# MULTIPLE STRATONOVICH INTEGRAL AND HU-MEYER FORMULA FOR LÉVY PROCESSES 

By Mercè Farré, Maria Jolis ${ }^{1}$ and Frederic Utzet ${ }^{1}$<br>Universitat Autònoma de Barcelona

In the framework of vector measures and the combinatorial approach to stochastic multiple integral introduced by Rota and Wallstrom [Ann. Probab. 25 (1997) 1257-1283], we present an Itô multiple integral and a Stratonovich multiple integral with respect to a Lévy process with finite moments up to a convenient order. In such a framework, the Stratonovich multiple integral is an integral with respect to a product random measure whereas the Itô multiple integral corresponds to integrate with respect to a random measure that gives zero mass to the diagonal sets. A general $\mathrm{Hu}-M e y e r$ formula that gives the relationship between both integrals is proved. As particular cases, the classical $\mathrm{Hu}-$ Meyer formulas for the Brownian motion and for the Poisson process are deduced. Furthermore, a pathwise interpretation for the multiple integrals with respect to a subordinator is given.

1. Introduction. Let $W=\left\{W_{t}, t \geq 0\right\}$ be a standard Brownian motion. Itô [10] defined the multiple stochastic integral of a function $f \in L^{2}\left(\mathbb{R}_{+}^{n}, \mathcal{B}\left(\mathbb{R}_{+}^{n}\right)\right.$, $\left.(d t)^{\otimes n}\right)$,

$$
I_{n}(f)=\int \cdots \int_{\mathbb{R}_{+}^{n}} f\left(t_{1}, \ldots, d t_{n}\right) d W_{t_{1}} \cdots d W_{t_{n}}
$$

taking care to ensure that the diagonal sets, like $\left\{\left(s_{1}, \ldots, s_{n}\right) \in \mathbb{R}_{+}^{n}, s_{1}=s_{2}\right\}$, do not contribute at all. For this reason the integral has very good properties and is easy to work with. However, for a function of the form

$$
\left(g_{1} \otimes \cdots \otimes g_{n}\right)\left(t_{1}, \ldots, t_{n}\right):=g\left(t_{1}\right) \cdots g\left(t_{n}\right)
$$

we have that, in general,

$$
I_{n}\left(g_{1} \otimes \cdots \otimes g_{n}\right) \neq I_{1}\left(g_{1}\right) \cdots I_{1}\left(g_{n}\right)
$$

That means the Itô multiple integral does not behave like the integral with respect to a product measure.

Many years later, Hu and Meyer [8] introduced (although they believed that this integral was already known [8], page 75) a multiple integral, $I_{n}^{S}(f)$, which

[^0]followed the ordinary rules of multiple integration. They called it the multiple Stratonovich integral. Furthermore, Hu and Meyer stated the relationship between the Itô and Stratonovich integrals, the celebrated Hu-Meyer formula, adding the contribution of the diagonals to the Itô integral: for a function $f\left(t_{1}, \ldots, t_{n}\right)$ symmetric with good properties,
$$
I_{n}^{S}(f)=\sum_{j=0}^{[n / 2]} \frac{n!}{(n-2 j)!j!2^{j}} I_{n-2 j}\left(\int_{\mathbb{R}_{+}^{j}} f\left(\cdot, t_{1}, t_{1}, t_{2}, t_{2}, \ldots, t_{j}, t_{j}\right) d t_{1} \cdots d t_{j}\right)
$$

This formula is simple because the quadratic variation of the Brownian motion is $t$, and the integral over coincidences of order three or superior are zero. Following their ideas, Solé and Utzet [28] proved a Hu-Meyer formula for the Poisson process. Again, in that case, the formula is relatively simple because the variations of any order of the process can always be written in terms of the Poisson process and $t$.

From another point of view, Engel [7], working with a general process with independent increments, related the (Itô) multiple stochastic integral with the theory of vector valued measures, and Masani [16], using also vector valued measures and starting from the Wiener's original ideas, developed both the Itô and Stratonovich integrals (with respect to the Brownian motion) and proved many profound results. The vector measures approach is no simple matter; Engel's work covers 82 pages, and Masani's covers 160. An important and clarifying contribution was made by Rota and Wallstrom [24] who used combinatorial techniques to show the features of the multiple stochastic integration. They did not really work with integrals, but with products of vector measures. However, the path towards a general theory of multiple stochastic integration had been laid. See also Pérez-Abreu [22] for an interesting generalization to Hilbert space valued random measures. Further, Vershik and Tsilevich [30], in a more algebraic context, constructed a Fock factorization for a Lévy process, and some important subspaces can be described through Rota and Wallstrom concepts. We should also mention the very complete survey by Peccati and Taqqu [21] in which a unified study of multiple integrals, moments, cumulants and diagram formulas, as well as applications to some new central limit theorems, is presented.

It is worth remarking that Rota and Walstrom's [24] combinatorial approach to multiple integration has been extended to the context of free probability in a very interesting and fertile field of research, started by Anshelevich (see [1-5] and the references therein). In fact, Rota and Walstrom's ideas fit very well with the combinatorics of free probability (see Nica and Speicher [19]) and noncommutative Lévy processes. Our renewed interest in Rota and Walstrom's paper [24] was motivated by Anshelevich's work.

In the present paper we use the powerful Rota and Wallstrom's [24] combinatorial machinery to study the Stratonovich integral (the integral with respect to the product random measure) with respect to a Lévy processes with finite moments up
to a convenient order. The key point is to understand how the product of stochastic measures works on the diagonal sets, and that leads to the diagonal measures defined by Rota and Wallstrom [24]. For a Lévy process those measures are related to the powers of the jumps of the process, and hence to a family of martingales introduced by Nualart and Schoutens [20], called Teugels martingales, which offer excellent properties. Specifically, these martingales have deterministic predictable quadratic variation and this makes it possible to easily construct an Itô multiple stochastic integral with respect to different integrators, which can be interpreted as an integral with respect to a random measure that gives zero mass to the diagonal sets. With all these ingredients we prove a general Hu-Meyer formula. The paper uses arduous combinatorics because of our need to work with stochastic multiple integrals with respect to the different powers of the jumps of the process, and such integrals can be conveniently handled through the lattice of the partitions of a finite set.

As in the Brownian case (see, e.g., [9, 12, 16, 27]), there are alternative methods to construct a multiple Stratonovich integral based on approximation procedures, and it is possible to relax the conditions on the integrator process by assuming more regularity on the integrand function. Such regularity is usually expressed in terms of the existence of traces of the function in a convenient sense. The advantage of using Lévy processes with finite moments lies in the fact that simple $L^{2}(\Omega)$ estimates for the multiple stochastic integral of simple functions can be obtained, and then the multiple Stratonovich integral can be defined in an $L^{2}$ space with respect to a measure that controls the behavior of the functions on the diagonal sets. In this way, the problem of providing a manageable definition of the traces is avoided.

We would like to comment that an impressive body of work on multiple stochastic integrals with respect to Lévy processes has been done by Kallenberg, Kwapien, Krakowiak, Rosinski, Szulga, Woyczinski and many others (see [13-15, 23] and the references therein). However, their approach is very different from ours, and assumes different settings to those used in this work. For this reason, we have only used a few results by those authors.

The paper is organized as follows. In Section 2 we review some combinatorics concepts and the basics of the stochastic measures as vector valued measures. In Section 3 we introduce the random measures induced by a Lévy process, and we identify the diagonal measures in such a case. In Section 4 we study the relationship between the product and Itô measures of a set, and we obtain a Hu-Meyer formula for measures. In Section 5 we define the multiple Itô stochastic integral and the multiple Stratonovich integral and also prove the general Hu-Meyer formula for integrals. In Section 6, as particular cases, we deduce the classical HuMeyer formulas for the Brownian motion and for the Poisson process. We also study the case where the Lévy process is a subordinator, and prove that both the multiple Itô stochastic integral and the multiple Stratonovich integral can be computed in a pathwise sense. Finally, in order to make the paper lighter, some of the combinatorial results are included as an Appendix.

## 2. Preliminaries.

2.1. Partitions of a finite set. We need some notation of the combinatorics of the partitions of a finite set; for details we refer to Stanley [29], Chapter 3, or Rota and Wallstrom [24].

Let $F$ be a finite set. A partition of $F$ is a family $\pi=\left\{B_{1}, \ldots, B_{m}\right\}$ of nonvoid subsets of $F$, pairwise disjoint, such that $F=\bigcup_{i=1}^{m} B_{i}$. The elements $B_{1}, \ldots, B_{m}$ are called the blocks of the partition. Denote by $\Pi(F)$ the set of all partitions of $F$, and write $\Pi_{n}$ for $\Pi(\{1, \ldots, n\})$. Given $\sigma, \pi \in \Pi(F)$, we write $\sigma \leq \pi$ if each block of $\sigma$ is contained in some block of $\pi$; we then say that $\sigma$ is a refinement of $\pi$. This relationship defines a partial order that is called the reversed refinement order, and it makes $\Pi(F)$ a lattice. We write $\widehat{0}=\{\{x\}, x \in F\}$, which is the minimal element, and $\widehat{1}=\{F\}$ the maximal one.

We say that a partition $\pi \in \Pi(F)$ is of type $\left(1^{r_{1}} 2^{r_{2}} \cdots n^{r_{n}}\right)$ if $\pi$ has exactly $r_{1}$ blocks with 1 element, exactly $r_{2}$ blocks with 2 elements, and so on. In the same way, for $\sigma \leq \pi, \# \sigma=m$ and $\# \pi=k$, we say that the segment $[\sigma, \pi]$ is of type ( $1^{r_{1}} 2^{r_{2}} \cdots m^{r_{m}}$ ) if there are exactly $r_{1}$ blocks of $\pi$ in $\sigma$; there are exactly $r_{2}$ blocks of $\pi$ that each one gives rise to 2 blocks of $\sigma$, etc. Necessarily,

$$
\sum_{j=1}^{m} r_{j}=k \quad \text { and } \quad \sum_{j=1}^{m} j r_{j}=m
$$

In that situation, the Möbius function of $[\sigma, \pi]$ is

$$
\mu(\sigma, \pi)=(-1)^{m-k}(2!)^{r_{3}} \cdots((m-1)!)^{r_{m}}
$$

We use the Möbius inversion formula, that in the context of the lattice of the partitions of a finite set, says that for two functions $f, g: \Pi(F) \longrightarrow \mathbb{R}$,

$$
g(\sigma)=\sum_{\pi \geq \sigma} f(\pi) \quad \forall \sigma \in \Pi(F)
$$

if and only if

$$
\begin{equation*}
f(\sigma)=\sum_{\pi \geq \sigma} \mu(\sigma, \pi) g(\pi) \quad \forall \sigma \in \Pi(F) \tag{1}
\end{equation*}
$$

(see [29], Proposition 3.7.2).
2.2. Diagonal sets induced by a partition. As we commented in the Introduction, we will introduce two random measures on a $n$-dimensional space, and the diagonal sets will play an essential role. Diagonal sets can be conveniently described through the partitions of the set $\{1, \ldots, n\}$. We use the notation introduced by Rota and Wallstorm [24].

Let $S$ be an arbitrary set, and consider $C \subset S^{n}$. Given $\pi \in \Pi_{n}$, we write $i \sim_{\pi} j$ if $i$ and $j$ belong to the same block of $\pi$. Put

$$
C_{\geq \pi}=\left\{\left(s_{1}, \ldots, s_{n}\right) \in C: s_{i}=s_{j} \text { if } i \sim_{\pi} j\right\}
$$

and

$$
C_{\pi}=\left\{\left(s_{1}, \ldots, s_{n}\right) \in C: s_{i}=s_{j} \text { if and only if } i \sim_{\pi} j\right\} .
$$

The sets $C_{\pi}$ are called diagonal sets. Note that $C_{\pi}=C \cap S_{\pi}^{n}$ and $C_{\geq \pi}=C \cap S_{\geq \pi}^{n}$.
For example, for $n=4$ and $\pi=\{\{1\},\{2\},\{3,4\}\}$, we have

$$
C_{\geq \pi}=\left\{\left(s_{1}, s_{2}, s_{3}, s_{4}\right) \in C: s_{3}=s_{4}\right\}
$$

and

$$
C_{\pi}=\left\{\left(s_{1}, s_{2}, s_{3}, s_{4}\right) \in C: s_{3}=s_{4}, s_{1} \neq s_{2}, s_{1} \neq s_{3}, s_{2} \neq s_{3}\right\} .
$$

The sets corresponding to the minimal and maximal partitions are specially important

$$
C_{\widehat{0}}=\left\{\left(s_{1}, \ldots, s_{n}\right) \in C: s_{i} \neq s_{j}, \forall i \neq j\right\}
$$

and

$$
C_{\widehat{1}}=\left\{\left(s_{1}, \ldots, s_{n}\right) \in C: s_{1}=\cdots=s_{n}\right\} .
$$

If $\sigma \neq \pi$, then

$$
\begin{equation*}
C_{\sigma} \cap C_{\pi}=\varnothing \quad \text { and } \quad\left(C_{\pi}\right)_{\sigma}=\varnothing \tag{2}
\end{equation*}
$$

The above notation $C_{\geq \pi}$ is coherent with the reversed refinement order

$$
\begin{equation*}
C_{\geq \pi}=\bigcup_{\sigma \geq \pi} C_{\sigma} \quad \text { (disjoint union). } \tag{3}
\end{equation*}
$$

In particular, $C=C_{\geq \widehat{0}}=\bigcup_{\sigma \in \Pi_{n}} C_{\sigma}$.
2.3. Random measures. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. In this paper, a random measure $\Phi$ on a measurable space $(S, \mathcal{S})$ is an $L^{2}(\Omega)$-valued $\sigma$ additive vector measure, that means, a map $\Phi: \mathcal{S} \rightarrow L^{2}(\Omega)$ such that for every sequence $\left\{A_{n}, n \geq 1\right\} \subset \mathcal{S}$, such that $A_{n} \cap A_{m}=\varnothing, n \neq m$,

$$
\Phi\left(\bigcup_{n=1}^{\infty} A_{n}\right)=\sum_{n=1}^{\infty} \Phi\left(A_{n}\right) \quad \text { convergence in } L^{2}(\Omega)
$$

The $\sigma$-additive vector measures defined on a $\sigma$-field inherit some basic properties of the ordinary measures, but not all. So, for a sake of easy reference, we write here a uniqueness property translated to our setting. The proof is the same as the one for ordinary measures.

