Multiplexing of Signals using Superframes*

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ABSTRACT

In this paper I intend to present the concept of superframes and its use, primarily, in multiplexing techniques. The signals are supposed band limited and three multiplexing schemes are considered: Time Division Multiple Access (TDMA), Frequency Division Multiple Access (FDMA) and Frequency Hoping Multiple Access (FHMA). The first two schemes give rise to tight superframes, whereas for FHMA, the associated superframes are more complex. For some such superframes the dual superframe is obtained in closed form. An example of a FHMA scheme is also presented.

Keywords: Frames, Encoding Scheme, Multiple Access

1. INTRODUCTION

The last fifteen years have witnessed an explosion of the harmonic and functional analysis associated to signal representation methods, particularly on single and multirate filter banks. These allow to decompose and then reconstruct a signal, performing at the same time a time-frequency or time-scale analysis of that signal.

The more general concept, for which a reconstruction formula always holds in a stable manner, is called *frame*. An abstract frame associated to a (separable) Hilbert space H and a countable index set $\mathbf{I}$ is defined as follows: A set $\mathcal{F} = \{f_i\ ,\ i \in \mathbf{I}\}$ of vectors of K $(H \subset K)$ is called frame if there are two positive numbers $0 < A \leq B < \infty$ such that for every $h \in H$,

$$A\|h\|^2 \leq \sum_{i\in\mathbf{I}} |<h, f_i>|^2 \leq B\|h\|^2 \tag{1}$$

Note we allow the frame set vectors to belong to a different (but bigger) Hilbert space K. The usual frame definition is given for $\{Pf_i, i \in \mathbf{I}\}$, where $P : K \to H$ is the orthogonal projector onto H. The numbers A, B are called *frame bounds*, and if they can be chosen equal, the frame is called *tight*. The frame theory has been thoroughly studied for both abstract Hilbert spaces and several concrete coherent sets (complex exponentials - see,[4] Weyl-Heisenberg and wavelet sets - see[8,12,10]); see also[5] for an excelent review of the theory and references therein.

Although generalizations to Banach spaces (see[6]) or uncountable index sets (see[1]) have been proposed, here we shall deal exclusively with the Hilbert space setting and extend it in a completely different direction through the use of supersets.

Suppose $\mathbf{I}$ is a countable index set and consider $(\mathcal{F}_1, \pi_1, \mathbf{I}) \cdots (\mathcal{F}_L, \pi_L, \mathbf{I})$, L indexed sets of vectors (not necessarily from the same Hilbert space), where $\pi_k : \mathbf{I} \to \mathcal{F}_k$ is the corresponding indexing map. A collection of such countable sets of vectors together with their corresponding indexing maps from a same index set is called a *superset*. For simplicity, when no danger of confusion can arise, we shall drop the explicit indexing notation. Thus $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is a superset when an indexing by a same index set $\mathbf{I}$ for each subset $\mathcal{F}_k$ of vectors of some Hilbert space H_k (or a bigger space K_k) is fixed.

This definition has been given with the following purpose in mind: direct sum. Note that one can immediately construct an indexed set in the direct sum space of H_k's simply by pointwisely computing direct sum of vectors:

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_L := \{f_i^1 \oplus \cdots \oplus f_i^L\ ,\ i \in \mathbf{I}\}\ \ ,\ \ f_i^k = \pi_k(i) \in \mathcal{F}_k \tag{2}$$

*Presented at SPIE2000 Meeting; Published in: "Wavelets Applications in Signal and Image Processing VIII", vol.4119, pp.118-130 (2000)

Then the superset $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is called a *superframe* if $\mathcal{F}$ is a frame for the space $H_1 \oplus \cdots \oplus H_L$ (the same goes for *super Riesz basis*, *super Riesz basis for its span* or *super s-Riesz basis*, but, since for the purposes of this paper these objects are not needed, we do not dig any further into their properties here). The frame condition (1) turns into the following. We say $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is a superframe if there are positive bounds $0 < A \leq B < \infty$ such that for every $h_k \in H_k$, $1 \leq k \leq L$,

$$A(\|h_1\|^2 + \cdots \|h_L\|^2) \leq \sum_{i \in \mathbf{I}} (\sum_{k=1}^{L} < h_k, f_i^k >)^2 \leq B(\|h_1\|^2 + \cdots \|h_L\|^2) \tag{3}$$

Suppose each component set $\mathcal{F}_k$ is a frame (or at least a Bessel sequence). We shall denote by E_k the range of its analysis operator $T_k : H_k \to l^2(\mathbf{I})$, $T_k(h) = \{< h, f_i^k > \}_{i \in \mathbf{I}}$, $E_k = Ran\, T_k$.

Superframes enjoy similar properties to the usual frame sets. In particular two properties are extremely useful, namely the geometric characterization and the duality, or frame reconstruction property. These are stated in the following theorem:

THEOREM 1.

I. The superset $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is a superframe if and only if the following two conditions hold true:

a) Each $\mathcal{F}_k$ is a frame, $1 \leq k \leq L$;

b) $E_1 \oplus \cdots \oplus E_L$ is a direct sum and closed subspace of $l^2(\mathbf{I})$, i.e. $E_k \cap (\sum_{l \neq k} E_l) = \{0\}$ and $\sum_{k=1}^{L} E_k$ is closed.

