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MULTIPLE SOLUTIONS FOR QUASI-LINEAR PDES INVOLVING THE CRITICAL SOBOLEV AND HARDY EXPONENTS

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Abstract. We use variational methods to study the existence and multiplicity of solutions for the following quasi-linear partial differential equation:

$$\begin{cases} -\triangle_p u = \lambda |u|^{r-2}u + \mu \frac{|u|^{q-2}}{|x|^s}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where λ and μ are two positive parameters and Ω is a smooth bounded domain in $\mathbf{R}^n$ containing 0 in its interior. The variational approach requires that $1 < p < n$, $p \le q \le p^*(s) \equiv \frac{n-s}{n-p}p$ and $p \le r \le p^* \equiv p^*(0) = \frac{np}{n-p}$, which we assume throughout. However, the situations differ widely with q and r, and the interesting cases occur either at the critical Sobolev exponent ($r = p^*$) or in the Hardy-critical setting ($s = p = q$) or in the more general Hardy-Sobolev setting when $q = \frac{n-s}{n-p}p$. In these cases some compactness can be restored by establishing Palais-Smale type conditions around appropriately chosen *dual sets*. Many of the results are new even in the case $p = 2$, especially those corresponding to singularities (i.e., when $0 < s \le p$).

1. Introduction

Consider the following quasi-linear partial differential equation:

$$(\mathrm{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u = \lambda |u|^{r-2}u + \mu \frac{|u|^{q-2}}{|x|^s}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where λ and μ are two positive parameters and Ω is a smooth bounded domain in $\mathbf{R}^n$ containing 0 in its interior. We shall assume throughout that $0 \le s \le p < n$.

The starting point of the variational approach to these problems is the following *Sobolev-Hardy inequality*, which is essentially due to Caffarelli, Kohn and Nirenberg [8]. Assume that $1 < p < n$ and that $q \le p^*(s) \equiv \frac{n-s}{n-p}p$; then there is a constant $C > 0$ such that

$$C\left(\int_\Omega \frac{|u|^q}{|x|^s}\right)^{\frac{p}{q}} dx \le \int_\Omega |\nabla u|^p \, dx \quad \text{for all } u \in H_0^{1,p}(\Omega).$$

We use $\mu_{s,q}(\Omega)$ to denote the best *Sobolev-Hardy* constant, i.e. the largest constant C satisfying the above inequality for all $u \in H_0^{1,p}(\Omega)$; that is,

$$\mu_{s,q}(\Omega) = \inf_{u \in H_0^{1,p}(\Omega), u \ne 0} \frac{\int_\Omega |\nabla u|^p \, dx}{\left(\int_\Omega \frac{|u|^q}{|x|^s}\right)^{\frac{p}{q}} dx}.$$

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In the important case where $q = p^*(s)$, we shall simply denote $\mu_{s,p^*(s)}$ as μ_s. Note that μ_0 is nothing but the best constant in the *Sobolev inequality* while μ_p is the best constant in the *Hardy inequality*, i.e.,

$$\mu_p(\Omega) = \inf_{u \in H_0^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega \frac{|u|^p}{|x|^p} dx}.$$

We shall always assume that $p \leq r \leq p^* \equiv p^*(0) = \frac{np}{n-p}$ for the non-singular term in such a way that the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{r} \int_\Omega |u|^r dx - \frac{\mu}{q} \int_\Omega \frac{|u|^q}{|x|^s} dx$$

is then well defined on the Sobolev space $H_0^{1,p}(\Omega)$. The (weak) solutions of the problem $(P_{\lambda,\mu})$ are then the critical points of the functional $E_{\lambda,\mu}$.

Another relevant parameter will be the first "eigenvalue" of the p-Laplacian $-\Delta_p$, defined as

$$\lambda_1(\Omega) \equiv \mu_{0,p}(\Omega) = \inf\{\int_\Omega |\nabla w|^p dx : w \in H_0^{1,p}(\Omega), \int_\Omega |w|^p dx = 1\}.$$

Here are the main results of this paper.

Theorem 1.1 (Hardy-Sobolev subcritical singular and non-singular terms)**.** *Suppose $1 < p \leq q < p^*(s)$ and $r < p^*$. Assume one of the following conditions holds:*

(1) *(High order singular term) $p < q$, $p \leq r$, $\lambda > 0$ and $\mu > 0$.*
(2) *(Low order singular term) $p = q$, $p < r$, $\lambda > 0$ and $\mu_{s,p} > \mu > 0$.*

Then $(P_{\lambda,\mu})$ has infinitely many solutions. Moreover, $(P_{\lambda,\mu})$ has an everywhere positive solution with least energy and another one that is sign-changing.

Theorem 1.2 (Hardy-critical singular term)**.** *Suppose $1 < p = q = p^*(s)$ (i.e., $s = p$).*

1. *(Subcritical non-singular term) If $r < p^*$, then $(P_{\lambda,\mu})$ has infinitely many solutions –at least one of them being positive– for any $\lambda > 0$ and $0 < \mu < \mu_p$.*
2. *(Critical non-singular term) If $r = p^*$ and Ω is star-shaped. Then $(P_{\lambda,\mu})$ has no non-trivial solution for any $\lambda > 0, \mu > 0$.*

Theorem 1.3 (Hardy-Sobolev critical singular term)**.** *Suppose $1 \leq p < q = p^*(s)$ (i.e., $s < p$).*

1. *(High order non-singular term) Assume $p < r < p^*$ and $\lambda > 0$, $\mu > 0$.*
   - *If $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$ (in particular if $n \geq p^2$), then $(P_{\lambda,\mu})$ has a positive solution.*
   - *If $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$ (in particular if $n > p^3 - p^2 + p$), then $(P_{\lambda,\mu})$ has also a sign-changing solution.*
2. *(Low order non-singular term) Assume $p = r < p^*$ and $0 < \lambda < \lambda_1$, $\mu > 0$.*
   - *If $n \geq p^2$, then $(P_{\lambda,\mu})$ has a positive solution.*
   - *If $n > p^3 - p^2 + p$, then $(P_{\lambda,\mu})$ has also a sign-changing solution.*
3. *(Sobolev-critical non-singular term) Assume $r = p^*$ and Ω is star-shaped, then $(P_{\lambda,\mu})$ has no non-trivial solution for any $\lambda > 0$ and any $\mu > 0$.*

Theorem 1.4 (Sobolev-critical non-singular term and subcritical singular term)**.** *Suppose $1 < p \leq q < p^*(s)$ and $r = p^*$.*

1. *(High order singular term) Assume that $p < q$ and $\lambda > 0,\ \mu > 0$.*
   - *If $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$ (in particular if $n \geq p^2 - (p-1)s$), then $(P_{\lambda,\mu})$ has a positive solution.*
   - *If $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$ (in particular if $n > p(p-1)(p-s)+p$), then $(P_{\lambda,\mu})$ has also a sign-changing solution.*
2. *(Low order non-singular term) Assume $p = q$ and $\lambda > 0$, $\mu_{s,p} > \mu > 0$.*
   - *If $n \geq p^2 - (p-1)s$, then $(P_{\lambda,\mu})$ has a positive solution.*
   - *If $n > p((p-1)(p-s)+1)$, then $(P_{\lambda,\mu})$ has also a sign-changing solution.*

The following tables summarize our results.

Table 1. Sobolev-subcritical non-singular term

Singular term	Parameters	Non-singular term	Dimension	# of solutions
(HS-subcritical)				Infinite
$p < q < p^*(s)$	$\lambda > 0;\ \mu > 0$	$1 \leq p \leq r < p^*$	$n > p$	One positive
$p = q < p^*(s)$	$\mu_{s,p} > \mu > 0;\ \lambda > 0$	$1 \leq p < r < p^*$	$n > p$	One positive
(H-critical) $p = q = p^*(s)$	$\lambda > 0;\ \mu_p > \mu > 0$	$1 \leq p < r < p^*$	$n > p$	Infinite (One positive)
(HS-critical) $p < q = p^*(s)$	$\lambda > 0;\ \mu > 0$	$1 \leq p < r < p^*$	$n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$	One positive
	—	$2 \leq p < r < p^*$	$n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$	Two
	$\lambda_1 > \lambda > 0;\ \mu > 0$	$1 \leq p = r < p^*$	$n \geq p^2$	One positive
	—	$2 \leq p = r < p^*$	$n > p^3 - p^2 + p$	Two

Table 2. Sobolev-critical non-singular term

Singular term	Parameters	Non-singular term	Dimension	# of solutions
$1 \leq p = q < p^*(s)$	$\mu_{s,p} > \mu > 0$	$r = p^*$	$n > p^2 - (p-1)s$	One positive
$2 \leq p = q < p^*(s)$	and $\lambda > 0$	—	$n > p((p-1)(p-s)+1)$	Two
$1 \leq p < q < p^*(s)$	$\lambda > 0;\ \mu > 0$	$r = p^*$	$n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$	One positive
$2 \leq p < q < p^*(s)$	—	—	$n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$	Two
$p \leq q = p^*(s)$	$\lambda > 0,\ \mu > 0$	$r = p^*$	$n > p$	None

2. A Pohozaev-type identity

In this section, we start by identifying the constraints on the problem of existence of solutions for $(P_{\lambda,\mu})$. Here is the main result.

Theorem 2.1. *If Ω is a star-shaped domain in $\mathbf{R}^n$, then problem $(P_{\lambda,\mu})$ has no solution in the doubly critical case: That is, for $r = p^*$ and $q = p^*(s) = \frac{n-s}{n-p}p$, the problem $(P_{\lambda,\mu})$ has no non-trivial solution.*

Assume Ω is a star-shaped domain. Then, if we let v denote the outwards normal to $\partial\Omega$, then $\langle x,\ v\rangle > 0$ on $\partial\Omega$. We assume we have the necessary regularity in the following operations; otherwise, we can use an approximation argument as in Guedda and Veron [20].

Multiplying the equation $(P_{\lambda,\mu})$ by $\langle x,\ \nabla u\rangle$ on both sides and integrate by parts, we get

$$\frac{p-1}{p}\int_{\partial\Omega}|\nabla u|^p\langle x,\ v\rangle dx + \frac{n-p}{p}\int_\Omega |\nabla u|^p dx = \mu\frac{n-s}{q}\int_\Omega \frac{|u|^q}{|x|^s}dx + \lambda\frac{n}{r}\int_\Omega |u|^r dx.$$

On the other hand, multiplying the equation by u and integrating, we get

$$\int_\Omega |\nabla u|^p = \mu \int_\Omega \frac{|u|^q}{|x|^s} dx + \lambda \int_\Omega |u|^r dx.$$

Putting the two identities together, we have

$$\frac{p-1}{p}\int_{\partial\Omega} |\nabla u|^p \langle x,\ v\rangle d\sigma = \mu(\frac{n-s}{q} - \frac{n-p}{p})\int_\Omega \frac{|u|^q}{|x|^s} dx + \lambda(\frac{n}{r} - \frac{n-p}{p})\int_\Omega |u|^r.$$

So if $r = \frac{np}{n-p} = p^*$ and $q = \frac{n-s}{n-p}p$, the problem has no non-trivial solution.

3. The extremal functions in the Hardy-Sobolev inequalities

In this section, we summarize the needed results concerning the Hardy-Sobolev inequalities. We first recall the *Hardy* inequality.

Lemma 3.1 ([13])**.** *Assume that $1 < p < n$ and $u \in H^{1,p}(\mathbf{R}^n)$. Then:*

(1) $\frac{u}{|x|} \in L^p(\mathbf{R}^n)$.
(2) *(Hardy Inequality)* $\int_{\mathbf{R}^n} \frac{|u|^p}{|x|^p} dx \le C_{n,p} \int_{\mathbf{R}^n} |\nabla u|^p dx$, *where* $C_{n,p} = (\frac{p}{n-p})^p$.
(3) *The constant $C_{n,p}$ is optimal.*

The following extension of the Hardy and Sobolev inequalities is essentially due to Caffarelli, Kohn and Nirenberg[8].

Lemma 3.2 (Sobolev-Hardy Inequality)**.** *Assume that $1 < p < n$ and that $p \le q \le p^*(s) := \frac{n-s}{n-p}p$. Then:*

(1) *There exists a constant $C > 0$ such that for any $u \in H_0^p(\Omega)$,*

$$(\int_\Omega \frac{|u|^q}{|x|^s})^p \ dx \le C(\int_\Omega |\nabla u|^p)^q \ dx.$$

(2) *The map $u \to \frac{u}{x^{s/q}}$ from $H_0^p(\Omega)$ into $L^q(\Omega)$ is compact provided $q < p^*(s)$.*

Proof. (1) For $s = 0$ or $s = p$, this is just the Sobolev (resp., the Hardy) inequality. Since $p^*(s) \ge p$, we have $0 \le s \le p$. We can therefore only consider the case where $0 < s < p$. By the Hardy, Sobolev and Hölder inequalities, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} \frac{|u|^{p^*(s)}}{|x|^s} &= \int_{\mathbf{R}^n} \frac{|u|^s}{|x|^s} \cdot |u|^{p^*(s)-s} \\
&\le (\int_{\mathbf{R}^n} |\frac{|u|^p}{|x|^p})^{\frac{s}{p}} (\int_{\mathbf{R}^n} |u|^{(p^*(s)-s)\frac{p}{p-s}})^{\frac{p-s}{p}} \\
&= (\int_{\mathbf{R}^n} |\frac{|u|^p}{|x|^p})^{\frac{s}{p}} (\int_{\mathbf{R}^n} |u|^{p^*})^{\frac{p-s}{p}} \\
&\le C_1(\int_{\mathbf{R}^n} |\nabla u|^p)^{\frac{s}{p}} (\int_{\mathbf{R}^n} |\nabla u|^p)^{\frac{p^*}{p}\cdot\frac{p-s}{p}} \\
&= C_1(\int_{\mathbf{R}^n} |\nabla u|^p)^{\frac{n-s}{n-p}}.
\end{aligned}$$

□

Remark 3.1. If Ω is the whole space, one can show that the conditions $p \le q = p^*(s) := \frac{n-s}{n-p}p$ are also necessary for the above inequality to hold. Indeed, a standard scaling argument shows that q must be equal to $p^*(s)$. On the other hand, if we insert into the inequality the following function (ρ and $\theta \in S^{n-1}$ being the polar coordinates),

$$u(x) = \begin{cases} 0 & \text{for } |x| \geq 1, \\ |x|^{\frac{p-n}{p}} \log \frac{1}{|x|} & \text{for } \varepsilon \leq |x| < 1, \\ \varepsilon^{\frac{p-n}{p}} \log \frac{1}{\varepsilon} & \text{for } |x| \leq \varepsilon, \end{cases}$$

and

$$\frac{du(x)}{d\rho} = \begin{cases} 0, & |x| \geq 1, \\ 0, & |x| \leq \varepsilon, \\ (1-\frac{n}{p})\rho^{-\frac{n}{p}} \log \frac{1}{\rho} - \rho^{-\frac{n}{p}}, & \varepsilon \leq |x| < 1, \end{cases}$$

we get

$$\int_{\mathbf{R}^n} |\nabla u|^p \sim \int_\varepsilon^1 \rho^{-1}(1 + (\frac{n}{p} - 1) \log \frac{1}{\rho})^p d\rho.$$

By L'Hospital's rule, we have

$$\lim_{\varepsilon \to 0} \frac{\int_\varepsilon^1 \rho^{-1}(1 + (\frac{n}{p} - 1) \log \frac{1}{\rho})^p d\rho}{\log^{1+p} \frac{1}{\varepsilon}} = \frac{\frac{n}{p} - 1}{1+p},$$

and also

$$\int_{\mathbf{R}^n} \frac{|u|^q}{|x|^s} \sim \int_\varepsilon^1 \rho^{-s} \log^q \frac{1}{\rho} \rho^{\frac{p-n}{p}q} \rho^{n-1} = \int_\varepsilon^1 \frac{1}{\rho} \log^q \frac{1}{\rho} \sim \log^{1+q} \frac{1}{\varepsilon}.$$

Thus from the inequality

$$\log^{1+\frac{1}{q}} \frac{1}{\varepsilon} \leq \log^{1+\frac{1}{p}} \frac{1}{\varepsilon},$$

we get that $q \geq p$.

The following is an extension of what is well known in the case $p = 2$ and $s = 0$.

Theorem 3.1. *Suppose $1 < p < n$, $0 \leq s < p$ and $q = p^*(s)$. Then the following hold:*

1. *$\mu_s(\Omega)$ is independent of Ω (and will henceforth be denoted by μ_s).*
2. *μ_s is attained when $\Omega = \mathbf{R}^n$ by the functions*

$$y_a(x) = (a \cdot (n-s)(\frac{n-p}{p-1})^{p-1})^{\frac{n-p}{p(p-s)}} (a + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}}$$

for some $a > 0$. Moreover the functions y_a are the only positive radial solutions of

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = \frac{u^{p^*(s)-1}}{|x|^s}$$

in $\mathbf{R}^n$. Hence,

$$\mu_s(\int_{\mathbf{R}^n} \frac{|y_a|^q}{|x|^s})^{\frac{p}{q}} = \|\nabla y_a\|_p^p = \int_{\mathbf{R}^n} \frac{|y_a|^q}{|x|^s} = \mu_s^{\frac{n-s}{p-s}}.$$

Proof. We prove (2). We show the best constants are attained at functions

$$u_s(x) = c(\lambda_0 + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \qquad (0 \leq s < p), \text{ where } \lambda_0 > 0 \text{ is a constant.}$$

For any f, let f^* be its *Schwarz* symmetrization (or rearrangement) [21]. Then we have

$$\int_{\mathbf{R}^n} |\nabla f^*|^p \leq \int_{\mathbf{R}^n} |\nabla f|^p \quad \text{and} \quad \int_{\mathbf{R}^n} \frac{|f^*|^q}{|x|^t} \geq \int_{\mathbf{R}^n} \frac{|f|^q}{|x|^t},$$

assuming the above integrals are well defined (refer to Lieb [21], [22]). By these inequalities, we may restrict our discussion to radial symmetric functions. Thus we may consider the following variational problem:

$$\text{Maximize } I(g) = \int_0^\infty |g(r)|^q r^{n-s-1} dr, \quad \text{when} \quad J(g) = \int_0^\infty |g'(r)|^p r^{n-1} dr = C.$$

where C is a given constant. The *Euler-Lagrange* equation is

$$(*) \qquad (r^{n-1}|u'(r)|^{p-2}u'(r))_r + kr^{n-s-1}|u|^{q-1} = 0.$$

It can be easily verified that the functions

$$u_s(x) = (\lambda + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \qquad (0 \leq s < p)$$

are solutions of $(*)$, where $\lambda > 0$. To continue, we need the following lemma of Bliss ([1], [2]).

