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MULTIPLE TESTING VERSUS MULTIPLE ESTIMATION. IMPROPER CONFIDENCE SETS. ESTIMATION OF DIRECTIONS AND RATIOS[1]

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0. Summary. The "S-method" of multiple comparison ([5]; [6], Section 3.5) was intended for multiple estimation, possibly combined with multiple testing. It is shown that if only multiple testing is desired a certain "modified S-method" is more powerful. While this result is of some theoretical interest, it is recommended after a discussion of the relative advantages of the two methods, that the new one generally not be used in applications. The multiple testing problems considered are related to estimating the direction of a vector or its unoriented direction–estimation problems which also have an inherent interest. A confidence set for a parameter point is called improper if the probability that it gives a trivially true statement is positive. The problems of estimating the direction and unoriented direction of a vector are reformulated to permit solution by proper confidence sets. In the case of the unoriented direction of a q-dimensional vector the confidence sets yield solutions of the problem of joint estimation of $q-1$ ratios and the problem of multiple estimation of all ratios in a certain infinite set. Specializing to the case $q = 2$ yields a proper confidence set as a substitute for Fieller's improper confidence set for a ratio.

1. Introduction. The reader interested only in Fieller's problem of estimating a ratio may proceed directly to the discussion following the Corollary near the end of Section 5. The reader not interested in multiple testing but in the estimation of directions and ratios may read through the sentence containing equation (3) and then skip to Section 3. We use the term "testing" to include the trichotomous procedure where if a hypothesis $\theta = 0$ is rejected by a two-tailed test we decide on one of the alternatives $\theta > 0$ or $\theta < 0$. "Estimation" refers to estimation by confidence intervals or other confidence sets.

The problems will be treated under the underlying assumptions Ω usually made in the analysis of variance,

$$\Omega: z_i = \sum_{j=1}^{p} x_{ji}\beta_j + e_i \quad (i = 1, \cdots, n), \text{ where } \{e_i\} \text{ are independently } N(0, \sigma^2);$$

here $\{z_1, \cdots, z_n\}$ is the sample, $\{\beta_j\}$ and σ^2 are unknown parameters, and $\{x_{ji}\}$ are known constants.

In the beginning, in Section 2, we consider multiple estimation and multiple testing of all members θ of a given q-dimensional space L of estimable functions, to be defined in a moment; at the end, in Section 5, we treat multiple estimation of the ratios of all pairs of θs in L, also the joint estimation of $q-1$ ratios. These esti-

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mation results for ratios are based on those for directions of vectors in Section 4. In the estimation problems of Sections 4 and 5, certain improper confidence sets arise in a natural way; we anticipate this in Section 3 by a general discussion of improper confidence sets.

By an estimable function we mean a linear function of the $\{\beta_j\}$, $\theta = \sum_{j=1}^{p} a_j \beta_j$, where the $\{a_j\}$ are known constants, for which there exists an unbiased estimate linear in the $\{z_i\}$. A q-dimensional space L of estimable functions consists of all linear combinations $\sum_{i=1}^{q} h_i \theta_i$, where $\{\theta_i\}$ are q linearly independent estimable functions, and $\{h_i\}$ are known constants. Many examples of Ω and L may be found in [6], chapters 3–6; there $\{z_i\}$, θ, $\{a_j\}$, $\hat{\sigma}^2$ are written $\{y_i\}$, ψ, $\{c_j\}$, s^2, respectively, while here the notation is that of [5] to which we make frequent references.

Since some confusion might arise from our using the same symbol θ for an estimable function and its value, we remark that the *function* $\theta = \sum_{j=1}^{p} a_j \beta_j = a'\beta$, where a and β are p-dimensional vectors, is defined by its coefficient vector a; its *value* $\theta = a'\beta_0$ depends on the true value β_0 of the vector β of parameters. By "the least-squares estimate $\hat{\theta}$" of an estimable function $\theta = a'\beta$ we mean $\hat{\theta} = a'\hat{\beta}$, where $\hat{\beta}$ is a least-squares estimate of β: $\hat{\theta}$ is unique although $\hat{\beta}$ is in general not. The variance of $\hat{\theta}$ will be of the form $\sigma_{\hat{\theta}}^2 = b^2\sigma^2$, where b^2 is a known constant; $\hat{\sigma}_{\hat{\theta}}^2$ will denote the estimate $b^2\hat{\sigma}^2$, where $\hat{\sigma}^2$ is the error mean square with $\nu = n-r$ df (degrees of freedom).

2. Multiple testing versus multiple estimation of all θ in L. A method of multiple estimation of the θ in L, which we call the S-method, can be based on the following theorem. We choose a confidence coefficient $1-\alpha$, or a related positive constant S, the relation being

$$\alpha = \Pr\{F(q,\nu) \geqq S^2/q\}, \tag{1}$$

where $F(q,\nu)$ denotes an F-variable with q and ν df. It will be more natural now to think of the S-method as defined by a given α, but later it will sometimes be more convenient to think of it as defined by a given S.

THEOREM 1. *Under the assumptions Ω the probability is $1-\alpha$ that the values of all estimable functions in L simultaneously satisfy the inequalities*

$$\hat{\theta} - S\hat{\sigma}_{\hat{\theta}} \leqq \theta \leqq \hat{\theta} + S\hat{\sigma}_{\hat{\theta}}, \tag{2}$$

where α and S are related by (1).

The application of the theorem to multiple estimation of the θ in a given q-dimensional space L of estimable functions consists in choosing a confidence level $1-\alpha$, determining S from (1), and making the statements (2) for as many θ in L as we please. The probability that all the statements will be correct is then $\geqq 1-\alpha$.

Theorem 1 was proved in [5] for the special case where L is a certain space of contrasts; the proof there is easily extended to the general case by using the results of the next paragraph about $\hat{\eta}$ and $\hat{\sigma}^2$. (To apply results of [5] we note that $\hat{\eta}$ and $\hat{\sigma}^2$ have the same joint distribution as in this paper if there we set $C = 1$, and replace

$k-1$ by q, and the word "contrast" by "estimable function in L".) A more complicated proof of the general case is given in ([6], Section 3.5).

Let $\hat{L}$ denote the set of least-squares estimates $\hat{\theta}$ of the θ in L; $\hat{L}$ is easily seen to be a q-dimensional space of linear functions of the $\{z_i\}$: If $\{\theta_1, \cdots, \theta_q\}$ is a basis for L, then $\{\hat{\theta}_1, \cdots, \hat{\theta}_q\}$ is a basis for $\hat{L}$, since for any θ in L there exist $\{h_i\}$ such that $\theta = \sum_1^q h_i \theta_i$, hence $\hat{\theta} = \sum_1^q h_i \hat{\theta}_i$, so the $\hat{\theta}_i$ span $\hat{L}$; furthermore they are linearly independent, since by taking expectations we find $\sum_1^q h_i \hat{\theta}_i = 0$ implies $\sum_1^q h_i \theta_i = 0$. Choose an orthogonal basis $\{\hat{\eta}_1, \cdots, \hat{\eta}_q\}$ for $\hat{L}$, normalized so that the covariance matrix of the vector $\hat{\eta} = (\hat{\eta}_1, \cdots, \hat{\eta}_q)'$ is $\sigma^2 I$, where I is the $p \times p$ identity matrix. (The construction is that used in deriving the canonical form of the linear hypothesis.) Let $\eta = E(\hat{\eta}) = (\eta_1, \cdots, \eta_q)'$. Then the $\{\eta_i\}$ form a basis for L, and so for any θ in L there exist $\{d_i\}$ such that $\theta = \sum_1^q d_i \eta_i = d'\eta$. Obviously, $\hat{\eta}$ and $\hat{\sigma}^2$ are statistically independent, $\hat{\eta}$ is $N(\eta, \sigma^2 I)$, and $\hat{\sigma}^2$ is $\sigma^2 \chi^2(v)/v$, where $\chi^2(v)$ denotes a chi-square variable with v df. We introduce a q-dimensional y-space of points $(y_1, \cdots, y_q)$ for graphing the parameter point or vector η, its estimate $\hat{\eta}$, and other quantities.

At various points in this paper we shall need to consider the hypothesis

$$H: \theta_1 = \theta_2 = \cdots = \theta_q = 0, \tag{3}$$

where the $\{\theta_i\}$ constitute a basis for L; H is equivalent to the statement that all θ in L have the value zero. Let us say for any θ in L that its estimate $\hat{\theta}$ is "significantly different from zero by the S-criterion" if the interval (2) fails to cover $\theta = 0$, that is, if $|\hat{\theta}| > S\hat{\sigma}_{\hat{\theta}}$. The relation (noted in [5] and [6], Section 3.5) of the S-method to the F-test is that some $\hat{\theta}$ in $\hat{L}$ will be significantly different from zero by the S-criterion if and only if the F-test rejects H at level α; here α and S are related by (1).

The following procedure considered in [5] implies a method of multiple testing: for any estimable function θ in L we may make one of the following three statements about its estimate $\hat{\theta}$: (i) $\hat{\theta}$ is not (by the S-criterion) significantly different from zero, (ii) $\hat{\theta}$ is significantly positive, or (iii) $\hat{\theta}$ is significantly negative, according as (i) $|\hat{\theta}| \leqq S\hat{\sigma}_{\hat{\theta}}$, (ii) $\hat{\theta} > S\hat{\sigma}_{\hat{\theta}}$, or (iii) $\hat{\theta} < -S\hat{\sigma}_{\hat{\theta}}$.

We may regard making one of the three statements (i), (ii), or (iii) for an estimable function in L as a test of the value θ of the function: If we make a statement (i) we accept the hypothesis $\theta = 0$; if (ii), the alternative $\theta > 0$; if (iii), the alternative $\theta < 0$. It will also be convenient to say we make decisions (i) $\theta = 0$, (ii) $\theta > 0$, or (iii) $\theta < 0$. Besides this trichotomous test or three-decision problem we shall wish later to refer to a dichotomous test or two-decision problem corresponding to (i) $\theta = 0$ or ($\bar{\text{i}}$) $\theta \neq 0$. Decision (i), like accepting a null hypothesis, has a different quality from (ii), (iii), or ($\bar{\text{i}}$), in that we generally do not really judge that θ is exactly zero, but rather that the evidence is not sufficiently discordant with the value $\theta = 0$ to reject this value. Furthermore, if we considered applications of this theory to real statistical problems we would find, leaving aside the question of how meaningful anyhow is the statement that θ is "exactly zero", that there usually would be some small ill-defined interval about zero in which we would prefer to take the action consequent on making decision (i) even though $\theta \neq 0$.

The method of multiple testing consists in selecting as many θ in L as we please and making this test of each. Its operating characteristic is partially characterized by certain probabilities P_1 and P_2, analogous to the probabilities of avoiding the two types of error in the Neyman–Pearson theory of testing hypotheses: P_1 is the probability of the event $\mathscr{E}_1$ that statement (i) would be made for all estimable functions θ in L whose true values are zero. We shall not define P_2 here; it is defined and evaluated in [5]. Since the hypothesis H in (3) is equivalent to $\eta = 0$, the value of P_1 is calculated in [5] to be

$$(4) \qquad \begin{aligned} P_1 &= \Pr\{F(q,\nu) \leqq S^2/q\} \quad \text{if} \quad H \quad \text{is true,} \\ &= \Pr\{F(q-1,\nu) \leqq S^2/(q-1)\} \quad \text{if} \quad H \quad \text{is false.} \end{aligned}$$

Heretofore the use of the S-method has been considered for multiple estimation alone or for a combination of multiple estimation with multiple testing. Now we shall consider its possible use for multiple testing alone.

It will simplify the typography in the rest of this paper, where values of q, ν, and α will be understood from the context, to write

$$(5) \qquad S_1 = [qF_\alpha(q,\nu)]^{\frac{1}{2}}, \qquad S_2 = [(q-1)F_\alpha(q-1,\nu)]^{\frac{1}{2}},$$

where $F_\alpha(q, \nu)$ denotes the upper α-point of $F(q, \nu)$. It is easy to show $S_1 > S_2$: First we prove an inequality we need below anyway, on the two values of $1-P_1$ given by (4). Let w_{q-1}, w_ν, and w_1 denote independent chi-square variables with $q-1$, ν, and 1 df respectively. Then $\Pr\{w_{q-1} > S^2 w_\nu/\nu\} < \Pr\{w_1 + w_{q-1} > S^2 w_\nu/\nu\}$, or

$$(6) \qquad \Pr\{F(q-1,\nu) > S^2/(q-1)\} < \Pr\{F(q,\nu) > S^2/q\}$$

for any S. Now putting $S = S_2$ gives $\alpha < \Pr\{F(q,\nu) > S_2{}^2/q\}$, whence

$$S_2{}^2/q < F_\alpha(q,\nu) = S_1{}^2/q, \quad \text{or} \quad S_1 > S_2.$$

We may define *significance level* and *size* as follows for any method of multiple testing which for each θ in L makes one of the three decisions (i), (ii), or (iii), or one of the two decisions (i) or ($\bar{\text{i}}$): Let L_0 be the set of those θ in L whose true value is zero, $L_0 = L_0(\eta)$. The event $\mathscr{E}_1$ for the method is that it makes decision (i) for all θ in L_0, and its probability is denoted by P_1. The probability $1-P_1$ of the complementary event is a function of η and σ^2, and any upper bound for $1-P_1$ will be called a significance level, and the least upper bound, the size of the method. These are then bounds on the probability that one of the undesired decisions (ii) or (iii), or else the undesired decision ($\bar{\text{i}}$), would be made for one or more θ in L_0.

