $C_0^\infty(\Omega, \boldsymbol{R}^2)$ under the norm

$$\|u\|_{\mathcal{D}^{1,2}} = (\|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2)^{1/2}.$$

As we are interested in radial solutions of (P), let us introduce $\mathcal{D}_{\mathrm{rad}}^{1,2}(\Omega, \boldsymbol{R}^2)$, the set of radial functions in $\mathcal{D}^{1,2}(\Omega, \boldsymbol{R}^2)$. By a *radial weak solution* of (P) we mean a function $u \in \mathcal{D}_{\mathrm{rad}}^{1,2}(\Omega, \boldsymbol{R}^2)$ such that

$$\int_\Omega (\nabla u \nabla \varphi + V'(u)\,\varphi)dx = 0 \qquad \text{for any } \varphi \in C_0^\infty(\Omega, \boldsymbol{R}^2).$$

It is well known that radial weak solutions of (P) correspond to critical points in $\mathcal{D}_{\mathrm{rad}}^{1,2}(\Omega, \boldsymbol{R}^2)$ of the energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega (V(u(x)) - V(0))\,dx.$$

For simplicity, we shall henceforth assume $V(0) = 0$.

By Radial Lemma (cf. [5, 14, 4]), any radial weak solution u of (P) is continuous and vanishes at infinity. Moreover, as a consequence of assumption (V3), u takes values in $\boldsymbol{R}^2 \setminus \{\bar{\xi}\}$. In other words, u describes a closed curve in the plane which starts and ends at the origin without crossing the singularity, thus the winding number of u around $\bar{\xi}$ makes sense (cf. Section 2). The nontrivial topological properties of the target space (namely, the fact that the fundamental group of $\boldsymbol{R}^2 \setminus \{\bar{\xi}\}$ is isomorphic to $\boldsymbol{Z}$) allow endowing the set of radial weak solutions of (P) with a topological classification. Therefore we shall look for radial weak solutions of (P) having prescribed winding number q around the singularity of V, for any integer q.

The multiplicity result we shall prove is the following:

THEOREM 1.1. – *Assume* (V1-V3). *For any* $q \in \boldsymbol{Z}$, (P) *has at least one weak radial solution* u_q *with winding number around* $\bar{\xi}$ *equal to* q. *Furthermore*:

i) $u_q \in C^2(\Omega)$;

ii) u_q *has exponential decay at infinity, together with its derivatives up to second order, namely*

$$|D^\alpha u_q(x)| \leq Ce^{-k|x|}, \qquad x \in \Omega$$

for some $C, k > 0$ *and for* $|\alpha| \leq 2$, *provided the following condition holds*:

$$\text{(V4)} \qquad 0 < \liminf_{\xi \to 0} \frac{V_{\xi_j}(\xi)}{\xi_j} \leq \limsup_{\xi \to 0} \frac{V_{\xi_j}(\xi)}{\xi_j} < +\infty, \qquad j = 1, 2.$$

REMARK 1.2. – As $V'(0) = 0$, (P) admits the trivial solution $u = 0$; our approach does not guarantee that the solution found in Theorem 1.1, for $q = 0$, is not trivial.

REMARK 1.3. – Condition (V4) is fulfilled, for example, if in a neighbourhood of zero V is a positive definite quadratic form.

REMARK 1.4. – In several differential problems involving a singular potential V solutions are classified by some topological invariant related to the singularities of V. For example, in planar dynamical systems solutions can be naturally classified by the winding number, as in our case (see [6, 7, 13] and references therein). In some recent papers [1-3], a suitable topological invariant (*e.g.*, the topological charge) is introduced in order to classify weak solutions of a quasilinear elliptic equation with a singular potential. Such an equation arises when looking for static solutions of a model equation, defined in a four dimensional space-time, which admits soliton-like solutions.

2. – Variational setting.

For the sake of simplicity, any positive constant depending only on N will be denoted by C_N.