II. Suppose $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is a superframe. Then there is a superframe $(\tilde{\mathcal{F}}_1, \ldots, \tilde{\mathcal{F}}_L)$ (called dual superframe *of $\mathcal{F}$) such that the following reconstruction formula holds for every $h_1 \in H_1, \ldots, h_L \in H_L$:*

$$h_k = \sum_{i \in \mathbf{I}} (\sum_{l=1}^{L} < h_l, f_i^l >) \tilde{f}_i^k = \sum_{i \in \mathbf{I}} (\sum_{l=1}^{L} < h_l, \tilde{f}_i^l >) f_i^k \tag{4}$$

Moreover, if $\tilde{E}_1, \ldots, \tilde{E}_L$ denote the coefficient ranges of the component sets of $\tilde{\mathcal{F}}$, then $\tilde{\mathcal{F}}$ is a dual superframe of $\mathcal{F}$ if and only if for every $1 \leq k \leq L$, $\tilde{\mathcal{F}}_k$ is a dual of $\mathcal{F}_k$ and $\tilde{E}_k$ is orthogonal (in $l^2(\mathbf{I})$-sense) to every E_l, with $l \neq k$. □

The second part of this result suggests to call two frames *orthogonal in the sense of superframes (or supersets)* when their coefficient ranges are orthogonal in $l^2(\mathbf{I})$.

Proofs of Theorem 1 (or variations) have been obtained independently by D.Larson and D.Han (see[11]) and myself (part of the results appeared for instance in[2,3]). The object considered here (superframe, or superset) has been also studied during the '40s and '50s in connection with geometries of selfadjoint projectors of a von Neumann algebra. However consequences to the frame theory are still not completely known. (Recently, together with Z.Landau, we have concentrated on supersets of Weyl-Heisenberg type where some results have been proved.) The purpose of this paper is to point out a connection with a signal transmission problem (namely, multiplexing, or Multiple Access) and to analyze several encoding-decoding schemes. The organization of the paper is the following: in Section 2 the multiplexing property is described in an abstract setting; in Section 3 the TDMA and FDMA schemes are presented; in Section 4 the FHMA scheme is analyzed; a numerical example is presented in Section 5 and is followed by the bibliography.

2. FORMAL MULTIPLEXING WITH SUPERFRAMES

By its very construction, the superframe is able to multiplex signals. For the purposes of this paper, a signal means a vector of a component Hilbert space. Then, the usual one channel encoding-decoding scheme using a frame is rendered in Figure 1. The usage of frame for encoding has the advantage of reducing the reconstruction noise by projecting it into the coefficients subspace. The channel noise reduction is proportional to the frame redundancy. This fact is shown in Figure 2.

The same Figure 2 suggests also a multiplexing procedure: suppose two frames $\mathcal{F}_1$ and $\mathcal{F}_2$ are chosen so that their corresponding coefficients space are transversal (in the sense of Theorem 1, i.e. closed direct sum); then the correspondence $(c^1, c^2) \in E_1 \times E_2 \to c = c^1 + c^2 \in E_1 + E_2$ is invertible. Thus the sequence c encodes full information

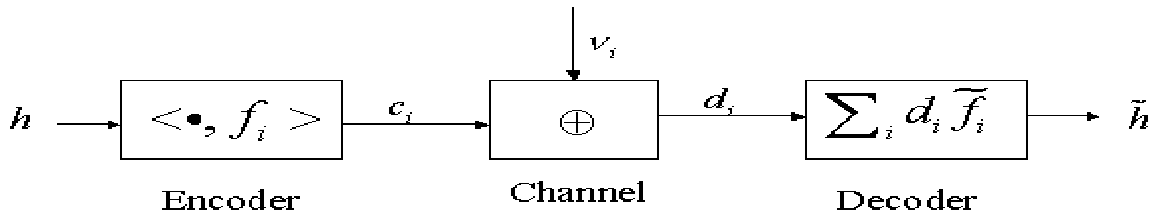

Figure 1. Single Access Encoding-Decoding Scheme using Frames

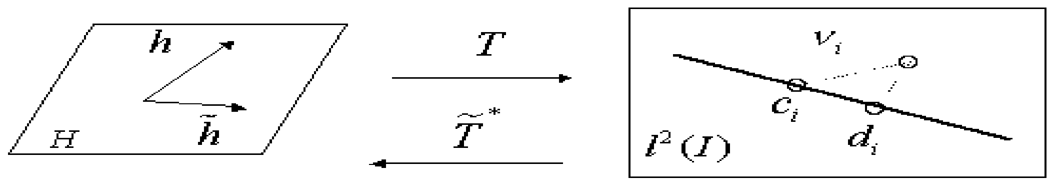

Figure 2. The operator-theoretic description of the decomposition-reconstruction formula

about the vectors $h_1 \in H_1$ and $h_2 \in H_2$ that generated coefficients c^1, respectively c^2. This fact is represented in Figure 3. The decoder has the task to recover the original signals h_1 and h_2 based on the sequence $c = c^1 + c^2 \in l^2(\mathbf{I})$. Since the reconstruction is supposed linear, there are two indexed subsets $\tilde{\mathcal{F}}_1 = \{\tilde{f}_i^1, i \in \mathbf{I}\}$ and $\tilde{\mathcal{F}}_2 = \{\tilde{f}_i^2, i \in \mathbf{I}\}$ of H_1, respectively H_2, such that the reconstructed vectors $\tilde{h}_1$, respectively $\tilde{h}_2$ are given by:

$$\tilde{h}_1 = \sum_{i \in \mathbf{I}} c_i \tilde{f}_i^1 \quad , \quad \tilde{h}_2 = \sum_{i \in \mathbf{I}} c_i \tilde{f}_i^2 \tag{5}$$

as pictured in Figure 4. Thus, the perfect reconstruction requirement (that is $\tilde{h}_1 = h_1$ and $\tilde{h}_2 = h_2$) implies $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ has to be a dual superframe of $(\mathcal{F}_1, \mathcal{F}_2)$ (implicitely it is assumed that (5) defines bounded operators).