Proposition 2.1. Let $\Phi$ and $\Psi$ be two random measures on $(S, \mathcal{S})$, and consider a family of sets $\mathcal{C} \subset \mathcal{S}$ closed under finite intersection and such that $\sigma(\mathcal{C})=\mathcal{S}$. Then

$$
\Phi=\Psi \text { on } \mathcal{C} \quad \Longrightarrow \quad \Phi=\Psi \text { on } \mathcal{S}
$$

2.4. Product and Itô stochastic measures. Assume that the measurable space $(S, \mathcal{S})$ satisfies that for every set $C \in \mathcal{S}^{\otimes n}$ and every $\pi \in \Pi_{n}$, we have $C_{\pi} \in \mathcal{S}^{\otimes n}$. As Rota and Wallstrom [24] point out, this condition is satisfied if $S$ is a Polish space and $\mathcal{S}$ its Borel $\sigma$-algebra. We extend the definition of good random measure introduced by Rota and Wallstorm [24] to a family of measures; specifically, we say that the random measures $\Phi_{1}, \ldots, \Phi_{k}$ over a measurable space $(S, \mathcal{S})$ are jointly good random measures if the finite additive product vector measure $\Phi_{1} \otimes \cdots \otimes \Phi_{k}$ defined on the product sets by

$$
\left(\Phi_{1} \otimes \cdots \otimes \Phi_{k}\right)\left(A_{1} \times \cdots \times A_{k}\right)=\prod_{j=1}^{k} \Phi_{i}\left(A_{i}\right), \quad A_{1}, \ldots, A_{k} \in \mathcal{S}
$$

can be extended to a (unique) $\sigma$-additive random measure on $\left(S^{n}, \mathcal{S}^{\otimes n}\right)$. This extension, obvious for ordinary measures, is in general not transferred to arbitrary vector measures (see Engel [7], Masani [16] and Kwapien and Woyczynski [15]).

Given a good random measure $\Phi$ (in the sense that the $n$-fold product $\Phi \otimes$ $\cdots \otimes \Phi=\Phi^{\otimes n}$ satisfies the above condition), the starting point of Rota and Wallstrom ([24], Definition 1) is to consider new random measures given by the restriction over the diagonal sets; specifically, for $\pi \in \Pi_{n}$ they define

$$
\Phi_{\pi}^{\otimes n}(C):=\Phi^{\otimes n}\left(C_{\geq \pi}\right) \quad \text { and } \quad \mathrm{St}_{\pi}^{[n]}(C):=\Phi^{\otimes n}\left(C_{\pi}\right) \quad \text { for } C \in \mathcal{S}^{\otimes n}
$$

The following definitions are the extension of these concepts to a family of random measures.

DEFINITION 2.2. Let $\Phi_{r_{1}}, \ldots, \Phi_{r_{n}}$ be jointly good random measures on $(S, \mathcal{S})$. For a partition $\pi \in \Pi_{n}$, define

$$
\begin{equation*}
\left(\Phi_{r_{1}} \otimes \cdots \otimes \Phi_{r_{n}}\right)_{\pi}(C)=\left(\Phi_{r_{1}} \otimes \cdots \otimes \Phi_{r_{n}}\right)\left(C_{\geq \pi}\right), \quad C \in \mathcal{S}^{\otimes n} \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{St}_{\pi}^{\left(r_{1}, \ldots, r_{n}\right)}(C)=\left(\Phi_{r_{1}} \otimes \cdots \otimes \Phi_{r_{n}}\right)\left(C_{\pi}\right), \quad C \in \mathcal{S}^{\otimes n} \tag{5}
\end{equation*}
$$

In agreement with the notation in Rota and Wallstrom [24], when $\Phi_{r_{1}}=\cdots=$ $\Phi_{r_{n}}=\Phi$, we simply write $\Phi_{\pi}^{\otimes n}$ for $(\Phi \otimes \cdots \otimes \Phi)_{\pi}$ and $\mathrm{St}_{\pi}^{[n]}$ for the corresponding measure given in (5). Since $C_{\geq \widehat{0}}=C$, then $\Phi_{\widehat{0}}^{\otimes n}=\Phi^{\otimes n}$, that is the product measure. The measure $\mathrm{St}_{\hat{0}}^{\left(r_{1}, \ldots, r_{n}\right)}$ is called the Itô multiple stochastic measure relative to $\Phi_{r_{1}}, \ldots, \Phi_{r_{n}}$.

As the ordinary multiple Itô integral, the Itô multiple stochastic measure gives zero mass to every diagonal set different from $C_{\widehat{0}}$ :

Proposition 2.3. Let $\pi \in \Pi_{n}$ such that $\pi>\widehat{0}$. For every $C \in \mathcal{S}^{\otimes n}$, we have

$$
\mathrm{St} \frac{r_{0}^{\left(r_{1}, \ldots, r_{n}\right)}\left(C_{\pi}\right)=0 \quad \text { a.s. } . ~}{\text {. }}
$$

Proof. From (2) we have $\left(C_{\pi}\right)_{\widehat{0}}=\varnothing$.
The basic result of Rota and Wallstrom [24], Proposition 1, is transferred to this situation:

PROPOSITION 2.4.

$$
\begin{equation*}
\left(\Phi_{r_{1}} \otimes \cdots \otimes \Phi_{r_{n}}\right)_{\pi}=\sum_{\sigma \geq \pi} \mathrm{St}_{\sigma}^{\left(r_{1}, \ldots, r_{n}\right)} \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{St}_{\pi}^{\left(r_{1}, \ldots, r_{n}\right)}=\sum_{\sigma \geq \pi} \mu(\pi, \sigma)\left(\Phi_{r_{1}} \otimes \cdots \otimes \Phi_{r_{n}}\right)_{\sigma} \tag{7}
\end{equation*}
$$

where $\mu(\pi, \sigma)$ is the Möbius function defined in Section 2.1.

Proof. The equality (6) is deduced from (3) and the definitions (4) and (5). The equality (7) follows from (6) and the Möebius inversion formula (1).
3. Random measures induced by a Lévy process. Let $X=\left\{X_{t}, t \in[0, T]\right\}$ be a Lévy process, that is, $X$ has stationary and independent increments, is continuous in probability, is cadlag and $X_{0}=0$. In all the paper we assume that $X$ has moments of all orders; however, if the interest is restricted to multiple integral up to order $n \geq 2$, then it is enough to assume that the process has moments up to order $2 n$.

Denote the Lévy measure of $X$ by $\nu$, and by $\sigma^{2}$ the variance of its Gaussian part. The existence of moments of $X_{t}$ of all orders implies that $\int_{\{|x|>1\}}|x| \nu(d x)<\infty$ and $\int_{\mathbb{R}}|x|^{n} v(d x)<\infty, \forall n \geq 2$. Write

$$
K_{1}=E\left[X_{1}\right],
$$

$$
\begin{equation*}
K_{2}=\sigma^{2}+\int_{\mathbb{R}} x^{2} v(d x) \quad \text { and } \quad K_{n}=\int_{\mathbb{R}} x^{n} v(d x)<\infty, \quad n \geq 3 \tag{8}
\end{equation*}
$$

From now on, take $S=[0, T]$ and $\mathcal{S}=\mathcal{B}([0, T])$. The basic random measure $\phi$ that we consider is the measure induced by the process $X$ itself, defined on the intervals by

$$
\begin{equation*}
\phi(] s, t])=X_{t}-X_{s}, \quad 0 \leq s \leq t \leq T, \tag{9}
\end{equation*}
$$

and extended to $\mathcal{B}([0, T])$. The measure $\phi$ is an independently scattered random measure, that is, if $A_{1}, \ldots, A_{n} \in \mathcal{B}([0, T])$ are pairwise disjoint, then $\phi\left(A_{1}\right), \ldots, \phi\left(A_{n}\right)$ are independent.

The random measures induced by the powers of the jumps of the process, $\Delta X_{t}=X_{t}-X_{t-}$, are also used. Consider the variations of the process $X$ (see

Meyer [17], page 319)

$$
\begin{align*}
& X_{t}^{(1)}=X_{t} \\
& X_{t}^{(2)}=[X, X]_{t}=\sum_{0<s \leq t}\left(\Delta X_{s}\right)^{2}+\sigma^{2} t  \tag{10}\\
& X_{t}^{(n)}=\sum_{0<s \leq t}\left(\Delta X_{s}\right)^{n}, \quad n \geq 3 .
\end{align*}
$$

The processes $X^{(1)}, \ldots, X^{(n)}, \ldots$ are Lévy processes such that

$$
\mathbb{E}\left[X_{t}^{(n)}\right]=K_{n} t \quad \forall n \geq 1
$$

So, the centered processes,

$$
Y_{t}^{(n)}=X_{t}^{(n)}-K_{n} t, \quad n \geq 1
$$

are square integrable martingales, called Teugels martingales (see Nualart and Schoutens [20]), with predictable quadratic covariation

$$
\left\langle Y^{(n)}, Y^{(m)}\right\rangle_{t}=K_{n+m} t, \quad n, m \geq 1
$$

Notation 3.1. We denote by $\phi_{n}$ the random measure induced by $X^{(n)}$, and for $n=1, \phi_{1}=\phi$ (we indistinctly use both $\phi_{1}$ and $\phi$ ). Every $\phi_{n}$ is a independently scattered random measure. For $A, B \in \mathcal{B}([0, T])$,

$$
\mathbb{E}\left[\phi_{n}(A) \phi_{m}(B)\right]=K_{n+m} \int_{A \cap B} d t+K_{n} K_{m} \int_{A} d t \int_{B} d t
$$

We stress the following property, which is the basis of all the paper, and is a consequence of Theorem 10.1.1 by Kwapien and Woyczynski [15].

THEOREM 3.2. For every $r_{1}, \ldots, r_{n} \geq 1$, the random measures $\phi_{r_{1}}, \ldots, \phi_{r_{n}}$ are jointly good random measures on $\left([0, T]^{n}, \mathcal{B}\left([0, T]^{n}\right)\right)$.
3.1. The diagonal measures. Rota and Wallstrom [24] define the diagonal measure of order $n$ of $\phi$ as the random measure on $[0, T]$ given by

$$
\begin{equation*}
\boldsymbol{\Delta}_{n}(A)=\phi^{\otimes n}\left(A_{\hat{1}}^{n}\right), \quad A \in \mathcal{B}([0, T]) . \tag{11}
\end{equation*}
$$

To identify the diagonal measures is a necessary step to study the stochastic multiple integral. In the case of a random measure generated by a Lévy process we show that the diagonal measures are the measures generated by the variations of the process.

Proposition 3.3. For every $A \in \mathcal{B}([0, T])$ and $n \geq 1$,

$$
\begin{equation*}
\boldsymbol{\Delta}_{n}(A)=\phi_{n}(A) \tag{12}
\end{equation*}
$$

where $\phi_{n}$ is the random measure induced by $X^{(n)}$.

Proof. Since both $\boldsymbol{\Delta}_{n}$ and $\phi_{n}$ are random measures, by Proposition 2.1 it is enough to check the equality for $A=(0, t]$. Consider an increasing sequence of equidistributed partitions of $[0, t]$ with the mesh going to 0 ; for example, take $t_{k}^{(m)}=t k / 2^{m}$ and let

$$
\mathcal{P}_{m}=\left\{t_{k}^{(m)}, k=0, \ldots, 2^{m}\right\}
$$

To shorten the notation, write $t_{k}$ instead of $t_{k}^{(m)}$. Consider the sets

$$
A_{m}=\left(0, t_{1}\right]^{n} \cup\left(t_{1}, t_{2}\right]^{n} \cup \cdots \cup\left(t_{2}{ }^{m}-1, t\right]^{n} .
$$

Random measures are sequentially continuous and $A_{m} \searrow(0, t]_{1}^{n}$, when $m \rightarrow \infty$, so we have that

$$
\boldsymbol{\Delta}_{n}((0, t])=\lim _{m} \sum_{k=0}^{2^{m}-1}\left(\phi\left(\left(t_{k}, t_{k+1}\right]\right)\right)^{n}=\lim _{m} \sum_{k=0}^{2^{m}-1}\left(X_{t_{k+1}}-X_{t_{k}}\right)^{n}
$$

in $L^{2}(\Omega)$. For $n=2$,

$$
\lim _{m} \sum_{k=0}^{2^{m}-1}\left(X_{t_{k+1}}-X_{t_{k}}\right)^{2}=[X, X]_{t}=\phi_{2}((0, t]) \quad \text { in probability }
$$

so the proposition is true in this case. For $n>2$, by Itô's formula,

$$
\begin{aligned}
& \sum_{k=0}^{2^{m}-1}\left(X_{t_{k+1}}-X_{t_{k}}\right)^{n} \\
& \quad=n \sum_{k=0}^{2^{m}-1} \int_{t_{k}}^{t_{k+1}}\left(X_{s-}-X_{t_{k}}\right)^{n-1} d X_{s} \\
& \quad+\frac{1}{2} n(n-1) \sum_{k=0}^{2^{m}-1} \int_{t_{k}}^{t_{k+1}}\left(X_{s}-X_{t_{k}}\right)^{n-2} d s \\
& \quad+\sum_{k=0}^{2^{m}-1} \sum_{t_{k}<s \leq t_{k+1}}\left[\left(X_{s}-X_{t_{k}}\right)^{n}-\left(X_{s-}-X_{t_{k}}\right)^{n}\right. \\
& \left.\quad-n\left(X_{s-}-X_{t_{k}}\right)^{n-1}\left(X_{s}-X_{s-}\right)\right]
\end{aligned}
$$

(a)

$$
=n \int_{0}^{t}\left(\sum_{k=0}^{2^{m}-1}\left(X_{s-}-X_{t_{k}}\right)^{n-1} \mathbf{1}_{\left(t_{k}, t_{k+1}\right]}(s)\right) d X_{s}
$$

(b)
(c)

$$
+\binom{n}{2} \int_{0}^{t}\left(\sum_{k=0}^{2^{m}-1}\left(X_{s-}-X_{t_{k}}\right)^{n-2} \mathbf{1}_{\left(t_{k}, t_{k+1}\right]}(s)\right) d[X, X]_{s}
$$

$$
+\sum_{j=3}^{n} \sum_{k=0}^{2^{m}-1} \sum_{t_{k}<s \leq t_{k+1}}\binom{n}{j}\left(X_{s-}-X_{t_{k}}\right)^{n-j}\left(\Delta X_{s}\right)^{j}
$$

For $j=3, \ldots, n-1$, the corresponding term in (c) is

$$
\begin{equation*}
\binom{n}{j} \int_{0}^{t}\left(\sum_{k=0}^{2^{m}-1}\left(X_{s-}-X_{t_{k}}\right)^{n-j} \mathbf{1}_{\left(t_{k}, t_{k+1}\right]}(s)\right) d X_{s}^{(j)} \tag{d}
\end{equation*}
$$

Hence, (a), (b) and (d) have the same structure

$$
\int_{0}^{t} H_{s}^{(m)} d Z_{s}
$$

where $H_{s}^{(m)}=\sum_{k=0}^{2^{m}-1}\left(X_{s-}-X_{t_{k}}\right)^{r} \mathbf{1}_{\left(t_{k}, t_{k+1}\right]}(s)$ is a predictable process and $Z$ is a semimartingale. Since $X_{s-}$ is left continuous,

$$
\lim _{m} H_{s}^{(m)}=0 \quad \text { a.s. }
$$

Moreover,

$$
\left|H_{s}^{(m)}\right| \leq C \sup _{0 \leq u \leq s}\left|X_{u}\right|^{r}
$$

and the process $\left\{\sup _{0 \leq u \leq s}\left|X_{u}\right|^{r}, s \in[0, t]\right\}$ is cadlag and adapted, and as a consequence, it is prelocally bounded (see pages 336 and 340 in Dellacherie and Meyer [6]). By the dominated convergence theorem for stochastic integrals (Dellacherie and Meyer [6], Theorem 14, page 338),

$$
\lim _{m} \int_{0}^{t} H_{s}^{(m)} d Z_{s}=0 \quad \text { in probability. }
$$

Finally, for $j=n$, the term in (c) is $\sum_{0<s \leq t}\left(\Delta X_{s}\right)^{n}=X_{t}^{(n)}$, and the proposition is proved.