For the S-method of multiple testing with $S = S_1$ of (5), we see from (4) and (6) that α is a significance level, and indeed α is the size (unless the assumption $\eta \neq 0$ is added to Ω). A more powerful method of multiple testing of the θ in L at the same significance level α is suggested by considering the probability $1-P_1$ for the S-method with $S = S_2$. This is α for all η except $\eta = 0$ (H true), where it is $> \alpha$ by (6). Can we somehow rule out the possibility that $\eta = 0$? If not, can we modify the S-method with $S = S_2$ to lower $1-P_1$ when $\eta = 0$ so that its value is then $\leqq \alpha$ without raising it elsewhere, in a desirable way so as to give a more powerful method of multiple testing than the unmodified S-method of size α (i.e., with $S = S_1$)?

Concerning the first possibility, we would not wish to rest on the argument that in general η cannot be "exactly zero", for the reason indicated above, namely, that in most applications there is some small neighborhood of $\eta = 0$ where we prefer to act as though η were zero. However, in some situations it may be acceptable to assume that there is some neighborhood of $\eta = 0$ in which η cannot lie. In applications where we are willing to assume η is bounded away from zero we may thus use the S-method of multiple testing with $S = S_2 < S_1$, and its size will then be α. Its operating characteristic is partially characterized by the probability $1-P_1$ which is exactly α no matter what the values of the parameters, and the probability P_2 given in [5].

Concerning the second possibility, an affirmative answer is implied by the following "modified S-method": First test the hypothesis H in (3) by the standard α-level F-test, in which H is rejected if the F-statistic is $> F_\alpha(q, \nu)$. If H is accepted we make decision (i) for all θ in L. If H is rejected we make decision (i), (ii), or (iii) for all θ in L according as $\hat{\theta}$ satisfies $|\hat{\theta}| \leqq S_2\hat{\sigma}_{\hat{\theta}}$, $\hat{\theta} > S_2\hat{\sigma}_{\hat{\theta}}$, or $\hat{\theta} < -S_2\hat{\sigma}_{\hat{\theta}}$, where S_2 is given by (5). (For the dichotomous method we make decisions (i) or ($\bar{\text{i}}$) according as $|\hat{\theta}|$ is $\leqq$ or $> S_2\hat{\sigma}_{\hat{\theta}}$.) It is easy to see that this modified S-method has the same size α as the (unmodified) S-method with $S = S_1$, and to show that it is more powerful by comparing the behavior of the two methods for any single chosen θ in L. To compare the behavior of the two methods globally for all θ in L we turn to the geometric picture in the q-dimensional y-space. The estimable functions θ in L may be represented by their coefficient vectors d, where $\theta = d'\eta$, and d is any vector in the y-space. Let $\mathscr{S}_1$ denote the $(1-\alpha)$-confidence sphere for η,

$$(7) \qquad |y-\hat{\eta}| \leqq S_1\hat{\sigma}.$$

Each of the two methods makes decision (i) for all d if and only if $\mathscr{S}_1$ covers the origin 0. If $\mathscr{S}_1$ does not cover 0 then the set $\mathscr{D}$ of d for which each method makes decision (ii) forms a cone of revolution of one nappe with vertex at 0 (not included in $\mathscr{D}$), axis along $\hat{\eta}$, and semivertex angle $\arccos(S\hat{\sigma}/|\hat{\eta}|)$, where $S = S_1$ for the S-method and $S = S_2$ for the modified S-method: This may be seen by interpreting as the signed length of the projection of $\hat{\eta}$ on d the left side of the inequality $\hat{\eta}'d/|d| > S\hat{\sigma}$, obtained by substituting $\hat{\theta} = d'\hat{\eta}$ and $\hat{\sigma}_{\hat{\theta}} = \hat{\sigma}|d|$ in $\hat{\theta} > S\hat{\sigma}_{\hat{\theta}}$. The cone $\mathscr{D}$ of coefficient vectors d for which the modified S-method makes decision (ii) is thus greater than that of the unmodified S-method by an amount corresponding to its greater semivertex angle. A similar statement applies to the cone $-\mathscr{D}$ of coefficient vectors for which the statement (iii) is made, $-\mathscr{D}$ being the reflection of $\mathscr{D}$ in 0. If the dichotomous methods were to be compared, the corresponding cone would be that of two nappes, $\mathscr{D}' = \mathscr{D}\cup(-\mathscr{D})$, which consists of the d for which decision ($\bar{\text{i}}$) is made.

The probabilities P_1 and P_2 for the modified S-method were calculated as triple integrals. To arrive at some numerical measure of the relative efficiency of the two methods, we may imagine replicating N times the experiment with n observations and using the S-method of multiple testing, or N' times and using the modified S-method, both of size α, and calculating the relative efficiency of the S- relative to the modified S-method to be $e = N'/N$, where N and N' give the same power P_2 in some sense. The ratio N'/N required to make P_2 equal to a given $1-\beta$ was found to depend on the unknown ratio $|\eta|/\sigma$ in a rather complicated way, so the greatest lower bound of P_2 was used, and this was set equal to $1-\beta$. This gave

$$\frac{N'}{N} = \left[\frac{\chi_\beta(q-1)+\chi_\alpha(q-1)}{\chi_\beta(q-1)+\chi_\alpha(q)}\right]^2, \tag{8}$$

where $\chi_\alpha(q)$ denotes the upper α-point of χ with q df. The referees suggested that these lengthy calculations be deleted since the results led the author to recommend that the modified S-method not be generally used in practice. Nevertheless the existence of a method of multiple testing more powerful than the S-method is of some theoretical interest, and its publication may save another worker from spending a long time on it. A table of the relative efficiency $e = N'/N$ showed that it does not depend very much on the values of α and β chosen, but it increases markedly with q (this table could be reconstituted from Table 1). However, with increasing q the number n of observations taken in actual experiments would in general increase, and the *number* $n(1-e)$ of observations saved, by using the modified S-method instead of the S-method, would then of course not decrease as rapidly as the proportion $1-e$ of observations saved. Indeed, if the number n were proportional to q, say n equals cq (approximately), then the number of observations saved would be $cq(1-e)$, and in each column of Table 1 we see this is practically constant for $q > 2$ or 3. We remark that in most of Table 1, $q(1-e) < .5$, and $q(1-e) \leqq .65$ everywhere in the table: Thus, if n were of the magnitude of about $5q$, the number of observations saved would, according to this criterion, be about two for most α, β-combinations listed in the table, and at most three, and for n less than $5q$ the saving would be smaller. We remark that the number n of observations taken in a multiple inference problem will usually be large compared with 2.

TABLE 1

Values of $q(1-e)$, where $e = N'/N$ is the relative efficiency of the S-relative to the modified S-method of multiple testing

q	α: .10			.05			.01		
	β: .7	.3	.1	.5	.3	.1	.5	.3	.1
2	.65	.58	.49	.58	.52	.45	.46	.43	.37
3	.55	.50	.44	.50	.46	.41	.42	.39	.35
4	.52	.48	.42	.48	.44	.40	.41	.39	.35
5	.50	.47	.42	.47	.44	.40	.41	.39	.35
6	.49	.46	.42	.46	.43	.40	.41	.39	.36
8	.48	.45	.42	.46	.43	.40	.41	.39	.37
10	.48	.45	.42	.46	.43	.40	.42	.40	.37
14	.47	.45	.43	.46	.44	.41	.43	.41	.38
20	.47	.46	.43	.46	.44	.42	.43	.42	.40

When we balance the small saving in the number of observations with the modified S-method as compared with the S-method of multiple testing, indicated by the preceding calculations, against the advantage of a certain follow-up procedure possible with the latter but not the former, we may decide the latter is preferable in most applications. Use of the S-method of multiple testing permits the following estimation procedure, generalizing that possible with an ordinary two-tailed α-level test of a hypothesis $\mu = \mu_0$ in the case where the test is related to a $(1-\alpha)$-confidence interval for μ (the test rejects the hypothesis if and only if the confidence interval fails to cover μ_0): For any θ in L for which we have made decisions (ii), (iii), or ($\bar{\text{i}}$) by the S-method we may give an interval (2), which will be consistent with the decision, namely, completely to the right of $\theta = 0$ in case (ii), to the left in case (iii), and not including $\theta = 0$ in case ($\bar{\text{i}}$). The interval statement (2) following decisions (ii), (iii), or ($\bar{\text{i}}$), is stronger than the corresponding decision, which it implies. For some of the θ for which we have made decision (i) we may also wish to give the intervals (2); these will also be consistent, in including $\theta = 0$. Here the interval statement, being implied by the decision, is formally weaker, but is really stronger in its interpretation, since, as remarked above, decision (i) is interpreted to mean that the evidence is insufficient to reject $\theta = 0$, whereas the interval (2) may be interpreted to mean that while the evidence is insufficient to reject $\theta = 0$ it is sufficient to reject all values further from zero than the ends of the interval. The probability that all these interval statements are correct is $\geqq 1-\alpha$. Such consistent subsequent interval estimation with overall confidence level $1-\alpha$ is not possible with the modified S-method at significance level α. (The writer remarks here that in the case of the ordinary two-tailed test he prefers this way of indicating "how strongly the hypothesis is rejected by the evidence" to calculation of the "P-value"—the minimum significance level at which the test would reject the hypothesis, for the observed sample—because he cannot find a satisfactory rationale for the "P-value" procedure in its operating characteristic.)

3. Improper confidence sets. In treating the estimation problems raised in Sections 4 and 5 we shall encounter confidence sets which have the property that with positive probability the confidence set gives a trivially true statement about the parameter point. The writer proposes to call such confidence sets *improper*. He will at this point make their definition precise, and then advance certain objections to their use, in order not to have to digress from the main developments in Sections 4 and 5.

Suppose the distribution of a sample point x depends on a certain set of parameters; we shall think of these as the coordinates of a point φ, the "original parameter point," and denote its domain of possible values by Φ, the "original parameter space." We consider confidence sets $\mathscr{A} = \mathscr{A}(x)$ for a parameter point τ which is a specified function of φ, $\tau = g(\varphi)$. An example of the notation is the following from Section 4: x is the point $(z_1, \cdots, z_n)$ and $\varphi = (\beta_1, \cdots, \beta_p, \sigma)$ under the Ω-assumptions of Section 1, τ is the unoriented direction of the vector η, which is the function of $(\beta_1, \cdots, \beta_p)$ defined in Section 2. Let T be the "space" (i.e., the range) of the parameter point τ, $T = g(\Phi)$, so for all x, $\mathscr{A}(x)$ is a subset of T. The confidence set $\mathscr{A}$

will be said to give a trivially true statement about τ for some $x = x_0$ if $\tau \in \mathscr{A}(x_0)$ is implied by $x = x_0$. In the examples in Sections 4 and 5 the trivially true statements will all be of the form $\tau \in T$, that is, $\mathscr{A}(x_0) = T$. In other situations a trivially true statement might be of the form that $\mathscr{A}(x_0)$ is a proper subset of T, depending on x_0, for example: Suppose $x = (x_1, \cdots, x_n)$ is a random sample of n from the uniform distribution on $(\varphi - \frac{1}{2}, \varphi + \frac{1}{2})$, Φ is the real line, $\tau \equiv \varphi$, and for some $x = x_0$, $\mathscr{A}(x_0)$ is the interval $(V - \frac{1}{2}, U + \frac{1}{2})$, where U and V are the smallest and largest coordinates of x_0. A confidence set $\mathscr{A}$ is defined to be improper if for some φ the probability is positive that $\mathscr{A}$ gives a trivially true statement about τ.

Before considering the objections to the use of improper confidence sets let us clarify the following terminology: We shall say $\mathscr{A}$ is a $(1-\alpha)$-confidence set for τ if $\Pr\{\tau \in \mathscr{A}(x)\} \geqq 1-\alpha$ for all φ in Φ. (We may say $1-\alpha$ is a *confidence level* and reserve the name *confidence coefficient* for the greatest lower bound of $\Pr\{\tau \in \mathscr{A}(x)\}$, so that a confidence level corresponds to a significance level in testing, and the confidence coefficient to the size.) The reader not convinced by the writer's objections may still be willing to call these confidence sets improper and feel as free to use them as he does improper fractions or improper integrals.

Let us imagine a statistician using an improper confidence set $\mathscr{A}$ when he is acting as a statistical consultant, and suppose $\mathscr{A}$ turns out to give a trivially true statement. Should he give the client the trivially true statement with some explanation like the following? "The method I'm using requires my making a trivially true statement like this some of the time in order to ensure that my long-run proportion of true statements in using $(1-\alpha)$-confidence sets will be at least equal to my claimed value of $1-\alpha$. The probability of my making such trivial statements depends on the unknown true values of the parameters in the problems." Then what can the unfortunate client do about the confidence statement giving him no information, except ignore it? And perhaps wonder what the more fortunate clients are really getting? Or should the statistician conceal from the client that the method gave a trivially true statement (thus invalidating the frequency interpretation of the confidence set), and tell him instead that more data are needed? Perhaps even more puzzled than the client receiving such an explanation would be the more fortunate one receiving a non-trivial statement resulting from the use of an improper confidence set, if the statistician felt obligated to give him a similar explanation about its operating characteristic.

4. Estimating the direction of a vector. We may relate the multiple testing problem we considered in Section 2 to the problem[2] of estimating the direction of a vector. If

[2] The problem of estimating the direction of a vector was encountered by Box [2] when trying to estimate the direction of the gradient of the response function in an experiment with q quantitative factors. His solution in the form of the improper confidence set $\mathscr{B}$ described below is really a confidence set for the unoriented direction, whereas it is the oriented direction which is of interest in this example. Problems of estimation of the unoriented direction are equivalent to problems of estimating ratios (Section 5), and these arise in the biological and social sciences. Actually the author was led to these problems along the following path: the confidence sphere $\mathscr{S}_1$ may be

we knew the direction of the vector η, or if we knew the direction was undefined (because $\eta = 0$), then we would know for all θ in L which of the three decisions (i) (ii), (iii) is correct, and conversely, since $\theta = d'\eta$. A similar statement applies to the unoriented direction of η and the two decisions (i) and ($\bar{\text{i}}$). We will represent the direction of a vector geometrically by a ray from the origin 0 (with the same direction), and the unoriented direction by a line through 0, in both cases with the point 0 deleted.