As in the standard frame theory there may be many duals. However there is one distingueshed dual (called the *standard dual superframe*) obtained as follows. Consider $(\mathcal{F}_1, \cdots, \mathcal{F}_L)$ a superframe for $(H_1, \ldots, H_L)$ indexed by $\mathbf{I}$, and set the *superframe operator* defined by:

$$S : H_1 \oplus \cdots \oplus H_L \to H_1 \oplus \cdots \oplus H_L \quad , \quad S = \begin{bmatrix} T_1^* T_1 & \cdots & T_1^* T_L \\ \vdots & & \vdots \\ T_L^* T_1 & \cdots & T_L^* T_L \end{bmatrix} \tag{6}$$

whose action is given by:

$$S(f_1 \oplus \cdots \oplus f_L) = (T_1^* T_1 f_1 + \cdots + T_1^* T_L f_L) \oplus \cdots \oplus (T_L^* T_1 f_1 + \cdots + T_L^* T_L f_L) \tag{7}$$

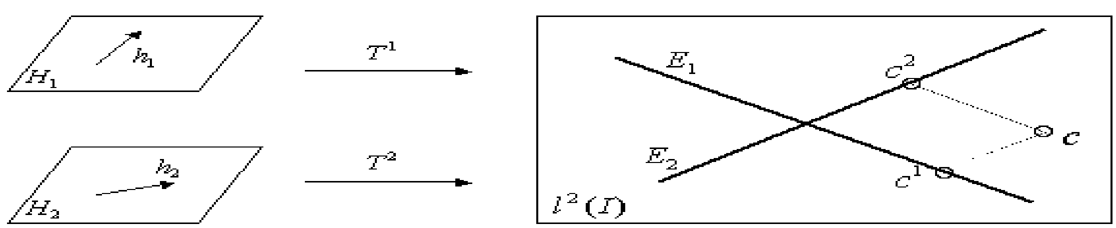

Figure 3. The formal representation for superframe coefficient spaces

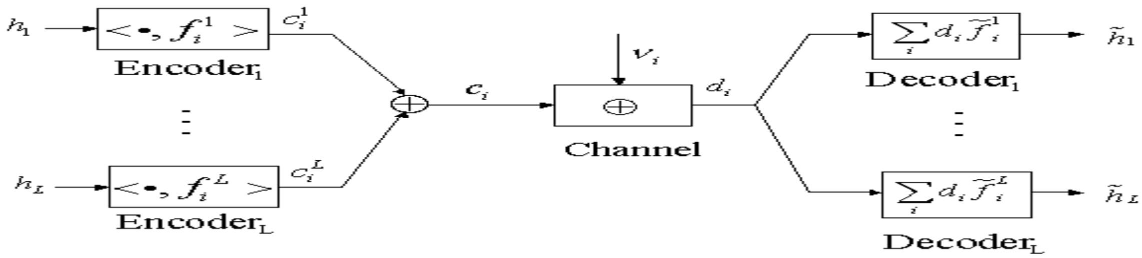

Figure 4. Multiple Access Encoding-Decoding Scheme using Superframes

where $T_k : H_k \to l^2(\mathbf{I})$ is the *analysis operator* of frame $\mathcal{F}_k$, $T_k(h) = \{< h, f_i^k >\}_{i\in\mathbf{I}}$, and $T_k^* : l^2(\mathbf{I}) \to H_k$ (the adjoint of the previous operator) is the *synthesis operator* given by $T_k^*(c) = \sum_{i\in\mathbf{I}} c_i f_i^k$. The superframe operator S is selfadjoint and positive, bounded above and below by the superframe bounds. Thus $S^{-1} : H_1 \oplus \cdots \oplus H_L \to H_1 \oplus \cdots \oplus H_L$ exists and is well-defined. Let $\pi_j : H_1 \oplus \cdots \oplus H_L \to H_j$ denote the canonical projection, $\pi_j(f_1 \oplus \cdots \oplus f_L) = f_j$, $1 \le j \le L$. Then the standard dual superframe $(\tilde{\mathcal{F}}_1, \cdots, \tilde{\mathcal{F}}_L)$ is defined by:

$$\tilde{\mathcal{F}}_k = \{\tilde{f}_i^k = \pi_k S^{-1}(f_i^1 \oplus \cdots \oplus f_i^L)\ ,\ i \in \mathbf{I}\} \tag{8}$$

One can easily check that $(\tilde{\mathcal{F}}_1, \cdots, \tilde{\mathcal{F}}_L)$ thus defined is a superframe and indeed a dual of $(\mathcal{F}_1, \cdots, \mathcal{F}_L)$ (i.e. the reconstruction formula (4) holds true).