Lemma 3.3. *Let $h(x) \geq 0$ be a measurable, real-valued function defined on $\boldsymbol{R}$ such that the integral $J_0 = \int_0^\infty h^{p_0}(x)dx$ is finite and given. Set $g(x) = \int_0^x h(t)dt$. Then $I_0 = \int_0^\infty g^{q_0}(x)x^{\alpha - q_0}dx$ attains its maximum value at the functions $h(x) = (\lambda x^\alpha + 1)^{-\frac{\alpha+1}{\alpha}}$, with p_0 and q_0 two constants satisfying $q_0 > p_0 > 1$, $\alpha = \frac{q_0}{p_0} - 1$, and $\lambda > 0$, a real number.*

By this lemma and with the change of variables $x = r^{\frac{p-n}{p-1}}$ we can deduce that $I(\cdot)$ attains its maximum at the functions

$$u_s(x) = (\lambda + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \qquad (0 \leq s < p).$$

Note that if

$$h(x) = (\lambda x^\alpha + 1)^{-\frac{\alpha+1}{\alpha}},$$

then

$$g(x) = \int_0^x h(t)dt = (\lambda + x^{-\alpha})^{-\frac{1}{\alpha}}.$$

And if $q = \frac{n-s}{n-p}p$, then $\alpha = \frac{q}{p} - 1 = \frac{p-s}{n-p}$. The theorem is thus proved. □

Remark 3.2. As expected, the compactness of the embedding $u \to \frac{u}{x^{s/q}}$ in Lemma 3.2.(2) above does not hold when $q = p^*(s)$. Indeed, let

$$f_k(x) = (\frac{1}{k})^{\frac{n-p}{p(p-s)}}(\frac{1}{k} + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}},$$

and set $\|\nabla f_k(x)\|_p^p = A$ and $\int_\Omega \frac{|f_k(x)|^{p^*(s)}}{|x|^s}dx = C$. Let

$$h_k(x) = f_k(x) - (\frac{1}{k})^{\frac{n-p}{p(p-s)}}(\frac{1}{k} + 1)^{\frac{p-n}{p-s}}$$

for $|x| \leq 1$, so that $h_k(x) \in H_0^{1,p}(B)$ and $\|h_k\|_{H_0^{1,p}(\Omega)} \to A^{\frac{1}{p}}$. Hence $\{h_k\}$ is bounded in $H_0^{1,p}(\Omega)$ and $\|\frac{h_k}{|x|^{s/p^*(s)}}\|_{L^{p^*(s)}} \to C^{1/p^*(s)}$. Now, $h_k(x) \to 0$ for $|x| \neq 0$ and 0 is the only possible cluster point of $\{\frac{h_k}{|x|^{s/p^*(s)}}\}$ in $L^{p^*(s)}(\Omega)$, which is impossible since $C \neq 0$.

4. The compactness lemmas

This section deals with the compactness properties of the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{r}\int_\Omega |u|^r dx - \frac{\mu}{q}\int_\Omega \frac{|u|^q}{|x|^s} dx.$$

We recall the following standard definition.

Definition 4.1. A C^1-functional E on Banach space X *satisfies the Palais-Smale condition at the level* c (in short $(\mathrm{PS})_c$), if every sequence $(u_n)_n$ satisfying $\lim_n E(u_n) = c$ and $\lim_n \|E'(u_n)\| = 0$ has a convergent subsequence.

Define the following function:

$$L(\mu, s) = \begin{cases} \frac{p-s}{p(n-s)}\left(\frac{\mu_s^{n-s}}{\mu^{n-p}}\right)^{\frac{1}{p-s}} & \text{if } s < p, \\ +\infty & \text{if } s = p \text{ and } \mu < \mu_p, \\ 0 & \text{if } s = p \text{ and } \mu \geq \mu_p. \end{cases}$$

Theorem 4.1. *Assume* $0 \leq s \leq p < n$, $p \leq q \leq p^*(s)$ *and* $p \leq r \leq p^*$.

(1) *If* $p \leq q < p^*(s)$ *and* $r < p^*$, *then for any* $\lambda > 0$ *and any* $\mu > 0$, *the functional* $E_{\lambda,\mu}$ *satisfies* $(\mathrm{PS})_c$ *for all* c.
(2) *If* $p \leq q = p^*(s)$ *and* $r < p^*$, *then for any* $\lambda > 0$ *and any* $\mu > 0$, *the functional* $E_{\lambda,\mu}$ *satisfies* $(\mathrm{PS})_c$ *for all* $c < L(\mu, s)$.
(3) *If* $p \leq q < p^*(s)$ *and* $r = p^*$, *then for any* $\lambda > 0$ *and any* $\mu > 0$, *the functional* $E_{\lambda,\mu}$ *satisfies* $(\mathrm{PS})_c$ *for all* $c < L(\lambda, 0) = \frac{1}{n}\left(\frac{\mu_0^n}{\lambda^{n-p}}\right)^{\frac{1}{p}}$.

Note that statement 2 above yields that $E_{\lambda,\mu}$ also satisfies $(\mathrm{PS})_c$ for all c when $p = q = q^*(s)$ (i.e., when $s = p$) as long as $\mu < \mu_p$. This solves a problem in [13] and [25], where only a certain *singular Palais-Smale condition* is established. On the other hand, when $\mu = 1$, $p = 2$ and $s = 0$, we recover the (by now well known) restricted compactness properties that appears in Yamabe-type problems, [4].

We first recall a few known results.

Lemma 4.1 ([25])**.** *Let* $x, y \in \mathbf{R}^n$, *and let* $\langle \cdot,\ \cdot \rangle_e$ *be the standard scalar product in* $\mathbf{R}^n$. *Then*

$$\langle |x|^{p-2}x - |y|^{p-2}y,\ x - y\rangle_e \geq \begin{cases} C_p |x-y|^p, & \text{if } p \geq 2, \\ C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2. \end{cases}$$

The following result of Brezis and Lieb ([4]) will be useful in the sequel.

Lemma 4.2. *Suppose* $f_n \to f$ *a.e. and* $\|f_n\|_p \leq C < \infty$ *for all* n *and for some* $0 < p < \infty$. *Then*

$$\lim_{n\to\infty} \{\|f_n\|_p^p - \|f_n - f\|_p^p\} = \|f\|_p^p.$$

Lemma 4.3. *Let* $(u_n)_n$ *be a bounded sequence in* $H_0^{1,p}(\Omega)$ *and let* $(q_n)_n$ *be a sequence such that* $p < q_n \leq p^*(s)$, $q_n \to p^*(s)$ *as* $n \to \infty$. *Then there exists a subsequence (without loss of generality still denoted by* $(u_n)_n$*) such that:*

(1) $u_n \to u$ *weakly in* $H_0^{1,p}(\Omega)$.
(2) $u_n \to u$ *in* $L^r(\Omega)$ *if* $1 < r < p^* = \frac{np}{n-p}$.
(3) $u_n \to u$ *almost everywhere.*
(4) $\frac{u_n}{x} \to \frac{u}{x}$ *weakly in* $L^p(\Omega)$.

(5) *For any* $f \in H_0^{1,p}(\Omega)$,

$$\int_\Omega \frac{|u_n|^{q_n-2}u_n}{|x|^{s_n}} f \to \int_\Omega \frac{|u|^{p^*(s)-2}u}{|x|^s} f.$$

(6) *If* $p \geq 2$, *then*

$$\int_\Omega |u_n|^{q_n} \leq \int_\Omega |u|^{q_n} + \int_\Omega |u_n - u|^{q_n} + o(1).$$

(7)

$$\int_\Omega \frac{|u_n - u|^{p^*(s)}}{|x|^s} = \int_\Omega \frac{|u_n|^{p^*(s)}}{|x|^s} - \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} + o(1).$$

Proof. These are standard applications of the Hardy-Sobolev embedding theorem and the Brezis-Lieb result. We just give the proofs of (5) and (6). Without loss of generality, we may assume that $q_n < p^*(s)$. For (5), it is clear that

$$\frac{|u_n|^{q_n-2}u_n}{|x|^{s\frac{q_n-1}{q_n}}} \to \frac{|u|^{p^*(s)-2}u}{|x|^{s\frac{p^*(s)-1}{p^*(s)}}} \ a.e.,$$

and that the integral

$$\begin{aligned}\int_\Omega \left|\frac{|u_n|^{q_n-1}}{|x|^{s(1-\frac{1}{p^*(s)})}}\right|^{\frac{p^*(s)}{p^*(s)-1}} &= \int_\Omega \frac{|u_n|^{p^*(s)\frac{q_n-1}{p^*(s)-1}}}{|x|^{s\frac{q_n-1}{p^*(s)-1}}} \cdot \frac{1}{|x|^{s(1-\frac{q_n-1}{p^*(s)-1})}} \\ &\leq \left(\int_\Omega \frac{|u_n|^{p^*(s)}}{|x|^s}\right)^{\frac{q_n-1}{p^*(s)-1}} \cdot \left(\int_\Omega \frac{1}{|x|^s}\right)^{\frac{p^*(s)-q_n}{p^*(s)-1}}\end{aligned}$$

is uniformly bounded in n. Since $f/|x|^{\frac{s}{p^*(s)}} \in L^{p^*(s)}(\Omega)$ for any $f \in H_0^{1,p}(\Omega)$, the conclusion follows.

In order to prove (6), we need the following easy lemma.

Calculus Lemma. *For every* $1 \leq q \leq 3$, *there exists a constant* C *(depending on* q*) such that for* $\alpha,\ \beta \in \mathbf{R}$ *we have*

$$|\ |\alpha+\beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})\ | \leq \begin{cases} C|\alpha||\beta|^{q-1} & \text{if } |\alpha| \geq |\beta|, \\ C|\alpha|^{q-1}|\beta| & \text{if } |\alpha| \leq |\beta|. \end{cases}$$

For $q \geq 3$, *there exists a constant* C *(depending on* q*) such that for* $\alpha,\ \beta \in \mathbf{R}$ *we have*

$$|\ |\alpha+\beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})\ | \leq C(|\alpha|^{q-2}\beta^2 + \alpha^2|\beta|^{q-2}).$$

From this inequality, we can actually deduce the following more convenient result for any $q \geq 1$:

$$|\ |\alpha+\beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})\ | \leq 2C(|\alpha|^{q-1}\beta + \alpha|\beta|^{q-1}).$$

Now, back to the proof of (6). Let $w_n = u_n - u$; then $w_n \to 0$ weakly in $H_0^{1,p}(\Omega)$. By the above calculus lemma,

$$\frac{|u_n|^{q_n}}{|x|^s} = \frac{|w_n + u|^{q_n}}{|x|^s} \leq \frac{|w_n|^{q_n}}{|x|^s} + \frac{|u|^{q_n}}{|x|^s} + C_1 \frac{|u||w_n|^{q_n-1}}{|x|^s} + C_2 \frac{|w_n||u|^{q_n-1}}{|x|^s}.$$

In view of (5), we only need to show that

$$\lim_n \int_\Omega \frac{|w_n||u|^{q_n-1}}{|x|^s} = \lim_n \int_\Omega \frac{|w_n||u|^{q_n-2}}{|x|^{s(1-\frac{1}{p^*(s)})}} \cdot \frac{|u|}{|x|^{\frac{s}{p^*(s)}}} = 0.$$

For that, we check that

$$\int_\Omega \Big(\frac{|w_n||u|^{q_n-2}}{|x|^{s(1-\frac{1}{p^*(s)})}}\Big)^{\frac{p^*(s)}{p^*(s)-1}} = \int_\Omega \frac{|w_n|^{\frac{p^*(s)}{p^*(s)-1}}}{|x|^{s\frac{1}{p^*(s)-1}}} \cdot \frac{|u|^{\frac{q_n-2}{p^*(s)-1}p^*(s)}}{|x|^{s\frac{q_n-2}{p^*(s)-1}}} \cdot \frac{1}{|x|^{s\frac{p^*(s)-q_n}{p^*(s)-1}}}$$
$$\le \Big(\int_\Omega \frac{|u_n|^{p^*(s)}}{|x|^s}\Big)^{\frac{1}{p^*(s)-1}} \Big(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\Big)^{\frac{q_n-2}{p^*(s)-1}} \Big(\int_\Omega \frac{1}{|x|^s}\Big)^{\frac{p^*(s)-q_n}{p^*(s)-1}}.$$

Hence it is uniformly bounded in n, and the claim follows. □

Lemma 4.4. *Let $E_n(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{q_n}\int_\Omega \frac{|u|^{q_n}}{|x|^s}dx - \frac{\lambda}{r}\int_\Omega |u|^r dx$ ($\lambda > 0, \mu > 0$), where q_n satisfy the conditions in the previous lemma and $1 < p \le r < p^*$. Assume the sequence $\{u_n\}$ satisfies $E_n(u_n) \to c$, $E_n'(u_n) \to 0$. Then, there exists a subsequence, still denoted by $\{u_n\}$, such that for some $u \in H_0^{1,p}(\Omega)$:*

1. *$u_n \to u$ weakly in $u \in H_0^{1,p}(\Omega)$.*
2. *$\nabla u_n \to \nabla u$ a.e.*
3. *$\int_\Omega |\nabla u_n - \nabla u|^p = \int_\Omega |\nabla u_n|^p - \int_\Omega |\nabla u|^p + o(1)$.*
4. *$|\nabla u_m|^{p-2}\nabla u_m \to |\nabla u|^{p-2}\nabla u$ weakly in $[L^{\frac{p}{p-1}}(\Omega)]^n$.*

Proof. Since

$$\lim_n E_n(u_n) = c \text{ and } \lim_n E_n'(u_n) = 0,$$

and

$$\langle E_n'(u_n),\ u_n\rangle = \int_\Omega |\nabla u_n|^p - \mu\int_\Omega \frac{|u_n|^{q_n}}{|x|^s} - \lambda\int_\Omega |u_n|^r,$$

we have

$$o(1)(1+\|u_n\|) + p|c| \ge pE_n(u_n) - \langle E_n'(u_n),\ u_n\rangle$$
$$= \begin{cases} \mu(1-\frac{p}{q_n})\int_\Omega \frac{|u|^{q_n}}{|x|^s}dx + \lambda(1-\frac{p}{r})\int_\Omega |u_n|^r dx, & r > p, \\ (1-\frac{p}{q_n})\int_\Omega \frac{|u|^{q_n}}{|x|^s}dx, & r = p. \end{cases}$$

Since Ω is bounded, we have

$$\int_\Omega |u_n|^p = \int_\Omega \frac{|u_n|^p}{|x|^{ps/q_n}} \cdot |x|^{ps/q_n}dx \le M\Big(\int_\Omega \frac{|u_n|^{q_n}}{|x|^s}\Big)^{p/q_n},$$

and

$$\|\nabla u_n\|_p^p = pE_n(u_n) + \mu\frac{p}{q_n}\int_\Omega \frac{|u_n|^{q_n}}{|x|^s}dx + \lambda\frac{p}{r}\int_\Omega |u_n|^r dx.$$

We conclude that $\{u_n\}$ is a bounded sequence in $H_0^{1,p}(\Omega)$.

We therefore can assume that $\{u_n\}$ satisfies all of the conclusions in Lemma 4.3. Now we use a technique initiated by Boccardo and Murat and already used by Garcia and Peral in a related context.