In this section we consider the problem of finding confidence sets for the direction and unoriented direction of a vector, first for the above vector η, and then more generally for a vector ξ. In the first case we are given estimates $\hat{\eta}$ and $\hat{\sigma}^2$ such that they are independent, $\hat{\eta}$ is $N(\eta, \sigma^2 I)$, and $\hat{\sigma}^2$ is $\sigma^2\chi^2(\nu)/\nu$. In the more general case we are given estimates $\hat{\xi}$ and $\hat{\sigma}^2$ satisfying the assumptions Ω_1 below.

Since a confidence set for the direction of a vector is a union of rays from 0, it will be a cone, while a confidence set for the unoriented direction will be a cone symmetrical about 0; *it will be understood henceforth that the confidence cones always have vertex at* 0 *but* 0 *is excluded.* When we consider the symmetry of the spherical normal distribution of the estimate $\hat{\eta}$, we are led to seek a confidence set for the direction of η in the form of a cone of revolution of one nappe with axis along $\hat{\eta}$, and for the unoriented direction of η, a similar cone of two nappes. (Questions arise if $\hat{\eta}$ or η should be zero, but we shall avoid these in reformulating the problem later.) Such confidence cones for the direction or unoriented direction of the vector η are also confidence cones for the point $\eta = (\eta_1, \cdots, \eta_q)$, and conversely for such confidence cones for the point η.

Assuming for the moment that $\eta \neq 0$, the following cone, which we shall denote by $\mathscr{B}$, may be shown to be a $(1-\alpha)$-confidence set for the unoriented direction of η. Let $\mathscr{S}_2$ denote the sphere $|y-\hat{\eta}| \leqq S_2\hat{\sigma}$, where S_2 is defined by (5). If $\mathscr{S}_2$ does not contain 0, $\mathscr{B}$ is the cone of two nappes circumscribed about $\mathscr{S}_2$; if $\mathscr{S}_2$ contains 0, $\mathscr{B}$ is the whole q-dimensional space with 0 deleted. This result can be deduced from [5], for it may be shown that $\Pr\{\eta \in \mathscr{B}\}$ is the P_1 of (4) above; it will also follow directly from our later calculations. The confidence cone $\mathscr{B}$ was given by Box [2]. The special case for $q = 2$ is equivalent to Fieller's solution of the problem of estimating a ratio which will be described in Section 5. The writer would be loathe to use $\mathscr{B}$ because it is an improper confidence set. Since he does not recommend $\mathscr{B}$ he will not disentangle the hair-splitting question of the probability that $\mathscr{B}$ gives a true statement about the unoriented direction of η when $\eta = 0$. The problem of finding a confidence cone for the (oriented) direction seems not to have been treated before under our assumptions.[3]

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said to generate the S-method of multiple testing (in the sense of the discussion in the paragraph below (25)). What are the maximal confidence sets which generate the S- and modified S-methods of multiple testing?

[3] G. S. Watson ([7] which see for further references) has treated it under different assumptions; he assumes one has observations not on the vector but only on its direction, and the distribution assumed for the observed direction is not that determined by a multivariate normal distribution of the corresponding vector.

In order to avoid improper confidence sets we propose to reformulate the problem by seeking a confidence set for the point η, instead of directly for the direction of η, of the following nature: It will be either a cone (not the whole space with 0 deleted), thus giving confidence bounds for the direction of η, or it will be a sphere containing 0. In the latter case the interpretation is that there is insufficient information to determine any bounds on the direction of η because at the desired confidence level we can only conclude that the point η is in the sphere containing 0. This is not a trivial statement, and may, like a proper confidence interval, or a proper confidence cone, that turns out too wide to be useful, suggest what further data must be gathered for a good chance of a usefully narrow determination.

Let us designate by $\mathscr{R}$ the confidence set we are seeking for η in the case of the (oriented) direction. We begin with the $(1-\alpha)$-confidence sphere $\mathscr{S}_1$ for η, $|y-\hat{\eta}| \leqq S_1\hat{\sigma}$, and if $\mathscr{S}_1$ covers 0, we will take $\mathscr{R}$ to be $\mathscr{S}_1$. If $\mathscr{S}_1$ does not cover 0, then for $\mathscr{R}$ we might consider using the cone circumscribed about $\mathscr{S}_1$, since it is fairly obvious that the resulting $\mathscr{R}$ would cover η with probability $\geqq 1-\alpha$, and $=\alpha$ if $\eta = 0$. However, it became intuitively evident to the writer while struggling with this problem that it should be possible as $|\hat{\eta}|/\hat{\sigma}$ increases to shrink the radius of the sphere centered at the point $\hat{\eta}$, about which the cone is circumscribed, toward a limiting value $S_2\hat{\sigma}$: This is suggested by the geometry and probability distribution, and the related Figure 1, underlying the calculations below for $\Pr\{\eta \in \mathscr{R}\}$. It is also suggested by the above improper $(1-\alpha)$-confidence set $\mathscr{B}$ for the unoriented direction of η.

We shall denote by $f\hat{\sigma}$ the radius of the inscribed sphere $\mathscr{S}$, where f is a function of $|\hat{\eta}|/\hat{\sigma}$, to be further specified, which decreases from S_1 to S_2 as $|\hat{\eta}|/\hat{\sigma}$ increases from S_1 to ∞. The function f is conveniently considered as a function of the F-statistic for testing H in (3), $F = |\hat{\eta}|^2/(q\hat{\sigma}^2)$, which under H is distributed as $F(q, v)$, so

$$f = f(F), \tag{9}$$

and $f(F)$ is monotone non-increasing from the value S_1 to the value S_2 as F increases from $F_\alpha(q, v)$ to ∞. A useful form of the function (9) might be $f = S_2 + cF^{-\mu}$, in which case

$$f(F) = S_2 + [F_\alpha(q, v)/F]^\mu(S_1 - S_2) \qquad (\mu > 0). \tag{10}$$

When $\mathscr{R}$ is the cone circumscribed about $\mathscr{S}$ it consists of the points y satisfying

$$\hat{\eta}'y/|y| \geqq (|\hat{\eta}|^2 - f^2\hat{\sigma}^2)^{\frac{1}{2}}, \qquad y \neq 0. \tag{11}$$

This may be obtained from a figure showing the following in the two dimensional section containing the vectors $\hat{\eta}$ and y: the sphere $\mathscr{S}$, the cone $\mathscr{R}$, the sphere with diameter $\hat{\eta}$, radii to the intersection of the two spheres from the center of $\mathscr{S}$, and the projection of $\hat{\eta}$ on y, whose signed length is $\hat{\eta}'y/|y|$.

We consider $\Pr\{\eta \in \mathscr{R}\}$ first in the case when $\eta = 0$. Then $\eta \in \mathscr{R}$ if and only if $\mathscr{R} = \mathscr{S}_1$, which happens if and only if $F \leqq F_\alpha(q, v)$, and the probability of this is $1-\alpha$, since H is true. Later, for the problem of estimating the unoriented direction we will use a confidence set $\mathscr{R}'$ for η which also turns out to be $\mathscr{S}_1$ if $\mathscr{S}_1$ covers 0, but a certain cone symmetric in 0 if $\mathscr{S}_1$ does not cover 0. It follows that also $\Pr\{\eta \in \mathscr{R}'\} = 1-\alpha$ when $\eta = 0$.

The calculation of $\Pr\{\eta \in \mathscr{R}\}$, and later, of $\Pr\{\eta \in \mathscr{R}'\}$, in the case when $\eta \neq 0$ will be made in the space (actually, quarter-space) of three independent random variables $u, v, \hat{\sigma}$, defined as follows: For $\eta \neq 0$ we resolve $\hat{\eta}$ into two components, the first along η, and the second orthogonal to η; u is the signed length of the first component (the sign is that of $\hat{\eta}'\eta$), and v is the length of the second component ($v \geqq 0$). Then u, v, and $\hat{\sigma}$ are independent, u is $N(|\eta|, \sigma^2)$, v is $\sigma\chi(q-1)$, and $\hat{\sigma}$ is $\sigma\nu^{-\frac{1}{2}}\chi(\nu)$. The calculation will be pictured in Figure 1, which is a cross-section of the $(u, v, \hat{\sigma})$-space

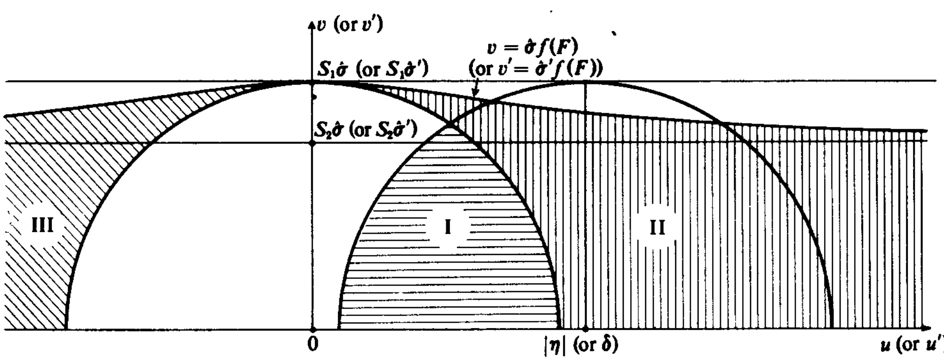

FIG. 1. Cross-section for $\hat{\sigma}$ = constant of events *I*, *II*, *III* in the u, v, $\hat{\sigma}$-space (or for $\hat{\sigma}'$ = constant in the u', v', $\hat{\sigma}'$-space).

for $\hat{\sigma}$ = constant. We shall also have to refer to the y-space of η and $\mathscr{R}$.

$\mathscr{R}$ will the sphere $\mathscr{S}_1$ if and only if $|\hat{\eta}|^2 \leqq S_1{}^2\hat{\sigma}^2$, or

$$u^2 + v^2 \leqq S_1{}^2\hat{\sigma}^2. \tag{12}$$

This is the inside of a cone with vertex at the origin of the $(u, v, \hat{\sigma})$-space, axis along the $\hat{\sigma}$-axis, and cross-section the inside of the semi-circle with center at 0 in Figure 1. In this case where $\mathscr{R} = \mathscr{S}_1$, $\mathscr{R}$ will cover η if and only if $|\hat{\eta}-\eta|^2 \leqq S_1{}^2\hat{\sigma}^2$, which we may write

$$(u-|\eta|)^2 + v^2 \leqq S_1{}^2\hat{\sigma}^2 \tag{13}$$

by considering the projections of $\hat{\eta}-\eta$ on and normal to η. The event (13) is the inside of a cone in the $(u, v, \hat{\sigma})$-space, which is the same as the cone (12) translated a distance $|\eta|$ in the positive u-direction. Its cross-section is the semi-circle with center at $u = |\eta|$ in Figure 1. $\mathscr{R}$ will be a sphere and cover η if and only if (12) and (13) hold: We shall call this the event *I*; its cross-section is shown horizontally shaded in Figure 1; the cross-section will be empty if and only if $\hat{\sigma} < |\eta|/(2S_1)$.

$\mathscr{R}$ will be a cone if and only if (12) is violated. The cone $\mathscr{R}$ is circumscribed about the sphere $\mathscr{S}$, $|y-\hat{\eta}| \leqq f\hat{\sigma}$, and will cover η if and only if

(14) $$u > 0$$

and

(15) $$v \leqq \hat{\sigma} f(F),$$

where

(16) $$F = (u^2 + v^2)/(q\hat{\sigma}^2).$$

Thus $\mathscr{R}$ will be a cone and cover η if and only if (14) and (15) hold and (12) does not: We shall call this the event II; its cross-section is shown vertically shaded in Figure 1. If we write the equation of its upper boundary $v = \hat{\sigma} f(F)$ in the form

(17) $$v\hat{\sigma}^{-1} = f(q^{-1}(u\hat{\sigma}^{-1})^2 + q^{-1}(v\hat{\sigma}^{-1})^2),$$

we see that II is a cone with vertex at the origin of the $(u, v, \hat{\sigma})$-space. Equation (17) may be solved for u,

(18) $$u = \hat{\sigma}[qf^{-1}(v\hat{\sigma}^{-1}) - (v^2\hat{\sigma}^{-2})]^{\frac{1}{2}}.$$

(If the function $f(F)$ had intervals of F where f is constant, the indeterminacy of f^{-1} would cause no real difficulty in the definition of the integrals where (18) is later used.) For the cross-section curve, where $\hat{\sigma}$ = constant, of the boundary surface (17) it may be verified that as u varies from 0 to ∞, v is non-increasing, the curve starting from $(0, S_1\hat{\sigma})$, remaining above the semicircle centered at 0 in Figure 1, and becoming asymptotic to the line $v = S_2\hat{\sigma}$.

For the case of the unoriented direction we will modify the confidence set $\mathscr{R}$ to the set $\mathscr{R}'$, which when $\mathscr{R}$ is a sphere is the same as $\mathscr{R}$, and when $\mathscr{R}$ is a cone is the two-napped cone consisting of $\mathscr{R}$ and its reflection in the origin of the y-space; from (11) we see that $\mathscr{R}'$ is then the cone

(19) $$|\hat{\eta}'y| \geqq |y|(|\hat{\eta}|^2 - f^2\hat{\sigma}^2)^{\frac{1}{2}}, \qquad y \neq 0.$$

(One would hope to use a smaller function $f(F)$ in this case, but it is indicated in the Appendix that, at least for $f(F)$ of the type (10), and $v = \infty$, no decrease of $f(F)$ of any practical importance is possible because the contribution of $\Pr\{III\}$ to $\Pr\{\eta \in \mathscr{R}'\} = \Pr\{\eta \in \mathscr{R}\} + \Pr\{III\}$ is negligible when $\Pr\{\eta \in \mathscr{R}\}$ is minimum; all these probabilities being functions of δ defined by (22)). Then $\mathscr{R}'$ will be a sphere and cover η if and only if event I happens. $\mathscr{R}'$ will be a cone and cover η if and only if (15) holds and (12) does not. This event is the union of II and its reflection III in the plane $u = 0$. The cross-section of III is shown slant-shaded in Figure 1.