3. TDMA, FDMA AND THEIR ASSOCIATED SUPERFRAMES

In this subsection we translate several Multiple Access (MA) techniques into the language of superframes. The basic scenario is the following: L Ω_0-band limited signals are to be encoded using a MA scheme (either TDMA, FDMA or FHMA). The single signal sampling is done at a higher-than-Nyquist rate, say $2\Omega > 2\Omega_0$, in order to avoid the use of "sinc" function at reconstruction. However, since there are L signals to be encoded simultaneously, the actual sampling rate would be L times larger than the one needed for only one signal.

First let us recall some known results on band-limited signals. Let $B^2_{\Omega_0}$ denote the space of Ω_0-band limited signals, i.e.

$$B^2_{\Omega_0} = \{f \in L^2(\mathbf{R})\quad,\quad supp\,\hat{f} \subset [-\Omega_0, \Omega_0]\} \tag{9}$$

where the Fourier transform is normalized by $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx$. Choose a function (window) $g \in L^2(\mathbf{R})$ such that $supp\,\hat{g} \subset [-\Omega, \Omega]$ and $\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}$ for all $|\omega| \le \Omega_0$. The following lemma characterizes the $\frac{\pi}{\Omega}$-translates of g:

LEMMA 2. *Consider a g as above. Then for every $f, h \in B^2_{\Omega_0}$ and $a \in \mathbf{R}$,*

1. *$f(x) =< f, g(\cdot - x) >$. In particular, $f(n\frac{\pi}{\Omega}) =< f, g(\cdot - n\frac{\pi}{\Omega}) >$.*
2. *$f(x) = \frac{\sqrt{2\pi}}{2\Omega} \sum_n f(n\frac{\pi}{\Omega}) g(x - n\frac{\pi}{\Omega})$.*
3. *$\{g(\cdot - n\frac{\pi}{\Omega} - a)\ ,\ n \in \mathbf{Z}\}$ is a tight frame for $B^2_{\Omega_0}$ with bound $\frac{2\Omega}{\sqrt{2\pi}}$.*
4. *$\sum_{n\in\mathbf{Z}} f(n\frac{\pi}{\Omega} + a)\overline{h(n\frac{\pi}{\Omega} + a)} = \frac{2\Omega}{\sqrt{2\pi}} < f, h >$* □

Now consider the case of L signals $f_0, \cdots, f_{L-1} \in B^2_{\Omega_0}$ and denote $T = \frac{\pi}{\Omega}$.

A. In the *Time Division Multiple Access* (TDMA) scheme, the encoded sequence is the following:

$$c_{nL+k} = f_k(nT + k\frac{T}{L})\ ,\ 0 \le k \le L\ ,\ n \in \mathbf{Z} \tag{10}$$

or, explicitely

$$c = (\ldots, f_0(nT), f_1(nT + \frac{T}{L}), \ldots, f_{L-1}(nT + (L-1)\frac{T}{L}), f_0((n+1)T), f_1((n+1)T + \frac{T}{L}), \ldots)$$

This transform suggests to use of the following frames:

$$\begin{aligned}
\mathcal{G}_0^{TD} &= \{\cdots, g(\cdot - nT), 0, \cdots, 0, g(\cdot - (n+1)T), 0, \cdots\} \\
\mathcal{G}_1^{TD} &= \{\cdots, 0, g(\cdot - nT - \frac{T}{L}), \cdots, 0, 0, g(\cdot - (n+1)T - \frac{T}{L}), \cdots\} \\
\cdots \\
\mathcal{G}_{L-1}^{TD} &= \{\cdots, 0, 0, \cdots, g(\cdot - nT - (L-1)\frac{T}{L}), 0, 0, \cdots, g(\cdot - (N+1)T - (L-1)\frac{T}{L}), \cdots\}
\end{aligned} \tag{11}$$

that compactly can be written as:

$$\mathcal{G}_k^{TD} = \{g_{nL+r}^k = a_{nL+r}^k g(\cdot - (nL+r)\frac{T}{L}) \quad , \quad 0 \le r \le L-1, n \in \mathbf{Z}\} \quad , \quad 0 \le k \le L-1 \tag{12}$$

where the coefficients a_{nL+r}^k are given by:

$$a_{nL+r}^k = \delta_{k,r} = \begin{cases} 1 & , \quad if\ k = r \\ 0 & , \quad otherwise \end{cases} \tag{13}$$

Then one can easily check that $(\mathcal{G}_0^{TD}, \mathcal{G}_1^{TD}, \ldots, \mathcal{G}_{L-1}^{TD})$ is a superframe for $(B_{\Omega_0}^2, B_{\Omega_0}^2, \ldots, B_{\Omega_0}^2)$. Note that each frame set is a subset of B_Ω^2, where $B_{\Omega_0}^2$ is naturally embedded. Moreover, the coding sequence c in (10) represents the coefficients sequence associated to the L-vector $(f_0, f_1, \cdots, f_{L-1})$ for the superframe $(\mathcal{G}_0^{TD}, \mathcal{G}_1^{TD}, \ldots, \mathcal{G}_{L-1}^{TD})$. Indeed this follows imediately from Lemma 2:

$$\sum_{k=0}^{L-1} < f_k, g_{nL+r}^k >=< f_r, g(\cdot - nT - r\frac{T}{L}) >= f_r(nT + r\frac{T}{L}) = c_{nL+r}$$