Define the functions

$$\tau_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ ks/|s| & \text{if } |s| \ge k. \end{cases}$$

We may assume also that $\tau_k(u_n - u) \to 0$ weakly in $H_0^{1,p}(\Omega)$ for any fixed positive k, since $\tau_k(u_n - u) \to 0$ a.e. and it is bounded. Then from the assumption we get

$$\begin{aligned} o(1) &= \langle E_n'(u_n) - (E_\lambda^p)'(u),\ \tau_k(u_n-u)\rangle + o(1) \\ &= \int_\Omega \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla \tau_k(u_n-u)\rangle_e \\ &\quad -\lambda \int_\Omega (\frac{|u_n|^{q_n-2}}{|x|^s}u_n - \frac{|u|^{p^*(s)-2}}{|x|^s}u)\tau_k(u_n-u). \end{aligned}$$

Since

$$\frac{|u_n|^{q_n-2}}{|x|^s}u_n \to \frac{|u|^{p^*(s)-2}}{|x|^s}u$$

in the weak star topology of $H^{-1,p'}(\Omega)$ (by Lemma 4.3), we have

$$|\int_\Omega (\frac{|u_n|^{q_n-2}}{|x|^s}u_n - \frac{|u|^{p^*(s)-2}}{|x|^s}u)\tau_k(u_n-u)| \le Ck,$$

and

$$\limsup_{n\to\infty}\int_\Omega \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla\tau_k(u_n-u)\rangle_e dx \le Ck.$$

Let $e_n(x) = \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u,\ \nabla\tau_k(u_n-u)\rangle_e$; then $e_n(x) \ge 0$ by Lemma 3.1, and is uniformly bounded in $L^1(\Omega)$. Take $0 < \theta < 1$ and split Ω into

$$S_n^k = \{x \in \Omega |\ |u_n - u| \le k\}, \quad G_n^k = \{x \in \Omega |\ |u_n - u| > k\}.$$

Then

$$\begin{aligned} \int_\Omega e_n^\theta dx &= \int_{S_n^k} e_n^\theta dx + \int_{G_n^k} e_n^\theta dx \\ &\le (\int_{S_n^k} e_n dx)^\theta |S_n^k|^{1-\theta} + (\int_{G_n^k} e_n dx)^\theta |G_n^k|^{1-\theta}. \end{aligned}$$

Now, for fixed k, $|G_n^k| \to 0$ as $n \to \infty$, and from the uniform boundedness in L^1 we get

$$\limsup_n \int_\Omega e_n^\theta dx \le (Ck)^\theta |\Omega|^{1-\theta}.$$

Letting $k \to 0$, we get that $e_n^\theta \to 0$ strongly in L^1. By Lemma 4.1,

$$\nabla u_n \to \nabla u \ \text{ in } L^q$$

for $1 < q < p$. By passing to a subsequence, we have

$$\nabla u_n \to \nabla u \ \text{ a.e.}$$

Thus (1) holds. As for (2), just apply Lemma 4.2. The proof of this lemma is thus complete. □

Proof of Theorem 4.1.(1): If $p \le q < p^*(s)$ and $r < p^*$, it is standard to show that the compactness of the Hardy-Sobolev embedding and of the Sobolev embedding imply that for any $\lambda > 0$ and any $\mu > 0$, the functional $E_{\lambda,\mu}$ satisfies $(\mathrm{PS})_c$ for all c. □

Proof of Theorem 4.1.(2): Recall that

$$L(\mu,s)=\begin{cases}\dfrac{p-s}{p(n-s)}\left(\dfrac{\mu_s^{n-s}}{\mu^{n-p}}\right)^{\frac{1}{p-s}} & \text{if } s<p,\\ +\infty & \text{if } s=p \text{ and } \mu<\mu_s,\\ 0 & \text{if } s=p \text{ and } \mu\ge\mu_s,\end{cases}$$

and assume that $p\le q=p^*(s)$ and $r<p^*$. We need to show that

$$E_{\lambda,\mu}(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\frac{\mu}{p^*(s)}\int_\Omega\frac{|u|^{p^*(s)}}{|x|^s}-\frac{\lambda}{r}\int_\Omega|u|^r dx$$

satisfies the Palais-Smale condition at any energy level less than $L(\mu,s)$.

For that, assume $\{u_n\}$ is a sequence in $H_0^{1,p}(\Omega)$ satisfying

$$E_{\lambda,\mu}(u_n)\to c<L(\mu,s)\ \text{ and }\ E'_{\lambda,\mu}(u_n)\to 0.$$

By Lemma 4.4, we may assume that $\{u_n\}$ satisfies the conclusions of both Lemma 4.2 and Lemma 4.3. For any $v\in C_0^\infty(\Omega)$,

$$\langle E'_{\lambda,\mu}(u_n),\ v\rangle=\int_\Omega(\langle|\nabla u_n|^{p-2}\nabla u_n,\ \nabla v\rangle-\lambda|u_n|^{r-2}u_nv-\mu\frac{|u_n|^{p^*(s)-2}u_n}{|x|^s}v)dx,$$

which converges as $n\to\infty$ to

$$0=\int_\Omega(\langle|\nabla u|^{p-2}\nabla u,\ \nabla v\rangle-\lambda|u|^{r-2}uv-\mu\frac{|u|^{p^*(s)-2}u}{|x|^s}v)dx=\langle E'_{\lambda,\mu}(u),\ v\rangle.$$

Hence $u\in H_0^{1,p}(\Omega)$ is a weak solution of (P_λ,μ). Choosing $v=u$, we have

$$0=\langle E'_{\lambda,\mu}(u),\ u\rangle=\int_\Omega(|\nabla u|^p-\lambda|u|^r-\mu\frac{|u|^{p^*(s)}}{|x|^s})dx,$$

and thus

$$E_{\lambda,\mu}(u)=\lambda(\frac{1}{p}-\frac{1}{r})\int_\Omega|u|^r+\mu(\frac{1}{p}-\frac{1}{p^*(s)})\int_\Omega\frac{|u|^{p^*(s)}}{|x|^s}dx\ge 0.$$

By Lemmas 4.3 and 4.4, we have

$$E_{\lambda,\mu}(u_n)=E_{\lambda,\mu}(u)+E_{0,\mu}(u_n-u)+o(1)$$

and

$$\begin{aligned}o(1)=\langle E'_{\lambda,\mu}(u_n),\ u_n-u\rangle &= \langle E'_{\lambda,\mu}(u_n)-E'_{\lambda,\mu}(u),\ u_n-u\rangle\\ &= \int_\Omega(|\nabla u_n-\nabla u|^p-\mu\frac{|u_n-u|^{p^*(s)}}{|x|^s})+o(1).\end{aligned}$$

If $s=p=p^*(s)$ and $\mu<\mu_p$, then

$$o(1)=\int_\Omega(|\nabla u_n-\nabla u|^p-\mu\frac{|u_n-u|^p}{|x|^p})+o(1)\le(1-\frac{\mu}{\mu_p})\int_\Omega(|\nabla u_n-\nabla u|^p+o(1);$$

that is, $u_n\to u$ strongly.

If $s<p$ (i.e., $p<p^*(s)$), we have, for large n,

$$\begin{aligned}E_{0,\mu}(u_n-u) &= E_{\lambda,\mu}(u_n)-E_{\lambda,\mu}(u)+o(1)\\ &\le E_{\lambda,\mu}(u_n)+o(1)\le c<L(\mu,s).\end{aligned}$$

Thus, for such n,

$$(\frac{1}{p}-\frac{1}{p^*(s)})\|\nabla u_n-\nabla u\|^p\le c<\frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}}(\frac{1}{\mu})^{\frac{n-p}{p-s}}.$$

By the Sobolev-Hardy inequality, we finally get

$$\begin{aligned} o(1) &= \int_\Omega (|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^{p^*(s)}}{|x|^s})dx \\ &\geq \int_\Omega |\nabla u_n - \nabla u|^p - \mu \mu_s^{-\frac{p^*(s)}{p}} (\int_\Omega |\nabla u_n - \nabla u|^p)^{\frac{p^*(s)}{p}} \\ &= (\int_\Omega |\nabla u_n - \nabla u|^p)[1 - \mu \mu_s^{-\frac{p^*(s)}{p}} (\int_\Omega |\nabla u_n - \nabla u|^p)^{\frac{p^*(s)-p}{p}}] \\ &\geq C \int_\Omega |\nabla u_n - \nabla u|^p \, dx. \end{aligned}$$

So again $u_n \to u$ in $H_0^{1,p}(\Omega)$ strongly. □

Proof of Theorem 4.1.(3): Suppose now that $p \leq q < p^*(s)$ and $r = p^*$; then we have compactness in the singular term but we will be dealing with a non-singular term involving the critical Sobolev exponent. We have again

$$\begin{aligned} E_0(u_n - u) &= E_{\lambda,\mu}(u_n) - E_{\lambda,\mu}(u) + o(1) \\ &\leq E_{\lambda,\mu}(u_n) + o(1) \leq c < L(\lambda, 0) = \frac{1}{n} \mu_0^{\frac{n}{p}} (\frac{1}{\lambda})^{\frac{n-p}{p}} \end{aligned}$$

Thus, for such n,

$$(\frac{1}{p} - \frac{1}{p^*}) \|\nabla u_n - \nabla u\|^p \leq c < \frac{1}{n} \mu_0^{\frac{n}{p}} (\frac{1}{\lambda})^{\frac{n-p}{p}},$$

so that this time we get, from the Sobolev inequality,

$$\begin{aligned} o(1) &= \langle E'_{\lambda,\mu}(u_n),\ u_n - u \rangle = \langle E'_{\lambda,\mu}(u_n) - E'_{\lambda,\mu}(u),\ u_n - u \rangle \\ &= \int_\Omega (|\nabla u_n - \nabla u|^p - \lambda \int_\Omega |u_n - u|^{p^*}) + o(1) \\ &= (\int_\Omega |\nabla u_n - \nabla u|^p)[1 - \lambda \mu_0^{\frac{-p^*}{p}} (\int_\Omega |\nabla u_n - \nabla u|^p)^{\frac{p^*}{p} - 1}] + o(1) \\ &\geq C \int_\Omega |\nabla u_n - \nabla u|^p \, dx. \end{aligned}$$

So again $u_n \to u$ in $H_0^{1,p}(\Omega)$ strongly. □

5. Min-max principles and dual sets associated to $E_{\lambda,\mu}$

For Banach spaces X and Y, we use $C(X, Y)$ to denote the space of all continuous maps from X to Y.

Definition 5.1. Let X be a Banach space and B be a closed subset of X. We say that a class $\mathcal{F}$ of compact subsets of X is a *homotopy-stable family with boundary* B provided that

(1) every set in $\mathcal{F}$ contains B, and
(2) for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t, x) = x$ for all (t, x) in $(\{0\} \times X) \cup ([0,1] \times B)$ we have that $\eta(\{1\} \times A) \in \mathcal{F}$.

We say that the class $\mathcal{F}$ is Z_2-homotopy stable if all sets in $\mathcal{F}$ are symmetric and if we only require stability under odd homotopies η (i.e., $\eta(t, -x) = -\eta(t, x)$).

We say that a closed set M is dual to the family $\mathcal{F}$ if

$$M \cap B = \emptyset \text{ and } M \cap A \neq \emptyset \text{ for all } A \in \mathcal{F}.$$

We shall need the following weakened version of the Palais-Smale condition.

Definition 5.2. A C^1-functional E on Banach space X *satisfies the Palais-Smale condition at level c and around the set M* (in short, $(\mathrm{PS})_{M,c}$), if every sequence $(u_n)_n$ satisfying $\lim_n E(u_n) = c$, $\lim_n \|E'(u_n)\| = 0$ and $\lim_n \mathrm{dist}(u_n, M) = 0$ has a convergent subsequence.

The following theorem of Ghoussoub [17] will be frequently used in the sequel.

Theorem 5.1. *Let E be a C^1-functional on X and consider a homotopy stable family $\mathcal{F}$ of compact subsets of X with a closed boundary B. Let M be a dual set to $\mathcal{F}$ such that*

$$\inf_{x\in M} E(x) = c := c(E, \mathcal{F}) = \inf_{A\in\mathcal{F}} \max_{x\in A} E(x).$$

If E satisfies $(\mathrm{PS})_{M,c}$, then $M \cap K_c \neq \emptyset$, where K_c is the set of all critical points of E at level c.

If $\mathcal{F}$ is only Z_2-homotopy stable, then the result still holds true as long as the functional E is even and the dual set M is symmetric.

Note that the above theorem includes the classical min-max principle which holds under the assumption that $\sup_{x\in B} E(x) < c$. It is enough to notice that in that case $M = \{x \in X; E(x) \geq c\}$ is a dual set.

Consider again the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{r}\int_\Omega |u|^r dx - \frac{\mu}{q}\int_\Omega \frac{|u|^q}{|x|^s} dx.$$

Recall that we assume that $1 < p < n$, $0 \leq s \leq p$, $0 \leq q \leq p^*(s) \equiv \frac{n-s}{n-p}p$ and that $p \leq r \leq p^* \equiv \frac{np}{n-p}$, so that E is a C^1-functional on the Sobolev space $H_0^{1,p}(\Omega)$.

<u>A first dual set</u>: Define the *Mountain Pass* class to be

$$\mathcal{F}_1 = \{\gamma \in C([0,1]; H_0^{1,p}(\Omega));\ \gamma(0) = 0,\ \gamma(1) \neq 0 \text{ and } E(\gamma(1)) \leq 0\},$$

which is clearly homotopy-stable with boundary $B = \{E \leq 0\}$. Let

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'(u), u\rangle = 0\}.$$

The following Nehari-type duality property is by now standard.

Theorem 5.2. *Assume $p \leq q \leq p^*(s)$ and one of the following cases:*

(1) *$p = r$, $p < q$ and $0 < \lambda < \lambda_1$, $\mu > 0$.*
(2) *$p < r \leq p^*$, $p = q$ and $0 < \mu < \mu_{s,p}$, $\lambda > 0$.*
(3) *$p < r \leq p^*$, $p < q$ and $\mu > 0$, $\lambda > 0$.*

The set M_1 is then closed, is dual to $\mathcal{F}_1$ and satisfies

$$\inf_{M_1} E_{\lambda,\mu} = c_1 := c(E_{\lambda,\mu}, \mathcal{F}_1)$$

Proof. By definition,

$$\langle E'_{\lambda,\mu}(u),\ u\rangle = \int_\Omega (|\nabla u|^p - \lambda|u|^r - \mu\frac{|u|^q}{|x|^s})dx.$$

Note that $B \cap M_1 = \emptyset$, since for every $u \in M_1$ we have

$$E_{\lambda,\mu}(u) = \lambda(\frac{1}{p} - \frac{1}{r}) \int_\Omega |u|^r dx + \mu(\frac{1}{p} - \frac{1}{q}) \int_\Omega \frac{|u|^q}{|x|^s} dx > 0,$$

under the assumption that either r or q is different from p. We also show that under this assumption, we have the estimate $c_1 \leq \inf_{u \in M_1} E_{\lambda,\mu}(u)$.

Let $u \neq 0$ in M_1, and consider the straight path $\gamma(t) = tu$. We have

$$E_{\lambda,\mu}(tu) = \frac{t^p}{p} \int_\Omega |\nabla u|^p - \frac{\lambda t^r}{r} \int_\Omega |u|^r - \mu \frac{t^q}{q} \int_\Omega \frac{|u|^q}{|x|^s}.$$

Since $\lim_{t\to\infty} E_{\lambda,\mu}(tu) = -\infty$, we have that $c_1 \leq \sup_{0 \leq t < \infty} E_{\lambda,\mu}(tu) = E_\lambda(t_0 u)$. From

$$\frac{dE_{\lambda,\mu}(tu)}{dt} = t^{p-1} \int_\Omega |\nabla u|^p - \lambda t^{r-1} \int_\Omega |u|^r - \mu t^{q-1} \int_\Omega \frac{|u|^q}{|x|^s}$$

and $\frac{dE_{\lambda,\mu}(tu)}{dt}(t_0) = 0$, we get

$$\int_\Omega |\nabla u|^p = t_0^{r-p} \cdot \lambda \int_\Omega |u|^r + \mu t_0^{q-p} \int_\Omega \frac{|u|^q}{|x|^s}.$$

Since $u \in M_1$, we should have

$$\int_\Omega |\nabla u|^p = \lambda \int_\Omega |u|^r + \mu \int_\Omega \frac{|u|^q}{|x|^s}.$$

Thus, t_0 must be equal to 1 as long as either r or q is distinct from p. This clearly shows that under any of the 3 conditions above, we have

$$c_1 \leq \inf_{u \in M_1} E_{\lambda,\mu}(u).$$

For the rest, we have to distinguish the 3 cases.

Case (1). $2 \leq p = r$, $p < q$ and $0 < \lambda < \lambda_1$.

To prove that M_1 is closed, use the Sobolev-Hardy inequality and the definition of λ_1 to find a constant $c > 0$ such that

$$\begin{aligned} \langle E'_{\lambda,\mu}(u),\ u\rangle &= (1 - \frac{\lambda}{\lambda_1}) \|u\|^p_{H_0^{1,p}(\Omega)} - c\|u\|^q_{H_0^{1,p}(\Omega)} \\ &= \|u\|^p_{H_0^{1,p}(\Omega)} (1 - \frac{\lambda}{\lambda_1} - c\|u\|^{q-p}_{H_0^{1,p}(\Omega)}). \end{aligned}$$

Choose some $\beta > 0$ such that if $\|u\| < \beta$, then $1 - \frac{\lambda}{\lambda_1} - c\|u\|^{q-p}_{H_0^{1,p}(\Omega)} > 0$. This means that we can find some constant $\beta > 0$ such that for any $u \in M_1$, we have $\|u\| \geq \beta$. So M_1 is closed.

To prove the intersection property, fix $\gamma \in \mathcal{F}_1$ joining 0 to υ, where $\upsilon \neq 0$ and $E_{\lambda,\mu}(\upsilon) \leq 0$. Note that since $\lambda < \lambda_1$, we have $\langle E'_{\lambda,\mu}(\gamma(t)), \gamma(t)\rangle > 0$ for t close to 0 (same proof as for the closedness of M_1). On the other hand, since $\upsilon \neq 0$, we have

$$\langle E'_{\lambda,\mu}(\upsilon), \upsilon\rangle < pE_{\lambda,\mu}(\upsilon) \leq 0.$$

It follows from the intermediate value theorem that there exists t_0 such that $\gamma(t_0) \in M_1$. This proves the duality, and consequently $c_1 \geq \inf\{E_{\lambda,\mu}(u) :\ u \in M_1\}$.

Case (2). $1 < p < r \leq p^*$, $p = q \leq p^*(s)$ and $0 < \mu < \mu_{s,q}$.

To prove that M_1 is closed, use the Sobolev-Hardy inequality with its best constant $\mu_{s,p}$ and the Sobolev inequality to get

$$\begin{aligned}\langle E'_{\lambda,\mu}(u),\ u\rangle &= \int_\Omega |\nabla u|^p - \lambda\int_\Omega |u|^r - \mu\int_\Omega \frac{|u|^q}{|x|^s}\\ &\geq \int_\Omega |\nabla u|^p - c(\int_\Omega |\nabla u|^p)^{\frac{r}{p}} - \frac{\mu}{\mu_{s,q}}\int_\Omega |\nabla u|^p\\ &= (1-\frac{\mu}{\mu_{s,q}})\|u\|^p_{H_0^{1,p}(\Omega)} - c\|u\|^r_{H_0^{1,p}(\Omega)}\\ &= \|u\|^p_{H_0^{1,p}(\Omega)}(1-\frac{\mu}{\mu_{s,q}} - c\|u\|^{r-p}_{H_0^{1,p}(\Omega)}).\end{aligned}$$

Choose some $\beta > 0$ such that if $\|u\| < \beta$, then $1 - \frac{\mu}{\mu_{s,q}} - c\|u\|^{r-p}_{H_0^{1,p}(\Omega)} > 0$. This means that we can find some constant $\beta > 0$ such that for any $u \in M_1$, we have $\|u\| \geq \beta$. So M_1 is closed.