We shall identify $\mathcal{D}^{1,2}_{\text{rad}}(\Omega, \boldsymbol{R}^2)$ with the «weighted» space $\mathcal{H}$, defined as the closure of $C_0^\infty((1, +\infty), \boldsymbol{R}^2)$ under the norm

$$\|u\| = \left(\int_1^{+\infty} r^{N-1}(|u_1'(r)|^2 + |u_2'(r)|^2)\, dr \right)^{1/2}.$$

Plainly, the energy functional E can be rewritten as

$$E(u) = \int_1^\infty r^{N-1}\left(\frac{1}{2}|u'(r)|^2 + V(u(r))\right) dr, \qquad u \in \mathcal{H}.$$

In the sequel, we shall need some embeddings properties of $\mathcal{H}$, which we briefly recall. By Radial Lemma, every $u \in \mathcal{H}$ is almost everywhere equal to some $U \in C([1, +\infty), \boldsymbol{R}^2)$, such that

$$|U(r)| \leqslant C_N r^{(2-N)/2} \|u\| \quad \text{for any } r \geqslant 1. \tag{2.1}$$

By identifying $u \in \mathcal{H}$ and U, (2.1) plainly implies that $\mathcal{H}$ is continuously embedded in the set of continuous and bounded functions from $[1, +\infty)$ to $\boldsymbol{R}^2$. It is easy to see that $\mathcal{H}$ is also continuously embedded in $H^1((a, b), \boldsymbol{R}^2)$, for any $1 < a < b$. Indeed:

$$\|u\|^2 = \int_1^{+\infty} r^{N-1} |u'(r)|^2\, dr \geqslant \int_a^b |u'(r)|^2\, dr;$$

on the other hand:

$$\|u\|^2 \geqslant C_N \left(\int_a^b |u(r)|^{2N/(N-2)}\, dr\right)^{(N-2)/N} \geqslant C_N (b-a)^{-2/N} \int_a^b |u(r)|^2\, dr$$

(we have taken into account the embedding $\mathcal{D}^{1,2}(\Omega, \boldsymbol{R}^2) \hookrightarrow L^{2N/(N-2)}(\Omega, \boldsymbol{R}^2)$, see [12]). As $H^1((a, b), \boldsymbol{R}^2)$ is compactly embedded in $L^\infty((a, b), \boldsymbol{R}^2)$, if $\{u_n\} \subset \mathcal{H}$ converges weakly in $\mathcal{H}$, then it converges pointwise in $[1, +\infty)$ and uniformly on any compact set contained in $[1, +\infty)$. Furthermore, as $H^1((a, b), \boldsymbol{R}^2)$ is continuously embedded in $C^{0,1/2}((a, b), \boldsymbol{R}^2)$, there exists a constant $c > 0$, depending on a, b such that

$$|u(r) - u(s)| \leqslant c|r - s|^{1/2} \|u\|_{H^1(a, b)}. \tag{2.2}$$

3. – Multiplicity result.

As the elements of $\mathcal{H}$ are continuous functions, it makes sense to consider the open subset of $\mathcal{H}$ defined by $\Lambda = \{u \in \mathcal{H}: u(r) \neq \xi\ \forall r > 1\}$, whose boundary is given by $\partial\Lambda = \{u \in \mathcal{H}: \exists \bar{r} > 1 \text{ s.t. } u(\bar{r}) = \xi\}$.

We aim to split Λ in the disjoint union of infinitely many components and then look for minima of the functional E in any of such components.

Up to a parametrization, $u \in \Lambda$ can be identified with a curve $\tilde{u}: [0, 1] \to \mathbf{R}^2 \setminus \{\bar{\xi}\}$ such that $\tilde{u}(0) = \tilde{u}(1) = 0$. Let us define Ind (u) = *the winding number of* $\tilde{u}$ *around* $\bar{\xi}$ (*e.g.*, cf. [11]). For any $q \in \mathbf{Z}$, let

$$\Lambda_q = \{u \in \Lambda : \mathrm{Ind}(u) = q\}.$$

For any q, Λ_q is not empty; by the properties of the winding number, Λ_q is an open connected subset of Λ and $\Lambda = \bigcup_{q \in \mathbf{Z}} \Lambda_q$.

We are able to state our multiplicity result.

THEOREM 3.1. – *For any* $q \in \mathbf{Z}$, *there exists* $u_q \in \Lambda_q$ *such that* $E(u_q) = \inf_{\Lambda_q} E$. *Moreover*

$$\lim_{q \to \pm\infty} E(u_q) = +\infty . \tag{3.1}$$

Let us first prove some useful lemmas.