The TDMA superframe defined by (11) is tight. Indeed for any $k \neq l$, the coefficient ranges E_k, E_l associated to $\mathcal{G}_k^{TD}$, respectively $\mathcal{G}_l^{TD}$, are orthogonal in $l^2(Z)$, because $a_{nL+r}^{k_1} a_{nL+r}^{k_2} = 0$, for $k_1 \neq k_2$. Moreover, each $\mathcal{G}_k^{TD}$ is tight as shown in Lemma 2. Thus the decoder is obtained from:

$$f_k(x) = \frac{\sqrt{2\pi}}{2\Omega} \sum_{n \in \mathbf{Z}} a_n^k c_n g(\cdot - n\frac{T}{L}) \tag{14}$$

that reduces to:

$$f_k(x) = \frac{\sqrt{2\pi}}{2\Omega} \sum_{n \in \mathbf{Z}} c_{nL+k} g(\cdot - nT - k\frac{T}{L}) \tag{15}$$

B. The *Frequency Division Multiple Access* (FDMA) encoder uses the following scheme:

$$(f_0, f_1, \ldots, f_{L-1}) \mapsto [f(t) = e^{i\omega_0 t} f_0(t) + e^{i\omega_1 t} f_1(t) + \cdots + e^{i\omega_{L-1} t} f_{L-1}(t)]|_{t = n\frac{T}{L}} \ , \ n \in \mathbf{Z} \tag{16}$$

that means the signals are shifted in frequency domain and then sampled at multiples of $\frac{T}{L}$. The frequencies (ω_k) are chosen linearly: $\omega_k = \omega_0 + k \cdot 2\Omega$. Thus the encoded sequence is:

$$c_n = e^{i\omega_0 n\frac{T}{L}} \sum_{k=0}^{L-1} e^{\frac{2\pi i}{L} nk} f_k(n\frac{T}{L}) \tag{17}$$

This multiplexing corresponds to the following superframe:

$$\mathcal{G}_k^{FD} = \{g_n^k = b_n^k g(\cdot - n\frac{T}{L}) \quad , \quad n \in \mathbf{Z}\} \tag{18}$$

where the complex numbers b_n^k are given by:

$$b_n^k = e^{-in\omega_0 \frac{T}{L}} e^{-\frac{2\pi i}{L} nk} \tag{19}$$

One can easily check this is a superframe for $(B^2_{\Omega_0}, B^2_{\Omega_0}, \ldots, B^2_{\Omega_0})$. Moreover, as in the TDMA case, this superframe is tight with bound $L\frac{2\Omega}{\sqrt{2\pi}}$ as we next show. First note that each component set $\mathcal{G}_k^{FD}$ is a tight frame for $B^2_{\Omega_0}$ because it is the union of L tight frames of the form $\{g(\cdot - nT - a), n \in \mathbf{Z}\}$ (hence the superframe bound). The only thing that remains to be proved is the orthogonality between any two component frames, in the sense of superframes. For every $f_k, f_l \in B^2_{\Omega_0}$ and $k \neq l$,

$$\begin{aligned} < T_k f_k, T_l f_l > &= \sum_{n\in\mathbf{Z}} < f_k, g_n^k >< g_n^l, f_l >= \sum_{n\in\mathbf{Z}} e^{\frac{2\pi i}{L} n(k-l)} f_k(n\frac{T}{L})\overline{f_l(n\frac{T}{L})} \\ &= \sum_{r=0}^{L-1} e^{\frac{2\pi i}{L} r(k-l)} \sum_{n\in\mathbf{Z}} f_k(nT + r\frac{T}{L})\overline{f_l(nT + r\frac{T}{L})} = \sum_{r=0}^{L-1} e^{\frac{2\pi i}{L} r(k-l)} \frac{2\Omega}{\sqrt{2\pi}} < f_k, f_l >= 0 \end{aligned}$$

Thus the ranges of analysis operators T_k, T_l of $\mathcal{G}_k^{FD}$, respectively $\mathcal{G}_l^{FD}$, are orthogonal, and the superframe $(\mathcal{G}_0^{FD}, \mathcal{G}_1^{FD}, \cdots, \mathcal{G}_{L-1}^{FD})$ is tight. Therefore the decoding is done by:

$$f_k(x) = \frac{\sqrt{2\pi}}{2\Omega L} \sum_{n\in\mathbf{Z}} c_n b_n^k g(x - n\frac{T}{L}) \tag{20}$$

4. THE FHMA SCHEME

The *Frequency Hoping Multiple Access* (FHMA) scheme is a variation of FDMA. The signals are multiplexed using a similar sampling formula as (16) but with randomly generated frequency hopes $(\omega_{k,n})_k$ at every n:

$$(f_0, f_1, \ldots, f_{L-1}) \mapsto [f(t) = e^{i\omega_{0,n}t} f_0(t) + e^{i\omega_{1,n}t} f_1(t) + \cdots + e^{i\omega_{L-1,n}t} f_{L-1}(t)]|_{t=n\frac{T}{L}} \ , \ n \in \mathbf{Z} \tag{21}$$

where $(\omega_{k,n})_k$ are obtained from a random permutation $\pi_n \in \mathcal{P}_L$ of $\{0, 1, \cdots, L-1\}$ via:

$$\omega_{k,n} = \omega_0 + 2\Omega\pi_n(k) \quad , \quad 0 \leq k \leq L-1 \ , \ n \in \mathbf{Z} \tag{22}$$