Diagonal measures associated to a random measure of the form $\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}$ are needed. This is an extension of the previous proposition, and it is a key result for the sequel.

THEOREM 3.4. Let $r_{1}, \ldots, r_{n} \geq 1, n \geq 2$, and $A \in \mathcal{B}([0, T])$. Then

$$
\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)\left(A_{\hat{1}}^{n}\right)=\boldsymbol{\Delta}_{r_{1}+\cdots+r_{n}}(A)=\phi_{r_{1}+\cdots+r_{n}}(A) .
$$

Proof. As in the proof of the last proposition and with the same notation, it suffices to prove that for all $t>0$

$$
\lim _{m} \sum_{k=0}^{2^{m}-1}\left(X_{t_{k+1}}^{\left(r_{1}\right)}-X_{t_{k}}^{\left(r_{1}\right)}\right) \cdots\left(X_{t_{k+1}}^{\left(r_{n}\right)}-X_{t_{k}}^{\left(r_{n}\right)}\right)=\phi_{r_{1}+\cdots+r_{n}}((0, t])
$$

in probability. This convergence follows from Proposition 3.3 by polarization.
4. The Hu-Meyer formula: Measures. The Hu-Meyer formula gives the relationship between the product measure $\phi^{\otimes n}$ and the Itô stochastic measures $\mathrm{St}_{0}^{\mathrm{r}}$. In this section we obtain this formula for measures and in the next one we extend it to the corresponding integrals.

The idea of Hu-Meyer formula is the following. Given $C \in \mathcal{B}\left([0, T]^{n}\right)$, we can decompose

$$
C=\bigcup_{\sigma \in \Pi_{n}} C_{\sigma}
$$

So

$$
\phi^{\otimes n}(C)=\sum_{\sigma \in \Pi_{n}} \phi^{\otimes n}\left(C_{\sigma}\right) .
$$

Next step is to express each $\phi^{\otimes n}\left(C_{\sigma}\right)$ as a multiple Itô stochastic measure. For example, take $n=3, \sigma=\{\{1\},\{2,3\}\}$ and $C=A^{3}$. Then,

$$
A_{\sigma}^{3}=\{(s, t, t), s, t \in A, s \neq t\}
$$

and we will prove that

$$
\phi^{\otimes 3}\left(A_{\sigma}^{3}\right)=\operatorname{St}_{0}^{(1,2)}\left(A^{2}\right)
$$

That is, both the product measure and the product set on the last two variables collapse to produce a diagonal measure, and since $s \neq t$, we get an Itô measure. To handle in general this property, we need some notation.

Given a partition $\sigma \in \Pi_{n}$ with blocks $B_{1}, \ldots, B_{m}$, we can order the blocks in agreement with the minimum element of each block. When necessary, we assume that the blocks have been ordered with that procedure, and we simply say that $B_{1}, \ldots, B_{m}$ are ordered. In that situation, we write

$$
\begin{equation*}
\bar{\sigma}=\left(\# B_{1}, \ldots, \# B_{m}\right) . \tag{13}
\end{equation*}
$$

We start considering a set $C=A^{n}$, with $A \in \mathcal{B}([0, T])$, and later we extend the Hu-Meyer formula to an arbitrary set $C \in \mathcal{B}\left([0, T]^{n}\right)$.

Theorem 4.1. Let $A \in \mathcal{B}([0, T])$. Then

$$
\begin{equation*}
\phi^{\otimes n}\left(A^{n}\right)=\sum_{\sigma \in \Pi_{n}} \mathrm{St}_{\frac{\mathrm{\sigma}}{\bar{\sigma}}}^{\bar{\sigma}}\left(A^{\# \sigma}\right) \tag{14}
\end{equation*}
$$

To prove this theorem we need two lemmas. The first one is an invariance-type property of product measures under permutations. We remember some standard notation.

Notation 4.2. We denote by $\mathfrak{G}_{n}$ the set of permutations of $1, \ldots, n$. Consider $p \in \mathfrak{G}_{n}$.

1. For a partition $\sigma \in \Pi_{n}$ with blocks $B_{1}, \ldots, B_{m}$, we write $p(\sigma)$ for the partition with blocks $W_{j}=p\left(B_{j}\right)=\left\{p(i), i \in B_{j}\right\}$. Note that in general the blocks $W_{1}, \ldots, W_{m}$ are not ordered, even when $B_{1}, \ldots, B_{m}$ are.
2. For a vector $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$, we write

$$
p(\mathbf{x})=\left(x_{p(1)}, \ldots, x_{p(n)}\right)
$$

Given $C \subset \mathbb{R}^{n}$, we put

$$
p(C)=\{p(\mathbf{x}), \text { for } \mathbf{x} \in C\}
$$

LEMMA 4.3. Let $p \in \mathfrak{G}_{n}$ and $r_{1}, \ldots, r_{n} \geq 1$. Then for every $C \in \mathcal{B}\left([0, T]^{n}\right)$,

$$
\begin{equation*}
\left(\phi_{r_{p(1)}} \otimes \cdots \otimes \phi_{r_{p(n)}}\right)(p(C))=\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)(C) \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{St}_{\hat{0}}^{p(\mathbf{r})}(p(C))=\mathrm{St}_{\hat{0}}^{\mathbf{r}}(C) \tag{16}
\end{equation*}
$$

Proof. Define the vector measure

$$
\Psi(C)=\left(\phi_{r_{p(1)}} \otimes \cdots \otimes \phi_{r_{p(n)}}\right)(p(C))
$$

For $C=A_{1} \times \cdots \times A_{n}$, we have that

$$
p\left(A_{1} \times \cdots \times A_{n}\right)=A_{p(1)} \times \cdots \times A_{p(n)}
$$

and it is clear that

$$
\Psi(C)=\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)(C) .
$$

Then, equality (15) follows from Proposition 2.1.
To prove (16), first note that, by definition, the Itô stochastic measure satisfies

$$
\operatorname{St}_{\widehat{0}}^{\mathbf{r}}(C)=\operatorname{St}_{\hat{0}}^{\mathbf{r}}\left(C_{\widehat{0}}\right)
$$

Moreover $(p(C))_{\widehat{0}}=p\left(C_{\widehat{0}}\right)$. So it suffices to prove (16) for a set $C=C_{\widehat{0}}$. Denote by $\mathcal{B}_{\hat{0}}^{n}$ the $\sigma$-algebra trace of $\mathcal{B}\left([0, T]^{n}\right)$ with $[0, T]_{\hat{0}}^{n}$, which is composed by all sets $C_{\widehat{0}}$, with $C \in \mathcal{B}\left([0, T]^{n}\right)$. This $\sigma$-algebra is generated (on [0,T] $]_{\hat{0}}^{n}$ ) by the family of rectangles $A_{1} \times \cdots \times A_{n}$, with $A_{1}, \ldots, A_{n}$ pairwise disjoint. By Proposition 2.1, we only need to check (16) for this type of rectangle, and the property reduces to (15).

The next lemma is an important step in proving Theorem 4.1. To have an insight into its meaning, consider the following example: let $n=4$ and $\sigma=$ $\{\{1\},\{2\},\{3,4\}\}$. With a slight abuse of notation, we can write

$$
A_{\geq \sigma}^{4}=\{(s, t, u, u): s, t, u \in A\}=A^{2} \times A_{\widehat{1}}^{2} .
$$

By Theorem 3.4,

$$
\begin{aligned}
\left(\phi_{r_{1}} \otimes \phi_{r_{2}} \otimes \phi_{r_{3}} \otimes \phi_{r_{4}}\right)\left(A_{\geq \sigma}^{4}\right) & =\phi_{r_{1}}(A) \phi_{r_{2}}(A)\left(\phi_{r_{3}} \otimes \phi_{r_{4}}\right)\left(A_{\hat{1}}^{2}\right) \\
& =\phi_{r_{1}}(A) \phi_{r_{2}}(A) \phi_{r_{3}+r_{4}}(A) .
\end{aligned}
$$

However, if you consider $\tau=\{\{1,3\},\{2\},\{4\}\}$, even though $\tau$ and $\sigma$ have the same number of blocks with 1 element and the same number of blocks with 2 elements (they have the same type), the computation of ( $\left.\phi_{r_{1}} \otimes \phi_{r_{2}} \otimes \phi_{r_{3}} \otimes \phi_{r_{4}}\right)\left(A_{\geq \tau}^{4}\right)$ is not so straightforward. The lemma gives such computation. Its proof demands some combinatorial results and it is transferred to Appendix A.3.

LEMMA 4.4. Let $r_{1}, \ldots, r_{n} \geq 1, \sigma \in \Pi_{n}$ with blocks $B_{1}, \ldots, B_{m}$ (ordered), and $A \in \mathcal{B}([0, T])$. Then

$$
\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)\left(A_{\geq \sigma}^{n}\right)=\prod_{j=1}^{m} \phi_{\sum_{i \in B_{j}} r_{i}}(A) .
$$

Proof of Theorem 4.1. By Proposition 2.4,

$$
\phi^{\otimes n}\left(A^{n}\right)=\sum_{\sigma \in \Pi_{n}} \operatorname{St}_{\sigma}^{[n]}\left(A^{n}\right) .
$$

So it suffices to prove that

$$
\operatorname{St}_{\sigma}^{[n]}\left(A^{n}\right)=\operatorname{St}_{\widehat{0}}^{\bar{\sigma}}\left(A^{\# \sigma}\right)
$$

By the second statement in Proposition 2.4 we have

$$
\begin{equation*}
\operatorname{St}_{\sigma}^{[n]}\left(A^{n}\right)=\sum_{\pi \in[\sigma, \widehat{1}]} \mu(\sigma, \pi) \phi_{\pi}^{\otimes n}\left(A^{n}\right)=\sum_{\pi \in[\sigma, \widehat{1}]} \mu(\sigma, \pi) \phi^{\otimes n}\left(A_{\geq \pi}^{n}\right) \tag{17}
\end{equation*}
$$

By Lemma 4.4,

$$
\begin{equation*}
\phi^{\otimes n}\left(A_{\geq \pi}^{n}\right)=\prod_{V \in \pi} \phi_{\# V}(A) \tag{18}
\end{equation*}
$$

Let $B_{1}, \ldots, B_{m}$ be the blocks of $\sigma \in \Pi_{n}$ (ordered) and write

$$
\bar{\sigma}=\left(\# B_{1}, \ldots, \# B_{m}\right)=\left(s_{1}, \ldots, s_{m}\right) .
$$

The partition $\pi \in[\sigma, \widehat{1}]$, with blocks $V_{1}, \ldots, V_{k}$, induces a unique partition of $\pi^{*} \in$ $\Pi_{m}$, with blocks $W_{1}, \ldots, W_{k}$ such that

$$
V_{i}=\bigcup_{j \in W_{i}} B_{j}
$$

(see Proposition A. 1 in the Appendix). Hence, for $i=1, \ldots, k$,

$$
\phi_{\# V_{i}}(A)=\phi_{\sum_{j \in W_{i}} \# B_{j}}(A)=\phi_{\sum_{j \in W_{i}} s_{j}}(A) .
$$

Thus, from (18) and Lemma 4.4,

$$
\begin{equation*}
\phi^{\otimes n}\left(A_{\geq \pi}^{n}\right)=\prod_{W_{i} \in \pi^{*}} \phi_{\sum_{j \in W_{i}} s_{j}}(A)=\left(\phi_{s_{1}} \otimes \cdots \otimes \phi_{s_{m}}\right)\left(A_{\geq \pi^{*}}^{m}\right) \tag{19}
\end{equation*}
$$

By (17) and (19) using again the bijection between $[\sigma, \widehat{1}]$ and $\Pi_{m}$ stated in Proposition A. 1 in the Appendix, and Proposition 2.4, we obtain

$$
\begin{aligned}
\mathrm{St}_{\sigma}^{[n]}\left(A^{n}\right) & =\sum_{\pi \in[\sigma, \widehat{1}]} \mu(\sigma, \pi)\left(\phi_{s_{1}} \otimes \cdots \otimes \phi_{s_{m}}\right)\left(A_{\geq \pi^{*}}^{m}\right) \\
& =\sum_{\rho \in \Pi_{m}} \mu(\widehat{0}, \rho)\left(\phi_{s_{1}} \otimes \cdots \otimes \phi_{s_{m}}\right)\left(A_{\geq \rho}^{m}\right)=\operatorname{St}_{\hat{0}}^{\bar{\sigma}}\left(A^{\# \sigma}\right) .
\end{aligned}
$$

In order to extend the Hu -Meyer formula for a general set in $\mathcal{B}\left([0, T]^{n}\right)$, we use a set function to express for an arbitrary set the contraction from $A^{n}$ to $A^{\# \sigma}$. That is, given a partition $\sigma \in \Pi_{n}$, with blocks $B_{1}, \ldots, B_{m}$ ordered, we want to contract a set $C \in \mathcal{B}\left([0, T]^{n}\right)$ into a set of $\mathcal{B}\left([0, T]^{\# \sigma}\right)$ according to the structure of the $\sigma$-diagonal sets. With this purpose, define the function

$$
\begin{align*}
q_{\sigma}:[0, T]^{\# \sigma} & \longrightarrow[0, T]^{n}  \tag{20}\\
\left(x_{1}, \ldots, x_{m}\right) & \longrightarrow\left(y_{1}, \ldots, y_{n}\right)
\end{align*}
$$

where $y_{i}=x_{j}$, if $i \in B_{j}$. For example, if $n=4$ and $\sigma=\{\{1\},\{2,4\},\{3\}\}$,

$$
q_{\sigma}\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}, x_{2}, x_{3}, x_{2}\right)
$$

Note that

$$
q_{\sigma}^{-1}\left(A^{n}\right)=A^{\# \sigma}
$$

See Appendix A. 4 for more details.
Theorem 4.5. Let $C \in \mathcal{B}\left([0, T]^{n}\right)$. Then

$$
\begin{equation*}
\phi^{\otimes n}(C)=\sum_{\sigma \in \Pi_{n}} \mathrm{St} \stackrel{\rightharpoonup}{0}_{\bar{\sigma}}\left(q_{\sigma}^{-1}(C)\right) \tag{21}
\end{equation*}
$$

Proof. We separate the proof in two steps. In the first one, we show that it is enough to prove the theorem for a rectangle of the form

$$
C=A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}
$$

where $A_{1}, \ldots, A_{\ell}$ are pairwise disjoint. In the second step we check formula (21) for those rectangles.