To prove the intersection property, fix $\gamma \in \mathcal{F}_1$ joining 0 to υ, where $\upsilon \neq 0$ and $E_{\lambda,\mu}(\upsilon) \leq 0$. Note that since $\mu < \mu_{s,q}$, we have $\langle E'_{\lambda,\mu}(\gamma(t)), \gamma(t)\rangle > 0$ for t close to 0 (same proof as for the closedness of M_1). On the other hand, since $\upsilon \neq 0$, we have $\langle E'_{\lambda,\mu}(\upsilon), \upsilon\rangle < pE_{\lambda,\mu}(\upsilon) \leq 0$. It follows from the intermediate value theorem that there exists t_0 such that $\gamma(t_0) \in M_1$. This proves the duality, and consequently $c_1 \geq \inf\{E_{\lambda,\mu}(u) :\ u \in M_1\}$.

Case (3). $2 \leq p < r \leq p^*$ and $\lambda > 0$.

To prove that M_1 is closed, again use the Sobolev-Hardy inequality and the Sobolev embedding to find constants $c' > 0,\ c'' > 0$ such that

$$\begin{aligned}\langle E'_{\lambda,\mu}(u),\ u\rangle &\geq \|u\|^p - c'\lambda\|u\|^r - c''\|u\|^q\\ &= \|u\|^p(1 - c'\lambda\|u\|^{r-p} - c''\|u\|^{q-p}).\end{aligned}$$

Since both r and q are distinct from p, we may choose $\gamma > 0$ such that for any $u \in H_0^{p,1}(\Omega)$ with $\|u\| < \gamma$, we have $1 - c_1\lambda\|u\|^{r-p} - c_2\|u\|^{q-p} > 0$. This means that $\|u\| \geq \gamma$ for any $u \in M_1$; hence M_1 is closed.

For the intersection property, consider any $\gamma \in \mathcal{F}_1$ joining 0 and υ. Since $p < r$, again the proof above of the closedness of M_1 yields that $\langle E_{\lambda,\mu}(\gamma(t)), \gamma(t)\rangle > 0$ for t close to 0. Also since $\upsilon \neq 0$, we have

$$\langle E'_{\lambda,\mu}(\upsilon), \upsilon\rangle < pE_{\lambda,\mu}(\upsilon) \neq 0.$$

Then again, by the intermediate value theorem, we conclude that there exists t_0 such that $\gamma(t_0) \in M_1$. This proves the duality and the inequality

$$c_1 \geq \inf\{E_{\lambda,\mu}(u),\ u \in M_1\}.$$

□

<u>Another dual set</u>: Denote by S_ρ the sphere $S_\rho = \{u \in H_0^{1,p}(\Omega);\ \|u\|_{H_0^{1,p}(\Omega)} = \rho\}$ and by $\mathcal{H}$ the set

$$\mathcal{H} = \{h : H_0^{1,p}(\Omega) \to H_0^{1,p}(\Omega)\ \text{ an odd homeomorphism}\}.$$

Let γ_{Z_2} denote the Krasnoselskii genus, defined for every closed symmetric subset D of $H_0^{1,p}(\Omega)$ as

$$\gamma_{Z_2}(D) = \inf\{n;\ \text{there exists an odd and continuous map } h : D \to \mathbf{R}^n \setminus \{0\}\},$$

and consider the class

$$\mathcal{F}_2 = \{A; A \text{ closed symmetric with } \ \gamma_{Z_2}(h(A) \cap S_\rho) \geq 2, \ \forall h \in \mathcal{H}\}.$$

It is easy to verify that $\mathcal{F}_2$ is a Z_2-homotopy stable class. Let

$$c_2 = \inf_{A \in \mathcal{F}_2} \sup_A E_{\lambda,\mu}.$$

We shall now consider an appropriate dual set to $\mathcal{F}_2$. First, we recall a few facts about the following weighted eigenvalue problem ($1 < p < \infty$):

$$(*) \qquad \begin{cases} -\triangle_p u = \lambda b(x)|u|^{p-2}u, \\ u \in H_0^{1,p}(\Omega), \qquad u \neq 0. \end{cases}$$

We will say that $\lambda \in \mathbf{R}$ is the eigenvalue and $u \in H_0^{1,p}(\Omega)$, $u \neq 0$, is the corresponding eigenfunction of the above problem if the equality

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lambda \int_\Omega b(x)|u|^{p-2} u \varphi dx$$

holds for any $\varphi \in H_0^{1,p}(\Omega)$. The following lemma is well known.

Lemma 5.1 ([25] [11]). *Assume $b(x) \geq 0$, $b(x) \in L^t(\Omega)$, and $|\{x \in \Omega : \ b(x) > 0\}| \neq 0$, where $t \geq 1$ if $p > n$, $t > 1$ if $p = n$ and $t > \frac{n}{p} > 1$ otherwise. Let $\lambda_0 = \inf\{\int_\Omega |\nabla v|^p; \ \int_\Omega b(x)|v|^p = 1\}$. Then:*

1. *$\lambda_0 > 0$ is the first eigenvalue of the problem $(*)$.*
2. *λ_0 is simple, and there exists precisely one pair of normalized eigenfunctions corresponding to λ_0 which do not change sign in Ω. Here, v being normalized means that $\int_\Omega b(x)|v|^p = 1$.*

We use the lemma to prove the following fact:

Lemma 5.2. *For $2 \leq p \leq r < p^*$, $p \leq q < p^*(s)$, $\lambda > 0, \mu > 0$ and any $u \in H_0^{1,p}(\Omega)$, $u \neq 0$, there exists a unique $v = v(u) \in H_0^{1,p}(\Omega)$ such that*

(a) $\int_\Omega (\lambda |u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) v^p = 1$, $v \geq 0$;

(b) $\|\nabla v\|_p^p = \inf\{\|\nabla \omega\|_p^p : \ \int_\Omega (\lambda |u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})|\omega|^p = 1\}$.

Furthermore, the map $u \to v(u)$ is continuous from $L^r(\Omega) \to H_0^{1,p}(\Omega)$.

Remark 5.1. It is quite unfortunate that the above lemma is not applicable –unless $p = 2$– whenever $r = p^*$ or when $q = p^*(s)$. This will create additional complications in the search for a second solution of the critical problems.

Proof. Since $\frac{np^*}{sp^* + (q-p)n} > \frac{n}{p}$, choose $\frac{n}{p} < t < \frac{np^*}{sp^* + (q-p)n}$; then $st \frac{p^*}{p^* - (q-p)t} < n$. Because

$$\int_\Omega \frac{|u|^{(q-p)t}}{|x|^{st}} \leq (\int_\Omega |u|^{p^*})^{\frac{t(q-p)}{p^*}} (\int_\Omega \frac{1}{|x|^{st \cdot \frac{p^*}{p^* - (q-p)t}}})^{\frac{p^* - (q-p)t}{p^*}},$$

we get that $|u|^{q-p}/|x|^s \in L^t(\Omega)$.

The functional $\psi(u) = \|u\|^p = \int_\Omega |\nabla u|^p dx$ is clearly weakly lower semicontinuous and coercive. Moreover, the constraint set

$$C = \{\omega \in H : \ \int_\Omega (\lambda |u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})|\omega|^p dx = 1\}$$

is weakly closed in $H_0^{1,p}(\Omega)$ and $\psi(\cdot)$ is bounded below on C. Therefore, by the direct methods of the calculus of variations (Struwe [26], p. 4), the infimum in (b) is achieved and this infimum is the first eigenvalue of $(*)$ and thus is simple. Any function where such an infimum is achieved is the eigenfunction corresponding to the first eigenvalue of $(*)$. By Lemma 5.1, it cannot changes sign in Ω. This gives the uniqueness of $v(u)$ and therefore its continuity for non-zero u. $\square$

Note that $(\nu_1(u), v(u))$ corresponds to the first eigenpair of the (weighted) eigenvalue problem

$$(**)\qquad \begin{cases} -\triangle_p v = \nu(\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})|v|^{p-2}v & \text{in } \Omega \\ \quad v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now let

$$M_2 = M_1 \cap \{u \in H_0^{1,p}(\Omega);\ \int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u = 0\}.$$

The following duality result was first noticed by G. Tarantello [28] in the case when $s = 0$ and $p = 2$.

Theorem 5.3. *Assume $p \leq q < p^*(s)$ and $r < p^*$. Then M_2 is a closed set that is dual to $\mathcal{F}_2$, and*

$$\inf_{M_2} E_{\lambda,\mu} = c_2 := c(E_{\lambda,\mu}, \mathcal{F}_2)$$

as long as we are in one of the following cases:

(1) $p = r$, $p < q$ *and* $0 < \lambda < \lambda_1$, $0 < \mu$.
(2) $p < r$, $p = q$ *and* $0 < \mu < \mu_{s,p}, 0 < \lambda$.
(3) $p < r$, $p < q$ *and* $0 < \mu$, $0 < \lambda$.

Proof. In the 3 cases, we get from Theorem 5.2 (and its proof) that M_1 is closed and that for any $u \neq 0$, there exists a unique $t(u) > 0$ such that $t(u)u \in M_1$. Clearly, $t(u) = t(|u|) = t(-u)$ and

$$E_{\lambda,\mu}(t(u)u) = \max_{t\geq 0} E_{\lambda,\mu}(tu).$$

The uniqueness of $t(u)$ and its properties tell us that the map $u \to t(u)$ is continuous on $H_0^{1,p}(\Omega)$ and that the map $u \to t(u)u$ defines an odd homeomorphism between S_ρ and M_1 which gives that $\gamma_{Z_2}(A \cap M_1) \geq 2$ for all $A \in \mathcal{F}_2$.

On the other hand, the map $h:\ A \cap M_1 \to \mathbf{R}$ given by

$$h(u) = \int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u dx$$

defines an odd and continuous map. Since $\gamma_{Z_2}(h(A \cap M_1)) \geq 2$, we get that $0 \in h(A \cap M_1)$ which means that $A \cap M_2 \neq \emptyset$ and M_2 is dual to $\mathcal{F}_2$. In particular, $c_2 \geq \inf_{u\in M_2} E_{\lambda,\mu}(u)$.

To prove the reverse inequality, take $u \in M_2$ and let $v(u)$ be such that

$$\int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u dx = 0.$$

Let $\omega(u)$ be a minimizer for the problem:

$$\mu_2 = \inf\{\psi(\omega);\ \omega \in H_0^{1,p},\ \int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}\omega = 0, \\ \int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})|\omega|^p = 1\}.$$

Since $u \in M_1$, we obtain

$$\mu_2 \leq \frac{\|\nabla u\|_p^p}{\int_\Omega (\lambda|u|^r + \mu\frac{|u|^q}{|x|^s})} = 1.$$

Define $A = \text{span}\{v(u), \omega(u)\} \in \mathcal{F}_2$. Then, clearly,

$$1 \geq \mu_2 \geq \frac{\|\nabla\omega\|_p^p}{\int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})|\omega|^p},\ \forall \omega \in A, \omega \neq 0.$$

For $\omega_0 \in A$ satisfying $E_{\lambda,\mu}(\omega_0) = \sup_A E_{\lambda,\mu} \geq c_2$, we have $\omega_0 \neq 0$ and $\omega_0 \in M_1$. From the above inequality, we derive

$$\int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})|\omega_0|^p \geq \|\nabla\omega_0\|_p^p.$$

This implies

$$\frac{1}{p}\int_\Omega (\lambda|u|^{r-p} + \mu\frac{|u|^{q-p}}{|x|^s})(|\omega_0|^p - |u|^p) \geq \frac{\|\nabla\omega_0\|_p^p}{p} - \frac{\|\nabla u\|_p^p}{p}.$$

Applying the inequality (valid for $t \geq p$ and $x, y \in \mathbf{R}$)

$$\frac{1}{t}(|x|^t - |y|^t) \geq \frac{1}{p}(|x|^p - |y|^p)|y|^{t-p}$$

with $t = r$ (resp. $t = q$), we conclude that

$$\frac{\lambda}{r}\int_\Omega |\omega_0|^r + \mu\frac{1}{q}\int_\Omega \frac{|\omega_0|^q}{|x|^s} - \frac{\lambda}{r}\int_\Omega |u|^r - \mu\frac{1}{q}\int_\Omega \frac{|u|^q}{|x|^s} \geq \frac{\|\nabla\omega_0\|_p^p}{p} - \frac{\|\nabla u\|_p^p}{p},$$

that is,

$$E_{\lambda,\mu}(u) \geq E_{\lambda,\mu}(\omega_0) \geq c_2.$$

This finishes the proof of the theorem. □

6. The solutions in the case of an HS-subcritical singular term

In this section, we consider the problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u = \lambda|u|^{r-2}u + \frac{|u|^{q-2}}{|x|^s}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $0 \leq s \leq p < n$, in the presence of a subcritical singular term ($1 < p \leq q < p^*(s)$) and a subcritical non-singular term ($1 < p \leq r < p^*$).

Theorem 6.1 (Hardy-Sobolev subcritical singular term)**.** *Suppose $1 < p \leq q < p^*(s)$ and $r < p^*$. Also assume one of the following conditions:*

1. *$p < q$, $p \leq r$ and $\lambda > 0$, $\mu > 0$.*
2. *$p = q$, $p < r$ and $\lambda > 0$, $\mu_{s,p} > \mu > 0$.*

Then the equation $(\mathrm{P}_{\lambda,\mu})$ *has infinitely many solutions. Moreover, it has an everywhere positive solution* u_1 *with minimal energy and a sign-changing solution* u_2 *that satisfies*

$$\int_\Omega (\lambda|u_2|^{r-p} + \mu\frac{|u_2|^{q-p}}{|x|^s})v(u_2)^{p-1}u_2 = 0,$$

where $v(u_2)$ *is the first eigenvector of the (weighted) eigenvalue problem*

$$\begin{cases} -\triangle_p v = \nu(\lambda|u_2|^{r-p} + \mu\frac{|u_2|^{q-p}}{|x|^s})|v|^{p-2}v & in\ \Omega, \\ \quad v = 0 \quad on \quad \partial\Omega. \end{cases}$$

Proof. Now that under these conditions the functional $E_{\lambda,\mu}$ satisfies $(\mathrm{PS})_c$ for any c. It is now enough to apply Theorem 5.1 to $\mathcal{F}_1$ and its dual set M_1 (resp., to $\mathcal{F}_2$ and its dual set M_2) to get a solution u_1 (resp. u_2) which minimizes the energy functional on M_1 (resp. M_2).

To obtain other solutions, we need the following result of Rabinowitz ([17]).

Lemma 6.1. *Let* E *be an even* C^1*-functional satisfying the Palais-Smale condition on a Banach space* $X = Y \oplus Z$ *with* $\dim(Y) < \infty$*. Assume* $E(0) = 0$*, as well as the following conditions:*

(1) *There is* $\rho > 0$ *such that* $\inf_{S_\rho(Z)} E \geq 0$.
(2) *There exists an increasing sequence* $\{Y_n\}_n$ *of finite dimensional subspaces of* X*, all containing* Y*, such that* $\lim_n dim(Y_n) = \infty$ *and for each* n, $\sup_{S_{R_n}(Y_n)} E \leq 0$ *for some* $R_n > \rho$.

Then E *has an unbounded sequence of critical values.*

We now show that the functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\mu}{q}\int_\Omega \frac{|u|^q}{|x|^s} - \frac{\lambda}{r}\int_\Omega |u|^r$$

satisfies the hypothesis of the lemma.

Without loss of generality, we assume that $\Omega = (0,\ 1)^n$. Let Y_k be the k-dimensional subspace of $X = H_0^{1,p}(\Omega)$, generated by the first k functions of the basis

$$\{(\sin k_1\pi x_1,\ \cdots,\ \sin k_n\pi x_n),\ k_i \in \mathbf{N},\ i = 1,\ \cdots,\ n\}.$$

Let Z_k denote the complement of Y_k in X, that is, the set generated by the base vectors not in Y_k. For any $u \in Y_{k-1}^c$, the topological complement of Y_{k-1},

$$\|u\|_p \leq C\|\nabla u\|_p/k^{\frac{1}{n}} \quad \text{(Peral [25])}.$$

Claim 1. For k sufficiently large, there exists $\rho > 0$ such that $E(u) \geq 1$ for all $u \in Z_{k-1}$ with $\|u\|_{H_0^{1,p}} = \rho$

Proof of Claim 1. We first consider the case where $p = q$, $p < r$ and $\mu < \mu_{s,p}$:

$$E(u) \geq (1 - \frac{\mu}{\mu_{s,p}})\frac{1}{p}\int_\Omega |\nabla u|^p - C\int_\Omega |u|^r.$$

By the Gagliardo-Nirenberg inequality,

$$(\int_\Omega |u|^r)^{\frac{1}{r}} \leq C_1(\int_\Omega |\nabla u|^p)^{\frac{a}{p}}(\int_\Omega |u|^p)^{\frac{1-a}{p}}$$

with $a = \frac{n}{p}(1 - \frac{p}{r})$. Hence, for $u \in \partial B_\rho \cap Y_{k-1}^c$,

$$\begin{aligned} E(u) &\geq C_2 \int_\Omega |\nabla u|^p - C_1 (\int_\Omega |\nabla u|^p)^{\frac{ra}{p}} (\int_\Omega |u|^p)^{\frac{r(1-a)}{p}} \\ &= \rho^p (C_2 - C_1 \rho^{ra} (\frac{C\rho}{k^{\frac{1}{n}}})^{r(1-a)} \rho^{-p}) \\ &= \rho^p (C_2 - C_3 \rho^{r-p} \frac{1}{k^{r(1-a)/n}}). \end{aligned}$$

Choosing $\rho = (\frac{C_2}{2C_3} k^{\frac{r(1-a)}{n}})^{\frac{1}{r-p}}$, we get that $E(u) \geq \frac{1}{2} C_2 \rho^p = C_4 k^{\frac{pr(1-a)}{n(r-p)}} \geq 1$, for k large enough. This completes the proof of Claim 1 in the first case.