Thus the associated frames to FHMA scheme are the following:

$$\begin{aligned} \mathcal{G}_0^{FH} &= \{e^{-in\omega_0\frac{T}{L}} e^{-\frac{2\pi i}{L} n\pi_n(0)} g(\cdot - n\frac{T}{L}) \ , \ n \in \mathbf{Z}\} \\ \mathcal{G}_1^{FH} &= \{e^{-in\omega_0\frac{T}{L}} e^{-\frac{2\pi i}{L} n\pi_n(1)} g(\cdot - n\frac{T}{L}) \ , \ n \in \mathbf{Z}\} \\ &\vdots \\ \mathcal{G}_{L-1}^{FH} &= \{e^{-in\omega_0\frac{T}{L}} e^{-\frac{2\pi i}{L} n\pi_n(L-1)} g(\cdot - n\frac{T}{L}) \ , \ n \in \mathbf{Z}\} \end{aligned} \tag{23}$$

or, more compactly,

$$\mathcal{G}_k^{FH} = \{g_n^k(x) = c_n^k g(\cdot - n\frac{T}{L}) \ , \ n \in \mathbf{Z}\} \quad , \quad c_n^k = e^{-in\Omega_0\frac{T}{L} - \frac{2\pi i}{L} n\pi_n(k)}$$

with $g \in B^2_\Omega$ and $\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}$ for $|\omega| \leq \Omega_0$.

By Lemma 2 each $\mathcal{G}_k^{FH}$ is a tight frame with bound $L\frac{2\Omega}{\sqrt{2\pi}}$, yet their collection $(\mathcal{G}_0^{FH}, \ldots, \mathcal{G}_{L-1}^{FH})$ may not be a superframe. In fact one can check that assuming independent and uniform distribution for $(\pi_n)_n$, the expected value of the superset lower bound is zero.

For the remaining of this section we shall restrict to the case when π_n is periodic, i.e. $\pi_{n+N_0} = \pi_n$, $\forall n$, for some N_0. Consider the period N_0 is a multiple of L (otherwise replace it by the least common multiple between N_0 and L). Suppose $N_0 = RL$. The goal is to find necessary and sufficient conditions for $(\mathcal{G}_0^{FH}, \ldots, \mathcal{G}_{L-1}^{FH})$ to be a superframe and to obtain its (standard) dual superframe in a closed form. We start by computing the frame operator, $S : B^2_{\Omega_0} \oplus \cdots \oplus B^2_{\Omega_0} \to B^2_{\Omega_0} \oplus \cdots \oplus B^2_{\Omega_0}$. It is more advantageous to compute its Fourier transform conjugate,

$\hat{S}:\mathcal{F}S\mathcal{F}^{-1}$, where $\mathcal{F}:B^2_{\Omega_0}\oplus\cdots\oplus B^2_{\Omega_0}\to L^2[-\Omega_0,\Omega_0]\oplus\cdots\oplus L^2[-\Omega_0,\Omega_0]$, $\mathcal{F}(f_0\oplus\cdots f_{L-1})=\hat{f}_0\oplus\cdots\oplus \hat{f_{L-1}}$. Then $\hat{S}$ acts as follows:

$$\hat{S}(\hat{f}_0\oplus\cdots\oplus \hat{f_{L-1}})=\bigoplus_{l=0}^{L-1}\sum_n(\sum_{k=0}^{L-1}<\hat{f}_k,\hat{g^k_n}>)\hat{g^l_n}$$

The l^{th} component, call it φ_l, has the following expansion:

$$\varphi_l(\omega)=\sum_n\sum_{k=0}^{L-1}\frac{1}{\sqrt{2\pi}}e^{\frac{2\pi i}{L}n(\pi_n(k)-\pi_n(l))}e^{-in\frac{T}{L}\omega}\int_{-\Omega_0}^{\Omega_0}e^{in\frac{T}{L}\xi}\hat{f}_k(\xi)d\xi$$

Setting $n=n'RL+r$, $0\le r\le RL-1$, and performing the summation over n' (for instance consider $\hat{f}_k$ in $C[-\Omega_0,\Omega_0]$ and apply the Poisson summation formula) the above expression turns into:

$$\varphi_l(\omega+\frac{2\Omega}{R}m)=\sum_{n=0}^{\rho-1}\sum_{k=0}^{L-1}[\frac{\Omega}{\pi R}\sum_{r=0}^{RL-1}e^{\frac{2\pi i}{L}r(\frac{n-m}{R}+\pi_r(k)-\pi_r(l))}]\hat{f}_k(\omega+\frac{2\Omega}{R}n) \tag{24}$$

where $\omega\in[-\Omega_0,-\Omega_0+\frac{2\Omega}{R}]$, $m\in\{0,1,\ldots,\rho-1\}$ and ρ is a positive integer such that $\omega+(\rho-1)\frac{2\Omega}{R}\le\Omega_0<\omega+\rho\frac{2\Omega}{R}$. Define the following objects:

$$\begin{aligned}
\rho &= \lceil\frac{R\Omega_0}{\Omega}\rceil\ \ \text{(the smallest integer greater than }\frac{R\Omega_0}{\Omega})\\
I_1 &= [-\Omega_0,\Omega_0-(\rho-1)\frac{2\Omega}{R}]\subset\mathbf{R}\\
I_2 &= [\Omega_0-(\rho-1)\frac{2\Omega}{R},-\Omega_0+\frac{2\Omega}{R}]\subset\mathbf{R}\\
\gamma^1_n &= [e^{-\frac{2\pi i}{L}n\pi_n(0)},e^{-\frac{2\pi i}{L}n(\pi_n(0)+\frac{1}{R})},\ldots,e^{-\frac{2\pi i}{L}n(\pi_n(0)+\frac{\rho-1}{R})},\ldots,e^{-\frac{2\pi i}{L}n\pi_n(L-1)},\ldots,e^{-\frac{2\pi i}{L}n(\pi_n(L-1)+\frac{\rho-1}{R})}]^T\in\mathbf{C}^{L\rho}\\
\gamma^2_n &= [e^{-\frac{2\pi i}{L}n\pi_n(0)},e^{-\frac{2\pi i}{L}n(\pi_n(0)+\frac{1}{R})},\ldots,e^{-\frac{2\pi i}{L}n(\pi_n(0)+\frac{\rho-2}{R})},\ldots,e^{-\frac{2\pi i}{L}n\pi_n(L-1)},\ldots,e^{-\frac{2\pi i}{L}n(\pi_n(L-1)+\frac{\rho-2}{R})}]^T\in\mathbf{C}^{L(\rho-1)}
\end{aligned}$$

and the unitary operator:

$$U:L^2[-\Omega_0,\Omega_0]\oplus\cdots\oplus L^2[-\Omega_0,\Omega_0]\to L^2(I_1,\mathbf{C}^{L\rho})\oplus L^2(I_2,\mathbf{C}^{L(\rho-1)})$$

$$\begin{aligned}
(f_0\oplus\cdots\oplus f_{L-1}) &\mapsto (\omega\mapsto[f_0(\omega),\cdots,f_0(\omega+(\rho-1)\tfrac{2\Omega}{R}),\cdots,f_{L-1}(\omega),\cdots,f_{L-1}(\omega+(\rho-1)\tfrac{2\Omega}{R})]^T\ ,\ \omega\in I_1)\oplus\\
&\oplus (\omega\mapsto[f_0(\omega),\cdots,f_0(\omega+(\rho-2)\tfrac{2\Omega}{R}),\cdots,f_{L-1}(\omega),\cdots,f_{L-1}(\omega+(\rho-2)\tfrac{2\Omega}{R})]^T\ ,\ \omega\in I_2)
\end{aligned}$$

Then, with this change of coordinates, the superframe operator acts multiplicatively on $L^2(I_1,\mathbf{C}^{L\rho})\oplus L^2(I_2,\mathbf{C}^{L(\rho-1)})$:

$$\begin{aligned}
&S^\sharp=U\mathcal{F}S\mathcal{F}^{-1}U^{-1}:L^2(I_1,\mathbf{C}^{L\rho})\oplus L^2(I_2,\mathbf{C}^{L(\rho-1)})\to L^2(I_1,\mathbf{C}^{L\rho})\oplus L^2(I_2,\mathbf{C}^{L(\rho-1)})\\
&S^\sharp(F_1\oplus F_2)=(A^1F_1\oplus A^2F_2)
\end{aligned} \tag{25}$$

where $A^1\in\mathbf{C}^{L\rho\times L\rho}$, $A^2\in\mathbf{C}^{L(\rho-1)\times L(\rho-1)}$ are constant matrices given by:

$$A^1_{l\rho+m,k\rho+n}=\frac{\Omega}{\pi R}\sum_{r=0}^{RL-1}e^{\frac{2\pi i}{L}r(\frac{n-m}{R}+\pi_r(k)-\pi_r(l))}\ \ ,\ \ 0\le n,m\le\rho-1\ ,\ 0\le l,k\le L-1 \tag{26}$$

and

$$A^2_{l\rho+m,k\rho+n}=\frac{\Omega}{\pi R}\sum_{r=0}^{RL-1}e^{\frac{2\pi i}{L}r(\frac{n-m}{R}+\pi_r(k)-\pi_r(l))}\ \ ,\ \ 0\le n,m\le\rho-1\ ,\ 0\le l,k\le L-2 \tag{27}$$

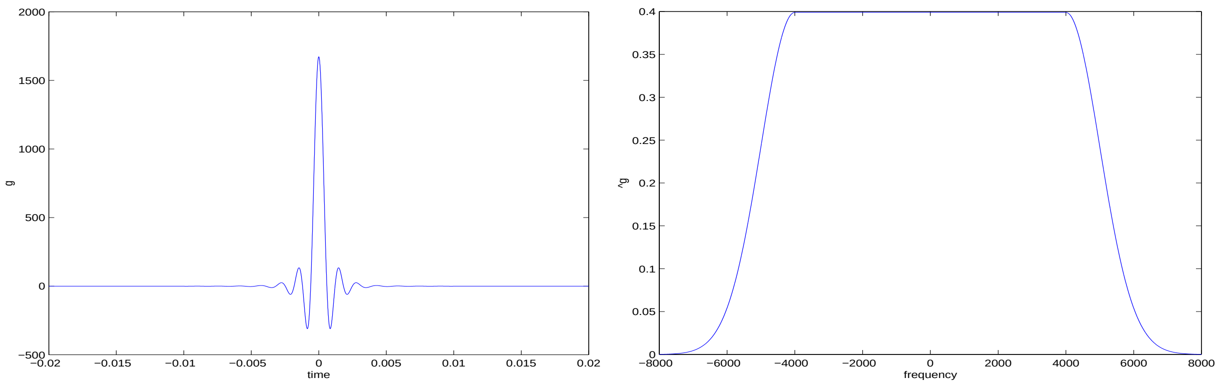

Figure 5. The encoding window in time (left) and frequency (right) domain.