LEMMA 3.2. – *E is coercive in the* $\mathcal{H}$ *norm and weakly lower semicontinuous in* Λ.

PROOF. – As V is nonnegative, E is coercive by definition. Let $u \in \Lambda$ and $\{u_n\} \subset \Lambda$ be such that u_n weakly converges to u; we aim to prove that $\liminf_{n \to \infty} E(u_n) \geq E(u)$. Let us assume that $E(u_n)$ is bounded (otherwise the claim is obvious); as a consequence, $\|u_n\|$ is bounded, hence $\|u_n\|_\infty \leq C$. Let $K \subset\subset (1, +\infty)$ and let us denote $2d = \inf_K |u(r) - \bar{\xi}| > 0$. Since u_n converges uniformly to u on K, there exists $\nu \in \mathrm{N}$ such that, for any $n \geq \nu$ and $r \in K$, $|u_n(r) - u(r)| \leq d$. Therefore, Mean Value Theorem applies and gives

$$\|V(u_n) - V(u)\|_{L^\infty(K)} \leq \sup_B |V'| \, \|u_n - u\|_{L^\infty(K)} ,$$

where $B = \{\xi \in \mathbf{R}^2 : \mathrm{dist}(\xi, u(K)) \leq d\}$. As a consequence, $V(u_n)$ converges uniformly to $V(u)$ on any compact subset of $(1, +\infty)$; in particular, for any $R > 0$:

$$\liminf_{n \to \infty} \int_1^{+\infty} V(u_n(r))\, dr \geq \liminf_{n \to \infty} \int_1^R V(u_n(r))\, dr = \int_1^R V(u(r))\, dr .$$

As $R \to +\infty$, we obtain

$$\liminf_{n \to \infty} \int_1^{+\infty} V(u_n(r))\, dr \geq \int_1^{+\infty} V(u(r))\, dr$$

whence the claim. ■

Assumption (V3) permits to control the behaviour of the functional E at the boundary of Λ.

LEMMA 3.3. – *Let* $\{u_n\} \subset \Lambda$ *be such that* $E(u_n)$ *is bounded. Then* u_n *weakly converges to some* $u \in \Lambda$ *(possibly up to a subsequence).*

PROOF. – As $E(u_n)$ is bounded, u_n is bounded in the $\mathcal{H}$ norm. As a consequence, it weakly converges to some $u \in \mathcal{H}$, up to a subsequence. By contradiction, let us assume $u \in \partial \Lambda$, hence there exists $\bar{r} > 1$ such that $u(\bar{r}) = \bar{\xi}$. By (2.2), there exists a constant $C > 0$ (depending on $\bar{r}$) such that

$$|u_n(r) - u_n(\bar{r})| \leq C|r - \bar{r}|^{1/2}$$

for any $r \in [(\bar{r}+1)/2, 2\bar{r}]$. Since $u_n(\bar{r}) \to \bar{\xi}$, we have

$$|u_n(r) - \bar{\xi}| \leq |u_n(r) - u_n(\bar{r})| + |u_n(\bar{r}) - \bar{\xi}| \leq C|r - \bar{r}|^{1/2} + o(1);$$

therefore, there exists $0 < \varrho < (\bar{r}+1)/2$ such that $|u_n(r) - \bar{\xi}| \leq \delta$ for $|r - \bar{r}| < \varrho$ and n sufficiently large. By (V3):

$$E(u_n) \geq \int_{|r-\bar{r}|<\varrho} V(u_n(r))\,dr \geq m \int_{|r-\bar{r}|<\varrho} |u_n(r) - \bar{\xi}|^{-2}\,dr \geq$$

$$\geq \bar{C} \int_{|r-\bar{r}|<\varrho} (|r - \bar{r}| + o(1))^{-1}\,dr$$

($\bar{C} > 0$ is a suitable constant) which implies $E(u_n) \to +\infty$ as $n \to \infty$, a contradiction. ■