For $\eta \neq 0$ we may express in terms of the probabilities of the disjoint events I, II, III, the probability that η is covered by $\mathscr{R}$ or $\mathscr{R}'$,

(20) $$\Pr\{\eta \in \mathscr{R}\} = \Pr\{I\} + \Pr\{II\},$$

(21) $$\Pr\{\eta \in \mathscr{R}'\} = \Pr\{I\} + \Pr\{II\} + \Pr\{III\}.$$

We remark first that $\Pr\{\eta \in \mathscr{R}\}$ and $\Pr\{\eta \in \mathscr{R}'\}$ are functions only of

(22) $$\delta = |\eta|/\sigma$$

(they depend of course also on α, q, ν, and the function $f(F)$, which are being held fixed); this may be seen by considering the events *I, II, III* in the space of the transformed variables $u' = u/\sigma$, $v' = v/\sigma$, $\hat{\sigma}' = \hat{\sigma}/\sigma$, whose joint distribution depends only on δ. Figure 1 will serve as a picture of the cross-section $\hat{\sigma}' =$ constant in the $(u', v', \hat{\sigma}')$-space if we replace the labels $u, v, S_1\hat{\sigma}, S_2\hat{\sigma}, |\eta|, v = \hat{\sigma}f(F)$ respectively by $u', v', S_1\hat{\sigma}', S_2\hat{\sigma}', \delta, v' = \hat{\sigma}'f(F)$, where F is the same function of $u', v', \hat{\sigma}'$ as of $u, v, \hat{\sigma}$ in (16). Hence the probabilities of the events *I, II, III* depend only on δ. We remark for later use that since the distribution of u is symmetrical about $u = |\eta|$, the distribution of $(u, v, \hat{\sigma})$ is symmetrical about the plane $u = |\eta|$, and the distribution of $(u', v', \hat{\sigma}')$ is symmetrical about the plane $u' = \delta$.

We note next that $h(\delta) = \Pr\{\eta \in \mathscr{R}\}$ has a saltus at $\delta = 0$, where it jumps from $1-\alpha$ to greater values. Contemplating Figure 1 (with the alternate labelling), we see that in the limit as $\delta \to 0$, the cross-section of the event *I* is the semi-circle centered at 0, and hence the limit of the event *I* is the cone (12) whose probability is $1-\alpha$. Thus as $\delta \to 0$, $\lim \Pr\{\eta \in \mathscr{R}\}$ exceeds $1-\alpha$ by $\lim \Pr\{II\}$. Similarly the saltus of $\Pr\{\eta \in \mathscr{R}'\}$ is $\lim \Pr\{II \cup III\}$.

Next we shall argue that $h(\delta) \to 1-\alpha$ as $\delta \to \infty$: Consider the conditional probability in the cross-section $\hat{\sigma}' = \text{constant} = c$. For $\delta > 2S_1\hat{\sigma}'$ the cross-section of *I* is empty. Since u' is $N(\delta, 1)$, the conditional probability in *II* or in $II \cup III$ approaches that in the strip $v' \leqq S_2\hat{\sigma}'$ as $\delta \to \infty$. The convergence is not uniform in c, but this causes no difficulty when we integrate against the probability density of $\hat{\sigma}'$, which drops off exponentially as $c \to \infty$. Thus $\lim h(\delta)$ equals the probability in the cone $v' \leqq S_2\hat{\sigma}'$, which is $1-\alpha$. The argument shows that as $\delta \to \infty$, $\lim \Pr\{II\} = \lim \Pr\{\eta \in \mathscr{R}\} = \lim \Pr\{\eta \in \mathscr{R}'\} = 1-\alpha$.

We shall conclude these easy calculations by establishing the following crude bound:

$$\Pr\{\eta \in \mathscr{R}\} > \tfrac{1}{2}[1-\alpha+\Pr\{F(q,\nu) \leqq q^{-1}(q-1)F_\alpha(q-1,\nu)\}]. \tag{23}$$

From Figure 1 we see that the cross-section of the event $I \cup II$ contains (a) the semi-infinite strip satisfying $v \leqq S_2\hat{\sigma}, u > |\eta|$, and (b) the quarter circle satisfying

$$(u-|\eta|)^2 + v^2 \leqq S_2{}^2\hat{\sigma}^2 \tag{24}$$

and $u < |\eta|$. From the symmetry of the distribution in the plane $u = |\eta|$, the probability in the cone whose cross-section is (a) is half that in the cone $v \leqq S_2\hat{\sigma}$, namely $1-\alpha$, and the probability in the cone whose cross-section is (b) is half that in the cone (24), namely, $\Pr\{F(q,\nu) \leqq q^{-1}(q-1)F_\alpha(q-1,\nu)\}$. This gives the bound (23); it is crude because we have ignored all the probability in $I \cup II$ outside the cone whose cross-section is (a)$\cup$(b). Numerical tabulation of the bound (23) showed that $\Pr\{\eta \in \mathscr{R}\} > 1-2\alpha$ for $\alpha \geqq .05$.

If at this point in our developments we consider the improper confidence set $\mathscr{B}$, described near the beginning of this section, for the unoriented direction of η, it is obvious that if $\eta \neq 0$, the probability of covering the unoriented direction is $1-\alpha$, since $\eta \in \mathscr{B}$ if and only if $(u, v, \hat{\sigma})$ falls in the cone $v \leqq S_2\hat{\sigma}$. The following remarks pertain to comparing the operating characteristics of $\mathscr{B}$ and $\mathscr{R}'$: The part of the probability $1-\alpha$ of coverage of η by $\mathscr{R}$ that comes from making the trivially true statement about the direction of η is the part in the cone

$$u^2+v^2 \leqq S_2{}^2\hat{\sigma}^2; \tag{25}$$

it is

$$\Pr\{F \leqq S_2{}^2/q\}, \tag{26}$$

where F is the F-statistic for testing the hypothesis (3), and is distributed as non-central F with q and ν df and non-centrality parameter given by the usual rule. If instead of the trivial statement about η our non-trivial statement that $\eta \in \mathscr{S}_1$ were made, the probability of coverage would be reduced by the probability inside the cone (25) and outside the cone (13): the cross-section of this solid could be obtained as the unshaded part of a semi-circle added in Figure 1 with center 0 and radius $S_2\hat{\sigma}$. This reduction of the probability of coverage may be regarded as the cost of replacing the trivially true statement by the non-trivial one.

All the probability calculations for $\mathscr{R}$ and $\mathscr{R}'$ made above, including the bound (23), are easily seen to be valid for the case where $f(F)$ is defined and monotone non-increasing for $F_\alpha(q,\nu) < F < \infty, f(F) \to S_2$ as $F \to \infty$, and $f(F) \to f_0$ as $F \to F_\alpha(q,\nu)$, where $S_1 \geqq f_0 \geqq S_2$; the integral expressions below for the probabilities are also easily modified for this case. The cross-section of the boundary (17) then starts in Figure 1 from $(\hat{\sigma}[S_1{}^2-f_0{}^2]^{\frac{1}{2}}, f_0\hat{\sigma})$ instead of $(0, S_1\hat{\sigma})$. A particular case of this more general $f(F)$ is $f(F) \equiv S_2$; this gives a confidence set which generates the modified S-method of multiple testing of size α. For this $f(F)$, when $\mathscr{R}$ is a cone it is that complementary to the cone $\mathscr{D}$, introduced below (7), with $S = S_2$, in the sense that $\mathscr{R}$ and $\mathscr{D}$ have the same axis and vertex but their semi-vertex angles are complementary. $\mathscr{R}$ generates[4] the modified S-method in the following sense: For any θ in L, say $\theta = d'\eta$ with d any fixed vector, the modified S-method makes decision (i) if and only if $\theta = 0$ for some $\eta \in \mathscr{R}$, decision (ii) if and only if $\theta > 0$ for all $\eta \in \mathscr{R}$, and decision (iii) if and only if $\theta < 0$ for all $\eta \in \mathscr{R}$. The confidence set $\mathscr{R}'$ for this $f(F)$ generates the dichotomous modified S-method, in which decision (i) is made if and only if $\theta = 0$ for some $\eta \in \mathscr{R}'$, decision ($\bar{\text{i}}$) if and only if $\theta \neq 0$ for all $\eta \in \mathscr{R}'$. For this confidence set $\mathscr{R}'$, and hence also for the $\mathscr{R}$ with the same $f(F)$, it can be seen that the hoped-for condition that the confidence coefficient is $\geqq 1-\alpha$ is violated, at least for large ν, by considering the limiting case $\nu = \infty$: Then σ is known, $\hat{\sigma} = \sigma$, and all the probability is in the two-dimensional space of Figure 1. If $|\eta| > 2S_1\sigma$ then I is empty and $II \cup III$ consists of the strip $v \leqq S_2\sigma$ with the part inside the semi-circle centered at 0 deleted, while the probability of the whole strip is $1-\alpha$. This shows $\Pr\{\eta \in \mathscr{R}') < 1-\alpha$ for $\delta > 2S_1$.

We shall now express $\Pr\{\eta \in \mathscr{R}\}$ and $\Pr\{\eta \in \mathscr{R}'\}$ in terms of triple integrals. These formulas are easily obtained by expressing the probabilities with the help of Figure

[4] The dichotomous (unmodified) S-method of multiple testing with $S = S_2$ is generated by the improper confidence set $\mathscr{R}$ discussed above. We remark that if a method of multiple testing is generated by a confidence set, the generating confidence set is in general not unique, but one might define a unique one as the maximal such set (which will in general be improper).

1 as triple integrals in the independent variables $u', v', \hat{\sigma}'$ defined below (22), and then dropping the primes. We shall use the following notation: The densities of $u', v', \hat{\sigma}'$ will be denoted by $p_1(u'), p_2(v'), p_3(\hat{\sigma}')$, so

$$(27) \qquad p_1(u) = \frac{e^{-\frac{1}{2}(u-\delta)^2}}{(2\pi)^{-\frac{1}{2}}}, \quad p_2(v) = \frac{e^{-\frac{1}{2}v^2} v^{q-2}}{2^{\frac{1}{2}(q-3)}\Gamma[\frac{1}{2}(q-1)]}, \quad p_3(\hat{\sigma}) = \frac{e^{-\frac{1}{2}\hat{\sigma}^2}\hat{\sigma}^{\nu-1}}{2^{\frac{1}{2}(\nu-2)}\Gamma(\frac{1}{2}\nu)},$$

the upper boundary (18) of II by $u' = U(v', \hat{\sigma}')$, so

$$(28) \qquad U(v,\hat{\sigma}) = \hat{\sigma}[qf^{-1}(v\hat{\sigma}^{-1}) - (v^2\hat{\sigma}^{-2})]^{\frac{1}{2}},$$

the left boundary of II by $u' = L(v', \hat{\sigma}')$, so

$$(29) \qquad L(v,\hat{\sigma}) = (S_1{}^2\hat{\sigma}^2 - v^2)^{\frac{1}{2}},$$

and hence the left boundary of I (when not empty) by $u' = \delta - L(v', \hat{\sigma}')$. Thus we find

$$(30) \qquad \Pr\{I\} = \int_{\frac{1}{2}\delta/S_1}^{\infty} \int_0^{L(\frac{1}{2}\delta,\hat{\sigma})} \int_{\delta - L(v,\hat{\sigma})}^{L(v,\hat{\sigma})} p(u,v,\hat{\sigma})\, du\, dv\, d\hat{\sigma},$$

$$(31) \qquad \Pr\{II\} = \left\{\int_0^{\infty}\int_0^{S_2\hat{\sigma}}\int_{L(v,\hat{\sigma})}^{\infty} + \int_{S_2\hat{\sigma}}^{S_1\hat{\sigma}}\int_{L(v,\hat{\sigma})}^{U(v,\hat{\sigma})}\right\} p(u,v,\hat{\sigma})\, du\, dv\, d\hat{\sigma},$$

$$(32) \qquad \Pr\{III\} = \left\{\int_0^{\infty}\int_0^{S_2\hat{\sigma}}\int_{-\infty}^{-L(v,\hat{\sigma})} + \int_{S_2\hat{\sigma}}^{S_1\hat{\sigma}}\int_{-U(v,\hat{\sigma})}^{-L(v,\hat{\sigma})}\right\} p(u,v,\hat{\sigma})\, du\, dv\, d\hat{\sigma},$$

where $p(u,v,\hat{\sigma}) = p_3(\hat{\sigma})p_2(v)p_1(u)$, the densities p_i are given by (27), $U(v,\hat{\sigma})$ by (28), $L(v,\hat{\sigma})$ by (29), and S_1 and S_2 by (5). $\Pr\{\eta \in \mathscr{R}\}$ and $\Pr\{\eta \in \mathscr{R}'\}$ are then found from (20) and (21).