First step. By Proposition 2.1, it suffices to prove the theorem for a rectangle $A_{1} \times \cdots \times A_{n}$. Since every rectangle can be written as a disjoint union of rectangles such that every two components are either equal or disjoint, we consider one of
this rectangles, $C=A_{1} \times \cdots \times A_{n}$, where for every $i, j, A_{i}=A_{j}$ or $A_{i} \cap A_{j}=\varnothing$. Now we show that the formula (21) applied to $C$ is invariant by permutations: specifically, we see that for any permutation $p \in \mathfrak{G}_{n}$

$$
\phi^{\otimes n}(p(C))=\phi^{\otimes n}(C) \quad \text { and } \quad \sum_{\sigma \in \Pi_{n}} \mathrm{St}_{\frac{\overline{0}}{\sigma}}\left(q_{\sigma}^{-1}(p(C))\right)=\sum_{\sigma \in \Pi_{n}} \mathrm{St}_{\frac{\sigma}{0}}^{\bar{\sigma}}\left(q_{\sigma}^{-1}(C)\right)
$$

The first equality is deduced from (15). For the second one, applying Proposition A.4(i), we have

$$
\mathrm{St}_{\hat{\sigma}}^{\bar{\sigma}}\left(q_{\sigma}^{-1}(p(C))\right)=\mathrm{St}_{\hat{\sigma}}^{\bar{\sigma}}\left(p_{1}^{-1}\left(q_{p(\sigma)}^{-1}(C)\right)\right)
$$

where $p_{1} \in \mathfrak{G}_{\# \sigma}$ is the permutation that gives the correct order of the blocks of $p(\sigma)$ (see the lines before Proposition A.4). By Lemma 4.3

$$
\operatorname{St} \hat{0}_{\hat{\sigma}}^{\overline{0}}\left(p_{1}^{-1}\left(q_{p(\sigma)}^{-1}(C)\right)\right)=\operatorname{St}_{0}^{p_{1}(\bar{\sigma})}\left(q_{p(\sigma)}^{-1}(C)\right)=\operatorname{St}_{\hat{0}}^{\overline{p(\sigma)}}\left(q_{p(\sigma)}^{-1}(C)\right)
$$

where the last equality is due to the fact that $p_{1}(\bar{\sigma})=\overline{p(\sigma)}$ by the definition of $p_{1}$ [see (37)]. Finally,

$$
\sum_{\sigma \in \Pi_{n}} \mathrm{St}_{\hat{0}}^{\bar{\sigma}}\left(q_{\sigma}^{-1}(C)\right)=\sum_{\sigma \in \Pi_{n}} \mathrm{St}_{\frac{0}{\overline{p(\sigma)}}}^{\overline{\mathrm{o}}}\left(q_{p(\sigma)}^{-1}(C)\right)
$$

because we are adding over all the set $\Pi_{n}=\left\{p(\sigma), \sigma \in \Pi_{n}\right\}$.
Second step. Consider

$$
C=A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}
$$

with $A_{1}, \ldots, A_{\ell}$ pairwise disjoint and $\sum_{i=1}^{\ell} r_{i}=n$. By Theorem 4.1,

$$
\begin{aligned}
\phi^{\otimes n}\left(A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}\right) & =\prod_{i=1}^{\ell} \phi^{\otimes r_{i}}\left(A_{i}^{r_{i}}\right)=\prod_{i=1}^{\ell} \sum_{\sigma_{i} \in \Pi_{r_{i}}} \mathrm{St}_{\hat{0}}^{\bar{\sigma}_{i}}\left(A_{i}^{\# \sigma_{i}}\right) \\
& =\sum_{\sigma_{1} \in \Pi_{r_{1}}, \ldots, \sigma_{\ell} \in \Pi_{r_{\ell}}} \mathrm{St}_{\hat{0}}^{\bar{\sigma}_{1}, \ldots, \bar{\sigma}_{\ell}}\left(A_{1}^{\# \sigma_{1}} \times \cdots \times A_{\ell}^{\# \sigma_{\ell}}\right)
\end{aligned}
$$

where the last equality is due to the fact that

$$
\left(A_{1}^{\# \sigma_{1}} \times \cdots \times A_{\ell}^{\# \sigma_{\ell}}\right)_{\widehat{0}}=\left(A_{1}^{\# \sigma_{1}}\right)_{\widehat{0}} \times \cdots \times\left(A_{\ell}^{\# \sigma_{\ell}}\right)_{\widehat{0}}
$$

and the definition of the Itô measure $\mathrm{St}_{\hat{0}}^{\bar{\sigma}_{1}, \ldots, \bar{\sigma}_{\ell}}$.
Let $\tau \in \Pi_{n}$ be the partition with blocks

$$
\begin{aligned}
F_{1} & =\left\{1, \ldots, r_{1}\right\} \\
F_{2} & =\left\{r_{1}+1, \ldots, r_{1}+r_{2}\right\} \\
& \vdots \\
F_{\ell} & =\left\{r_{1}+\cdots+r_{\ell-1}+1, \ldots, n\right\} .
\end{aligned}
$$

There is a bijection between the elements $\sigma \in \Pi_{n}$, with $\sigma \leq \tau$, and $\left(\sigma_{1}, \ldots, \sigma_{\ell}\right) \in$ $\Pi_{r_{1}} \times \cdots \times \Pi_{r_{\ell}}$ such that

$$
\bar{\sigma}=\left(\bar{\sigma}_{1}, \ldots, \bar{\sigma}_{\ell}\right) \quad \text { and } \quad q_{\sigma}^{-1}\left(A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}\right)=A_{1}^{\# \sigma_{1}} \times \cdots \times A_{\ell}^{\# \sigma_{\ell}}
$$

where we use equality (36) in the Appendix. Then,

$$
\begin{aligned}
\phi^{\otimes n}\left(A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}\right) & =\sum_{\sigma \in \Pi_{n}, \sigma \leq \tau} \operatorname{St} \mathrm{\sigma}_{0}^{\bar{\sigma}}\left(q_{\sigma}^{-1}\left(A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}\right)\right) \\
& =\sum_{\sigma \in \Pi_{n}} \operatorname{St}_{\frac{\sigma}{0}}^{\bar{\sigma}}\left(q_{\sigma}^{-1}\left(A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}\right)\right)
\end{aligned}
$$

where the last equality is due to the fact that if $\sigma \not \leq \tau$, then $q_{\sigma}^{-1}\left(A_{1}^{r_{1}} \times \cdots \times A_{\ell}^{r_{\ell}}\right)=$ $\varnothing$ [see (36)].
5. Multiple Itô and Stratonovich integral, and the corresponding $\mathbf{H u}$ Meyer formula. We extend Theorem 4.5 to integrals with respect to the random measures involved. We first define an Itô-type multiple integral and an integral with respect to the product measure.
5.1. Multiple Itô stochastic integral. We generalize the multiple Itô integral with respect to the Brownian motion (Itô [10]; see also [11]) to a multiple integral with respect to the Lévy processes $X^{\left(r_{1}\right)}, \ldots, X^{\left(r_{n}\right)}$. As we will prove, that integral can be interpreted as the integral with respect to the Itô stochastic measure. The ideas used to construct this integral are mainly Itô's; however, the fact that these processes (in general) are not centered obstructs the classical isometry property, being substituted by an inequality.

Write $L_{n}^{2}=L^{2}\left([0, T]^{n}, \mathcal{B}\left([0, T]^{n}\right),(d t)^{\otimes n}\right)$. Denote by $\mathcal{E}_{n}^{\text {Ito }}$ the set of the socalled Itô-elementary functions, having the form

$$
f\left(t_{1}, \ldots, t_{n}\right)=\sum_{i_{1}, \ldots, i_{n}=1}^{m} a_{i_{1}, \ldots, i_{n}} \mathbf{1}_{A_{i_{1}} \times \cdots \times A_{i_{n}}}\left(t_{1}, \ldots, t_{n}\right),
$$

where $A_{1}, \ldots, A_{m} \in \mathcal{B}([0, T])$ are pairwise disjoint, and $a_{i_{1}, \ldots, i_{n}}$ is zero if two indices are equal. It is well known (see Itô [10]) that $\mathcal{E}_{n}^{\text {Ito }}$ is dense in $L_{n}^{2}$. Consider $f \in \mathcal{E}_{n}^{\text {Ito }}$ and define the multiple Itô integral of $f$ with respect to $X^{\left(r_{1}\right)}, \ldots, X^{\left(r_{n}\right)}$ by

$$
I_{n}^{\left(r_{1}, \ldots, r_{n}\right)}(f)=\sum_{i_{1}, \ldots, i_{n}=1}^{m} a_{i_{1}, \ldots, i_{n}} \phi_{r_{1}}\left(A_{i_{1}}\right) \cdots \phi_{r_{n}}\left(A_{i_{n}}\right)
$$

Lemma 5.1. Let $f \in \mathcal{E}_{n}^{\text {Ito }}$ and $\mathbf{r}=\left(r_{1}, \ldots, r_{n}\right)$. Then

$$
\mathbb{E}\left[\left(I_{n}^{\mathbf{r}}(f)\right)^{2}\right] \leq \alpha_{\mathbf{r}} \int_{[0, T]^{n}} f^{2}\left(t_{1}, \ldots, t_{n}\right) d t_{1} \cdots d t_{n}
$$

where $\alpha_{\mathbf{r}}$ is a constant that depends on $r_{1}, \ldots, r_{n}$ but not on $f$.

Proof. The proof follows exactly the same steps as that of Theorem 4.1 in Engel [7]. The key point is that the measures $\phi_{r_{i}}$ can be written as

$$
\phi_{r_{i}}(A)=\bar{\phi}_{r_{i}}(A)+K_{r_{i}} \int_{A} d t
$$

where $\bar{\phi}_{r_{i}}$ is the centered and independently scattered random measure corresponding to $Y^{\left(r_{i}\right)}$.

The extension of the multiple Itô stochastic integral to $L_{n}^{2}$, stated below, is proved as in the Brownian case (see Itô [10]).

THEOREM 5.2. The map

$$
\begin{aligned}
I_{n}^{\mathbf{r}}: \mathcal{E}_{n}^{\mathrm{Ito}} & \longrightarrow L^{2}(\Omega), \\
f & \longrightarrow I_{n}^{\mathbf{r}}(f)
\end{aligned}
$$

can be extended to a unique linear continuous map from $L_{n}^{2}$ to $L^{2}(\Omega)$. In particular, $I_{n}^{\mathbf{r}}(f)$ satisfies the inequality

$$
\begin{equation*}
\mathbb{E}\left[\left(I_{n}^{\mathbf{r}}(f)\right)^{2}\right] \leq \alpha_{\mathbf{r}} \int_{[0, T]^{n}} f^{2}\left(t_{1}, \ldots, t_{n}\right) d t_{1} \cdots d t_{n} \tag{22}
\end{equation*}
$$

As in the Brownian case, it is useful to express the multiple integral in terms of iterated integrals of the form

$$
\int_{0}^{T}\left(\int_{0}^{t_{i_{1}}-} \cdots\left(\int_{0}^{t_{i_{n-1}}-} f\left(t_{i_{1}}, \ldots, t_{i_{n}}\right) d X_{t_{i_{n}}}^{\left(r_{i_{n}}\right)}\right) \cdots d X_{t_{i_{2}}}^{\left(r_{i_{2}}\right)}\right) d X_{t_{i_{1}}}^{\left(r_{i_{1}}\right)}
$$

where $i_{1}, \ldots, i_{n}$ is a permutation of $1, \ldots, n$. This integral is properly defined for $f \in L_{n}^{2}$. This can be checked using the decomposition of $X^{\left(r_{i}\right)}$ as a special semimartingale $X_{t}^{\left(r_{i}\right)}=K_{r_{i}} t+Y_{t}^{\left(r_{i}\right)}$, where, as we said in Section 3, $Y^{\left(r_{i}\right)}$ is a square integrable martingale with predictable quadratic variation $\left\langle Y^{\left(r_{i}\right)}, Y^{\left(r_{i}\right)}\right\rangle_{t}=K_{2 r_{i}} t$. The previous iterated integral then reduces to a linear combination of iterated integrals of type

$$
\int_{0}^{T}\left(\int_{0}^{t_{i_{1}}-} \cdots\left(\int_{0}^{t_{i_{n-1}}-} f\left(t_{1}, \ldots, t_{n}\right) d Z_{t_{i_{n}}}^{(n)}\right) \cdots d Z_{t_{i_{2}}}^{(2)}\right) d Z_{t_{i_{1}}}^{(1)}
$$

being $Z_{t}^{(j)}$ either $t$ or $Y_{t}^{\left(r_{j}\right)}$. Hence, at each iteration, the integrability condition

$$
\mathbb{E}\left[\int_{0}^{t} g^{2} d\left\langle Z^{(i)}, Z^{(i)}\right\rangle\right]<\infty
$$

of a predictable process $g$ with respect to $Z^{(i)}$ can be easily verified.
Next proposition gives the precise expression of the multiple integral as a sum of iterated integrals. Since we are integrating with respect to different processes, we need to separate the space $[0, T]^{n}$ into simplexs.