We turn to the case where $p < q$: Since $q < p^*(s)$, choose $\varepsilon > 0$ such that $q < \frac{n-s-\varepsilon}{n-p} p$; then

$$\int_\Omega \frac{|u|^q}{|x|^s} \leq C_0 (\int_\Omega |u|^{q\frac{n}{n-s-\varepsilon}})^{\frac{n-s-\varepsilon}{n}}.$$

If $q \frac{n}{n-s-\varepsilon} \leq r$, then

$$\int_\Omega |u|^{q\frac{n}{n-s-\varepsilon}} \leq C_1 (\int_\Omega |u|^r)^{\frac{q\frac{n}{n-s-\varepsilon}}{r}}.$$

Thus

$$\int_\Omega \frac{|u|^q}{|x|^s} \leq C_2 (\int_\Omega |u|^r)^{\frac{q}{r}}.$$

If $q \frac{n}{n-s-\varepsilon} \geq r$, then

$$\int_\Omega |u|^r \leq C_3 (\int \Omega |u|^{q\frac{n}{n-s-\varepsilon}})^{\frac{n-s-\varepsilon}{n} \cdot \frac{r}{q}}.$$

Because of these relationships, we could combine the last two terms of the functional E together. In this sense, we may assume that

$$E(u) \geq \frac{1}{p} \int_\Omega |\nabla u|^p - C \int_\Omega |u|^r,$$

and the rest is as in case (1). □

Let $Y = Y_k$ with the k chosen in Claim 1. We now show the following.

Claim 2. In both cases, there exist for each finite dimensional subspace $Y_k \subset H_0^{1,p}(\Omega)$, positive constants $C_1,\ C_2$ (depending on Y_k) such that

$$\sup_{u \in \partial B_R(Y_k)} E(u) \leq C_1 R^p - C_2 R^r.$$

Indeed, for any $u \in H_0^{1,p}(\Omega)$ and any $R > 0$, we have

$$E(Ru) \leq \frac{R^p}{p} \|u\|^p_{H_0^{1,p}(\Omega)} - \frac{R^r}{r} \|u\|_r^r.$$

Since Y_k is a finite dimensional space, it is closed and the two norms $\|\cdot\|_r$ and $\|\cdot\|_{H_0^{1,p}(\Omega)}$ on Y_k are equivalent. This implies the Claim.

Now we can apply Lemma 6.1 to conclude that $E_{\lambda,\mu}$ has an unbounded sequence of critical values. Theorem 6.1 is proved. □

7. The solutions in the case of a Hardy-critical singular term

Theorem 7.1 (Hardy-critical singular term)**.** *Suppose* $1 < p = q = p^*(s)$ *(i.e., $s = p$).*

(1) *If $p < r < p^*$ (high order non-singular term), then* $(\mathrm{P}_{\lambda,\mu})$ *has infinitely many solutions –at least one of them being positive– for any $\lambda > 0$ and any $0 < \mu < \mu_p$.*
(2) *If $r = p^*$ (critical non-singular term) and Ω is star-shaped, then* $(\mathrm{P}_{\lambda,\mu})$ *has no non-trivial solution for any $\lambda > 0, \mu > 0$.*

Proof. If $r < p^*$, then by Theorem 4.1.2 the functional $E_{\lambda,\mu}$ satisfies $(\mathrm{PS})_c$ for any c as long as $\mu < \mu_p$. Since $p < r$, the proof is the same as in Theorem 6.1.(2), while the second case of the theorem is covered in section 2. □

Remark 7.1. The case when $p = q = r$ is really an eigenvalue problem. There are solutions for $(P_{\lambda,\mu})$ as long as λ is an eigenvalue of the problem $-\Delta_p u - \frac{\mu|u|^{p-2}u}{|x|^p} = \lambda|u|^{p-2}u$ in $H_0^{1,p}(\Omega)$.

If $0 \le \mu < \mu_p$, one can show that there is an infinite number of eigenvalues for the above problem. Indeed, these correspond to the critical levels of the restriction of the functional

$$\tilde{E}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \frac{\mu}{p}\int_\Omega \frac{|u|^p}{|x|^p} \, dx$$

to the submanifold $\{u; \int_\Omega |u|^p \, dx = 1\}$. But a slight variation of Theorem 4.1.(2) shows that in this case $\tilde{E}$ has $(\mathrm{PS})_c$ for any c, and therefore a standard application of Ljusternik-Schnirelmann theory applied to the genus $\gamma_{\mathbf{Z}_2}$ will yield the result.

8. A positive solution in the case of a Hardy-Sobolev critical singular term

In this section, we consider the first solution for the problem $(\mathrm{P}_{\lambda,\mu})$ with the critical *Sobolev-Hardy* exponent.

Theorem 8.1 (Hardy-Sobolev critical singular term)**.** *Suppose* $1 < p < q = p^*(s)$ *(i.e., $s < p$) in the equation:*

$$(\mathrm{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u = \lambda|u|^{r-2}u + \mu\frac{|u|^{p^*(s)-2}}{|x|^s}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

- *If $r < p^*$, then* $(\mathrm{P}_{\lambda,\mu})$ *has a solution that is strictly positive everywhere on Ω, under any one of the following conditions:*
  (1) *$p = r < p^*$ and $n \ge p^2$, $0 < \lambda < \lambda_1$ and $\mu > 0$.*
  (2) *$p < r < p^*$, λ is large enough and $\mu > 0$.*
  (3) *$p < r < p^*$ and $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$, $\lambda > 0, \mu > 0$.*
- *If $r = p^*$ and Ω is star-shaped, then* $(\mathrm{P}_{\lambda,\mu})$ *has no non-trivial solution for any $\lambda > 0, \mu > 0$.*

Proof. Note that the last case ($r = p^*$) was covered in section 2. Now if $r < p^*$, then Theorem 5.2 asserts that any one of the 3 conditions yields that the set

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'_{\lambda,\mu}(u), u\rangle = 0\},$$

is closed, that it is dual to the Mountain Pass class

$$\mathcal{F}_1 = \{\gamma \in C([0,1]; H_0^{1,p}(\Omega));\ \gamma(0)=0,\ \gamma(1)\neq 0 \text{ and } E_{\lambda,\mu}(\gamma(1)) \leq 0\},$$

and that

$$\inf_{M_1} E_{\lambda,\mu} = c_1 := c(E_{\lambda,\mu}, \mathcal{F}_1).$$

On the other hand, Theorem 4.1.(2) yields that $E_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition for any

$$c < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Therefore, we should be able to apply Theorem 5.1 and obtain our desired assertion, if only we can prove the following case. □

Lemma 8.1. *In any one of the above three cases, we have*

$$c_1 < \frac{p-s}{p(n-s)} (\frac{\mu_s^{n-s}}{\mu^{n-p}})^{\frac{1}{p-s}}.$$

Proof. We may assume without loss of generality that $\mu = 1$. We first consider the following case:

Case (1). $p < r$ and λ is large.

In order to estimate the energy level c_1, we consider the functions

$$g(t) = E_{\lambda,\mu}(tv_\varepsilon) = \frac{t^p}{p}\int_\Omega |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^r}{r}\int_\Omega |v_\varepsilon|^r$$

and

$$\bar{g}(t) = \frac{t^p}{p}\int_\Omega |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)},$$

where v_ε is the extremal function defined in the appendix. Note that $\lim_{t\to\infty} g(t) = -\infty$ and $g(t) > 0$ when t is close to 0, so that $\sup_{t\geq 0} g(t)$ is attained for some $t_\varepsilon > 0$. From

$$0 = g'(t_\varepsilon) = t_\varepsilon^{p-1}(\int_\Omega |\nabla v_\varepsilon|^p - t_\varepsilon^{p^*(s)-p} - \lambda t_\varepsilon^{r-p}\int_\Omega |v_\varepsilon|^r)$$

we have

$$\int_\Omega |\nabla v_\varepsilon|^p = t_\varepsilon^{p^*(s)-p} + \lambda t_\varepsilon^{r-p}\int_\Omega |v_\varepsilon|^r > t_\varepsilon^{p^*(s)-p},$$

and therefore

$$t_\varepsilon \leq (\int_\Omega |\nabla v_\varepsilon|^p)^{\frac{1}{p^*(s)-p}}.$$

Thus

$$\int_\Omega |\nabla v_\varepsilon|^p \leq t_\varepsilon^{p^*(s)-p} + \lambda(\int_\Omega |\nabla v_\varepsilon|^p)^{\frac{r-p}{p^*(s)-p}}(\int_\Omega |v_\varepsilon|^r).$$

Choose ε small enough so that by (1) and (6) of Lemma 11.1 we have $t_\varepsilon^{p^*(s)-p} \geq \frac{\mu_s}{2}$. That is, we get a lower bound for t_ε, which is independent of ε.

Now we estimate $g(t_\varepsilon)$. The function $\bar{g}(t)$ attains its maximum at

$$t = (\int_\Omega |\nabla v_\varepsilon|^p)^{\frac{1}{p^*(s)-p}}$$

and is increasing in the interval

$$[0,\ \ (\int_\Omega |\nabla v_\varepsilon|^p)^{\frac{1}{p^*(s)-p}}].$$

By Lemma 11.1, we have

$$\begin{aligned} g(t_\varepsilon) &= \bar{g}(t_\varepsilon) - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &\leq \bar{g}((\int_\Omega |\nabla v_\varepsilon|^p)^{\frac{1}{p^*(s)-p}}) - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &= (\frac{1}{p} - \frac{1}{p^*(s)})(\int_\Omega |\nabla v_\varepsilon|^p)^{\frac{p^*(s)}{p^*(s)-p}} - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &\leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - \frac{\lambda}{r}(\frac{\mu_s}{2})^{\frac{r}{p^*(s)-p}} \int_\Omega |v_\varepsilon|^r. \end{aligned}$$

So for λ large enough, we have

$$g(t_\varepsilon) < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Case (2). $p < r < p^*$ and $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$, $\lambda > 0$.

Note first that the above condition is equivalent to $\max\{p, p^* - \frac{p}{p-1}\} < r < p^*$ and $\lambda > 0$.

For any $\lambda > 0$, the above estimate on $g(t_\varepsilon)$ and Lemma 6.1 yield

$$g(t_\varepsilon) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{p-1}{p-s}(n - \frac{r(n-p)}{p})}),$$

so that if r is chosen in such a way that

$$\frac{n-p}{p-s} > \frac{p-1}{p-s}(n - \frac{r(n-p)}{p}),$$

i.e. $r > p^* - \frac{p}{p-1}$, then

$$g(t_\varepsilon) < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Case (3). $p = r$, $0 < \lambda < \lambda_1$ and $n \geq p^2$.

We still use the function $g(t)$. Since $\lambda < \lambda_1$, we have $g(t) > 0$ when t is close to 0, and $\lim_{t\to\infty} g(t) = -\infty$. So again $g(t)$ attains its maximum at some $t_\varepsilon > 0$. From

$$g'(t) = t^{p-1}(\int_\Omega |\nabla v_\varepsilon|^p - t^{p^*(s)-p} - \lambda \int_\Omega |v_\varepsilon|^p) = 0$$

we get

$$t_\varepsilon = (\int_\Omega |\nabla v_\varepsilon|^p - \lambda \int_\Omega |v_\varepsilon|^p)^{\frac{1}{p^*(s)-p}}.$$

Thus

$$\begin{aligned} g(t_\varepsilon) &= (\frac{1}{p} - \frac{1}{p^*(s)})(\int_\Omega |\nabla v_\varepsilon|^p - \lambda \int_\Omega |v_\varepsilon|^p)^{\frac{p^*(s)}{p^*(s)-p}} \\ &= \begin{cases} \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{p(p-1)}{p-s}}), & p > p^*(1 - \frac{1}{p}), \\ \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{n-p}{p-s}} |\log \varepsilon|), & p = p^*(1 - \frac{1}{p}). \end{cases} \end{aligned}$$

In the case where $p > p^*(1-\frac{1}{p})$, we require that $\frac{n-p}{p-s} > \frac{p(p-1)}{p-s}$, but both are equivalent to $p^2 < n$. In the case where $p = p^*(1-\frac{1}{p})$ we have $p^2 = n$, and the proof the lemma is now complete. □

Remark 8.1. (1) If $p^2 \leq n$, then $p^* - \frac{p}{p-1} \leq p^*(1-\frac{1}{p}) \leq p$, so that $r > p^* - \frac{p}{p-1}$ whenever $p < r$. In this case, r can take any value between p and p^*.

(2) If $p^2 > n$, then $p < p^*(1-\frac{1}{p}) < p^* - \frac{p}{p-1}$, and then we require that $p^* - \frac{p}{p-1} < p < p^*$.

9. A sign changing solution in the Hardy-Sobolev critical case

In this section, we extend the arguments of Tarantello [28] to establish the following.

Theorem 9.1 (Hardy-Sobolev critical singular term)**.** *Suppose* $2 \leq p < q = p^*(s)$ *and* $r < p^*$ *in the equation*

$$(\mathrm{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u = \lambda |u|^{r-2}u + \mu \frac{|u|^{p^*(s)-2}}{|x|^s} u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Assume any one of the following conditions:

(1) $p = r < p^*$, $n > p^3 - p^2 + p$, $\mu > 0$ *and* $0 < \lambda < \lambda_1$.
(2) $p < r < p^*$, $\mu > 0$ *and* λ *large enough.*
(3) $p < r < p^*$, $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$ *and* $\mu > 0, \lambda > 0$.

Then $(\mathrm{P}_{\lambda,\mu})$ *has also a changing-sign solution* u *that satisfies*

$$\int_\Omega (\lambda |u|^{r-p} + \mu \frac{|u|^{p^*(s)-p}}{|x|^s}) v(u)^{p-1} u = 0,$$

where $v(u)$ *is the first eigenvector of the (weighted) eigenvalue problem*

$$\begin{cases} -\triangle_p v = \nu(\lambda |u|^{r-p} + \mu \frac{|u|^{p^*(s)-p}}{|x|^s})|v|^{p-2}v \quad \text{in} \quad \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Proof. We assume without loss of generality that $\mu = 1$. Theorem 5.2 asserts that for any $q < p^*(s)$, any one of the 3 conditions yields that the closed set

$$M_2^q = M_1^q \cap \{u \in H_0^{1,p}(\Omega);\ \int_\Omega (\lambda |u|^{r-p} + \frac{|u|^{q-p}}{|x|^s}) v(u)^{p-1} u = 0\}.$$

is dual to the class $\mathcal{F}_2$, and that

$$\inf_{M_2^q} E_{\lambda,q} = c_{2,q} := c(E_{\lambda,q}, \mathcal{F}_2),$$

where for each $q < q^*(s)$, the sets M_1^q (resp. M_2^q) denote the dual sets associated to the functional

$$E_{\lambda,q} := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{q}\int_\Omega \frac{|u|^q}{|x|^s} - \frac{\lambda}{r}\int_\Omega |u|^r.$$

E_λ will denote $E_{\lambda,p^*(s)}$ and M_1 (resp. M_2) will denote $M_1^{p^*(s)}$ (resp. $M_2^{p^*(s)}$). Note that by Theorem 5.2, M_1 is dual to $\mathcal{F}_1$, but the same cannot be said about M_2 and $\mathcal{F}_2$ unless $p = 2$. Therefore, to establish the theorem above, we shall resort to a limiting argument as $q \to p^*(s)$.

Lemma 9.1. *Under any one of the 3 conditions in Theorem 9.1, we have:*

(1) $c_{i,q} \to c_i$ $(i = 1, 2)$ *as* $q \to p^*(s)$.
(2) *There exist* $\sigma > 0$ *and* $\delta_0 > 0$ *such that for* $0 < |q - p^*(s)| < \delta_0$, *we have* $c_{2,q} \leq c_{1,q} + \frac{1}{n} S^{\frac{n}{p}} - \sigma$.

Proof. (1) First we prove that $(c_{1,q})_q$ and $(c_{2,q})_q$ are uniformly bounded in q. We shall only show it for $c_{2,q}$. For any $u \in M_2^q$,

$$\int_\Omega |\nabla u|^p - \int_\Omega \frac{|u|^q}{|x|^s} - \lambda \int_\Omega |u|^r = 0.$$

Thus

$$E_{\lambda,q}(u) = \lambda(\frac{1}{p} - \frac{1}{r}) \int_\Omega |u|^r + (\frac{1}{p} - \frac{1}{q}) \int_\Omega \frac{|u|^q}{|x|^s} dx \geq 0,$$

i.e., $c_{2,q} \geq \inf_{M_2^q} E_{\lambda,q} \geq 0$. Since now

$$\begin{aligned} E_{\lambda,q}(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{1}{q} \int_\Omega \frac{|u|^q}{|x|^s} - \frac{\lambda}{r} \int_\Omega |u|^r \\ &\leq \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda}{r} \int_\Omega |u|^r \equiv E(u), \end{aligned}$$

and for any $u_0,\ v_0 \in H_0^{1,p}(\Omega)$,

$$\begin{aligned} &\lim_{\alpha\to\infty,\beta\to\infty} E(\alpha u_0 + \beta v_0) \\ &\quad = \lim_{\alpha\to\infty,\beta\to\infty} (\frac{1}{p} \int_\Omega |\nabla(\alpha u_0 + \beta v_0)|^p - \frac{\lambda}{r} \int_\Omega |\alpha u_0 + \beta v_0|^r) = -\infty, \end{aligned}$$

$E(\alpha u_0 + \beta v_0)$ attains its maximum at some finite α_0 and β_0. This means

$$0 \leq c_{2,q} \leq E(\alpha_0 u_0 + \beta_0 v_0),$$

which is independent of q.