Computer calculation of tables of $\Pr\{I\}$, $\Pr\{II\}$, and $\Pr\{III\}$ is discussed in the Appendix for the limiting case $\nu = \infty$, with $f(F)$ of the form (10). These tables give a partial determination of the operating characteristics of the confidence sets $\mathscr{R}$ and $\mathscr{R}'$, and permit numerical evaluation of the confidence coefficients. It is found there that for $\alpha = .10$ and $.05$ the confidence coefficient of $\mathscr{R}$ is $1-\alpha$ if $\mu = 1$ when $q = 2$, $\mu = 3/2$ when $q = 3$, and $\mu = 2$ when $q \geqq 4$, and that if we are willing to accept confidence coefficients of .898 and .949 instead of .900 and .950 we may use $\mu = 3/2$ when $q = 2$, and $\mu = 2$ when $q = 3$. It is also indicated that the values $\mu = 1.35$ for $q = 2$ and $\mu = 1.75$ for $\mu = 3$ give confidence coefficients of .900 and .950. These numerically determined confidence coefficients are expected to be correct within .001. It is also found that the confidence coefficients of $\mathscr{R}'$ are to three decimals the same as for $\mathscr{R}$ with the same μ. It is recommended that these values of μ be used also with finite μ, pending computer calculations of (30)–(32).

We turn now to the more general case of estimating the direction or unoriented direction of a vector $\xi = (\xi_1, \cdots, \xi_q)$ under the following assumptions, which we shall denote by Ω_1. (These might be regarded as a special case of Ω and L, but we use the ξ-notation for greater clarity in this problem.)

Ω_1: We have estimates $\{\hat{\xi}_i\}$ which are jointly normal. The q-variate distribution of $\hat{\xi} = (\hat{\xi}_1, \cdots, \hat{\xi}_q)$ has mean ξ and covariance matrix $\sigma^2 B$, where $B = (b_{ij})$ is a known non-singular matrix. We have also an independent estimate $\hat{\sigma}^2$ of σ^2 such that $\hat{\sigma}^2$ is distributed as $\sigma^2\chi^2(\nu)/\nu$.

The problem of finding a confidence set for the direction or unoriented direction of ξ we shall reformulate as we did that for η. Indeed we shall solve it by transforming to ξ our previous solution for η.

There exists a non-singular matrix M such that $MBM' = I$, the identity matrix. Let $\hat{\eta} = M\hat{\xi}$, $\eta = M\xi$, so that $\hat{\eta}$ is $N(\eta, \hat{\sigma}^2 I)$. Then the resulting $\hat{\eta}$ together with $\hat{\sigma}^2$ have the same joint distribution as before, and we need only transform our confidence sets $\mathscr{R}$ and $\mathscr{R}'$ from the y-space of η to the x-space of ξ, where

$$x = M^{-1}y. \tag{33}$$

The sphere $\mathscr{S}_1$, $(y-\hat{\eta})'(y-\hat{\eta}) \leqq S_1\hat{\sigma}^2$ transforms into the ellipsoid $\tilde{\mathscr{S}}_1$,

$$(x-\hat{\xi})'B^{-1}(x-\hat{\xi}) \leqq S_1{}^2\hat{\sigma}^2, \tag{34}$$

and similarly the sphere $\mathscr{S}$, $|y-\hat{\eta}| \leqq f\hat{\sigma}$, transforms into the ellipsoid $\tilde{\mathscr{S}}$,

$$(x-\hat{\xi})'B^{-1}(x-\hat{\xi}) \leqq f^2\hat{\sigma}^2. \tag{35}$$

Here $f = f(F)$, where F is the F-statistic for testing the hypothesis $H: \xi = 0$, which may be written $F = \hat{\xi}'B^{-1}\hat{\xi}/(q\hat{\sigma}^2)$. If the ellipsoid $\tilde{\mathscr{S}}_1$ covers the origin of the x-space, i.e., if $F \leqq F_\alpha(q, \nu)$, the confidence set $\tilde{\mathscr{R}}$ for ξ is the ellipsoid (34). Otherwise $\tilde{\mathscr{R}}$ is the cone (with vertex at the origin) circumscribed about the ellipsoid (35); this consists of the points x satisfying

$$\hat{\xi}'B^{-1}x \geqq (x'B^{-1}x)^{\frac{1}{2}}(\hat{\xi}'B^{-1}\hat{\xi} - f^2\hat{\sigma}^2)^{\frac{1}{2}}, \qquad x \neq 0, \tag{36}$$

which is found by transforming (11).

To estimate the unoriented direction of ξ we use the transform $\tilde{\mathscr{R}}'$ of $\mathscr{R}'$: If $F \leqq F_\alpha(q, \nu)$, $\tilde{\mathscr{R}}'$ is the ellipsoid (34). Otherwise $\tilde{\mathscr{R}}'$ is the cone of two nappes obtained by reflecting $\tilde{\mathscr{R}}$ in the origin; it satisfies the inequality

$$|\hat{\xi}'B^{-1}x| \geqq (x'B^{-1}x)^{\frac{1}{2}}(\hat{\xi}'B^{-1}\hat{\xi} - f^2\hat{\sigma}^2)^{\frac{1}{2}}, \qquad x \neq 0. \tag{37}$$

Obviously $\Pr\{\xi \in \tilde{\mathscr{R}}\} = \Pr\{\eta \in \mathscr{R}\}$, and similarly for $\tilde{\mathscr{R}}'$ and $\mathscr{R}'$.

Finally, let us note the following easy result about the operating characteristic of the method: For the confidence sets $\mathscr{R}$, $\mathscr{R}'$, $\tilde{\mathscr{R}}$, $\tilde{\mathscr{R}}'$ the probability that they will not be ellipsoids is the well-known power of the α-level F-test for testing the hypothesis $\eta = 0$ or $\xi = 0$; the non-centrality parameter is precisely the δ of (22) previously introduced in the case of $\mathscr{R}$ and $\mathscr{R}'$, while for $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{R}}'$ it may be written $\delta = (\xi'B^{-1}\xi)^{\frac{1}{2}}/\sigma$.

5. Estimation of ratios. Estimation of ratios is related to the estimation of the unoriented direction of a vector, since the ratios of all pairs of $\{\xi_1, \cdots, \xi_q\}$ (except those with zero denominators) are determined by the unoriented direction of the vector $\xi = (\xi_1, \cdots, \xi_q)'$, and conversely, provided $\xi \neq 0$. We begin with the estimation of a single ratio, and introduce a third notation which will be useful later for multiple estimation of ratios. Suppose then we consider the estimation of a ratio θ/φ subject to the following assumptions

Ω_2: The estimates $\hat{\theta}$ and $\hat{\varphi}$ have a non-singular bivariate normal distribution with means θ and φ, variances $\sigma_{\hat{\theta}}{}^2 = b_{11}\sigma^2$ and $\sigma_{\hat{\varphi}}{}^2 = b_{22}\sigma^2$, and covariance $\sigma_{\hat{\theta}\hat{\varphi}} = b_{12}\sigma^2$, where the $\{b_{ij}\}$ are known constants. The estimate $\hat{\sigma}^2$ is independent of $\hat{\theta}$ and $\hat{\varphi}$ and is distributed as $\sigma^2\chi^2(\nu)/\nu$.

We shall denote the estimates of $\sigma_{\hat\theta}^2, \sigma_{\hat\varphi}^2, \sigma_{\theta\varphi}$ by

(38) $$\hat\sigma_{\hat\theta}^2 = b_{11}\hat\sigma^2, \qquad \hat\sigma_{\hat\varphi}^2 = b_{22}\hat\sigma^2, \qquad \hat\sigma_{\theta\varphi} = b_{12}\hat\sigma^2.$$

Fieller's [3, 4] confidence set[5] for the estimation of the ratio $\lambda = \theta/\varphi$ can be derived by considering the normal variable $w = \hat\theta - \lambda\hat\varphi$, and building the ratio $t = [\sigma_w^{-1}(w-E(w))]/[\hat\sigma\sigma^{-1}]$, which for $\varphi \neq 0$ has the t-distribution with ν df. From $\Pr\{t^2 \leqq S^2\} = 1-\alpha$, with $S^2 = F_\alpha(1,\nu)$, we then find that if[6] $\varphi \neq 0$ the values of λ satisfying

(39) $$\lambda^2(\hat\varphi^2 - S^2\hat\sigma_{\hat\varphi}^2) - 2\lambda(\hat\theta\hat\varphi - S^2\hat\sigma_{\theta\varphi}) + (\hat\theta^2 - S^2\hat\sigma_{\hat\theta}^2) \leqq 0$$

form a $(1-\alpha)$-confidence set for λ. Later we shall also be interested in the set of λ satisfying (39) for other values of S^2; let us call the set $\mathfrak{F} = \mathfrak{F}(S^2)$, so that Fieller's confidence set is $\mathfrak{F}(F_\alpha(1,\nu))$. Write

(40) $$a = \hat\varphi^2 - S^2\hat\sigma_{\hat\varphi}^2, \quad b = \hat\theta\hat\varphi - S^2\hat\sigma_{\theta\varphi}, \quad c = \hat\theta^2 - S^2\hat\sigma_{\hat\theta}^2, \quad \Delta = b^2 - ac.$$

If $\Delta > 0$ and $a > 0$ then $\mathfrak{F}$ is the interval $r_- \leqq \lambda \leqq r_+$, where $r_\pm = (b \pm \Delta^{\frac{1}{2}})/a$. If $\Delta > 0$ and $a < 0$ then $\mathfrak{F}$ is the outside of the interval (r_-, r_+). We shall ignore the cases $a = 0$ and $\Delta = 0$ since these will occur with probability zero. If $\Delta < 0$ and $a < 0$ then $\mathfrak{F}$ is the whole line $-\infty < \lambda < +\infty$. If $\Delta < 0$ and $a > 0$ then $\mathfrak{F}$ would be the empty set, but we shall show this is impossible: write Δ in the form

(41) $$\Delta = S^2\hat\sigma^2(-2b_{12}\hat\theta\hat\varphi + S^2b_{12}^2\hat\sigma^2 + b_{11}\hat\varphi^2 + b_{22}\hat\theta^2 - S^2b_{11}b_{22}\hat\sigma^2).$$

The impossibility of $\Delta < 0$ and $a > 0$ does not depend on the assumptions Ω_2 but is true for an arbitrary set of real numbers $S, \hat\sigma, \hat\theta, \hat\varphi, b_{11}, b_{22}, b_{12}$ in (41) subject to the conditions $b_{11} > 0, b_{22} > 0, b_{11}b_{22} - b_{12}^2 > 0$: This may be seen by writing (41) in the form

$$\Delta = S^2\hat\sigma^2[b_{22}(\hat\theta - b_{12}b_{22}^{-1}\hat\varphi)^2 + b_{22}^{-1}(b_{11}b_{22} - b_{12}^2)(\hat\varphi^2 - b_{22}S^2\hat\sigma^2)]$$

and noting $\hat\varphi^2 - b_{22}S^2\hat\sigma^2 = a$.

Under the assumptions Ω_2 the confidence set $\mathfrak{F}$ is the whole λ-space $(-\infty < \lambda < +\infty)$ with positive probability, and so $\mathfrak{F}$ is improper. However, that $\mathfrak{F}$ may be the outside of an interval should not disturb us, since if φ/θ lies in an interval containing

[5] Fieller ([4] 176) says his solution is implied by a result of Bliss ([1] 325). However, Bliss does not seem to note that the confidence set may fail to be a finite interval.

[6] The question as to what happens when the denominator of a ratio we are estimating is actually zero arises here and later. The statement that the ratio is in any non-empty set of real values is then, strictly speaking, false. We might still ask to learn more about the operating characteristic of a confidence set for the ratio in this case beyond the triviality that the probability that it gives a false statement is one. The writer believes that for all the confidence sets considered, with at least the claimed probability, "they behave as though they were trying to give an informative statement," for example, in the case of estimating a single ratio $\lambda = \theta/\varphi$, the statement in quotes means they give a set for λ which includes all λ with $|\lambda| > M$ for some M, or a set for (θ, φ) containing (0, 0). However, he finds the analysis of this problem too nit-picking for his taste because (i) our assertions are mathematically correct for any non-zero denominator, however small, and (ii) there is the usual question of how operationally meaningful is the statement that φ is "exactly zero" in a real application.

zero, then θ/φ lies in a set which is the outside of an interval if $\varphi \neq 0$. Later we shall see geometrically that the three possibilities for $\mathfrak{J}$ (a finite interval, the outside of a finite interval, or the whole line) arise because $\mathfrak{J}$ is the section $\varphi = 1$ of an improper confidence set in the θ, φ-plane either in the form of two opposite sectors bounded by two lines through the origin, or else the whole plane. From the later geometrical picture it will also be clear that under Ω_2 the probability that $\mathfrak{J}(F_\alpha(1, \nu))$ is the whole λ-space is $\Pr\{F \leq \frac{1}{2}F_\alpha(1, \nu)\}$, where F is the F-statistic for testing $H: \theta = \varphi = 0$, and has a non-central F-distribution with 2 and ν df, and non-centrality parameter calculated in the usual way. A proper confidence set for the reformulation of this problem will be given by the Corollary near the end.