Proposition 5.3. Let $f \in L_{n}^{2}$. Then

$$
I_{n}^{\left(r_{1}, \ldots, r_{n}\right)}(f)=\sum_{p \in \mathfrak{G}_{n}} \int \cdots \int_{p\left(\Sigma_{n}\right)} f\left(t_{1}, \ldots, t_{n}\right) d X_{t_{1}}^{\left(r_{1}\right)} \cdots d X_{t_{n}}^{\left(r_{n}\right)}
$$

where $\Sigma_{n}=\left\{0<t_{1}<\cdots<t_{n}<T\right\}$, and the integrals on the right-hand side are interpreted as iterated integrals.

Proof. By linearity and density arguments, it suffices to consider a function

$$
f=\mathbf{1}_{A_{1} \times \cdots \times A_{n}},
$$

where $A_{i}=\left(s_{i}, t_{i}\right]$ are pairwise disjoint, and a computation gives the result.
When $r_{1}=\cdots=r_{n}=1$, we write $I_{n}(f)$ instead of $I_{n}^{(1, \ldots, 1)}(f)$; in that case, the multiple Itô integral enjoys nicer properties.

## PROPOSITION 5.4.

1. Let $f \in L_{n}^{2}$. Then

$$
I_{n}(f)=I_{n}(\tilde{f})
$$

where $\tilde{f}$ is the symmetrization of $f$

$$
\begin{equation*}
\tilde{f}=\frac{1}{n!} \sum_{p \in \mathfrak{G}_{n}} f \circ p \tag{23}
\end{equation*}
$$

2. Assume $\mathbb{E}\left[X_{t}\right]=0$. For $f, g \in L_{n}^{2}$,

$$
\mathbb{E}\left[I_{n}(f) I_{m}(g)\right]=\delta_{n, m} K_{2}^{n} n!\int_{[0, T]^{n}} \tilde{f} \tilde{g} d \mathbf{t}
$$

where $\delta_{n, m}=1$, if $n=m$, and 0 otherwise.
3. Let $f \in L_{n}^{2}$ be a symmetric function. Then

$$
I_{n}(f)=n!\int_{0}^{T}\left(\int_{0}^{t_{1}-} \cdots\left(\int_{0}^{t_{n-1}-} f\left(t_{1}, \ldots, t_{n}\right) d X_{t_{n}}\right) \cdots d X_{t_{2}}\right) d X_{t_{1}}
$$

We now state the relationship between the Itô stochastic measure $\mathrm{St}_{\hat{0}}^{\mathrm{r}}$ and the Itô multiple integral $I_{n}^{\mathbf{r}}$.

Proposition 5.5. Let $C \in \mathcal{B}\left([0, T]^{n}\right)$ and $\mathbf{r}=\left(r_{1}, \ldots, r_{n}\right)$. Then

$$
\begin{equation*}
\operatorname{St}_{\hat{0}}^{\mathbf{r}}(C)=I_{n}^{\mathbf{r}}\left(\mathbf{1}_{C}\right) \tag{24}
\end{equation*}
$$

Proof. By (22) the map $C \mapsto I_{n}^{\mathbf{r}}\left(\mathbf{1}_{C}\right)$ defines a vector measure on $\mathcal{B}\left([0, T]^{n}\right)$. On the left-hand side of (24), the Itô measure satisfies

$$
\begin{equation*}
\mathrm{St}_{\widehat{0}}^{\mathbf{r}}(C)=\mathrm{St}_{\widehat{0}}^{\mathbf{r}}\left(C_{\widehat{0}}\right) \tag{25}
\end{equation*}
$$

Now, look at right-hand side of (24). For $\pi \in \Pi_{n}$, we have that $C_{\pi}=C \cap[0, T]_{\pi}^{n}$. For all $\pi>\widehat{0}$,

$$
\mathbb{E}\left[\left(I_{n}^{\mathbf{r}}\left(\mathbf{1}_{C_{\pi}}\right)\right)^{2}\right] \leq \alpha_{n} \int_{[0, T]^{n}} \mathbf{1}_{C_{\pi}} d t_{1} \cdots d t_{n} \leq \alpha_{n} \int_{[0, T]^{n}} \mathbf{1}_{[0, T]_{\pi}^{n}} d t_{1} \cdots d t_{n}=0
$$

Hence,

$$
\begin{equation*}
I_{n}^{\mathbf{r}}\left(\mathbf{1}_{C}\right)=I_{n}^{\mathbf{r}}\left(\mathbf{1}_{C_{\widehat{0}}}\right) . \tag{26}
\end{equation*}
$$

From (25) and (26), it suffices to prove (24) for a set $C=C_{\widehat{0}}$. As in the proof of the second part of Lemma 4.3, this can be reduced to check that equality for a rectangle $\left(s_{1}, t_{1}\right] \times \cdots \times\left(s_{n}, t_{n}\right]$, with the intervals pairwise disjoint. This follows from the fact that both sides of (24) are equal to $\phi_{r_{1}}\left(\left(s_{1}, t_{1}\right]\right) \cdots \phi_{r_{n}}\left(\left(s_{n}, t_{n}\right]\right)$.

The property $I_{n}(f)=I_{n}(\tilde{f})$ is lost when the integrators are different. However, from Proposition 5.5 and (16) we can deduce the following useful property:

PROPOSITION 5.6. Let $f \in L_{n}^{2}$, and $\mathbf{r}=\left(r_{1}, \ldots, r_{n}\right)$, where $r_{1}, \ldots, r_{n} \geq 1$. Consider $p \in \mathfrak{G}_{n}$. Then

$$
I_{n}^{\mathbf{r}}(f)=I_{n}^{p(\mathbf{r})}\left(f \circ p^{-1}\right)
$$

5.2. Multiple Stratonovich integral and Hu-Meyer formula. Given a map $f:[0, T]^{n} \rightarrow \mathbb{R}$, the integral with respect to the product measure $\phi^{\otimes n}$ is called the multiple Stratonovich integral, and denoted by $I_{n}^{S}(f)$. Its basic property is that the integral of a product function factorizes

$$
I_{n}^{S}\left(g_{1} \otimes \cdots \otimes g_{n}\right)=I_{1}^{S}\left(g_{1}\right) \cdots I_{1}^{S}\left(g_{n}\right)
$$

where

$$
I_{1}^{S}(g)=I_{1}(g)=\int_{0}^{T} g(t) d X_{t}
$$

In order to construct this integral, we consider ordinary simple functions of the measurable space $\left([0, T]^{n}, \mathcal{B}\left([0, T]^{n}\right)\right)$. Specifically, denote by $\mathcal{E}_{n}^{\text {Strato }}$ the set of functions with the form

$$
f=\sum_{i=1}^{k} a_{i} \mathbf{1}_{C_{i}}
$$

where $C_{i} \in \mathcal{B}\left([0, T]^{n}\right), i=1, \ldots, k$. For such $f$, define the multiple Stratonovich integral by

$$
I_{n}^{S}(f)=\sum_{i=1}^{k} a_{i} \phi^{\otimes n}\left(C_{i}\right)
$$

The integral of a simple function does not depend on its representation, and it is linear. Moreover,

Proposition 5.7. Let $f \in \mathcal{E}_{n}^{\text {Strato }}$. Then we have the Hu -Meyer formula

$$
\begin{equation*}
I_{n}^{S}(f)=\sum_{\sigma \in \Pi_{n}} I_{\# \sigma}^{\bar{\sigma}}\left(f \circ q_{\sigma}\right), \tag{27}
\end{equation*}
$$

where the function $q_{\sigma}:[0, T]^{\# \sigma} \rightarrow[0, T]^{n}$ is introduced in (20), $\bar{\sigma}=\left(\# B_{1}, \ldots\right.$, $\left.\# B_{m}\right)$ is the vector whose components are the sizes of the ordered blocks of $\sigma$, and $I_{m}^{\left(s_{1}, \ldots, s_{m}\right)}$ is the multiple Itô integral of order $m$ with respect to the measures $\phi_{s_{1}}, \ldots, \phi_{s_{m}}$.

Proof. By linearity, it suffices to consider $f=\mathbf{1}_{C}$, where $C \in \mathcal{B}\left([0, T]^{n}\right)$. A generic term on the right-hand side of (27) is

$$
I_{\# \sigma}^{\bar{\sigma}}\left(\mathbf{1}_{C} \circ q_{\sigma}\right)
$$

and

$$
\mathbf{1}_{C} \circ q_{\sigma}=\mathbf{1}_{q_{\sigma}^{-1}(C)} .
$$

Hence, by Proposition 5.5,

$$
I_{\# \sigma}^{\bar{\sigma}}\left(\mathbf{1}_{C} \circ q_{\sigma}\right)=\operatorname{St}_{\widehat{0}}^{\bar{\sigma}}\left(q_{\sigma}^{-1}(C)\right),
$$

and (27) follows from Theorem 4.5.
Let $\sigma \in \Pi_{n}$, with $\# \sigma=m$, and denote by $\lambda_{\sigma}$ the image measure of the Lebesgue measure $(d t)^{\otimes m}$ by the function $q_{\sigma}:[0, T]^{m} \rightarrow[0, T]^{n}$. The image measure theorem implies that for $f:[0, T]^{n} \rightarrow \mathbb{R}$ measurable, positive or $\lambda_{\sigma}$-integrable,

$$
\begin{equation*}
\int_{[0, T]^{n}} f\left(t_{1}, \ldots, t_{n}\right) d \lambda_{\sigma}\left(t_{1}, \ldots, t_{n}\right)=\int_{[0, T]^{m}} f\left(q_{\sigma}\left(t_{1}, \ldots, t_{m}\right)\right) d t_{1} \cdots d t_{m} \tag{28}
\end{equation*}
$$

Define on $\mathcal{B}\left([0, T]^{n}\right)$ the measure

$$
\Lambda_{n}=\sum_{\sigma \in \Pi_{n}} \lambda_{\sigma}
$$

and write $L^{2}\left(\Lambda_{n}\right)$ for $L^{2}\left([0, T]^{n}, \mathcal{B}\left([0, T]^{n}\right), \Lambda_{n}\right)$.
In order to extend the multiple Stratonovich integral we need the following inequality of norms:

LEMMA 5.8. Let $f \in \mathcal{E}_{n}^{\text {Strato }}$. Then

$$
\begin{equation*}
\mathbb{E}\left[\left(I_{n}^{S}(f)\right)^{2}\right] \leq C \int_{[0, T]^{n}} f^{2} d \Lambda_{n} \tag{29}
\end{equation*}
$$

where $C$ is a constant.

Proof. By (27), (22) and (28),

$$
\begin{aligned}
\mathbb{E}\left[\left(I_{n}^{S}(f)\right)^{2}\right] & \leq C \sum_{\sigma \in \Pi_{n}} \mathbb{E}\left[\left(I_{\# \sigma}^{\bar{\sigma}}\left(f \circ q_{\sigma}\right)\right)^{2}\right] \\
& \leq C \sum_{\sigma \in \Pi_{n}} \int_{[0, T]^{\# \sigma}}\left(f \circ q_{\sigma}\right)^{2} d t_{1} \cdots d t_{\# \sigma} \\
& =C \sum_{\sigma \in \Pi_{n}} \int_{[0, T]^{n}} f^{2} d \lambda_{\sigma} \\
& =C \int_{[0, T]^{n}} f^{2} d \Lambda_{n} .
\end{aligned}
$$

The main result of the paper is the following theorem:
THEOREM 5.9. The map $I_{n}^{S}: \mathcal{E}_{n}^{\text {Strato }} \rightarrow L^{2}(\Omega)$ can be extended to a unique linear continuous map from $L^{2}\left(\Lambda_{n}\right)$ to $L^{2}(\Omega)$, and we have the Hu -Meyer formula

$$
\begin{equation*}
I_{n}^{S}(f)=\sum_{\sigma \in \Pi_{n}} I_{\# \sigma}^{\bar{\sigma}}\left(f \circ q_{\sigma}\right) . \tag{30}
\end{equation*}
$$

Proof. The extension of $I_{n}^{S}$ to a continuous map on $L^{2}\left(\Lambda_{n}\right)$ is proved using a density argument and inequality (29). To prove the $\mathrm{Hu}-\mathrm{Meyer}$ formula, let $f \in L^{2}\left(\Lambda_{n}\right)$ and $\left\{f_{k}, k \geq 1\right\} \subset \mathcal{E}_{n}^{\text {Strato }}$ such that $\lim _{k} f_{k}=f$ in $L^{2}\left(\Lambda_{n}\right)$. For every $\sigma \in \Pi_{n}$, we have $\lim _{k} f_{k} \circ q_{\sigma}=f \circ q_{\sigma}$ in $L_{\# \sigma}^{2}$; hence, from Theorem 5.2 the Itô integrals on the right-hand side of (30) converge, and the formula follows from Proposition 5.7.

REMARKS 5.10.
(1) Let $g_{1}, \ldots, g_{n} \in L^{2 n}([0, T], d t)$. Then $g_{1} \otimes \cdots \otimes g_{n} \in L_{n}^{2}\left(\Lambda_{n}\right)$ and

$$
I_{n}^{S}\left(g_{1} \otimes \cdots \otimes g_{n}\right)=I_{1}^{S}\left(g_{1}\right) \cdots I_{1}^{S}\left(g_{n}\right)
$$

This result is easily checked for simple functions $g_{1}, \ldots, g_{n}$ and extended to the general case by a density argument.
(2) In order to prove the Hu-Meyer formula for $I_{n}^{S}$ it is enough to assume that the process $X$ has moments up to order $2 n$.
(3) For $\sigma \in \Pi_{n}, \sigma>\widehat{0}$, the measure $\lambda_{\sigma}$ is singular with respect to the Lebesgue measure on $[0, T]^{n}$. For example, for $n=2$ and $\sigma=\widehat{1}$, let $D=\{(t, t), t \in$ $[0, T]\}$ be the diagonal of $[0, T]^{2}$. Then $\lambda_{\widehat{1}}$ is concentrated in $D$, that has zero Lebesgue measure, but $\lambda_{\hat{1}}$ is nonzero

$$
\lambda_{\widehat{\mathrm{I}}}(D)=\int_{[0, T]^{2}} \mathbf{1}_{D}(s, t) d \lambda_{\widehat{\mathrm{I}}}(s, t)=\int_{[0, T]} \mathbf{1}_{D}(t, t) d t=T
$$

(4) As in the Brownian case (see $[9,12,16,27]$ and the references therein), there are other procedures to construct the multiple Stratonovich integral. The main difficulty in every approach is that the usual condition $f \in L_{n}^{2}$ in Itô's theory is not sufficient to guarantee the multiple Stratonovich integrability of $f$. The reason is that one needs to control the behavior of $f$ on the diagonal sets $[0, T]_{\sigma}^{n}$ that have zero Lebesgue measure when $\sigma>\widehat{0}$. We solve this difficulty using the norm induced by the measure $\Lambda_{n}$, which seems to be appropriate for dealing with the diagonal sets, avoiding in this way the difficulty of a manageable definition of the traces.