This implies the existence of constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \leq \int_\Omega \frac{|u_{1,q}|^q}{|x|^s} \leq C_2.$$

Similar estimates also hold for $\|\nabla u_{1,q}\|_p$ and $\|u_{1,q}\|_r$. Notice that for every $u \neq 0$, there exist unique $t_q(u) > 0$ and $t(u) > 0$ such that

$$t(u)u \in M_1 \quad \text{and} \quad t_q(u)u \in M_1^q.$$

Furthermore, $t_q(u) \to t(u)$ as $q \to p^*(s)$. Set $s_q = t(u_{1,q})$ so that $s_q u_{1,q} \in M_1$. We have

$$\begin{aligned} c_1 &\leq E_\lambda(s_q u_{1,q}) \\ &= \frac{1}{p} \int_\Omega |\nabla s_q u_{1,q}|^p - \frac{\lambda}{r} \int_\Omega |s_q u_{1,q}|^r - \frac{1}{p^*(s)} \int_\Omega \frac{|s_q u_{1,q}|^{p^*(s)}}{|x|^s} \\ &= (\frac{1}{p} - \frac{1}{p^*(s)}) \int_\Omega |\nabla s_q u_{1,q}|^p + \lambda(\frac{1}{p^*(s)} - \frac{1}{r}) \int_\Omega |s_q u_{1,q}|^r. \end{aligned}$$

Since $u_{1,q} \in M_1^q$, we have

$$\begin{aligned} E_{\lambda,q}(u_{1,q}) &= \frac{1}{p}\int_\Omega |\nabla u_{1,q}|^p - \frac{\lambda}{r}\int_\Omega |u_{1,q}|^r - \frac{1}{q}\int_\Omega \frac{|u_{1,q}|^q}{|x|^s} \\ &= (\frac{1}{p}-\frac{1}{q})\int_\Omega |\nabla u_{1,q}|^p + \lambda(\frac{1}{q}-\frac{1}{r})\int_\Omega |u_{1,q}|^r. \end{aligned}$$

Thus

$$\begin{aligned} c_1 &\leq E_\lambda(s_q u_{1,q}) \\ &= (\frac{1}{p}-\frac{1}{q})\int_\Omega s_q^p|\nabla u_{1,q}|^p + (\frac{1}{q}-\frac{1}{p^*(s)})\int_\Omega s_q^p|\nabla u_{1,q}|^p \\ &\quad +\lambda(\frac{1}{p^*(s)}-\frac{1}{r})\int_\Omega |s_q u_{1,q}|^r \\ &= (\frac{1}{p}-\frac{1}{q})\int_\Omega |\nabla u_{1,q}|^p + (\frac{1}{p}-\frac{1}{q})(s_q^p-1)\int_\Omega |\nabla u_{1,q}|^p \\ &\quad +(\frac{1}{q}-\frac{1}{p^*(s)})\int_\Omega s_q^p|\nabla u_{1,q}|^p + \lambda(\frac{1}{p^*(s)}-\frac{1}{r})\int_\Omega |s_q u_{1,q}|^r \\ &= E_{\lambda,q}(u_{1,q}) + (\frac{1}{p}-\frac{1}{q})(s_q^p-1)\int_\Omega |\nabla u_{1,q}|^p \\ &\quad +(\frac{1}{q}-\frac{1}{p^*(s)})\int_\Omega s_q^p|\nabla u_{1,q}|^p \\ &\quad \lambda(\frac{1}{q}-\frac{1}{r})(s_q^r-1)\int_\Omega |u_{1,q}|^r + \lambda(\frac{1}{p^*(s)}-\frac{1}{q})\int_\Omega s_q^r|u_{1,q}|^r. \end{aligned}$$

Note that $s_q \to 1$ as $q \to p^*(s)$; therefore

$$c_1 \leq c_{1,q} + o(1).$$

To obtain the reverse inequality, set $t_q = t_q(u_1) > 0$. Thus, $t_q u_1 \in M_1^q$, $t_q \to 1$ as $q \to p^*(s)$, and

$$\begin{aligned} c_{1,q} \leq E_{\lambda,q}(t_q u_1) &= E_\lambda(t_q u_1) + \frac{1}{p^*(s)}\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} - \frac{1}{q}\int_\Omega \frac{|u|^q}{|x|^s} \\ &= E_\lambda(u_1) + o(1) = c_1 + o(1). \end{aligned}$$

This completes part (1) of the lemma.

We prove part (2) by estimating $\sup_{A_\varepsilon} E_{\lambda,q}$, where $A_\varepsilon = \text{span}\{u_1, v_\varepsilon\} \in \mathcal{F}_2$ and u_1 is the first solution of the critical problem. For that, we need the smoothness of u_1; but this cannot be guaranteed unless $p = 2$. However, an easy approximation argument, and the fact that $\sup_{t\geq 0} E_\lambda(tu) \to \sup_{t\geq 0} E(tu_1)$ as $u \to u_1$ strongly, allow us to assume that u_1 has the required smoothness.

Therefore we may suppose that $u_1,\ \nabla u_1 \in L^\infty(\Omega)$. We shall consider the case where $r > p$ first.

Case (1). Assume $p < r < p^*$ and $\lambda > 0$.

For $\varepsilon > 0$ and q sufficiently close to $p^*(s)$, and by the calculus lemma,

$$
\begin{aligned}
&E_{\lambda,q}(\alpha\ u_1\ +\beta v_\varepsilon\ ) \\
&\quad= \frac{1}{p}\int_\Omega |\nabla(\alpha u_1+\beta v_\varepsilon)|^p - \frac{\lambda}{r}\int_\Omega |\alpha u_1+\beta v_\varepsilon|^r - \frac{1}{q}\int_\Omega \frac{|\alpha u_1+\beta v_\varepsilon|^q}{|x|^s} \\
&\quad\le \frac{1}{p}\int_\Omega |\nabla(\alpha u_1)|^p - \frac{\lambda}{r}\int_\Omega |\alpha u_1|^r - \frac{1}{q}\int_\Omega \frac{|\alpha u_1|^q}{|x|^s} \\
&\qquad+\ \frac{1}{p}\int_\Omega |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{r}\int_\Omega |\beta v_\varepsilon|^r - \frac{1}{q}\int_\Omega \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
&\qquad+\ A_1[\int_\Omega |\nabla\alpha u_1||\nabla\beta v_\varepsilon|^{p-1} + |\nabla\alpha u_1|^{p-1}|\nabla\beta v_\varepsilon|] \\
&\qquad+\ B_1[\int_\Omega |\alpha u_1||\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1}|\beta v_\varepsilon|] \\
&\qquad+\ C_1[\int_\Omega |\alpha u_1|\frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1}\frac{|\beta v_\varepsilon|}{|x|^s}] \\
&\quad\le \frac{1}{p}\int_\Omega |\nabla(\alpha u_1)|^p - \frac{\lambda}{r}\int_\Omega |\alpha u_1|^r - \frac{1}{q}\int_\Omega \frac{|\alpha u_1|^q}{|x|^s} \\
&\qquad+\ \frac{1}{p}\int_\Omega |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{r}\int_\Omega |\beta v_\varepsilon|^r - \frac{1}{q}\int_\Omega \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
&\qquad+\ A_2(|\alpha|^p+|\beta|^p)\varepsilon^{\frac{n-p}{p(p-s)}} + B_2(|\alpha|^r+|\beta|^r)\varepsilon^{\frac{p-1}{p-s}(n-\frac{(r-1)(n-p)}{p})} \\
&\qquad+\ C_2(|\alpha|^q+|\beta|^q)\varepsilon^{\frac{n-p}{p(p-s)}}.
\end{aligned}
$$

Therefore, for ε sufficiently small,

$$\lim_{\alpha,\beta\to\infty} E_{\lambda,q}(\alpha u_1+\beta v_\varepsilon) = -\infty.$$

So we may assume that α and β are in a bounded set.

As in the study of the first solution, let us consider the function

$$g(t) = E_\lambda(tv_\varepsilon) = \frac{t^p}{p}\int_\Omega |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^r}{r}\int_\Omega |v_\varepsilon|^r$$

again. As in the previous section, we have

$$g(t_\varepsilon) \le \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} + C\varepsilon^{\frac{n-p}{p-s}} - \frac{\lambda}{r}(\frac{\mu_s}{2})^{\frac{r}{p^*(s)-p}}\int_\Omega |v_\varepsilon|^r.$$

If now $r > p^* - \frac{1}{p-1} > p^* - 1$, we have

$$\begin{aligned} E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) &\leq E_{\lambda,q}(\alpha u_1) + E_\lambda(\beta v_\varepsilon) + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\ &\quad + A_3\varepsilon^{\frac{n-p}{p(p-s)}} + B_3\varepsilon^{\frac{p-1}{p-s}(n-\frac{(r-1)(n-p)}{p})} \\ &\leq c_{1,q} + \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} + C\varepsilon^{\frac{n-p}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} \\ &\quad - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} - \frac{\lambda}{r}(\frac{\mu_s}{2})^{\frac{r}{p^*(s)-p}}\int_\Omega |v_\varepsilon|^r \\ &\quad + A_3\varepsilon^{\frac{n-p}{p(n-s)}} + B_3\varepsilon^{\frac{p-1}{p-s}(n-\frac{(r-1)(n-p)}{p})} \\ &\leq c_{1,q} + \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} + C\varepsilon^{\frac{n-p}{p-s}} + A_3\varepsilon^{\frac{n-p}{p(p-s)}} \\ &\quad + B_3\varepsilon^{\frac{p-1}{p-s}(n-\frac{(r-1)(n-p)}{p})} - C_4\varepsilon^{\frac{p-1}{p-s}(n-\frac{r(n-p)}{p})} \\ &\quad + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}. \end{aligned}$$

Choose ε small enough so that

$$C\varepsilon^{\frac{n-p}{p-s}} + A_3\varepsilon^{\frac{n-p}{p(p-s)}} + B_3\varepsilon^{\frac{p-1}{p-s}(n-\frac{(r-1)(n-p)}{p})} - C_4\varepsilon^{\frac{p-1}{p-s}(n-\frac{r(n-p)}{p})} \leq -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} < \sigma \quad \text{for} \quad 0 < |q - p^*(s)| < \delta_0.$$

Thus the case when $r > p$ is established.

Case (2). $r = p$, $p^3 - p^2 + p < n$ and $0 < \lambda < \lambda_1$.

The assumption $p^3 - p^2 + p < n$ implies that $p^2 < n$, $p > p^*(1 - \frac{1}{p})$ and $p - 1 < p^*(1 - \frac{1}{p})$. We assume that α and β are in a bounded set, and we estimate $E_\lambda(\alpha u_1 + \beta v_\varepsilon)$. Again let

$$g(t) = E_\lambda(tv_\varepsilon) = \frac{t^p}{p}\int_\Omega |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^p}{p}\int_\Omega |v_\varepsilon|^p;$$

then the maximum $g(t_\varepsilon)$ of $g(t)$ satisfies

$$g(t_\varepsilon) = \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{p(p-1)}{p-s}}).$$

Thus we have

$$
\begin{aligned}
&E_{\lambda,q}(\alpha\ u_1\ \ +\beta v_\varepsilon\ )\\
&\quad= \frac{1}{p}\int_\Omega |\nabla(\alpha u_1+\beta v_\varepsilon)|^p - \frac{\lambda}{p}\int_\Omega |\alpha u_1+\beta v_\varepsilon|^p - \frac{1}{q}\int_\Omega \frac{|\alpha u_1+\beta v_\varepsilon|^q}{|x|^s}\\
&\quad\le \frac{1}{p}\int_\Omega |\nabla(\alpha u_1)|^p - \frac{\lambda}{p}\int_\Omega |\alpha u_1|^p - \frac{1}{q}\int_\Omega \frac{|\alpha u_1|^q}{|x|^s}\\
&\qquad+\ \frac{1}{p}\int_\Omega |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{p}\int_\Omega |\beta v_\varepsilon|^p - \frac{1}{q}\int_\Omega \frac{|\beta v_\varepsilon|^q}{|x|^s}\\
&\qquad+\ A_1[\int_\Omega |\nabla\alpha u_1||\nabla\beta v_\varepsilon|^{p-1} + |\nabla\alpha u_1|^{p-1}|\nabla\beta v_\varepsilon|]\\
&\qquad+\ B_1[\int_\Omega |\alpha u_1||\beta v_\varepsilon|^{p-1} + |\alpha u_1|^{p-1}|\beta v_\varepsilon|]\\
&\qquad+\ C_1[\int_\Omega |\alpha u_1|\frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1}\frac{|\beta v_\varepsilon|}{|x|^s}]\\
&\quad\le \frac{1}{p}\int_\Omega |\nabla(\alpha u_1)|^p - \frac{\lambda}{p}\int_\Omega |\alpha u_1|^p - \frac{1}{q}\int_\Omega \frac{|\alpha u_1|^q}{|x|^s}\\
&\qquad+\ \frac{1}{p}\int_\Omega |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{p}\int_\Omega |\beta v_\varepsilon|^p - \frac{1}{q}\int_\Omega \frac{|\beta v_\varepsilon|^q}{|x|^s}\\
&\qquad+\ A_2(|\alpha|^p+|\beta|^p)\varepsilon^{\frac{n-p}{p(p-s)}} + B_2(|\alpha|^p+|\beta|^p)\varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}}\\
&\qquad+\ C_2(|\alpha|^q+|\beta|^q)\varepsilon^{\frac{n-p}{p(p-s)}}\\
&\quad\le E_{\lambda,q}(\alpha u_1) + E_\lambda(\beta v_\varepsilon) + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\\
&\qquad+\ A_3\varepsilon^{\frac{n-p}{p(p-s)}} + B_3\varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}}\\
&\quad\le c_{1,q} + \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} + C\varepsilon^{\frac{n-p}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} - O(\varepsilon^{\frac{p(p-1)}{p-s}})\\
&\qquad+\ A_3\varepsilon^{\frac{n-p}{p(n-s)}} + B_3\varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}}\\
&\quad\le c_{1,q} + \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} + C\varepsilon^{\frac{n-p}{p-s}} + A_3\varepsilon^{\frac{n-p}{p(p-s)}} + B_3\varepsilon^{\frac{(np)(p-1)}{p(p-s)}}\\
&\qquad-\ C_4\varepsilon^{\frac{p(p-1)}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}.
\end{aligned}
$$

Since $p^3 - p^2 + p < n$, we may choose ε small enough so that

$$C\varepsilon^{\frac{n-p}{p-s}} + A_3\varepsilon^{\frac{n-p}{p(p-s)}} + B_3\varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} - C_4\varepsilon^{\frac{p(p-1)}{p-s}} \le -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} < \sigma \quad\text{for}\quad 0 < |q - p^*(s)| < \delta_0.$$

The proof of the lemma is now complete. □

Proof of Theorem 9.1. In order to get the second solution, we shall consider the second solutions $u_{2,q}$ of the problems corresponding to $q < p^*(s)$ and we will find a limit as $q \to p^*(s)$. The location of $u_{2,q}$ on the dual sets M_2^q will be crucial for the compactness.

Since $c_{2,q}$ is bounded uniformly in q, there is $K > 0$ such that

$$\|\nabla u_{2,q}\|_p \leq K \text{ whenever } 0 < |q - p^*(s)| < \delta_0.$$

For $x \in \Omega$, define $(u_{2,q})^+(x) = \max\{u_{2,q}(x), 0\}$ and $(u_{2,q})^-(x) = \max\{-u_{2,q}(x), 0\}$. Since $u_{2,q} \in M_2^q$, both $(u_{2,q})^+$ and $(u_{2,q})^-(x)$ are non-zero and belong to $H_0^{1,p}(\Omega)$. In addition,

$$\|\nabla (u_{2,q})^{\pm}\| \leq K \text{ whenever } 0 < |q - p^*(s)| < \delta_0.$$

Thus, we can find q_n such that $q_n \to p^*(s)$ as $n \to +\infty, u^+, u^- \in H_0^{1,p}$ and

$$(u_{2,q_n})^{\pm} \rightharpoonup u^{\pm} \quad \text{weakly in } H_0^{1,p} \text{ as } n \to +\infty.$$

We claim that $u^+ \neq 0$ and $u^- \neq 0$. To shorten notation, set $u_n^{\pm} = (u_{2,q_n})^{\pm}, c_{1,n} = c_{1,q_n}, E_n = E_{\lambda,q_n}$ and $\Gamma_n = M_1^{q_n}$. Since u_n is the solution of the corresponding sub-critical problem, we have that $u_n^{\pm} \in \Gamma_n$. In particular,

$$E_n(u_n^{\pm}) \geq c_{1,n}.$$

From Lemma 9.1, we also know that

$$E_n(u_n^+) + E_n(u_n^-) = E_n(u_n) = c_{2,q_n} \leq c_{1,n} + \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} - \sigma$$

for n large. Necessarily,

$$E_n(u_n^{\pm}) \leq \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} - \sigma$$

for n large. From the fact that $u_n^{\pm} \in \Gamma_n$ and $c_1^n \to c_1 > 0$, we derive

$$K_1 \leq \int_\Omega \frac{|u_n^{\pm}|^{q_n}}{|x|^s} \leq K_2$$

with suitable positive constants K_1 and K_2.