We consider now joint estimation of a set of ratios. Under the assumptions and notation of Ω_1 suppose first that we wish to estimate jointly the $q-1$ ratios $\xi_i/\xi_q (i = 1, \cdots, q-1)$. The solution will be a confidence set in the x-space of ξ, which is either an ellipsoid containing $\xi = 0$, or else a set of contours along which the $q-1$ functions ξ_i/ξ_q remain constant. Indeed, the $\tilde{\mathscr{R}}'$ found in Section 4 is such a confidence set, the contours being lines through 0 (with 0 deleted) when $\tilde{\mathscr{R}}'$ is a cone; however, when $\tilde{\mathscr{R}}'$ is a cone we should delete from it its intersection with the plane $x_q = 0$ since there the ξ_i/ξ_q are not defined. To get a $(q-1)$-dimensional picture when $\tilde{\mathscr{R}}'$ is a cone, and to help solve a later multiple estimation problem, we note that if $\xi_q \neq 0$, $\xi \in \tilde{\mathscr{R}}'$ is equivalent to the $(q-1)$-tuple $(\xi_1/\xi_q, \cdots, \xi_{q-1}/\xi_q)$ being in the set of $(x_1, \cdots, x_{q-1})$ for which $(x_1, \cdots, x_{q-1}, 1)$ is in the cross-section of the cone $\tilde{\mathscr{R}}'$ by the plane $x_q = 1$. This is a special case of the following method of joint estimation of the q ratios $\{\xi_i/\varphi\}$, where $\varphi = \sum_1^q g_i \xi_i$, and $\{g_i\}$ is a given set of constants not all zero: If $\mathscr{S}_1$ covers the origin (i.e., if $H: \xi = 0$ is accepted by the α-level F-test) we say $\xi \in \mathscr{S}_1$; else we say the value of the q-tuple $(\xi_1/\varphi, \cdots, \xi_q/\varphi)$ is in the set of $(x_1, \cdots, x_q)$ which is the cross-section of $\tilde{\mathscr{R}}'$ by $\sum_1^q g_i x_i = 1$, or of (37) by $g'x = 1$. To see that the resulting statement is equivalent to $\xi \in \tilde{\mathscr{R}}'$, if $g'\xi \neq 0$, note that if $\tilde{\mathscr{R}}'$ is a cone, and $g'\xi \neq 0$, then $\xi \in \tilde{\mathscr{R}}'$ if and only if the q-tuple $(\xi_1/\varphi, \cdots, \xi_q/\varphi)$ takes on the value that $(x_1/g'x, \cdots, x_q/g'x)$ has along some line (with 0 deleted) in $\tilde{\mathscr{R}}'$, and this is the same as the value of $(x_1, \cdots, x_q)$ where the line intersects the plane $g'x = 1$.

That these confidence sets for the joint estimation of a set of ratios can turn out to be any kind of conic section seems something of a mathematical curiosity to the writer, but he feels that, for $q > 2$, they will be of less direct practical use than the following method of multiple estimation derived from them, which gives the inside or outside of easily calculated intervals for all ratios in a certain set. The set will be that of all θ/φ with θ and φ in a q-dimensional space L of estimable functions, and for which the value of φ is not zero. In the above context of estimating ratios, L would be the space of all linear combinations of $\xi_1, \cdots, \xi_q$ under the assumptions Ω_1. In the analysis-of-variance context the space L would be that defined in Section 1 under the assumptions Ω; in that case we would take as the $\{\xi_i\}$ any basis $\{\theta_i\}$ for L. We shall denote the resulting assumptions by Ω'.

To state the theorem embodying the method of multiple estimation of ratios, we shall employ the following further notation: For any θ and φ in L we denote by $\hat{\theta}$

and $\hat{\varphi}$ their least-squares estimates; these satisfy Ω_2, and we shall denote their estimated variances and covariance by the $\hat{\sigma}_\theta^2, \hat{\sigma}_\varphi^2, \hat{\sigma}_{\theta\varphi}$ defined by (38). The F-statistic for testing the hypothesis $H : \xi = 0$ will be denoted by F, so that H will be accepted by the α-level F-test if and only if $F \leqq F_\alpha(q, v)$, or, if and only if the ellipsoid $\mathscr{S}_1$ of (34) covers 0. If the confidence set $\tilde{\mathscr{R}}'$ for η in Section 4 is constructed for a nominal confidence coefficient $1-\alpha$ and for a chosen function $f(F)$ satisfying the conditions below (9), $1-\alpha_0$ will denote a lower bound for $h'(\delta) = \Pr\{\eta \in \mathscr{R}'\}$. Then $1-\alpha_0 \leqq 1-\alpha$ since $h'(0) = 1-\alpha$, or if $\eta = 0$ is excluded, since $h'(\delta) \to 1-\alpha$ as $\delta \to \infty$.

For applying Theorem 2 and its Corollary a very conservative value of $1-\alpha_0$ would be the bound (23); this is valid for any $f(F)$ satisfying the conditions below (9). It is suggested that $f(F)$ of the form (9) be used with $\mu = 3/2$ for $q = 2$ and $\mu = 2$ for $q > 2$. This is expected to give a good approximation of $1-\alpha_0$ to the nominal $1-\alpha$ as explained in the Appendix, where it is shown that in the case $v = \infty$, this choice of μ numerically justifies $1-\alpha_0 = .898$ if $1-\alpha = .900$, and $1-\alpha_0 = .949$ if $1-\alpha = .950$. See also the first paragraph beginning after (32), where other choices of μ are mentioned.

THEOREM 2. *If the hypothesis $H : \xi = 0$ is accepted by the α-level F-test, state that ξ is in the ellipsoid $\mathscr{S}_1$ of* (34), *which covers $\xi = 0$. If H is rejected, then for as many θ/φ as desired, with θ and φ in L, state that the value of $\lambda = \theta/\varphi$ satisfies the quadratic inequality* (39) *with $S = f(F)$ if the value of φ is not zero. Under the assumptions Ω' the probability is $\geqq 1-\alpha_0$ that all the statements will be true.*

We shall be able to discuss some of the implications of this theorem better after we have proved it. Since $\Pr\{\xi \in \tilde{\mathscr{R}}'\} \geqq 1-\alpha_0$, it will suffice to prove that if $\xi \in \tilde{\mathscr{R}}'$ and $\tilde{\mathscr{R}}' \neq \tilde{\mathscr{S}}_1$, then the values of all θ/φ, for which θ and φ are in L and for which the value of φ is not zero, satisfy (39) with $S = f(F)$. Actually it will suffice to prove this for all θ and φ in L with $\sigma_\theta^2 = \sigma^2$, or $b_{11} = 1$, since under the transformation $\theta = c\theta', \varphi = \varphi', \lambda = c\lambda'$ $(c \neq 0)$, the inequality (39) remains invariant. We transform from the x-space of ξ to the y-space of η by (33). Choose any θ and φ in L with $\sigma_\theta^2 = \sigma^2$, say $\theta = d'\eta$ and $\varphi = g'\eta$, and hold d and g fixed. Since $\sigma_\theta^2 = d'd\sigma^2$, $d'd = 1$. If $g'\eta = \varphi \neq 0$ and we write $\lambda = d'(\eta/\varphi)$, we see that λ is the (signed length of the) projection of the vector η/φ on the unit vector d. We assume in the remainder of this proof that $\eta \in \mathscr{R}'$, the cone (19), and $g'\eta \neq 0$. Then the point η/φ lies in the section of $\mathscr{R}'$ by the plane $g'y = 1$, by the reasoning that led to the conic-section outcome of our confidence sets above. The problem has now been reduced to showing that for all points y_0 in the section of $\mathscr{R}'$ by the plane $g'y = 1$, the projection λ of y_0 on d satisfies (39) with $S = f(F)$.

To solve this problem we introduce the cone $\mathscr{R}^*$ (including 0) which consists of all lines through 0 normal to some direction in $\mathscr{R}'$. An inequality for $\mathscr{R}^*$ may be obtained as follows: $\mathscr{R}'$ consists of all y satisfying $|\cos(\hat{\eta}, y)| \geqq \cos\kappa$, where κ is the semi-vertex angle of $\mathscr{R}'$, $\sin\kappa = f\hat{\sigma}/|\hat{\eta}|$. Hence $\mathscr{R}^*$ consists of all y satisfying $|\cos(\hat{\eta}, y)| \leqq \cos(\frac{1}{2}\pi - \kappa)$ or $y = 0$, which is equivalent to $(\hat{\eta}'y)^2 \leqq |y|^2(f\hat{\sigma})^2$, or

$$y'(\hat{\eta}\hat{\eta}' - f^2\hat{\sigma}^2 I)y \leqq 0. \tag{42}$$

Now suppose that y_0 is in the section of $\mathscr{R}'$ by $g'y = 1$, that $\lambda = d'y_0$, and consider the vector $\lambda g - d$: Since $(\lambda g - d)'y_0 = \lambda g'y_0 - d'y_0 = 0$, $\lambda g - d$ is orthogonal to y_0 and hence lies in $\mathscr{R}^*$, so by (42),

$$(43) \qquad (\lambda g - d)'(\hat{\eta}\hat{\eta}' - f^2\hat{\sigma}^2 I)(\lambda g - d) \leqq 0.$$

We may write this

$$(44) \qquad \lambda^2[(g'\hat{\eta})^2 - f^2\hat{\sigma}^2 g'g] - 2\lambda[(d'\hat{\eta})(g'\hat{\eta}) - f^2\hat{\sigma}^2 d'g] + [(d'\hat{\eta})^2 - f^2\hat{\sigma}^2 d'd] \leqq 0.$$

From $\theta = d'\eta$, $\varphi = g'\eta$, we may identify the coefficients in (44) by calculating $g'\hat{\eta} = \hat{\varphi}$, $\hat{\sigma}^2 g'g = \hat{\sigma}_\varphi{}^2$, $\hat{\sigma}^2 d'g = \hat{\sigma}_{\theta\varphi}$, etc., and so we find (44) may be written in the form (39) with $S = f$.

Concerning Theorem 2 we point out that if $q > 2$ the sets given by the theorem for estimating $\lambda = \theta/\varphi$ when $\mathscr{S}_1$ does not cover 0 will be the whole line, $-\infty < \lambda < +\infty$, for some θ and φ in L, but never for all θ and φ in L. This is because they are derived from the proper confidence set $\mathscr{R}'$ for η when $\mathscr{R}' \neq \mathscr{S}_1$ by projecting the section of the cone $\mathscr{R}'$ by the plane $g'y = 1$ on the vector d, where $\theta = d'\eta$ and $\varphi = g'\eta$ (we may assume $d \neq 0$ and $g \neq 0$), and then "multiplying" the projection by $|d|$: It may be seen clearly in the case $q = 3$ from the three-dimensional picture, in which $\mathscr{R}'$, when not the ellipsoid $\mathscr{S}_1$, is a cone of revolution of two nappes with vertex at 0 and semi-vertex angle $\kappa < \frac{1}{2}\pi$, and $\kappa > 0$ with probability one. The section of the cone $\mathscr{R}'$ by $g'y = 1$ will be bounded by an ellipse, parabola, or hyperbola, depending on $\mathscr{R}'$ and g. We can ignore the parabolic case, since for random $\mathscr{R}'$ and fixed g it happens with probability zero. Given $\mathscr{R}'$, if g is such that the section is elliptical, then its projection on d for any d is a finite interval. If g is such that the section is hyperbolic, then for some d the projection is the outside of an interval, and for some d it is the whole line, the only other possibility being the case where d has the same direction as g, say $d = cg$, when $\theta/\varphi = c$, and the projection is a single point giving $\lambda = c$. However, in the case $q = 2$ the corresponding two-dimensional picture shows that for no θ and φ in L can the set for $\lambda = \theta/\varphi$ given by the theorem be the whole line: $\mathscr{R}'$ when not the circle $\mathscr{S}_1$ consists of two opposite sectors of vertex angle $2\kappa < \pi$ (and $\kappa > 0$ with probability one) bounded by two lines through the origin. The elliptical sections degenerate into intervals of finite positive length, the hyperbolic sections into the outsides of such intervals, and their projections (excluding again the case $d = cg$) will have respectively the same character.

In the case $q = 2$, which we now consider in more detail, there is no real multiple estimation problem for θ/φ, for θ and φ in L: We may assume θ and φ are linearly independent members of L, else φ is the zero element of L or θ/φ is a known constant. It is easy to verify that for any linearly independent θ and φ in L and any linearly independent $\tilde{\theta}$ and $\tilde{\varphi}$ in L, $\tilde{\lambda} = \tilde{\theta}/\tilde{\varphi}$ is a 1 : 1 function of $\lambda = \theta/\varphi$. For the case $q = 2$ then we may as well take the assumptions in the form Ω_2, and Theorem 2 has the following

COROLLARY. *Under the assumptions* Ω_2 *calculate the value of*

$$(45)\qquad F = (\hat{\theta}, \hat{\varphi})B^{-1}(\hat{\theta}, \hat{\varphi})'/(2\hat{\sigma}^2) \quad or$$
$$= [b_{22}\hat{\theta}^2 - 2b_{12}\hat{\theta}\hat{\varphi} + b_{11}\hat{\varphi}^2]/[2(b_{11}b_{22} - b_{12}^2)\hat{\sigma}^2],$$

the F-statistic for testing the hypothesis $\theta = \varphi = 0$. *If* $F \leqq F_\alpha(2, \nu)$ *state that* (θ, φ) *is in the ellipse*

$$(46)\qquad (\theta - \hat{\theta}, \varphi - \hat{\varphi})B^{-1}(\theta - \hat{\theta}, \varphi - \hat{\varphi})' \leqq 2F_\alpha(2, \nu)\hat{\sigma}^2, \quad or$$
$$b_{22}(\theta - \hat{\theta})^2 - 2b_{12}(\theta - \hat{\theta})(\varphi - \hat{\varphi}) + b_{11}(\varphi - \hat{\varphi})^2 \leqq 2F_\alpha(2, \nu)(b_{11}b_{22} - b_{12}^2)\hat{\sigma}^2,$$

which covers the origin of the θ, φ-*plane. If* $F > F_\alpha(2, \nu)$ *solve for* λ *the equation* $a\lambda^2 - 2b\lambda + c = 0$, *where*

$$a = \hat{\varphi}^2 - f^2 b_{22}\hat{\sigma}^2, \qquad b = \hat{\theta}\hat{\varphi} - f^2 b_{12}\hat{\sigma}^2, \qquad c = \hat{\theta}^2 - f^2 b_{11}\hat{\sigma}^2,$$

and $f = f(F)$. *There will be two real unequal roots* (*with probability one*); *denote them by* $r_- < r_+$. *If* $a > 0$ *state that* $r_- \leqq \theta/\varphi \leqq r_+$ *if* $\varphi \neq 0$; *if* $a < 0$ *state* $\theta/\varphi \leqq r_-$ *or* $\geqq r_+$ *if* $\varphi \neq 0$. *Then the probability of a true statement is* $\geqq 1 - \alpha_0$.