PROOF OF THEOREM 3.1. – Let $q \in \boldsymbol{Z}$ and let $\{v_n\} \subset \Lambda_q$ be a minimizing sequence, namely $E(v_n) \to \inf_{\Lambda_q} E =: E_q$ as $n \to \infty$. By Lemma 3.3, v_n weakly converges to some $u_q \in \Lambda$; we still have to prove $u_q \in \Lambda_q$. We claim that there exists $d > 0$ such that $m_n := \inf_{(1,+\infty)} |v_n(r) - \bar{\xi}| \geq d$. By contradiction, assume that $m_n \to 0$ as $n \to \infty$. By Radial Lemma, there exist R, $c > 0$ such that $|v_n(r) - \bar{\xi}| \geq c$ for any n and for any $r > R$. For n sufficiently large, there exists $1 < r_n \leq R$, such that $m_n = |v_n(r_n) - \bar{\xi}|$. We can plainly assume that r_n converges to some $\bar{r} \leq R$. As v_n converges to u, uniformly on compact sets, letting $n \to \infty$ gives $u_q(\bar{r}) = \bar{\xi}$, which contradicts $u_q \in \Lambda$ and proves the claim. Now, by Radial Lemma again, it easily follows $\sup_r |v_n(r) - u_q(r)| < d$ for n sufficiently large; thus $\mathrm{Ind}(v_n) = \mathrm{Ind}(u_q)$ for n large, that is $u_q \in \Lambda_q$ and

$$E_q \leq E(u_q) \leq \liminf_{n\to\infty} E(v_n) = E_q\,.$$

We are left to prove (3.1). We confine ourselves to positive integers (the proof is the same in the other case). Let us notice that the sequence E_q is nondecreasing. Indeed, if $u_q \in \Lambda_q$, then there exists $[t_1, t_2] \subset [1, +\infty]$ such that $u_q(t_1) = u_q(t_2)$ and $\mathrm{Ind}(u_{q|[t_1,t_2]}) = 1$. If we set $v(r) = u_q(r)$ for $r \leq t_1$ and $v(r) = u_q(r - t_1 + t_2)$ for $r > t_1$, then $v \in \Lambda_{q-1}$ and

$$E_{q-1} \leq E(v) = \int_1^{t_1} \left(\frac{1}{2} |u_q'(r)|^2 + V(u_q(r)) \right) r^{N-1}\,dr +$$

$$+ \int_{t_2}^{+\infty} \left(\frac{1}{2} |u_q'(r)|^2 + V(u_q(r)) \right) (r + t_1 - t_2)^{N-1}\,dr \leq E(u_q) = E_q\,.$$

If we assume E_q bounded, then u_q weakly converges in Λ and uniformly on compact sets, by Lemma 3.3. Arguing as in the first part of the proof, it is easy to see that the sequence Ind (u_q) is definitively constant, a contradiction. ∎

4. – Regularity results.

In the present section we adapt to our setting the arguments in [4, Section 4].

LEMMA 4.1. – *Assume* (V1). *Any radial weak solution of* (P) *is a classical solution.*

PROOF. – Let u a radial weak solution of (P). By Radial Lemma, we have $u \in C^0(\Omega, \boldsymbol{R}^2) \cap L^\infty(\Omega, \boldsymbol{R}^2)$; by (V1), $V'(u) \in C^0(\Omega, \boldsymbol{R}^2) \cap L^\infty(\Omega, \boldsymbol{R}^2)$. Since u is a weak solution of (P), $\Delta u \in L^1_{\mathrm{loc}}(\Omega, \boldsymbol{R}^2)$, hence

$$\Delta u(x) = V'(u(x)) \text{ a.e. in } \Omega .$$

The right–hand side being continuous, we get $u \in C^2(\Omega, \boldsymbol{R}^2)$. ∎

LEMMA 4.2. – *Assume* (V1), (V2) *and* (V4). *Then any radial solution of (P) has exponential decay at infinity, together with its derivatives up to second order, namely*

$$|D^\alpha u(x)| \leq Ce^{-k|x|}, \qquad x \in \Omega$$

for some C, $k > 0$ *and for* $|\alpha| \leq 2$.