When the function $f \in L^{2}\left(\Lambda_{n}\right)$ is symmetric, the Hu-Meyer formula can be considerably simplified. We show that we can assume that symmetry on $f$ without loss of generality.

Proposition 5.11. Let $f \in L^{2}\left(\Lambda_{n}\right)$. Then $I_{n}^{S}(f)=I_{n}^{S}(\tilde{f})$, where $\tilde{f}$ is the symmetrization of $f$ [see (23)].

Proof. The proof is straightforward for $f=\mathbf{1}_{C}, C \in \mathcal{B}\left([0, T]^{n}\right)$, using Lemma 4.3. By linearity the equality $I_{n}^{S}(f)=I_{n}^{S}(\tilde{f})$ is extended to $\mathcal{E}_{n}^{\text {Strato }}$, and by density to $L^{2}\left(\Lambda_{n}\right)$.

Next we show the Hu-Meyer formula for a symmetric function $f$. In general (for $f$ symmetric), the function $f \circ q_{\sigma}$ is nonsymmetric, but as we will see in the proof of the next theorem, its multiple Itô integral depends only on the block structure of $\sigma$ (the type of $\sigma$ ). For example, with $n=3, f\left(t_{1}, t_{2}, t_{3}\right)=t_{1} t_{2} t_{3}$ and $\sigma=\{\{1\},\{2,3\}\}$, we have that

$$
f\left(q_{\sigma}\left(t_{1}, t_{2}\right)\right)=t_{1} t_{2}^{2}
$$

that is nonsymmetric. Its integral is

$$
I_{\# \sigma}^{\bar{\sigma}}\left(f \circ q_{\sigma}\right)=I_{2}^{(1,2)}\left(f \circ q_{\sigma}\right)=I_{2}^{(1,2)}\left(t_{1} t_{2}^{2}\right) .
$$

Take $\pi=\{\{1,3\},\{2\}\}$. Then $f\left(q_{\pi}\left(t_{1}, t_{2}\right)\right)=t_{1}^{2} t_{2}$ and

$$
I_{\# \pi}^{\bar{\pi}}\left(f \circ q_{\pi}\right)=I_{2}^{(2,1)}\left(t_{1}^{2} t_{2}\right)=I_{2}^{(1,2)}\left(t_{1} t_{2}^{2}\right),
$$

where the last equality is due to Proposition 5.6.

We use the following notation: given nonnegative integers $r_{1}, \ldots, r_{k}$ such that $\sum_{i=1}^{k} i r_{i}=n$, we write

$$
\left[r_{1}, r_{2}, \ldots, r_{k}\right]=(\underbrace{1, \ldots, 1}_{r_{1}}, \underbrace{2, \ldots, 2}_{r_{2}}, \ldots)
$$

Note that this corresponds to $\bar{\sigma}$ when

$$
\sigma=\left\{\{1\}, \ldots,\left\{r_{1}\right\},\left\{r_{1}+1, r_{1}+2\right\}, \ldots,\left\{r_{1}+2 r_{2}-1, r_{1}+2 r_{2}\right\}, \ldots\right\}
$$

We also write $q_{r_{1}, \ldots, r_{k}}$ for $q_{\sigma}$, with $\sigma$ the above partition.
THEOREM 5.12. Let $f \in L_{n}^{2}\left(\Lambda_{n}\right)$ be a symmetric function. Then

$$
\begin{equation*}
I_{n}^{S}(f)=\sum \frac{n!}{r_{1}!(2!)^{r_{2}} r_{2}!\cdots(k!)^{r_{k}} r_{k}!} I_{r_{1}+\cdots+r_{k}}^{\left[r_{1}, \ldots, r_{k}\right]}\left(f \circ q_{r_{1}, \ldots, r_{k}}\right), \tag{31}
\end{equation*}
$$

where the sum is extended over all nonnegative integers $r_{1}, \ldots, r_{k}$ such that $\sum_{i=1}^{k} i r_{i}=n$, for $k=1, \ldots, n$.

Proof. Let $f \in L^{2}\left(\Lambda_{n}\right)$ symmetric. For every $\sigma \in \Pi_{n}$ and $p \in \mathfrak{G}_{n}$,

$$
\begin{aligned}
I_{\# p(\sigma)}^{\overline{p(\sigma)}}\left(f \circ q_{p(\sigma)}\right) & ={ }_{(\mathrm{a})} I_{\# p(\sigma)}^{\overline{p(\sigma)}}\left(f \circ p^{-1} \circ q_{\sigma} \circ p_{1}^{-1}\right) \\
& ={ }_{(\mathrm{b})} I_{\# p(\sigma)}^{\overline{p(\sigma)}}\left(f \circ q_{\sigma} \circ p_{1}^{-1}\right)={ }_{(\mathrm{c})} I_{\# \sigma}^{\bar{\sigma}}\left(f \circ q_{\sigma}\right),
\end{aligned}
$$

where (a) is due to Proposition A.4(ii), the equality (b) follows from the symmetry of $f$ and (c) from Proposition 5.6 and the fact that $p_{1}$ gives the correct order of $p(\sigma)$ [see (37)]. This implies that all the partitions that have the same number of blocks of 1 element, the same number with two elements, etc. (i.e., they have the same type) give the same Itô multiple integral in the Hu-Meyer formula. To obtain (31) it suffices to count the number of partitions of $\{1, \ldots, n\}$ with $r_{1}$ blocks with 1 element, $r_{2}$ blocks with 2 elements, $\ldots, r_{k}$ blocks with $k$ elements, which is

$$
\frac{n!}{r_{1}!(2!)^{r_{2}} r_{2}!\cdots(k!)^{r_{k} r_{k}!}}
$$

Final remark. One may expect that by decomposing the Lévy process into a sum of two independent processes, one with the small jumps and the other with the large ones, the assumption of the existence of moments could be avoided. However, this decomposition introduces dramatic changes to the context of the work, and such an extension is beyond the scope and purposes of the present paper.

## 6. Special cases.

6.1. Brownian motion. When $X=W$ is a standard Brownian motion,

$$
\phi_{2}([0, t])=t \quad \text { and } \quad \phi_{n}=0, \quad n \geq 3
$$

It follows that in the Hu -Meyer formula only the partitions with all blocks of cardinality 1 or 2 give a contribution, and all the Itô integrals are a mixture of multiple stochastic Brownian integrals and Lebesgue integrals. We can organize the sum according the number of blocks of two elements. For a partition having $j$ blocks of 2 elements, and $f \in L_{n}^{2}\left(\Lambda_{n}\right)$ symmetric, the multiple Itô integral is

$$
\begin{aligned}
& I_{n-j}^{[n-2 j, j]}(f) \\
& \quad=\int_{[0, T]^{n-j}} f\left(s_{1}, \ldots, s_{n-2 j},\right. \\
& \left.\quad t_{1}, t_{1}, \ldots, t_{j}, t_{j}\right) d W_{s_{1}} \cdots d W_{s_{n-2 j}} d t_{1} \cdots d t_{j} \\
& \quad=I_{n-2 j}\left(\int_{[0, T]^{j}} f\left(\cdot, t_{1}, t_{1}, \ldots, t_{j}, t_{j}\right) d t_{1} \cdots d t_{j}\right),
\end{aligned}
$$

where the last equality is due to a Fubini-type theorem. Therefore,

$$
\begin{align*}
I_{n}^{S}(f)= & \sum_{j=0}^{[n / 2]} \frac{n!}{(n-2 j)!j!2^{j}}  \tag{32}\\
& \quad \times I_{n-2 j}\left(\int_{[0, T]^{j}} f\left(\cdot, t_{1}, t_{1}, t_{2}, t_{2}, \ldots, t_{j}, t_{j}\right) d t_{1} \cdots d t_{j}\right)
\end{align*}
$$

which is the classical $\mathrm{Hu}-\mathrm{Meyer}$ formula (see [8]).
On the other hand, in the measure $\Lambda_{n}$ only participate the measures $\lambda_{\sigma}$ corresponding to the partitions above mentioned. Consider the measure $\ell_{2}=\lambda_{\hat{1}}$ on $[0, T]^{2}$, that is, for a positive or $\ell_{2}$ integrable function $h$,

$$
\int_{[0, T]^{2}} h(s, t) d \ell_{2}(s, t)=\int_{[0, T]} h(t, t) d t
$$

Given the partition $\sigma \in \Pi_{n}$,

$$
\sigma=\{\{1\}, \ldots,\{n-2 j\},\{n-2 j+1, n-2 j+2\}, \ldots,\{n-1, n\}\},
$$

we have

$$
\lambda_{\sigma}=(d t)^{\otimes(n-2 j)} \otimes \ell_{2}^{\otimes j}
$$

6.2. Poisson process. Let $N_{t}$ be a standard Poisson process with intensity 1 , and consider the process $X_{t}=N_{t}-t$. For every $n \geq 2$,

$$
X_{t}^{(n)}=N_{t}=X_{t}+t
$$

and hence, a multiple Itô integral can be reduced to a linear combination of multiple integrals where all the integrators are $d X$ or $d t$. For $f \in L_{n}^{2}\left(\Lambda_{n}\right)$ symmetric,
each integral $I_{r_{1}+\ldots+r_{k}}^{\left[r_{1}, \ldots, r_{k}\right]}\left(f \circ q_{r_{1}, \ldots, r_{k}}\right)$ in (31) can be expressed in terms of the number of Lebesgue integrals that appear

$$
\begin{aligned}
& I_{r_{1}+\ldots+r_{k}}^{\left[r_{1}, \ldots, r_{k}\right]}\left(f \circ q_{r_{1}, \ldots, r_{k}}\right) \\
& =\sum_{j=0}^{r_{2}+\cdots+r_{k}} I_{r_{1}+\cdots+r_{k}-j}\left(\int _ { [ 0 , T ] ^ { j } } \left(\sum_{\substack{l_{1}, \ldots, l_{j}=r_{1}+1 \\
\text { different }}}^{r_{1}+\cdots+r_{k}}\left(f \circ q_{r_{1}, \ldots, r_{k}}\right)\right.\right. \\
& \left.\left.\times\left(t_{1}, \ldots, t_{r_{1}+\cdots+r_{k}}\right)\right) d t_{l_{1}} \cdots d t_{l_{j}}\right),
\end{aligned}
$$

and the Hu -Meyer formula of Solé and Utzet [28] can be deduced from this expression.
6.3. Gamma process and subordinators. A subordinator is a Lévy process with increasing paths. An important example of a subordinator with moments of all orders is the Gamma process, denoted by $\left\{G_{t}, t \geq 0\right\}$, which is the Lévy process corresponding to an exponential law of parameter 1. Its Lévy measure is

$$
v(d x)=\frac{e^{-x}}{x} \mathbf{1}_{\{x>0\}}(x) d x
$$

The law of $G_{t}$ is Gamma with mean $t$ and scale parameter equal to one. A Gamma process can be represented as the sum of its jumps, that are all positive,

$$
G_{t}=\sum_{0<s \leq t} \Delta G_{s}
$$

The Lévy measure of $G^{(n)}$ is (see Schoutens [26])

$$
v_{n}(d x)=\frac{e^{-x^{1 / n}}}{n x} \mathbf{1}_{\{x>0\}}(x) d x, \quad n \geq 1
$$

and the Teugels martingales are

$$
Y_{t}^{(n)}=\sum_{\{0<s \leq t\}}\left(\Delta G_{s}\right)^{n}-(n-1)!t, \quad n \geq 1
$$

In this case, unlike the Brownian motion and the Poisson process, the Hu-Meyer formula does not simplify, due to the fact that the diagonal measures cannot be expressed in a simple way in terms of, say, the process and a deterministic measure. However, for a Gamma process, and in general, for a subordinator without drift (see below for the definition) with moments of all orders, both the multiple Itô and Stratonovich integrals can be computed pathwise integrating with respect to an ordinary measure. This is a multivariate extension of the property that states that the stochastic integral and the pathwise Lebesgue-Stieljes integral with respect to
a semimartingale of bounded variation are equal; such property was proved for the integral with respect to a Lévy process of bounded variation by Millar [18] under weak conditions, and part of our proof follows his scheme.

Let $X=\left\{X_{t}, t \geq 0\right\}$ be a subordinator. The Lévy-Itô representation of $X$ takes the form

$$
X_{t}=\gamma_{0} t+\sum_{0<s \leq t} \Delta X_{s}
$$

with $\gamma_{0} \geq 0$ (see Sato [25], Theorems 21.5 and 19.3). The number $\gamma_{0}$ is called the drift of the subordinator, and we will assume that $\gamma_{0}=0$. Consider the sequence of stopping times $\left\{T_{k}, k \geq 1\right\}$ with disjoint graphs that exhaust the jumps of $X: \Delta X_{T_{k}} \neq 0, \forall k \geq 1$, and $X$ only has jumps on these times (see, e.g., Dellacherie and Meyer [6], Theorem B, page XIII, for a construction of this sequence). Denote by $J_{n}$ the set of $n$-tuples $\left(T_{i_{1}}, \ldots, T_{i_{n}}\right)$, with $T_{i_{j}} \leq T$, and all entries different. For $r_{1}, \ldots, r_{n} \geq 1$, define a measure on $[0, T]^{n}$ by

$$
m_{r_{1}, \ldots, r_{n}}=\sum_{\left(T_{i_{1}}, \ldots, T_{i_{n}}\right) \in J_{n}}\left(\Delta X_{T_{i_{1}}}\right)^{r_{1}} \cdots\left(\Delta X_{T_{i_{n}}}\right)^{r_{n}} \delta_{\left(T_{i_{1}}, \ldots, T_{T_{n}}\right)},
$$

where $\delta_{a}$ is a Dirac measure at point $a$, with the convention that the sum is 0 if $J_{n}=\varnothing$. We have the following property:

Proposition 6.1. Let $X=\left\{X_{t}, t \geq 0\right\}$ be a subordinator without drift and with moments of all orders. With the preceding notation, for every $f \in L_{n}^{2}$,

$$
\begin{equation*}
I_{n}^{\left(r_{1}, \ldots, r_{n}\right)}(f)=\int_{[0, T]^{n}} f d m_{r_{1}, \ldots, r_{n}} \quad \text { a.s. } \tag{33}
\end{equation*}
$$

Proof. First, note two facts:
(a) $m_{r_{1}, \ldots, r_{n}}$ is a finite measure

$$
\begin{aligned}
m_{r_{1}, \ldots, r_{n}}\left([0, T]^{n}\right) & =\sum_{\left(T_{i_{1}}, \ldots, T_{i_{n}}\right) \in J_{n}}\left(\Delta X_{T_{i_{1}}}\right)^{r_{1}} \cdots\left(\Delta X_{T_{i_{n}}}\right)^{r_{n}} \\
& \leq \sum_{T_{i_{1}} \leq T, \ldots, T_{i_{n}} \leq T}\left(\Delta X_{T_{i_{1}}}\right)^{r_{1}} \cdots\left(\Delta X_{T_{i_{n}}}\right)^{r_{n}} \\
& =X_{T}^{\left(r_{1}\right)} \cdots X_{T}^{\left(r_{n}\right)}<\infty .
\end{aligned}
$$

(b) If the intervals $\left(s_{1}, t_{1}\right], \ldots,\left(s_{n}, t_{n}\right]$ are pairwise disjoint, then (33) is true for $f=\mathbf{1}_{\left(s_{1}, t_{1}\right] \times \cdots \times\left(s_{n}, t_{n}\right]}$. The proof is straightforward.