At this point we insert a discussion giving direct access in this lengthy paper to the reader interested only in Fieller's problem of estimating the ratio $\lambda = \theta/\varphi$ of two means. (The author is grateful to the referee who suggested the multiple entry.) The problem is treated under the assumptions Ω_2, near the beginning of Section 5, on the estimates $\hat{\theta}$, $\hat{\varphi}$, and an estimate $\hat{\sigma}^2$ of an unknown variance σ^2. Fieller's $(1-\alpha)$-confidence set for λ is given by the quadratic inequality (39) with the $\hat{\sigma}$'s defined by (38) and the constant $S^2 = F_\alpha(1, \nu)$, the upper α point of $F(1, \nu)$. The nature of the confidence set is explained below (40) through the first two sentences of the next paragraph.

The difficulties in this estimation problem may be seen intuitively as follows: If the estimate $(\hat{\theta}, \hat{\varphi})$ is too close to $(0, 0)$, as measured by the F-statistic (45), the ratio θ/φ is poorly determined; thus for small F the confidence ellipse (46) for (θ, φ) includes $(0, 0)$, in the neighborhood of which θ/φ takes on all possible values $-\infty < \theta/\varphi < +\infty$. Indeed, for values of $F \leqq \frac{1}{2}F_\alpha(1, \nu)$, Fieller's confidence set gives the trivial statement $-\infty < \theta/\varphi < +\infty$. (The objections to such "improper" confidence sets are discussed in Section 3.) The above Corollary reformulates Fieller's problem of finding a confidence set for θ/φ as that of finding one for (θ, φ) of the following nature: The confidence set for (θ, φ) implies non-trivial bounds for the value of θ/φ or else is an ellipse containing $(0, 0)$. Specifically, if $F \leqq F_\alpha(2, \nu)$ the statement is made that (θ, φ) is in the ellipse (46) containing $(0, 0)$; otherwise (θ, φ) is stated to be in a set for which θ/φ is inside, or else outside, the finite interval (r_-, r_+) defined in the Corollary. While Fieller's confidence set is based on the quadratic inequality (39) with $S^2 = S_1{}^2 = F_\alpha(1, \nu)$, the interval (r_-, r_+) of the Corollary corresponds to the inequality (39) with a different value of S^2: It would be easy to justify $S^2 = S_2{}^2 = 2F_\alpha(2, \nu)$ from the $(1-\alpha)$-confidence ellipse for (θ, φ). Actually it corresponds to (39) with $S^2 = f^2(F)$, where $f^2(F)$ is a monotone function of F which decreases from $S_2{}^2$ for $F = F_\alpha(2, \nu)$ toward a limiting value of $S_1{}^2$ as

$F \to \infty$. A suitable function $f^2(F)$ is given by the square of (10) with $q = 2$. If the constant μ in (10) is taken as 3/2, the true confidence level $1-\alpha_0$ (a lower bound for the probability that the confidence set covers (θ, φ)) will be close to the nominal $1-\alpha$. (Thus if $\nu = \infty$, then within .001, $1-\alpha_0 = .949$ for $1-\alpha = .95$, and $1-\alpha_0 = .898$ for $1-\alpha = .90$, and these approximations are expected to hold up well for finite ν. Other possible choices of μ are mentioned in the first paragraph beginning after (32), where the confidence coefficient of $\mathscr{R}'$ equals that of the confidence set of the Corollary.)

We conclude by remarking on the results that would follow from applying our methods to the improper confidence set $\mathscr{B}$ of Section 4 for estimating the unoriented direction of the vector η, instead of to our proper $\mathscr{R}'$. Suppose we begin with the problem of estimating the unoriented direction of the vector ξ under the assumptions Ω_1, transform to the y-space of η by (33), use $\mathscr{B}$ for the unoriented direction of η, transform back to the x-space, and let $\tilde{\mathscr{B}}$ be the transform of $\mathscr{B}$. Then we find that $\tilde{\mathscr{B}}$ is an improper confidence set for the unoriented direction of ξ whose probability of covering ξ is exactly $1-\alpha$ if $\xi \neq 0$, which we assume henceforth. We may describe $\tilde{\mathscr{B}}$ as follows: If the ellipsoid $\tilde{\mathscr{S}}_2$, obtained by replacing S_1 by S_2 in (34) covers the origin 0, then $\tilde{\mathscr{B}}$ is the whole ξ-space with 0 deleted; the probability of this outcome, which leads to a trivially true statement, is given by (26). If $\tilde{\mathscr{S}}_2$ does not cover 0, $\tilde{\mathscr{B}}$ is the cone with vertex at 0 circumscribed about $\tilde{\mathscr{S}}_2$; it is determined by the inequality (36) with f replaced by S_2. A confidence set for the joint estimation of $(\xi_1/\varphi, \cdots, \xi_q/\varphi)$, where $\varphi = g'\xi$ with g given and non-zero, is the set of points $(x_1, \cdots, x_q)$ which is the section of $\tilde{\mathscr{B}}$ by $g'x = 1$; if $g'\xi \neq 0$ the probability of covering is exactly $1-\alpha$. For the multiple estimation of all $\lambda = \theta/\varphi$, with θ and φ in L, we proceed as in the proof of Theorem 2 and let $\theta = d'\eta$, $\varphi = g'\eta$, assume $|d| = 1$ and $g \neq 0$, and if $\eta \in \mathscr{B}$, interpret λ as the projection on d of a point y_0 in the section of $\mathscr{B}$ by $g'y = 1$. If $\mathscr{S}_2$, $|y - \hat{\eta}| \leqq S_2\hat{\sigma}$, does not cover 0, we define $\mathscr{B}^*$ by (42) with f replaced by S_2, argue as before that $\lambda g - d \in \mathscr{B}^*$ and hence (44) is satisfied with $f = S_2$, and (39) with $S = S_2$. If $\mathscr{S}_2$ covers 0, $|\hat{\eta}| \leqq S_2\hat{\sigma}$, hence for *any* vector y, $|\hat{\eta}'y| \leqq |\hat{\eta}|\cdot|y| \leqq S_2\hat{\sigma}|y|$, or $(\hat{\eta}'y)^2 \leqq S_2{}^2\hat{\sigma}^2 y'y$, which may be written

$$y'(\hat{\eta}\hat{\eta}' - S_2{}^2\hat{\sigma}^2 I)y \leqq 0. \tag{47}$$

If we now substitute $y = \lambda g - d$ in (47) and proceed as we did from (43) to (39) but with $f = S_2$ we get (39) with $S = S_2$. This shows that the probability that all $\lambda = \theta/\varphi$, for which the value of φ is not zero, simultaneously satisfy (39) is $\geqq 1-\alpha$; it may be shown the probability is $= 1-\alpha$. The probability of getting the trivially true statement for all λ is (26). In the special case $q = 2$ the multiple estimation of all θ/φ is, as we remarked above, equivalent to the estimation of a single θ/φ under Ω_2, and the result of using $\mathscr{B}$ instead of $\mathscr{R}'$ would be the confidence set of Fieller derived at the beginning of this section.

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APPENDIX ON COMPUTER CALCULATION OF CONFIDENCE COEFFICIENTS AND PARTIAL OPERATING CHARACTERISTICS OF $\mathscr{R}$ AND $\mathscr{R}'$

We consider here the problem of the choice of the exponent μ which determines the function $f(F)$ in (10). Numerical determination of the confidence coefficients for the resulting $\mathscr{R}$ and $\mathscr{R}'$ involves the calculation for different δ of the three probabilities Pr $\{I\}$, Pr $\{II\}$, and Pr $\{III\}$, and each of these gives information about the operating characteristic. These results apply to $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{R}}'$ as well as to $\mathscr{R}$ and $\mathscr{R}'$. Thus, all for the case $\nu = \infty$, Table 2 of Pr $\{I\}$ gives the probability that $\tilde{\mathscr{R}}$ (also $\tilde{\mathscr{R}}'$) will be an ellipsoid covering ξ; Table 3 of Pr $\{II\}$, that $\tilde{\mathscr{R}}$ will be a cone covering ξ; Table 6 of Pr $\{III\}$, that $\tilde{\mathscr{R}}'$ will be a cone with ξ not in the nappe containing its estimate $\hat{\xi}$; Table 4 gives Pr $\{\xi \in \tilde{\mathscr{R}}\}$, and this plus Pr $\{III\}$ gives Pr $\{\xi \in \tilde{\mathscr{R}}'\}$. I mention here that the programmer estimated the probabilities listed in Tables 2–6 to be correct within .001; also that the rows for $q = 6, 8, 13, 16$ were deleted from Table 4 and Table 6 because they could be reconstituted by linear interpolation between the remaining rows.

TABLE 2
Pr $\{I\}$ *for* $\nu = \infty$

q	δ									
	0^+	.2	.5	1	2	3	4	5	6	8
					$\alpha = .10$					
2	.900	.884	.848	.752	.427	.121	.006	.000	.000	.000
3	.900	.887	.859	.780	.492	.173	.024	.000	.000	.000
4	.900	.889	.865	.797	.536	.215	.040	.002	.000	.000
5	.900	.890	.869	.808	.569	.250	.055	.004	.000	.000
6	.900	.891	.871	.817	.595	.281	.069	.007	.000	.000
8	.900	.892	.875	.829	.634	.332	.097	.013	.001	.000
10	.900	.893	.878	.837	.662	.373	.124	.020	.001	.000
13	.900	.894	.881	.845	.692	.424	.161	.032	.003	.000
16	.900	.894	.883	.851	.714	.463	.196	.046	.005	.000
20	.900	.895	.885	.857	.736	.505	.237	.065	.009	.000
					$\alpha = .05$					
2	.950	.940	.918	.851	.568	.216	.033	.000	.000	.000
3	.950	.942	.925	.871	.628	.278	.059	.003	.000	.000
4	.950	.943	.929	.883	.669	.328	.083	.008	.000	.000
5	.950	.944	.931	.891	.698	.369	.105	.013	.000	.000
6	.950	.945	.933	.897	.721	.404	.127	.018	.001	.000
8	.950	.945	.935	.905	.754	.461	.167	.030	.002	.000
10	.950	.946	.937	.910	.778	.505	.205	.043	.004	.000
13	.950	.946	.939	.916	.803	.558	.255	.064	.008	.000
16	.950	.947	.940	.920	.820	.598	.299	.087	.013	.000
20	.950	.947	.941	.923	.837	.639	.350	.117	.021	.000

TABLE 3
Pr $\{II\}$ *for* $\nu = \infty$

μ	q	δ												
		0^+	.2	.5	1	2	3	4	5	6	8	16	32	64
							$\alpha = .10$							
1	2	.031	.047	.080	.173	.489	.789	.903	.916	.912	.907	.902	.900	.900
	3	.026	.038	.065	.141	.424	.739	.887	.913	.912	.907	.902	.900	.900
	4	.023	.033	.057	.123	.380	.698	.873	.911	.911	.907	.902	.900	.900
2	2	.030	.044	.077	.167	.478	.774	.887	.902	.902	.900	.900		
	3	0.25	.036	.062	.137	.415	.726	.874	.902	.902	.901	.900		
	4	0.22	.032	.054	.119	.372	.688	.861	.900	.902	.901	.900		
	5	.020	.029	.049	.106	.340	.654	.848	.899	.903	.901	.900		
	6	.019	.027	.045	.097	.314	.625	.835	.896	.903	.901	.900		
	8	.017	.024	.040	.084	.276	.575	.808	.891	.903	.901	.900		
	10	.015	.022	.036	.076	.248	.534	.782	.885	.902	.902	.900		
	13	.014	.019	.032	.066	.218	.485	.746	.873	.901	.902	.900		
	16	.013	.018	.029	.060	.196	.446	.712	.861	.900	.902	.900		
	20	.011	.016	.026	.054	.174	.404	.672	.842	.896	.903	.900		
							$\alpha = .05$							
1	2	.017	.026	.047	.113	.392	.741	.922	.958	.958	.955	.951	.950	
	3	.014	.021	.038	.091	.331	.679	.898	.954	.957	.955	.951	.950	
	4	.013	.019	.033	.078	.291	.630	.875	.949	.957	.955	.951	.950	
2	2	.016	.025	.046	.110	.386	.733	.913	.949	.951	.950	.950		
	3	.013	.021	.037	.089	.327	.673	.890	.947	.952	.951	.950		
	4	.012	.018	.032	.076	.287	.624	.868	.943	.951	.951	.950		
	5	.011	.016	.029	.068	.257	.584	.847	.938	.951	.951	.950		
	6	.010	.015	.027	.061	.235	.550	.826	.934	.951	.951	.950		
	8	.009	.014	.023	.053	.201	.494	.786	.922	.950	.951	.950		
	10	.009	.012	.021	.047	.178	.449	.749	.910	.948	.951	.950		
	13	.008	.011	.019	.041	.153	.398	.700	.889	.945	.951	.950		
	16	.007	.010	.017	.037	.135	.358	.656	.867	.940	.952	.950		
	20	.007	.009	.015	.033	.119	.317	.605	.838	.933	.952	.950		