Arguing by contradiction, assume, for example, that $u^+ = 0$. From the above and the fact that $u_n^{\pm} \in \Gamma_n$, we obtain

$$\frac{1}{p}\|\nabla u_n^+\|_p^p - \frac{1}{q_n}\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \leq \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} - \sigma + o(1)$$

and

$$\|\nabla u_n^+\|_p^p - \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} = o(1).$$

Consequently,

$$\begin{aligned}
&\mu_s\Big(\int_\Omega \frac{|u_n^+|^{p^*(s)}}{|x|^s}\Big)^{\frac{p}{p^*(s)}} \\
&\quad \leq \|\nabla u_n^+\|_p^p = \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} + o(1) = \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^{\frac{q_n}{p^*(s)}s}} \cdot \frac{1}{|x|^{s(1-\frac{q_n}{p^*(s)})}} \\
&\quad \leq \Big(\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s}\Big)^{\frac{q_n}{p^*(s)}}\Big(\int_\Omega \frac{1}{|x|^s}\Big)^{\frac{p^*(s)-q_n}{p^*(s)}} + o(1).
\end{aligned}$$

Since $\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s}$ is bounded away from zero, we conclude that

$$\Big(\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s}\Big)^{\frac{q_n-p}{p^*(s)}} \geq \Big(\int_\Omega \frac{1}{|x|^s}\Big)^{\frac{q_n-p^*(s)}{p^*(s)}}\mu_s + o(1).$$

That is,

$$\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \geq \mu_s^{\frac{n-s}{p-s}} + o(1).$$

Thus, we have

$$\begin{aligned}\frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} + o(1) &\leq \frac{p-s}{p(n-s)}\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s}\\ &= \frac{1}{p}\|\nabla u_n^+\|_p^p - \frac{1}{q_n}\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} + o(1) \leq \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} - \sigma + o(1).\end{aligned}$$

This is a contradiction, and we conclude that $u^+ \neq 0$. Similarly, $u^- \neq 0$.

Set $u = u^+ - u^-$; that is, u changes sign in Ω and

$$u_n := u_{q_n} \rightharpoonup u \text{ weakly in } H_0^{1,p}(\Omega).$$

So, $\langle E_\lambda'(u),\ w\rangle = 0$ for any $w \in H_0^{1,p}(\Omega)$, i.e. u is a weak solution of (P_λ). Now, we prove that a subsequence of $\{u_n\}$ converges to u strongly in $H_0^{1,p}(\Omega)$ and conclude that u is a solution of $(\mathrm{P}_{\lambda,p^*(s)})$ that is located on M_2.

Since $\{E(u_n)\}$ is bounded and $E_n'(u_n) \to 0$, we may assume that the conclusions of Lemma 4.4 hold for the sequence $(u_n)_n$.

Note that $u \in M_1$; hence $E(u) \geq c_1$. Set $u_n = u + w_n$, with $w_n \rightharpoonup 0$ weakly in $H_0^{1,p}$. We have

$$\begin{aligned}c_{1,n} \quad &+ \quad \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} - \sigma \geq E_n(u+w_n)\\ &\geq \quad \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q_n}\int_\Omega \frac{|u|^{q_n}}{|x|^s} - \frac{\lambda}{r}\|u\|_r^r + \frac{1}{p}\|\nabla w_n\|_p^p - \frac{1}{q_n}\int_\Omega \frac{|w_n|^{q_n}}{|x|^s} + o(1)\\ &\geq \quad c_1 + \frac{1}{p}\|\nabla w_n\|_p^p - \frac{1}{q_n}\frac{|w_n|^{q_n}}{|x|^s} + o(1).\end{aligned}$$

Since $|c_{1,n} - c_1| = o(1)$, we derive

$$\frac{1}{p}\|\nabla w_n\|_p^p - \frac{1}{q_n}\int_\Omega \frac{|w_n|^{q_n}}{|x|^s} \leq \frac{p-s}{p(n-s)}\mu_s^{\frac{n-s}{p-s}} - \sigma + o(1).$$

Furthermore,

$$0 = \langle E_n'(u_n), u_n\rangle = \langle E'(u), u\rangle + \|\nabla w_n\|_p^p - \int_\Omega \frac{|w_n|^{q_n}}{|x|^s} + o(1);$$

i.e.

$$\|\nabla w_n\|_p^p - \int_\Omega \frac{|w_n|^{q_n}}{|x|^s} = o(1).$$

The last two relations show that the sequence $\|\nabla w_n\|_p$ cannot be bounded away from zero, and therefore a subsequence of $\{w_n\}$ converges strongly to zero. □

10. Sobolev critical non-singular term

In this section, we prove Theorem 1.4. We reformulate it as follows.

Theorem 10.1. *Suppose $1 < p \leq q < p^*(s)$ and $r = p^*$ in the equation*

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u = \lambda |u|^{p^*-2}u + \mu\frac{|u|^{q-2}}{|x|^s}u \quad \text{in } \Omega,\\ u|_{\partial\Omega} = 0.\end{cases}$$

Then $(\mathrm{P}_{\lambda,\mu})$ has a positive solution if any one of the following conditions holds:

(1) $p < q$, $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$ *and* $\lambda > 0, \mu > 0$.
(2) $p = q$, $n \geq p^2 - (p-1)s$ *and* $\lambda > 0$, $\mu_{s,p} > \mu > 0$.

If one of the following conditions holds:

(1') $p < q$, $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$ *and* $\lambda > 0, \mu > 0$,
(2') $p = q$, $n > p((p-1)(p-s)+1)$ *and* $\lambda > 0$, $\mu_{s,p} > \mu > 0$,

then $(\mathrm{P}_{\lambda,\mu})$ *has also a sign-changing solution.*

Remark 10.1. The existence of a positive solution under condition (2) above has already been noticed in [13] in the case where $p = q$.

By scaling, we can always assume that $\lambda = 1$. The corresponding functional is again

$$E_\mu(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*} - \frac{\mu}{q}\int_\Omega \frac{|u|^q}{|x|^s}.$$

Recall that under any one of the above conditions, the set

$$M_1 = \{u \in H_0^{1,p}(\Omega);\ u \neq 0,\ \langle E'_\mu(u),\ u\rangle = 0\}$$

is dual to the class

$$\mathcal{F}_1 = \{\gamma \in C([0,1];\ H_0^{1,p}(\Omega));\ \gamma(0) = 0,\ \gamma(1) \neq 0 \text{ and } E_\mu(\gamma(1)) \leq 0\}.$$

Moreover, the energy level $c_1 = \inf_{A\in\mathcal{F}_1}\sup_{u\in A} E_\mu(u)$ is equal to $\inf_{u\in M_1} E_\mu(u)$.

By Theorem 4.1.(3), E_μ satisfies $(\mathrm{PS})_c$ for any $c < \frac{1}{n}{\mu_0}^{\frac{n}{p}}$. So, the existence of the first positive solution follows immediately from the following estimates.

Lemma 10.1. *If $r = p^*$, then $c_1 < \frac{1}{n}\mu_0^{\frac{n}{p}}$ in any one of the following three cases:*

(1) $q = p$, $0 < \mu < \mu_{s,p}$ *and* $n \geq p^2 - (p-1)s$.
(2) $p < q < p^*(s)$ *and* μ *large enough.*
(3) $p < q < p^*(s)$ *and* $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$.

Proof. Take v_ε to be the function as in Lemma 11.2 of the appendix. Then, as in the proof of Lemma 9.1, we consider:

Case $q > p$. We have

$$\begin{aligned}\max_{0\leq t<\infty} E_\mu(tv_\varepsilon) &\leq \frac{1}{n}\mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - \frac{\mu}{q}(\frac{\mu_0}{s})^{\frac{q}{p^*-p}}\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\ &= \frac{1}{n}\mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-q)}).\end{aligned}$$

where we require that $q > \frac{n-s}{n-p}(p-1)$. The estimate then follows.

Case $q = p$. We have

$$\begin{aligned}\max_{0\leq t<\infty} E_\mu(tv_\varepsilon) &= \frac{1}{n}(\int_\Omega |\nabla v_\varepsilon|^p - \frac{|v_\varepsilon|^p}{|x|^s})^{\frac{n}{p}} \\ &= \begin{cases} \frac{1}{n}\mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-p)}), & p > \frac{n-s}{n-p}(p-1), \\ \frac{1}{n}\mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{n-p}{p}}|\log\varepsilon|), & p = \frac{n-s}{n-p}(p-1). \end{cases}\end{aligned}$$

Since

$$\frac{(n-p)(p-1)}{p^2}(p^*(s)-p) = \frac{(p-s)(p-1)}{p},$$

the conclusions now follow immediately.

For the sign changing solution, we shall proceed as in the case of the Hardy-Sobolev critical exponent. First we find appropriate sign changing solutions for the sub-critical problem, i.e. when $r < p^*$, and then we pass to the limit as $r \to p^*$.

Write again

$$E_{\mu,r}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{r}\int_\Omega |u|^r - \frac{\mu}{q}\int_\Omega \frac{|u|^q}{|x|^s},$$

$$\mathcal{F}_1^r = \{\gamma \in C([0,1];\ H_0^{1,p}(\Omega));\ \gamma(0) = 0,\ \gamma(1) \neq 0 \text{ and } E_{\mu,r}(\gamma(1)) \leq 0\},$$

$$M_1^r = \{u \in H_0^{1,p}(\Omega);\ u \neq 0,\ \langle (E_{\mu,r})'(u),\ u\rangle = 0\},$$

and

$$c_{1,r} = \inf_{A\in\mathcal{F}_1}\sup_{u\in A} E_{\mu,r}(u).$$

Then, as previously, we know that M_1^r is dual to $\mathcal{F}_1^r$ and $c_{1,r} = \inf_{u\in M_1} E_\mu(u)$. Also define

$$c_{2,r} = \inf_{A\in\mathcal{F}_2}\sup_{u\in A} E_{\mu,r}(u),$$

where $\mathcal{F}_2$ is defined in section 5. We write c_2 (resp. E_μ) for c_{2,p^*} (resp E_{μ,p^*}). □

Lemma 10.2. *Under either one of the following conditions,*

(1) $p = q$, $0 < \mu < \mu_{s,p}$ *and* $n > p(p-1)(p-s) + p$,
(2) $p < q$, $\mu > 0$ *and* $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$,

we have

(i) $c_{i,r} \to c_i$ *($i = 1, 2$) as* $r \to p^*$,
(ii) $c_{2,r} \leq c_{1,r} + \frac{1}{n}\mu_0^{\frac{n}{p}} - \sigma$ *for some* $\sigma > 0$ *and* r *sufficiently close to* p^*.

Proof. For the first conclusion, the proof is exactly the same as in the last section. For the second one, we can assume that the first solution u_1 is smooth and $\nabla u_1 \in L^\infty(\Omega)$.

For $\varepsilon > 0$ and q sufficiently close to p^*, apply the calculus lemma to obtain

$$\begin{aligned} E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) \quad &\leq \quad E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) \\ &\quad + A_1[\int_\Omega |\nabla\alpha u_1||\nabla\beta v_\varepsilon|^{p-1} + |\nabla\alpha u_1|^{p-1}|\nabla\beta v_\varepsilon|] \\ &\quad + B_1[\int_\Omega |\alpha u_1||\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1}|\beta v_\varepsilon|] \\ &\quad + C_1[\int_\Omega |\alpha u_1|\frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1}\frac{|\beta v_\varepsilon|}{|x|^s}] \\ &\leq \quad E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) + \ A_2(|\alpha|^p + |\beta|^p)\varepsilon^{\frac{n-p}{p^2}} \\ &\quad + B_2(|\alpha|^r + |\beta|^r)\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\ &\quad + C_2(|\alpha|^q + |\beta|^q)\varepsilon^{\frac{n-p}{p^2}}. \end{aligned}$$

Again, we note that for ε sufficiently small,

$$\lim_{\alpha,\beta\to\infty} E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) = -\infty.$$

So we may assume that α and β are in a bounded set.

Case 1. Assume $p < q$, $0 < \mu$ and $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$.

Then, by the calculus lemma,

$$\begin{aligned}
&E_{\mu,r}(\alpha u_1+\beta v_\varepsilon)\\
&\quad\le E_{\mu,r}(\alpha u_1)+E_\mu(\beta v_\varepsilon)+\frac{|\beta|^{p^*}}{p^*}-\frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r\\
&\qquad+\ A_3\varepsilon^{\frac{n-p}{p^2}}+B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})}\\
&\quad\le c_{1,r}+\frac{1}{n}S^{\frac{n}{p}}+C\varepsilon^{\frac{n-p}{p}}+\frac{|\beta|^{p^*}}{p^*}-\frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r-\frac{\mu}{q}(\frac{S}{2})^{\frac{q}{p^*-p}}\int_\Omega\frac{|v_\varepsilon|^q}{|x|^s}\\
&\qquad+\ A_3\varepsilon^{\frac{n-p}{p^2}}+B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})}\\
&\quad\le c_{1,r}+\frac{1}{n}S^{\frac{n}{p}}+C\varepsilon^{\frac{n-p}{p}}+A_3\varepsilon^{\frac{n-p}{p^2}}+B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})}\\
&\qquad-\ C_4\varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-q)}+\frac{|\beta|^{p^*}}{p^*}-\frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r,
\end{aligned}$$

where we require $q > \frac{n-s}{n-p}(p-1)$. From

$$\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p}) > \frac{(n-p)(p-1)}{p^2}(p^*(s)-q)$$

we get $q > r-1-\frac{ns}{n-p}$. From

$$\frac{n-p}{p^2} > \frac{(n-p)(p-1)}{p^2}(p^*(s)-q),$$

we get that $q > p^*(s)-\frac{1}{p-1}$. Since

$$p^*(s)-\frac{1}{p-1}\ge p^*(s)-1 = p^*-1-\frac{ps}{n-p} > r-1-\frac{ns}{n-p},$$

the hypothesis $q > p^*(s)-\frac{1}{p-1}$ (i.e. $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$) is sufficient to allow us to choose ε small enough so that

$$C\varepsilon^{\frac{n-p}{p}}+A_3\varepsilon^{\frac{n-p}{p^2}}+B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})}-C_4\varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-q)}\le -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*}}{p^*}-\frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r < \sigma \quad\text{for}\quad 0<|r-p^*|<\delta_0.$$

Thus we have proved the case for $q > p$.

Case 2. $q = p$, $n > p(p-1)(p-s)+p$ and $0<\mu<\mu_{s,p}$.

We assume that α and β are in a bounded set, and we estimate $E_\lambda(\alpha u_1 + \beta v_\varepsilon)$:

$$\begin{aligned}
E_{\mu,r}(\alpha\ u_1 & + \beta v_\varepsilon\ ) \\
&\le E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) \\
&\quad + A_1[\int_\Omega |\nabla \alpha u_1||\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1}|\nabla \beta v_\varepsilon|] \\
&\quad + B_1[\int_\Omega |\alpha u_1|\frac{|\beta v_\varepsilon|^{p-1}}{|x|^s} + |\alpha u_1|^{p-1}\frac{|\beta v_\varepsilon|}{|x|^s}] \\
&\quad + C_1[\int_\Omega |\alpha u_1||\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1}|\beta v_\varepsilon|] \\
&\le E_{\mu,r}(\alpha u_1) + E_\mu(\beta v_\varepsilon) + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r \\
&\quad + A_3\varepsilon^{\frac{n-p}{p^2}} + B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
&\le c_{1,r} + \frac{1}{n}S^{\frac{n}{p}} + C\varepsilon^{\frac{n-p}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r - O(\varepsilon^{\frac{(p-s)(p-1)}{p}}) \\
&\quad + A_3\varepsilon^{\frac{n-p}{p^2}} + B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
&\le c_{1,r} + \frac{1}{n}S^{\frac{n}{p}} + C\varepsilon^{\frac{n-p}{p}} + A_3\varepsilon^{\frac{n-p}{p^2}} + B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
&\quad - C_4\varepsilon^{\frac{(p-s)(p-1)}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r.
\end{aligned}$$

Note that we have required that $p > \frac{n-s}{n-p}(p-1)$. By the assumption, we can choose ε small enough so that

$$C\varepsilon^{\frac{n-p}{p}} + A_3\varepsilon^{\frac{n-p}{p^2}} + B_3\varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} - C_4\varepsilon^{\frac{(p-s)(p-1)}{p}} \le -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r}\int_\Omega |v_\varepsilon|^r < \sigma \quad \text{for} \quad 0 < |q - p^*| < \delta_0.$$

The proof of this lemma is now complete. □

The rest of the proof of the theorem is now very similar to Theorem 9.1. The details are left for the interested reader. □

11. Appendix: Estimates on the extremal Sobolev-Hardy functions

Assume, without loss of generality, that $0 \in \Omega$, and let

$$U_\varepsilon(x) = (\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}}.$$

$U_\varepsilon(x)$ is a function in $H^{1,p}(\mathbf{R}^n)$ where the best constant in the Sobolev-Hardy inequality is attained. They are, modulo translation and dilations, the unique positive ones where the best constant is achieved. (See section 2.)

Let $0 \le \phi(x) \le 1$ be a function in $C_0^\infty(\Omega)$ defined as

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \le R, \\ 0 & \text{if } |x| \ge 2R, \end{cases}$$

where $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \phi(x)U_\varepsilon(x)$. For $\varepsilon \to 0$, the behavior of u_ε has to be the same as that of U_ε but we need precise estimates of the error terms.

Lemma 11.1. *Assume $0 \le s < p$, $p \ge 2$ and $q = \frac{n-s}{n-p}p$. By taking*

$$v_\varepsilon = \frac{u_\varepsilon}{(\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s})^{\frac{1}{q}}}$$

so that $\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} = 1$, we have the following estimates:

(1) $\|\nabla v_\varepsilon\|_p^p = \mu_s + O(\varepsilon^{\frac{n-p}{p-s}})$,

(2) $\int_\Omega |\nabla v_\varepsilon|^\alpha = O(\varepsilon^{\frac{(n-p)\alpha}{p(p-s)}})$, *for* $\alpha = 1,\ 2,\ p-2,\ p-1$.