PROOF Let $u = (u_1, u_2)$ be a radial solution of (P). By Lemma 4.1, $u \in C^2$; as a radial function, u_j $(j = 1, 2)$ satisfies

$$-u_j'' - \frac{N-1}{r} u_j' + V_{\xi_j}(u(r)) = 0 . \tag{4.1}$$

Let $w_j = r^{N-1} u_j^2$; then

$$w_j'' \geq 2\left(\frac{V_{\xi_j}(u(r))}{u_j(r)} + \frac{(N-1)(N-3)}{4r^2}\right) w_j .$$

By (V4), since $u(r) \to 0$ as $r \to +\infty$, there exist r_0, $c > 0$ such that $w_j'' \geq c^2 w_j$. It is easy to see that the function $z_j = e^{-cr}(w_j' + cw_j)$ is nondecreasing in $[r_0, +\infty)$. If $z_j(\bar{r}) > 0$ for some $\bar{r} > r_0$, taking into account (2.1) yields

$$z_j(\bar{r})\, e^{cr} \leq w_j'(r) + cw_j(r) \leq c_1 + c_2 r + c_3 r^{N/2} |u_j'(r)|$$

for any $r > \bar{r}$ (here and in the sequel, $c_1, c_2, \ldots$ are positive constants). Thus, for r sufficiently large:

$$r^{N-1} |u_j'(r)|^2 \geq c_4 r^{-1} e^{2cr} ,$$

a contradiction, since $r^{(N-1)/2}u_j'(r)$ is in $L^2(1, +\infty)$. As a consequence, $z_j(r) \leqslant 0$ in $[r_0, +\infty)$, whence $(e^{cr}w_j)' \leqslant 0$ in $[r_0, +\infty)$. Then for some $C > 0$ $|u_j(r)| \leqslant \leqslant Cr^{-(N-1)/2}e^{-cr}$ in $[r_0, +\infty)$. Next, by (V4) there exist $m_2 > m_1 > 0$ such that $m_1|u_j(r)| \leqslant |V_{\xi_j}(u(r))| \leqslant m_2|u_j(r)|$ for r large. Taking into account (4.1), the exponential decay of u_j and arguing exactly as in [4, Section 4], the exponential decay of u_j' and u_j'' can be obtained. ∎

5. – Proof of the main result and additional remarks.

PROOF OF THEOREM 1.1. – By Theorem 3.1, for any $q \in \boldsymbol{Z}$, (P) admits a radial weak solution $u_q \in \Lambda_q$ such that $E(u_q) = \inf_{\Lambda_q} E$. By Lemma 4.1 and 4.2, respectively, i) and ii) in Theorem 1.1 hold. ∎

REMARK 5.1. – Let us consider the quasilinear problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V'(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x| \to \infty} u(x) = 0, \end{cases} \tag{5.1}$$

where $\Omega = \boldsymbol{R}^N \setminus \overline{B}_1(0)$ and $1 < p < N$. As before, we assume that V is a C^1 real map, defined in $\boldsymbol{R}^2 \setminus \{\bar{\xi}\}$ $(\bar{\xi} \neq 0)$, which has a global minimum at $\xi = 0$. As concernes the behaviour of V around the singularity, we now assume that $V(\xi) \geqslant m|\xi - \bar{\xi}|^{-p'}$ in a neighbourhood of $\bar{\xi}$, with $m > 0$ and p' is the exponent conjugate to p.

It is natural to give a variational formulation of (5.1) in $\mathcal{D}^{1,p}_{\mathrm{rad}}(\Omega, \boldsymbol{R}^2)$, the closure of the set of radial functions in $C_0^\infty(\Omega, \boldsymbol{R}^2)$ under the norm $\|\nabla u\|^p_{L^p}$. In such a space, Radial Lemma holds: any $u \in \mathcal{D}^{1,p}_{\mathrm{rad}}(\Omega, \boldsymbol{R}^2)$ can be identified with a continuous function and $|u(x)| \leqslant C|x|^{1-N/p}\|\nabla u\|^p_{L^p}$ (cf. [8]). One can therefore repeat the arguments in Section 3 and prove that for any $q \in \boldsymbol{Z}$, (5.1) has at least one weak radial solution u_q with winding number around $\bar{\xi}$ equal to q.

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