We restrict our considerations to $\mathscr{R}$ until the end of this Appendix, where we include $\mathscr{R}'$. Ideally we might like to choose sup μ for which $\inf_{\delta>0} \Pr\{\eta \in \mathscr{R}\} = 1-\alpha$ (we know this cannot be $> 1-\alpha$ since $\Pr\{\eta \in \mathscr{R}\} \to 1-\alpha$ as $\delta \to \infty$), and ask for a table of this μ for use in applications—a triple-entry table according to α, q, ν. However, the calculation of $\Pr\{\eta \in \mathscr{R}\}$ for a single value of δ is formidable because of the triple integrations. A more feasible procedure is to choose a few values of μ to be considered, perhaps exponents easy for the potential user to calculate with, and to try these sequentially until a satisfactory one is found. I did this for the case $\nu = \infty$, because I do not have the resources for the calculation of the triple integrals, and they degenerate to double integrals in this case. I suggest that the values of μ thus found be used also in the case of finite ν, to approximate the desired confidence coefficient $1-\alpha$ and the partial operating characteristics, until such time if any when the triple integral calculations are made for finite ν. If these calculations for finite ν should ever be made, then the values of μ adopted here for $\nu = \infty$ should be tried first, and one might then be satisfied simply to adopt these also for finite ν if the confidence coefficient did not drop below $1-\alpha$; if it did drop below, smaller μ would have to be tried, at least if the amount of the drop were considered of practical importance. (I conjecture it would not drop below $1-\alpha$, on the basis of speculations too tenuous to be recounted here.) For finite ν one could safely use the crude bound (23) for the confidence coefficient, but how crude this is for the values of μ recommended here (the bound depends on ν but not on μ) is shown for the case $\nu = \infty$ in the following tabulation:

q:	2	3	6	11	21
bound (23) when $\alpha = .10$:	.821	.848	.870	.879	.886
bound (23) when $\alpha = .05$:	.902	.919	.932	.938	.942

TABLE 4

$\Pr\{\eta \in \mathscr{R}\} = \Pr\{I\} + \Pr\{II\}$ *for* $\nu = \infty$

μ	q	δ												
		0^+	.2	.5	1	2	3	4	5	6	8	16	32	64
								$\alpha = .10$						
1	2	.931	.930	.929	.925	.917	.910	.909	.916	.912	.907	.902	.900	.900
	3	.926	.925	.924	.921	.916	.912	.911	.913	.912	.907	.902	.900	.900
	4	.923	.922	.921	.919	.916	.913	.913	.913	.911	.907	.902	.900	.900
2	2	.930	.928	.925	.919	.905	.894	.893	.902	.902	.900	.900		
	3	.925	.923	.921	.917	.907	.900	.898	.902	.902	.901	.900		
	4	.922	.921	.919	.916	.908	.903	.901	.902	.902	.901	.900		
	6	.918	.918	.917	.914	.909	.906	.904	.903	.903	.901	.900		
	10	.915	.915	.914	.912	.910	.908	.906	.905	.904	.902	.900		
	20	.911	.911	.911	.910	.910	.910	.909	.907	.905	.903	.900		
								$\alpha = .05$						
1	2	.967	.966	.966	.964	.960	.957	.955	.958	.958	.955	.951	.950	
	3	.964	.964	.963	.962	.960	.958	.957	.957	.957	.955	.951	.950	
	4	.963	.962	.962	.961	.959	.958	.957	.957	.957	.955	.951	.950	
2	2	.966	.965	.964	.961	.955	.949	.946	.949	.951	.950	.950		
	3	.963	.963	.962	.960	.955	.951	.949	.950	.952	.951	.950		
	4	.962	.962	.961	.959	.955	.952	.951	.951	.951	.951	.950		
	6	.960	.960	.959	.958	.956	.954	.952	.952	.952	.951	.950		
	10	.958	.958	.958	.957	.956	.955	.954	.953	.952	.951	.950		
	20	.956	.956	.956	.956	.956	.956	9.55	.954	.953	.952	.950		

TABLE 5
Supplementary values of Pr $\{\eta \in \mathscr{R}\}$ *for determining confidence coefficient of* $\mathscr{R}$ *for* $\nu = \infty$

q	μ	δ 2.8	3.0	3.2	3.4	3.6	3.8	4.0
				$\alpha = .10$				
2	1		.910		.909	.908	.908	.909
	$\frac{1}{2}$	.902	.901	.900	.899	.898	.898	.899
	2	.896	.894	.893	.893	.892	.893	.893
3	$\frac{1}{2}$			.904	.904	.903	.903	.903
	2		.900	.899	.898	.898	.898	.898
4	2		.903	.902	.901	.901	.901	.901
				$\alpha = .05$				
2	1		.957		.956	.955	.955	.955
	$\frac{1}{2}$		.952		.951	.950	.950	.949
	2		.949		.947	.946	.946	.946
3	$\frac{1}{2}$						.952	.952
	2		.951		.950	.950	.949	.949
4	2		.952	.952	.951	.951	.951	.951

The values of μ recommended below give .900 (or .898) for the confidence coefficient in the second row of the table, and .950 (or .949) in the last row. In judging the crudeness one should of course regard the complements of all these probabilities.

For $\nu = \infty$, as we remarked in Section 4, all the probability in the $u, v, \hat{\sigma}$-space lies in the plane $\hat{\sigma} = \sigma$. If accordingly in the triple integrals (30)–(32) we suppress the integration with respect to $p_3(\hat{\sigma})\, d\hat{\sigma}$, let $\hat{\sigma} = 1$, transform $u = w + \delta$, and use formula (10) for $f(F)$, we find

$$\Pr\{I\} = \int_0^{[S_1{}^2 - (\frac{1}{2}\delta)^2]^{1/2}} p_2(v) \int_{-(S_1{}^2 - v^2)^{1/2}}^{-\delta + (S_1{}^2 - v^2)^{1/2}} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}w^2}\, dw\, dv$$

for $\delta < 2S_1$, while $\Pr\{I\} = 0$ for $\delta \geqq 2S_1$, and

$$\Pr\{II\} = \int_0^{S_2} p_2(v) \int_{-\delta + (S_1{}^2 - v^2)^{1/2}}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}w^2}\, dw\, dv$$

$$+ \int_{S_2}^{S_1} p_2(v) \int_{-\delta + (S_1{}^2 - v^2)^{1/2}}^{-\delta + [S_1{}^2((S_1 - S_2)(v - S_2)^{-1})^{1/\mu} - v^2]^{1/2}} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}w^2}\, dw\, dv,$$

where $S_1 = \chi_\alpha(q)$, $S_2 = \chi_\alpha(q-1)$, and $p_2(v)$ is given by (27). $\Pr\{III\}$ is programmed for the computer like $\Pr(II)$, but with $-\delta$ entered in place of δ.

I began with $\mu = 1$, intending to continue with $\mu = 2$ or $\frac{1}{2}$, depending on the outcome for $\mu = 1$. Table 2 of $\Pr\{I\}$, which does not depend on μ, was calculated,

TABLE 5—*continued*

q	μ	δ 4.2	4.4	4.6	4.8	5.0	5.4	rel. min. near $\delta = 4$	conf. coef.
					$\alpha = .10$				
2	1	.910	.913	.915		.916		.908	.900
	$\frac{3}{2}$	.900	.904					.898	.898
	2	.895	.898	.901		.902		.892	.892
3	$\frac{3}{2}$	.903	.904					.903	.900
	2	.899	.899	.900		.902		.898	.898
4	2	.901	.901	.901	.902	.902	.902	.901	.900
					$\alpha = .05$				
2	1	.955	.955	.955	.956	.958		.955	.950
	$\frac{3}{2}$	.949	.949	.950		.952		.949	.949
	2	.946	.946	.947		.949		.946	.946
3	$\frac{3}{2}$	.952	.952	.952				.952	.950
	2	.949	.949	.949	.950	.950		.949	.949
4	2	.951	.951	.951	.951	.951	.951	.951	.950

then $\Pr\{II\}$ for the α and δ shown in Table 3, and $q = 2\text{–}6, 8, 10, 13, 16, 20$. (The rows for $q > 4$ were later deleted from Table 3 because we shall see there is no future need for them.) For these α, δ, q, a table of $\Pr\{\eta \in \mathscr{R}\} = \Pr\{I\} + \Pr\{II\}$ was then constructed. (Rows for $q \leqq 4$ are shown in Table 4.) The resulting table showed for these α, δ, q that $\Pr\{\eta \in \mathscr{R}\} \geqq 1 - \alpha$ for all δ; possible doubt for the case $q = 2$ was dispelled by the use of supplementary values of δ in Table 5.

I therefore continued with $\mu = 2$. $\Pr\{II\}$ is given in Table 3, and the resulting $\Pr\{\eta \in \mathscr{R}\}$ in Table 4. Table 4 (including rows for $q = 5, 8, 13, 16$) showed that $\Pr\{\eta \in \mathscr{R}\} \geqq 1 - \alpha$ except in the cases $q = 2$ or 3; possible doubt about the case $q = 4$ was again dispelled by the supplementary Table 5. For $q = 2$ or 3, the relative minimum of $\Pr\{\eta \in \mathscr{R}\}$ near $\delta = 4$ is explored in Table 5. The next-to-the-last column in Table 5 is based on the preceding ones, and the last column is also based on those together with Table 4 and the known $\lim \Pr\{\eta \in \mathscr{R}\} = 1 - \alpha$ as $\delta \to \infty$.

I decided to try no larger values of μ, but to suggest the use of $\mu = 2$ in practice when $q \geqq 4$ because of the following considerations: Recall that κ denotes the semivertex angle of the confidence set $\mathscr{R}$ when it is a cone. I analyzed the effect on κ of increasing μ beyond 2 for $\alpha \leqq .10$ and $q \geqq 4$, and made the somewhat arbitrary assumption that $\kappa \leqq 30°$ in most applications where the direction of η is determined closely enough to be useful. I calculated that for $\kappa \leqq 30°$, when μ is increased from 2 to ∞, κ decreases by at most 1.3 %, and this did not seem worth trying for. (Actually it is not possible to increase μ all the way to ∞ without lowering the confidence coefficient below $1-\alpha$, as we found at the end of the last paragraph above (27). On the other hand, I found that the corresponding bound for increasing μ from 1 to 2 is 2.8 %, and that this is nearly attained.)

Since for $q = 2$ or 3 the confidence coefficient fell below $1-\alpha$ with $\mu = 2$, I decided to try also $\mu = 3/2$ for these q. In order to determine the confidence coefficient for $\mu = 3/2$ it is necessary to calculate $\Pr\{\eta \in \mathscr{R}\}$ only for δ in the set where $\Pr\{\eta \in \mathscr{R}\} < 1-\alpha$ for $\mu = 2$, since $\Pr\{\eta \in \mathscr{R}\}$ is a strictly decreasing function of μ, because $\mathscr{R}$ is a decreasing set for increasing μ.

Table 5 shows that the largest of the three values 1, 3/2, 2 for μ that gives confidence coefficient $1-\alpha$ is $\mu = 1$ for $q = 2$, and $\mu = 3/2$ for $q = 3$. If one is willing to settle for a confidence coefficient of .898 instead of .900, and of .949 instead of .950,

TABLE 6
Pr {*III*} *for* $v = \infty$
(*Linear interpolation between tabled values of q is satisfactory*)

μ	q	δ: 0^+	.2	.5	1	2
		$\alpha = .10$				
1	2	.031	.021	.010	.003	.000
	3	.026	.017	.009	.003	.000
	4	.023	.015	.008	.003	.000
2	2	.030	.019	.010	.003	.000
	3	.025	.016	.008	.002	.000
	4	.022	.015	.008	.002	.000
	6	.019	.013	.007	.002	.000
	10	.015	.010	.006	.002	.000
	20	.011	.008	.005	.002	.000
		$\alpha = .05$				
1	2	.017	.011	.005	.001	.000
	3	.014	.009	.005	.001	.000
	4	.013	.008	.004	.001	.000
2	2	.016	.010	.005	.001	.000
	3	.013	.009	.004	.001	.000
	4	.012	.008	.004	.001	.000
	6	.010	.007	.004	.001	.000
	10	.009	.006	.003	.001	.000
	20	.007	.005	.003	.001	.000

Table 5 shows he may adopt $\mu = 3/2$ for $q = 2$, and $\mu = 2$ for $q = 3$. If the use of values of μ other than 1, 3/2, 2 is entertained, I mention that interpolation on the relative minimum of $\Pr\{\eta \in \mathscr{R}\}$ for these three values of μ indicates that $\mu = 1.35$ for $q = 2$, and $\mu = 1.75$ for $q = 3$, are safe values. In any case, I recommend adopting $\mu = 2$ for $q \geqq 4$.

If the partial operating characteristics of $\mathscr{R}$ or $\mathscr{R}'$ are desired for $\mu = 3/2$ (or for $1 < \mu < 2$) and $q = 2$ or 3, they may be obtained as follows: Table 2 for $\Pr\{I\}$ does not depend on μ; in Table 6 for $\Pr\{III\}$ the difference between $\mu = 1$ and $\mu = 2$ is negligible; in Table 3 for $\Pr\{II\}$ and Table 4 for $\Pr\{\eta \in \mathscr{R}\}$ the difference between $\mu = 1$ and $\mu = 2$ is $\leqq .016$, and it is suggested that linear interpolation in $2/\mu$ be used (this predicts fairly well the values for $\mu = 3/2$ in Table 5, where the maximum difference of .016 is attained).

That the confidence coefficient of $\mathscr{R}'$ is the same to three decimals as that of $\mathscr{R}$, for $1 \leqq \mu \leqq 2$ and $\alpha = .10$ or .05, may be seen from Table 6 of $\Pr\{III\}$ and Table 4 and Table 5 of $\Pr\{\eta \in \mathscr{R}\}$, since $\Pr\{\eta \in \mathscr{R}'\} = \Pr\{\eta \in \mathscr{R}\} + \Pr\{III\}$, and the tables indicate that $\Pr\{\eta \in \mathscr{R}\} < 1 - \alpha$ implies $\delta > 2$, while $\Pr\{III\} = .000$ for $\delta \geqq 2$.

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