We separate the proof of the proposition in two steps.
Step 1. Formula (33) is true for every map $f:[0, T]^{n} \rightarrow \mathbb{R} \mathcal{B}_{0}$-measurable and bounded, where $\mathcal{B}_{0}$ is the $\sigma$-field on $[0, T]^{n}$ generated by the rectangles $\left(s_{1}, t_{1}\right] \times$ $\cdots \times\left(s_{n}, t_{n}\right]$, with $\left(s_{1}, t_{1}\right], \ldots,\left(s_{n}, t_{n}\right]$ pairwise disjoint.

To prove this claim we use a convenient monotone class theorem. Denote by $\mathcal{H}$ the family of functions that satisfy (33); it is a vector space such that:
(i) $1 \in \mathcal{H}$.
(ii) If $f_{m} \in \mathcal{H}, 0 \leq f_{m} \leq K$ for some constant $K$, and $f_{m} \nearrow f$, then $f \in \mathcal{H}$.

To see (i), consider the dyadic partition of $[0, T]$ with mesh $2^{-k}$, write

$$
B_{j}=\left((j-1) T 2^{-k}, j T 2^{-k}\right], \quad j=1, \ldots, 2^{k}
$$

and define

$$
f_{k}=\sum_{\substack{j_{1}, \ldots, j_{n} \\ \text { different }}} \mathbf{1}_{B_{j_{1}} \times \cdots \times B_{j_{n}}}
$$

By the remark (b) at the beginning of the proof,

$$
I_{n}^{\left(r_{1}, \ldots, r_{n}\right)}\left(f_{k}\right)=\int_{[0, T]^{n}} f_{k} d m_{r_{1}, \ldots, r_{n}}
$$

Moreover, $f_{k} \nearrow 1$ out off the diagonal sets $[0, T]_{\sigma}^{n}$, with $\sigma \neq \widehat{0}$, and then $f_{k} \nearrow 1$ a.e. with respect to the Lebesgue measure, and in $L_{n}^{2}$. Therefore,

$$
\lim _{k} I_{n}^{\left(r_{1}, \ldots, r_{n}\right)}\left(f_{k}\right)=I_{n}^{\left(r_{1}, \ldots, r_{n}\right)}(1)
$$

On the other hand, for every $\omega$, the measure $m_{r_{1}, \ldots, r_{n}}$ does not charge on any of the above mentioned diagonal sets. Thus, the convergence $f_{k} \nearrow 1$ is also $m_{r_{1}, \ldots, r_{n}}$-a.e. By the monotone convergence theorem,

$$
\lim _{k} \int_{[0, T]^{n}} f_{k} d m_{r_{1}, \ldots, r_{n}}=\int_{[0, T]^{n}} 1 d m_{r_{1}, \ldots, r_{n}}
$$

and (i) follows.
Point (ii) is deduced directly from the monotone convergence theorem and taking into account that under the conditions in (ii) we have $f_{m} \rightarrow f$ in $L_{n}^{2}$.

Again by remark (b) above, the indicator of a set $\left(s_{1}, t_{1}\right] \times \cdots \times\left(s_{n}, t_{n}\right]$, with $\left(s_{1}, t_{1}\right], \ldots,\left(s_{n}, t_{n}\right]$ pairwise disjoint, is in $\mathcal{H}$, and this family of sets is closed by intersection. By the monotone class theorem, it follows that all bounded $\mathcal{B}_{0}-$ measurable functions are in $\mathcal{H}$.

Step 2. Extension of (33) to all $f \in L_{n}^{2}$. First, note that $\mathcal{B}_{0}$ is the $\sigma$-field generated by the Borelian sets $B \in \mathcal{B}\left([0, T]^{n}\right)$ such that $B \subset[0, T]_{0}^{n}$. Then, given $B \in \mathcal{B}\left([0, T]^{n}\right)$, the indicator $\mathbf{1}_{B \cap[0, T]_{0}^{n}}$ is $\mathcal{B}_{0}$ measurable. Let $f \in L_{n}^{2}$, and assume $f \geq 0$. There is a sequence of simple (and then bounded) functions such that $0 \leq f_{m} \nearrow f$. Define $f_{m}^{0}=f_{m} \mathbf{1}_{[0, T]_{0}^{n}}$, which is $\mathcal{B}_{0}$ measurable, and $f_{m}^{0} \nearrow f$ a.e. with respect to the Lebesgue measure. The convergence is also in $L_{n}^{2}$, and then $\lim _{m} I_{n}\left(f_{m}^{0}\right)=I(f)$. On the other hand, $f_{m}^{0} \nearrow f, m_{r_{1}, \ldots, r_{n}}$-a.e. so

$$
\lim _{m} \int_{[0, T]^{n}} f_{m}^{0} d m_{r_{1}, \ldots, r_{n}}=\int_{[0, T]^{n}} f d m_{r_{1}, \ldots, r_{n}}
$$

By Step 1, we get the result. For a general $f \in L_{n}^{2}$, decompose $f=f^{+}-f^{-}$.

Finally, for a subordinator without drift and with moments of all orders, the multiple Stratonovich measure can be identified with the $n$-fold product measure of $\phi=\sum_{k} \Delta X_{T_{k}} \delta_{T_{k}}$. So for $f \in \mathcal{E}_{n}^{\text {Srato }}$, by definition,

$$
I_{n}^{S}(f)=\int_{[0, T]^{n}} f d \phi^{\otimes n}
$$

Using similar arguments as in the previous proposition, but easier, it is proved that

$$
I_{n}^{S}(f)=\int_{[0, T]^{n}} f d \phi^{\otimes n} \quad \forall f \in L^{2}\left(\Lambda_{n}\right)
$$

Then, the $\mathrm{Hu}-\mathrm{Meyer}$ formula can be transferred to a pathwise context.

## APPENDIX

A.1. The isomorphism $[\sigma, \widehat{1}] \simeq \Pi_{\# \sigma}$. Fix a partition $\sigma \in \Pi_{n}$, with blocks $B_{1}, \ldots, B_{m}$. Let $\pi \geq \sigma$, with blocks $V_{1}, \ldots, V_{k}$; each block $V_{i}$ is the union of some of the blocks $B_{1}, \ldots, B_{m}$. Hence, we can consider the partition $\pi^{*} \in \Pi_{m}$ that gives the relationship between the $V_{i}$ 's and the $B_{j}$ 's, that is, $\pi^{*}$ has blocks $W_{1}, \ldots, W_{k}$ defined by

$$
V_{i}=\bigcup_{j \in W_{i}} B_{j}, \quad i=1, \ldots, k
$$

Proposition A.1. Let $\sigma \in \Pi_{n}$ with $\# \sigma=m$. With the above notation, the map

$$
\begin{aligned}
{[\sigma, \widehat{1}] } & \longrightarrow \Pi_{m} \\
\pi & \mapsto \pi^{*}
\end{aligned}
$$

is a bijection and, for $\pi, \tau \in[\sigma, \widehat{1}]$,

$$
\pi \leq \tau \quad \Longleftrightarrow \quad \pi^{*} \leq \tau^{*}
$$

Moreover,

$$
\mu^{(n)}(\sigma, \pi)=\mu^{(m)}\left(\widehat{0}, \pi^{*}\right)
$$

where $\mu^{(r)}$ is the Môbius function on $\Pi_{r}$.
The proof is straightforward.
A.2. Permutations and partitions. Let $p:\{1, \ldots, n\} \longrightarrow\{1, \ldots, n\}$ be a permutation. This application induces a bijection on $\Pi_{n}$, and a bijection on $\mathbb{R}^{n}$. Specifically:

1. For a subset $B \subset\{1, \ldots, n\}$ we denote by $p(B)$ the image of $B$ by $p$

$$
p(B)=\{p(j), \text { for } j \in B\}
$$

Given a partition $\sigma \in \Pi_{n}$, with blocks $B_{1}, \ldots, B_{m}$, let $p(\sigma)$ be the partition with blocks $W_{1}, \ldots, W_{m}$ defined by $W_{j}=p\left(B_{j}\right)$. Note that in general the blocks $W_{1}, \ldots, W_{m}$ are not ordered. The application

$$
\begin{aligned}
p: \Pi_{n} & \longrightarrow \Pi_{n} \\
\sigma & \mapsto p(\sigma)
\end{aligned}
$$

is a bijection and for $\sigma, \tau \in \Pi_{n}$,

$$
\sigma \leq \tau \quad \Longleftrightarrow \quad p(\sigma) \leq p(\tau)
$$

This last property is clear, because if $V \in \tau$, and $V=B_{r_{1}} \cup \cdots \cup B_{r_{k}}$, then

$$
p(V)=p\left(B_{r_{1}}\right) \cup \cdots \cup p\left(B_{r_{k}}\right) .
$$

Further, this application is compatible with the relationship introduced in Section 2.2

$$
\begin{equation*}
i \sim_{\sigma} j \quad \Longleftrightarrow p(i) \sim_{p(\sigma)} p(j) \tag{34}
\end{equation*}
$$

2. For a vector $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$, we write

$$
p(\mathbf{x})=\left(x_{p(1)}, \ldots, x_{p(n)}\right),
$$

and the application $\mathbf{x} \mapsto p(\mathbf{x})$ determines a bijection on $\mathbb{R}^{n}$, that we also denote by $p$. For a set $C \subset \mathbb{R}^{n}$, we write

$$
p(C)=\{p(\mathbf{x}), \text { for } \mathbf{x} \in C\}
$$

In particular, for $A_{1}, \ldots, A_{n} \subset \mathbb{R}$,

$$
p\left(A_{1} \times \cdots \times A_{n}\right)=A_{p(1)} \times \cdots \times A_{p(n)} .
$$

Notice that if we look for the position of a particular set, say $A_{1}$, in $p\left(A_{1} \times \cdots \times\right.$ $A_{n}$ ), we find it at place $p^{-1}(1)$

$$
\begin{gathered}
A_{i_{1}} \times \cdots \times \longdiv { \boxed { \mathrm { A } _ { 1 } } \times \cdots \times A _ { i _ { n } } } \\
\uparrow \\
p^{-1}(1) .
\end{gathered}
$$

This last observation gives some light to the next property:

Proposition A.2. Consider $p \in \mathfrak{G}_{n}, C \subset \mathbb{R}^{n}$ and $\sigma \in \Pi_{n}$.
(i) $p\left(C_{\sigma}\right)=(p(C))_{p^{-1}(\sigma)}$.
(ii) $p\left(C_{\geq \sigma}\right)=(p(C))_{\geq p^{-1}(\sigma)}$. In particular, $p\left(A_{\geq \sigma}^{n}\right)=A_{\geq p^{-1}(\sigma)}^{n}$.

Proof. (i) Let $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in p\left(C_{\sigma}\right) \subset p(C)$. Write

$$
p^{-1}(\mathbf{x})=\left(x_{p^{-1}(1)}, \ldots, x_{p^{-1}(n)}\right)=\left(y_{1}, \ldots, y_{n}\right)=\mathbf{y} \in C_{\sigma} .
$$

Therefore,

$$
\mathbf{y} \in C \quad \text { and } \quad y_{i}=y_{j} \quad \Longleftrightarrow \quad i \sim_{\sigma} j
$$

The condition on the right is equivalent to $p^{-1}(i) \sim_{p^{-1}(\sigma)} p^{-1}(j)$ [see (34)]. So, returning to the $\mathbf{x ' s}$,

$$
\mathbf{x} \in p(C) \quad \text { and } \quad x_{p^{-1}(i)}=x_{p^{-1}(j)} \quad \Longleftrightarrow \quad p^{-1}(i) \sim_{p^{-1}(\sigma)} p^{-1}(j)
$$

Call $p^{-1}(i)=r$ and $p^{-1}(j)=s$. We have

$$
\mathbf{x} \in p(C) \quad \text { and } \quad x_{r}=x_{s} \quad \Longleftrightarrow \quad r \sim_{p^{-1}(\sigma)} s
$$

Hence, $\mathbf{x} \in(p(C))_{p^{-1}(\sigma)}$.
The reciprocal inclusion is analogous.
(ii) Applying (i),

$$
\begin{aligned}
p\left(C_{\geq \sigma}\right) & =p\left(\bigcup_{\pi \geq \sigma} C_{\pi}\right)=\bigcup_{\pi \geq \sigma} p\left(C_{\pi}\right)=\bigcup_{\pi \geq \sigma}(p(C))_{p^{-1}(\pi)} \\
& =\bigcup_{\tau \geq p^{-1}(\sigma)}(p(C))_{\tau}=(p(C))_{\geq p^{-1}(\sigma)} .
\end{aligned}
$$

Consider a partition $\sigma \in \Pi_{n}$ with blocks $B_{1}, \ldots, B_{m}$ (ordered). If the elements of each block are consecutive numbers, then,

$$
\begin{equation*}
A_{\geq \sigma}^{n}=\underset{j=1}{m} A_{\widehat{1}}^{\# B_{j}} \tag{35}
\end{equation*}
$$

When $\sigma$ does not fulfill the previous condition, the expression (35) is not valid. However, since we are interested in computing $\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)\left(A_{\geq \sigma}^{n}\right)$, thanks to Lemma 4.3, we fortunately can permute both the set and the product measure to make things work. The next proposition is essential for this purpose.