(3) *if* $r > p^*(1-\frac{1}{p})$, *then*

$$C_1\varepsilon^{(\frac{p-1}{p-s})(n-\frac{r(n-p)}{p})} \le \|v_\varepsilon\|_r^r \le C_2\varepsilon^{(\frac{p-1}{p-s})(n-\frac{r(n-p)}{p})},$$

(4) *if* $r = p^*(1-\frac{1}{p})$, *then*

$$C_1\varepsilon^{\frac{(n-p)r}{p(p-s)}}|\log\varepsilon| \le \|v_\varepsilon\|_r^r \le C_2\varepsilon^{\frac{(n-p)r}{p(p-s)}}|\log\varepsilon|,$$

(5) *if* $r < p^*(1-\frac{1}{p})$, *then*

$$C_1\varepsilon^{\frac{(n-p)r}{p(p-s)}} \le \|v_\varepsilon\|_r^r \le C_2\varepsilon^{\frac{(n-p)r}{p(p-s)}},$$

(6) *if* $p < r < p^*$, *then* $\|v_\varepsilon\|_r^r \to 0$ *(as* $\varepsilon \to 0$*),*

(7)

$$\int_\Omega \frac{|v_\varepsilon|^{q-1}}{|x|^s} = O(\varepsilon^{\frac{p-1}{p}(\frac{n-p}{p-s})}).$$

(8)

$$\int_\Omega \frac{|v_\varepsilon|}{|x|^s} = O(\varepsilon^{\frac{n-p}{p(p-s)}}).$$

where C_1, $C_2 > 0$ are constants.

Proof. Let

$$k(\varepsilon) = (\varepsilon\cdot(n-s)(\frac{n-p}{p-1})^{p-1})^{\frac{n-p}{p(p-s)}}.$$

Then $y_\varepsilon(x) = k(\varepsilon)U_\varepsilon(x)$ is the extremal function in the Sobolev-Hardy inequality. Furthermore,

$$k(\varepsilon)^p\|\nabla U_\varepsilon(x)\|_p^p = \|\nabla y_\varepsilon(x)\|_p^p = \mu_s^{\frac{n-s}{p-s}}$$

The gradient of $u_\varepsilon(x)$ is given by

$$\begin{aligned}\nabla u_\varepsilon(x) &= (\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}}\nabla\phi(x) + \frac{p-n}{p-1}\cdot\frac{x\phi(x)}{(\varepsilon+|x|^{\frac{p-s}{p-1}})^{\frac{n-s}{p-s}}|x|^{\frac{p-2}{p-1}}}\\ &= \begin{cases}\frac{p-n}{p-1}\cdot\frac{x}{(\varepsilon+|x|^{\frac{p-s}{p-1}})^{\frac{n-s}{p-s}}|x|^{\frac{p-2}{p-1}}} & \text{if } |x|\le R,\\ 0 & \text{if } |x| \ge 2R.\end{cases}\end{aligned}$$

Thus we have

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^p &= O(1) + \int_{|x|\le R}|\nabla U_\varepsilon(x)|^p dx = O(1) + \int_{\mathbf{R}^n}|\nabla U_\varepsilon(x)|^p dx\\ &= O(1) + \|\nabla U_\varepsilon(x)\|_p^p,\end{aligned}$$

and

$$\int_\Omega \frac{|u_\varepsilon|^p}{|x|^s} = O(1) + \int_{\mathbf{R}^n} \frac{|U_\varepsilon|^q}{|x|^s} = O(1) + \int_{\mathbf{R}^n} \frac{|y_\varepsilon|^q}{|x|^s} \cdot k^{-q}(\varepsilon) = O(k^{-q}(\varepsilon)).$$

From this we further get

$$\begin{aligned}
\|\nabla v_\varepsilon\|_p^p &= \frac{\|\nabla u_\varepsilon\|_p^p}{(\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s})^{\frac{p}{q}}} = \frac{O(1) + \|\nabla U_\varepsilon\|_p^p}{(\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s})^{\frac{p}{q}}} \\
&= \frac{O(1) + \mu_s^{\frac{n-s}{p-s}} k(\varepsilon)^{-p}}{(\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s})^{\frac{p}{q}}} = \frac{O(1) + \mu_s^{\frac{n-s}{p-s}} k(\varepsilon)^{-p}}{O(1) + k(\varepsilon)^{-p} \mu_s^{\frac{p(n-s)}{(p-s)q}}} \\
&= O(k(\varepsilon)^p) + \mu_s^{\frac{n-s}{p-s} - \frac{p(n-s)}{(p-s)q}} = \mu_s + O(\varepsilon^{\frac{n-p}{p-s}}).
\end{aligned}$$

(1) is thus proved. For (2), let ω_n denote the surface area of the $(n-1)$-sphere S^{n-1} in $\mathbf{R}^n$; then

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^\alpha &= O(1) + \int_{|x|\le R} (\frac{n-p}{p-1})^\alpha \frac{|x|^\alpha}{(\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{\alpha(n-s)}{p-s}} |x|^{\frac{(p-2)\alpha}{p-1}}} dx \\
&= O(1) + \omega_n \int_0^R (\frac{n-p}{p-1})^\alpha \frac{r^\alpha \cdot r^{n-1}}{(\varepsilon + r^{\frac{p-s}{p-1}})^{\frac{\alpha(n-s)}{p-s}} r^{\frac{(p-2)\alpha}{p-1}}} dr \\
&\le O(1) + \omega_n \int_0^R (\frac{n-p}{p-1})^\alpha r^{\alpha+n-1-\frac{\alpha(n-s)}{p-1} - \frac{\alpha(p-2)}{p-1}} dr,
\end{aligned}$$

and the order of r in the integrand is

$$\begin{aligned}
\alpha + n - 1 - \frac{\alpha(n-s)}{p-1} - \frac{\alpha(p-2)}{p-1} &= \frac{pn - p - n + 1 - \alpha n + \alpha s + \alpha}{p-1} \\
&= \frac{pn - n - \alpha n + \alpha s + \alpha}{p-1} - 1 > -1
\end{aligned}$$

for $\alpha = 1,\ p-2,\ p-1$ and $\alpha = 2$ if $p \ge 3$. Thus

$$\int_\Omega |\nabla u_\varepsilon|^\alpha = O(1),$$

and we conclude that

$$\int_\Omega |\nabla v_\varepsilon|^\alpha = O(\varepsilon^{\frac{(n-p)\alpha}{p(p-s)}}).$$

For (3), (4) and (5),

$$\begin{aligned}
\|u_\varepsilon\|_r^r &= O(1) + \omega_n \int_0^R (\varepsilon + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s} r} x^{n-1} dx \\
&= O(1) + \omega_n \varepsilon^{-\frac{n-p}{p-s} r + \frac{p-1}{p-s} n} \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s} r} x^{n-1} dx
\end{aligned}$$

If $r = p^*(1-\frac{1}{p})$, then $-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n = 0$, and

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} \frac{1}{x} dx \\ &= O(1) + O(|\log \varepsilon|).\end{aligned}$$

So we get

$$\|v_\varepsilon\|_r^r = O(|\log \varepsilon| \varepsilon^{\frac{n-p}{p(p-s)} \cdot r}).$$

If $r < p^*(1-\frac{1}{p})$, then $-\frac{n-p}{p-1}r + n - 1 > -1$. We conclude that

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \int_0^R (\varepsilon + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &\leq O(1) + \omega_n \int_0^R x^{-\frac{n-p}{p-1}r + n - 1} dx = O(1)\end{aligned}$$

and

$$\|v_\varepsilon\|_r^r = O(\varepsilon^{\frac{n-p}{p(p-s)}r}).$$

If $r > p^*(1-\frac{1}{p})$, then $-\frac{n-p}{p-1}r + n - 1 < -1$ and $-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n < 0$. We have

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n} \int_1^\infty (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &= O(\varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n}),\end{aligned}$$

and

$$\|v_\varepsilon\|_r^r = O(\varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n + \frac{n-p}{p(p-s)}r}) = O(\varepsilon^{\frac{p-1}{p-s}(n - \frac{r(n-p)}{p})}).$$

(3), (4) and (5) are thus proved.

For (7) and (8), we have

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^s} dx &= O(1) + \int_{|x| \leq R} (\varepsilon + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} |x|^{-s} dx \\ &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{-s} \cdot r^{n-1} dr \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} \varepsilon^{-\frac{n-p}{p-s}\alpha + (n-s)\frac{p-1}{p-s}} dr.\end{aligned}$$

If $\alpha = q - 1$, then $-\frac{n-p}{p-s}\alpha + (n-s)\frac{p-1}{p-s} = -1$. We have

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^s} dx &= O(1) + O(\varepsilon^{-1}) \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} dr \\ &= O(1) + O(\varepsilon^{-1}) \omega_n \int_0^\infty (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} dr = O(\varepsilon^{-1}),\end{aligned}$$

since $n - s - \frac{n-p}{p-1}(q-1) = \frac{s-p}{p-1} < 0$. Then

$$\int_\Omega \frac{|v_\varepsilon|^\alpha}{|x|^s} dx = O(\varepsilon^{-1}) \cdot \varepsilon^{\frac{n-p}{p(p-s)}(q-1)} = O(\varepsilon^{\frac{p-1}{p}(\frac{n-p}{p-s})}).$$

If $\alpha = 1$, since $-\frac{n-p}{p-1} + n - s > -\frac{n-p}{p-1} + n - p = (n-p)(1 - \frac{1}{p-1}) \geq 0$ for $p \geq 2$, we have

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|}{|x|^s} dx &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} r^{n-s-1} dr \\ &\leq O(1) + \omega \int_0^R r^{-\frac{n-p}{p-1}+n-s-1} dr = O(1),\end{aligned}$$

and furthermore

$$\int_\Omega \frac{|v_\varepsilon|}{|x|^s} dx = O(\varepsilon^{\frac{n-p}{p(p-s)}}).$$

(7) and (8) are thus proved. □

Note that the above results are well known for the extremal functions associated to the Sobolev embedding, that is, when $s = 0$. In the following lemma, we prove additional properties in the case where $s > 0$.

Lemma 11.2. *For $0 \leq t < p$, we have*

$$\int_\Omega \frac{|v_\varepsilon|^\alpha}{|x|^t} = \begin{cases} O(\varepsilon^{\frac{n-p}{p^2}\alpha} |\log \varepsilon|), & \alpha = \frac{n-t}{n-p}(p-1), \\ O(\varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(t)-\alpha)}), & \alpha > \frac{n-t}{n-p}(p-1), \\ O(\varepsilon^{\frac{n-p}{p^2}\alpha}), & \alpha < \frac{n-t}{n-p}(p-1). \end{cases}$$

Proof. As above, let ω_n denote the surface area of the $(n-1)$ sphere S^{n-1} in $\mathbf{R}^n$. Then

$$\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^t} = O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr.$$

Case 1. $\alpha = \frac{n-t}{n-p}(p-1)$ (then $\frac{p-n}{p-1}\alpha + n - t = 0$). Then

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^t} &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} (1 + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} \frac{1}{r} dr = O(|\log \varepsilon|).\end{aligned}$$

Case 2. $\alpha > \frac{n-t}{n-p}(p-1)$ (then $\frac{p-n}{p-1}\alpha + n - t - 1 < -1$). Then

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^t} &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} (1 + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \varepsilon^{\frac{p-n}{p}\alpha + (n-s)\frac{p-1}{p}} \\ &= O(1) + O(\varepsilon^{\frac{p-n}{p}\alpha + (n-s)\frac{p-1}{p}}).\end{aligned}$$

Case 3. $\alpha < \frac{n-t}{n-p}(p-1)$ (i.e. $\frac{p-n}{p-1}\alpha + n - t - 1 > -1$). Then

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^t} &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \\ &= O(1) + \omega_n \int_0^R r^{\frac{p-n}{p-1}\alpha + n-t-1} dr \\ &= O(1).\end{aligned}$$

Now, the conclusion follows immediately. □

References

[1] T. Aubin: *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, Springer-Verlag, 1982. MR **85j:**58002

[2] G. Bliss: *An Integral Inequality*, J. London Math. Soc., 5 (1930) 40-46.

[3] L. Boccardo and F. Murat: *Almost Everywhere Convergence of the Gradients of Solutions to Elliptic and Parabolic Equations,* Nonlinear Analysis, TMA 19, (1992) 581-597. MR **93h:**35061

[4] H. Brezis, E. Lieb: *A Relation Between Point Convergence of Functions and Convergence of Functionals*, Proc. Amer. Math. Soc. 88(1983), 486-490. MR **84c:**28003

[5] H. Brezis and L. Nirenberg: *Positive Solutions of Nonlinear Elliptic Equations involving Critical Exponents*, Communications on Pure and Applied Mathematics, 34 (1983), 437-477. MR **84h:**35059

[6] H. Brezis and L. Nirenberg: *A Minimization Problem with Critical Exponent and Non-zero Data*, in *"Symmetry in Nature"*, Scuola Norm. Sup. Pisa (1989), 129-140.

[7] L. Caffarelli, B. Gidas and J. Spruck: *Asymptotic Symmetry and Local Behavior of Semi-linear Elliptic Equations with Critical Sobolev Growth*, Comm. Pure Appl. Math. 42 (1989) 271-477. MR **90c:**35075

[8] L. Caffarelli, R. Kohn and L. Nirenberg: *First Order Interpolation Inequality with Weights*, Compositio Math. 53 (1984) 259-275. MR **86c:**46028

[9] G. Cerami, S. Solimini, and M. Struwe: *Some Existence Results for Super-linear Elliptic Boundary Value Problems Involving Critical Exponents*, J. Functional Analysis 69, 9 (1986) 289-306. MR **88b:**35074

[10] K. S. Chou and C. W. Chu: *On the Best Constant for a Weighted Sobolev-Hardy Inequality*, J. London Math. Soc. (2) 48 (1993) 137-151. MR **94h:**46052

[11] P. Drabek: *Strongly Nonlinear Degenerated and Singular Elliptic Problems*, Pitman Research Notes in Mathematics Series 343 (1996) 112-146. MR **97k:**35091

[12] P. Drabek and S. I. Pohozaev: *Positive Solutions for the p-Laplacian: Application of the Fibering Method*, Proceeding of the Royal Society of Edinburgh, 127A (1997) 703-726. MR **98h:**35061

[13] J. P. Garcia Azorero and I. Peral Alonso: *Hardy Inequalities and Some Critical Elliptic and Parabolic Problems*, J. of Differential Equations 144 (1998) 441-476. MR **99f:**35099

[14] J. Garcia Azorero and I. Peral Alonso: *Multiplicity of Solutions for Elliptic Problems with Critical Exponent or with a Non-symmetric Term*, Trans. Amer. Math. Soc. 323 (1991), 877-895. MR **99g:**35108

[15] J. Garcia Azorero and I. Peral Alonso: *Some Results about the Existence of a Second Positive Solution in a Quasi-linear Critical Problem*, Indiana University Math. Journal 43 (1994), 941–957. MR **95i:**35093

[16] J. Garcia Azorero and I. Peral Alonso: *Existence and Non-uniqueness for the p-Laplacian: Nonlinear Eigenvalues.* Commun. in PDE, 12 (1987), 1389-1430. MR **89e:**35058

[17] N. Ghoussoub: *Duality and Perturbation Methods in Critical Point Theory*, Cambridge University Press (1993). MR **95a:**58021

[18] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag (1983). MR **86c:**35035

[19] M. Guedda, L. Veron: *Local and Global Properties of Solutions of Quasi-linear Elliptic Equations*, Journal of Differential Equations 76 (1988) 159-189. MR **89k:**35078

[20] M. Guedda, L. Veron: *Quasi-linear Elliptic Equations Involving Critical Sobolev Exponents*, Nonlinear Anal TMA, 13 (8) (1989) 879-902. MR **90h:**35100

[21] E. H. Lieb: *Sharp Constants in the Hardy-Littlewood-Sobolev and Related Inequalities*, Ann. of Math. 118 (1983) 349-374. MR **86i:**42010

[22] E. H. Lieb and M. Loss: *Analysis*, Graduate Studies in Mathematics Vol. 14 Amer. Math. Soc., Providence, RI, (1997). MR **98b:**00004

[23] M. Otani: *Existence and Nonexistence of Nontrivial Solutions of Some Nonlinear Degenerated Elliptic Equations*, Journal of Functional Analysis 76 (1988) 140-159. MR **89c:**35067

[24] E. S. Noussair, C. A. Swanson, J. Yang: *Quasi-linear Elliptic Problems with Critical Exponents*, Nonlin. Anal. 20 (1993) 285-301. MR **94b:**35106

[25] I. Peral: *Multiplicity of Solutions for the p-Laplacian*, Lecture Notes at the Second School on Nonlinear Functional Analysis and Applications to Differential Equations at ICTP of Trieste (April 21 - May 9, 1997).

[26] M. Struwe: *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Second Edition, Springer-Verlag (1996). MR **98f:**49002

[27] G. Talenti: *Best Constant in Sobolev Inequalities*, Ann. Math. Pura Appl. 110 (1976), 353–372. MR **57:**3846

[28] G. Tarantello: *Nodal Solutions of Semi-linear Elliptic Equations with Critical Exponent*, Differential and Integral Equations, 5 (1992) 25-42. MR **92k:**35109

[29] P. Tolksdorf: *Regularity for a More General Class of Quasi-linear Elliptic Equations*, Journal of Differential Equations 51 (1984) 126-150. MR **85g:**35047

[30] J. Yang: *Positive Solutions of Quasi-linear Elliptic Obstacle Problems with Critical Exponents*, Nonlinear Analysis, TMA 25 (1995) 1283-1306. MR **96m:**35120

[31] C. Yuan: *On Non-homogeneous Quasi-linear PDEs Involving the p-Laplacian and the Critical Sobolev Exponent*, Ph.D. Dissertation, UBC 1998.

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