Proposition A.3. Let $A \subset \mathbb{R}$ and $\sigma \in \Pi_{n}$ be a partition, with blocks $B_{1}, \ldots, B_{m}$ (ordered). There is a permutation $p \in \mathfrak{G}_{n}$ such that

$$
p\left(A_{\geq \sigma}^{n}\right)=\underset{j=1}{m} A_{\widehat{1}}^{\# B_{j}}
$$

Proof. Write $s_{j}=\# B_{j}, j=1, \ldots, n$, and let $p^{\prime} \in \mathfrak{G}_{n}$ such that

$$
\begin{aligned}
p^{\prime}\left(B_{1}\right) & =\left\{1, \ldots, s_{1}\right\} \\
p^{\prime}\left(B_{2}\right) & =\left\{s_{1}+1, \ldots, s_{1}+s_{2}\right\}
\end{aligned}
$$

Take $p=\left(p^{\prime}\right)^{-1}$ and apply Proposition A.2(ii).
A.3. Proof of Lemma 4.4. We prove the following lemma:

LEMMA 4.4. Let $r_{1}, \ldots, r_{n} \geq 1, \sigma \in \Pi_{n}$ with blocks $B_{1}, \ldots, B_{m}$ (ordered), and $A \in \mathcal{B}[0, T]$. Then

$$
\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)\left(A_{\geq \sigma}^{n}\right)=\prod_{j=1}^{m} \phi_{\sum_{i \in B_{j}} r_{i}}(A) .
$$

Proof. Let $B_{1}, \ldots, B_{m}$ be the blocks of $\sigma$ ordered. If $\sigma$ is such that $A_{\geq \sigma}^{n}=$ $\times_{j=1}^{m} A_{\widehat{1}}^{\# B_{j}}$, by Theorem 3.4,

$$
\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)\left(A_{\geq \sigma}^{n}\right)=\prod_{j=1}^{m}\left(\bigotimes_{i \in B_{j}} \phi_{r_{i}}\left(A_{\hat{1}}^{\# B_{j}}\right)\right)=\prod_{j=1}^{m} \phi_{\sum_{i \in B_{j}} r_{i}}(A) .
$$

For the general case, let $p$ be the permutation given by Proposition A. 3 and write $V_{j}=p^{-1}\left(B_{j}\right), j=1, \ldots, m$. By Proposition A. 3 (first), and $\# B_{j}=\# V_{j}$ (second), we have

By Lemma 4.3 and the first part of the proof,

$$
\begin{aligned}
\left(\phi_{r_{1}} \otimes \cdots \otimes \phi_{r_{n}}\right)\left(A_{\geq \sigma}^{n}\right) & =\left(\phi_{r_{p(1)}} \otimes \cdots \otimes \phi_{r_{p(n)}}\right)\left(p\left(A_{\geq \sigma}^{n}\right)\right) \\
& =\left(\phi_{u_{1}} \otimes \cdots \otimes \phi_{u_{n}}\right)\left(\underset{j=1}{m} A_{\widehat{1}}^{\# V_{j}}\right) \\
& =\prod_{j=1}^{m} \phi_{\sum_{i \in V_{j}} u_{i}}(A),
\end{aligned}
$$

where $u_{i}=r_{p(i)}$. For every $j=1, \ldots, m$,

$$
\sum_{i \in V_{j}} u_{i}=\sum_{i \in V_{j}} r_{p(i)}=\sum_{i \in p^{-1}\left(B_{j}\right)} r_{p(i)}=\sum_{i \in B_{j}} r_{i} .
$$

A.4. The function $\boldsymbol{q}_{\boldsymbol{\sigma}}$. Given a partition $\sigma \in \Pi_{n}$, with blocks $B_{1}, \ldots, B_{m}$ (ordered), the function $q_{\sigma}$ [see (20)] is defined by

$$
\begin{aligned}
q_{\sigma}:[0, T]^{m} & \longrightarrow[0, T]^{n}, \\
\left(x_{1}, \ldots, x_{m}\right) & \rightarrow\left(y_{1}, \ldots, y_{n}\right),
\end{aligned}
$$

where $y_{i}=x_{j}$, if $i \in B_{j}$. This function is a bijection between $[0, T]^{m}$ and $[0, T]_{\geq \sigma}^{n}$, and it is Borel measurable because

$$
\begin{equation*}
q_{\sigma}^{-1}\left(A_{1} \times \cdots \times A_{n}\right)=\left(\bigcap_{i \in B_{1}} A_{i}\right) \times \cdots \times\left(\bigcap_{i \in B_{m}} A_{i}\right) \tag{36}
\end{equation*}
$$

Given a partition $\sigma \in \Pi_{n}$, with blocks (ordered) $B_{1}, \ldots, B_{m}$ and a permutation $p \in \mathfrak{G}_{n}$, as we commented, the blocks of $p(\sigma)$ in general are not ordered. It is convenient to consider the permutation $p_{1} \in \mathfrak{G}_{m}$ that gives the correct order of the blocks of $p(\sigma)$, that means, $p_{1}(1)=i$ if $p\left(B_{i}\right)$ is the first block of $p(\sigma), p_{1}(2)=j$ if $p\left(B_{j}\right)$ is the second block, and so on; in other words,

$$
p\left(B_{p_{1}(1)}\right), \ldots, p\left(B_{p_{1}(m)}\right)
$$

are the blocks of $p(\sigma)$ ordered. Remember that we defined [see (13)] the $m$ dimensional vector $\bar{\sigma}=\left(\# B_{1}, \ldots, \# B_{m}\right)$. Then

$$
\begin{equation*}
p_{1}(\bar{\sigma})=\overline{p(\sigma)} \tag{37}
\end{equation*}
$$

Proposition A.4. Consider $\sigma \in \Pi_{n}$, with $\# \sigma=m, p \in \mathfrak{G}_{n}$, and let $p_{1} \in \mathfrak{G}_{m}$ be the permutation that gives the correct order of the blocks of $p(\sigma)$.
(i) For $A_{1}, \ldots, A_{n} \in \mathcal{B}([0, T])$,

$$
q_{\sigma}^{-1}\left(p\left(A_{1} \times \cdots \times A_{n}\right)\right)=p_{1}^{-1}\left(q_{p(\sigma)}^{-1}\left(A_{1} \times \cdots \times A_{n}\right)\right)
$$

(ii) $p^{-1} \circ q_{\sigma}=q_{p(\sigma)} \circ p_{1}$.

Proof. (i) Let $B_{1}, \ldots, B_{m}$ the blocks of $\sigma$ (ordered). We have

$$
\begin{aligned}
q_{\sigma}^{-1}\left(p\left(A_{1} \times \cdots \times A_{n}\right)\right) & =q_{\sigma}^{-1}\left(A_{p(1)} \times \cdots \times A_{p(n)}\right) \\
& =\left(\bigcap_{i \in B_{1}} A_{p(i)}\right) \times \cdots \times\left(\bigcap_{i \in B_{m}} A_{p(i)}\right) \\
& =\left(\bigcap_{i \in p\left(B_{1}\right)} A_{i}\right) \times \cdots \times\left(\bigcap_{i \in p\left(B_{m}\right)} A_{i}\right) \\
& =G_{1} \times \cdots \times G_{m},
\end{aligned}
$$

where

$$
G_{j}=\bigcap_{i \in p\left(B_{j}\right)} A_{i}, \quad j=1, \ldots, m
$$

Since $p_{1}$ gives the correct order of $p\left(B_{1}\right), \ldots, p\left(B_{m}\right)$,

$$
\begin{aligned}
q_{p(\sigma)}^{-1}\left(A_{1} \times \cdots \times A_{n}\right) & =\left(\bigcap_{i \in p\left(B_{p_{1}(1)}\right)} A_{i}\right) \times \cdots \times\left(\bigcap_{i \in p\left(B_{p_{1}(m)}\right)} A_{i}\right) \\
& =G_{p_{1}(1)} \times \cdots \times G_{p_{1}(m)}=p_{1}\left(G_{1} \times \cdots \times G_{m}\right),
\end{aligned}
$$

and then

$$
p_{1}^{-1}\left(q_{p(\sigma)}^{-1}\left(A_{1} \times \cdots \times A_{n}\right)\right)=G_{1} \times \cdots \times G_{m}
$$

(ii) Consider $\mathbf{y}=\left(y_{1}, \ldots, y_{n}\right) \in[0, T]^{n}$. Since $\{\mathbf{y}\}=\left\{y_{1}\right\} \times \cdots \times\left\{y_{n}\right\}$,

$$
\left(p^{-1} \circ q_{\sigma}\right)^{-1}(\{y\})=q_{\sigma}^{-1}(\{p(\mathbf{y})\})={ }_{(*)} p_{1}^{-1}\left(q_{p(\sigma)}^{-1}(\{\mathbf{y}\})\right)=\left(q_{p(\sigma)} \circ p_{1}\right)^{-1}(\{y\}),
$$

where the equality $\left({ }^{*}\right)$ is due to part (i).

## REFERENCES

[1] Anshelevich, M. (2000). Free stochastic measures via noncrossing partitions. Adv. Math. 155 154-179. MR1789851
[2] Anshelevich, M. (2002). Free stochastic measures via noncrossing partitions. II. Pacific J. Math. 207 13-30. MR1973343
[3] Anshelevich, M. (2002). Itô formula for free stochastic integrals. J. Funct. Anal. 188 292315. MR1878639
[4] Anshelevich, M. (2004). q-Lévy processes. J. Reine Angew. Math. 576 181-207. MR2099204
[5] Anshelevich, M. (2005). Linearization coefficients for orthogonal polynomials using stochastic processes. Ann. Probab. 33 114-136. MR2118861
[6] Dellacherie, C. and Meyer, P.-A. (1980). Probabilités et Potentiel. Chapitres V à VIII, Revised ed. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics] 1385. Hermann, Paris. MR566768
[7] Engel, D. D. (1982). The multiple stochastic integral. Mem. Amer. Math. Soc. 38 v+82. MR660396
[8] Hu, Y. Z. and MEYER, P. A. (1988). Sur les intégrales multiples de Stratonovitch. In Séminaire de Probabilités, XXII. Lecture Notes in Math. 1321 72-81. Springer, Berlin. MR960509
[9] Hu, Y. Z. and Meyer, P. A. (1993). Sur l'approximation des intégrales multiples de Stratonovich. In Stochastic Processes: A Festschrift in Honor of G. Kallianpur (S. Cambanis et al., eds.) 141-147. Springer, New York.
[10] ITô, K. (1951). Multiple Wiener integral. J. Math. Soc. Japan 3 157-169. MR0044064
[11] ITô, K. (1956). Spectral type of the shift transformation of differential processes with stationary increments. Trans. Amer. Math. Soc. 81 253-263. MR0077017
[12] Johnson, G. W. and Kallianpur, G. (1990). Some remarks on Hu and Meyer's paper and infinite-dimensional calculus on finitely additive canonical Hilbert space. Theory Probab. Appl. 34 679-689. MR1036713
[13] Kallenberg, O. and Szulga, J. (1989). Multiple integration with respect to Poisson and Lévy processes. Probab. Theory Related Fields 83 101-134. MR1012497
[14] Krakowiak, W. and Szulga, J. (1988). A multiple stochastic integral with respect to a strictly $p$-stable random measure. Ann. Probab. 16 764-777. MR929077
[15] Kwapień, S. and Woyczyński, W. A. (1992). Random Series and Stochastic Integrals: Single and Multiple. Birkhäuser, Boston, MA. MR1167198
[16] Masani, P. R. (1997). The homogeneous chaos from the standpoint of vector measures. Philos. Trans. R. Soc. Lond. Ser. A 355 1099-1258. MR1460708
[17] MEyER, P. A. (1976). Un cours sur les intégrales stochastiques. In Séminaire de Probabilités, X. Lecture Notes in Math. 511 245-400. Springer, Berlin. MR0501332
[18] Millar, P. W. (1972). Stochastic integrals and processes with stationary independent increments. In Proc. Sixth Berkeley Sympos. Math. Statist. Probab. Vol. III: Probab. Theory 307-331. Univ. California Press, Berkeley, CA. MR0402922
[19] Nica, A. and Speicher, R. (2006). Lectures on the Combinatorics of Free Probability. London Mathematical Society Lecture Note Series 335. Cambridge Univ. Press, Cambridge. MR2266879
[20] Nualart, D. and Schoutens, W. (2000). Chaotic and predictable representations for Lévy processes. Stochastic Process. Appl. 90 109-122. MR1787127
[21] Peccati, G. and TaQQu, M. S. (2010). Wiener Chaos: Moments, Cumulants and Diagram Formulae. A Survey with Computer Implementation. Springer, Berlin.
[22] Pérez-Abreu, V. (1991). On the $L^{2}$-theory of product stochastic measures and multiple Wiener-Itô integrals. Stoch. Anal. Appl. 9 53-70. MR1096732
[23] Rosiński, J. and Woyczyński, W. A. (1984). Products of random measures, multilinear random forms, and multiple stochastic integrals. In Measure Theory, Oberwolfach 1983 (Oberwolfach, 1983). Lecture Notes in Math. 1089 294-315. Springer, Berlin. MR786707
[24] Rota, G.-C. and Wallstrom, T. C. (1997). Stochastic integrals: A combinatorial approach. Ann. Probab. 25 1257-1283. MR1457619
[25] Sato, K.-I. (1999). Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics 68. Cambridge Univ. Press, Cambridge. MR1739520
[26] Schoutens, W. (2000). Stochastic Processes and Orthogonal Polynomials. Lecture Notes in Statistics 146. Springer, New York. MR1761401
[27] Solé, J. L. and Utzet, F. (1990). Stratonovich integral and trace. Stochastics Stochastics Rep. 29 203-220. MR1041036
[28] Solé, J. L. and Utzet, F. (1991). Intégrale multiple de Stratonovich pour le processus de Poisson. In Séminaire de Probabilités, XXV. Lecture Notes in Math. 1485 270-283. Springer, Berlin. MR1187785
[29] Stanley, R. P. (1997). Enumerative Combinatorics. Vol. 1. Cambridge Studies in Advanced Mathematics 49. Cambridge Univ. Press, Cambridge. MR1442260
[30] Vershik, A. M. and Tsilevich, N. V. (2003). Fock factorizations and decompositions of the $L^{2}$ spaces over general Lévy processes. Uspekhi Mat. Nauk 58 3-50. MR1998773

Department of Mathematics Edifici C
Universitat Autònoma de Barcelona
08193 BELLATERRA (BARCELONA)
SPAIN
E-MAIL: farre@mat.uab.cat
mjolis@mat.uab.cat
utzet@mat.uab.cat


[^0]:    Received March 2009; revised November 2009.
    ${ }^{1}$ Supported by Grant MTM2009-08869 Ministerio de Ciencia e Innovación and FEDER.
    AMS 2000 subject classifications. Primary 60G51, 60H99; secondary 05A18, 60G57.
    Key words and phrases. Lévy processes, Stratonovich integral, Hu-Meyer formula, random measures, Teugels martingales.