Note A^2 is a submatrix of A^1 and both are hermitian matrices. A straightforward computation shows that:

$$U\mathcal{F}(P_{\Omega_0} g_n^k) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-in(\omega_0+\omega)\frac{T}{L}} \gamma_n^1 & , \quad for\ \omega \in I_1 \\ \frac{1}{\sqrt{2\pi}} e^{-in(\omega_0+\omega)\frac{T}{L}} \gamma_n^2 & , \quad for\ \omega \in I_2 \end{cases}$$

where P_{Ω_0} is the orthogonal projection onto $B^2_{\Omega_0}$ from B^2_{Ω}. Note also that $\gamma^{1,2}_{n+RL} = \gamma_n^{1,2}$. All these constructions and remarks imply the following result:

THEOREM 3. *I. The FH superset $(\mathcal{G}_0^{FH}, \ldots, \mathcal{G}_{L-1}^{FH})$ defined by (23) is a superframe if and only if $det(A^1) \neq 0$. Moreover, the superframe bounds are the smallest and largest eigenvalues of A^1.*

II. The standard dual superframe $(\mathcal{G}_0^{\tilde{F}H}, \ldots, \mathcal{G}_{L-1}^{\tilde{F}H})$ is given by $\mathcal{G}_k^{\tilde{F}H} = \{e^{-in\omega_0\frac{T}{L}} \tilde{g_n^k}(\cdot - n\frac{T}{L}), n \in \mathbf{Z}\}$ with:

$$U\mathcal{F}\tilde{g_n^k} = \begin{cases} \frac{1}{\sqrt{2\pi}}(A^1)^{-1}\gamma_n^1 & , \quad for\ \omega \in I_1 \\ \frac{1}{\sqrt{2\pi}}(A^2)^{-1}\gamma_n^2 & , \quad for\ \omega \in I_2 \end{cases} \tag{28}$$

Moreover, the dual sequence is RL-periodic, i.e. $\tilde{g^k_{pRL+n}} = \tilde{g_n^k}$.

5. AN EXAMPLE

In this section we present the dual windows for a FHMA scheme. We have considered four signal, $L = 4$, and permutations of period sixteen, $R = 4$. The signals have $\Omega_0 = 4000$, and the oversampling is at a rate twice the Nyquist rate, $\Omega = 8000$. The encoding window is plotted in Figure 5. Using (28) for the decoding windows, we obtained $16 \times 4 = 64$ windows, 16 for each signal. We present several such dual windows in Figures 6-16. The windows indexed by multiples of L (here 4) are real-valued and identical for all k's (the imaginary component shown in Figures 6, 8 and 10 are due to numerical integration); the other windows are complex-valued. Unfortunately, the frequency domain discontinuities make the windows to decay like $\frac{1}{t}$ in time domain. The decay can be improved for the real-valued windows by conveniently tailoring the frequency profile outside the $[-\Omega_0, \Omega_0]$ band, as we have done for g in Figure 5.

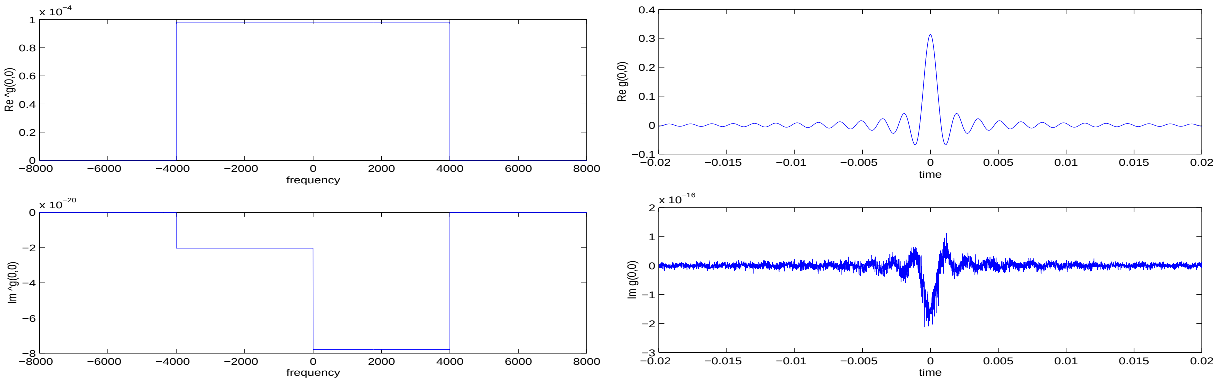

Figure 6. The decoding window g_0^0 (i.e. $n = 0, k = 0$) in frequency (left) and time (right) domain.

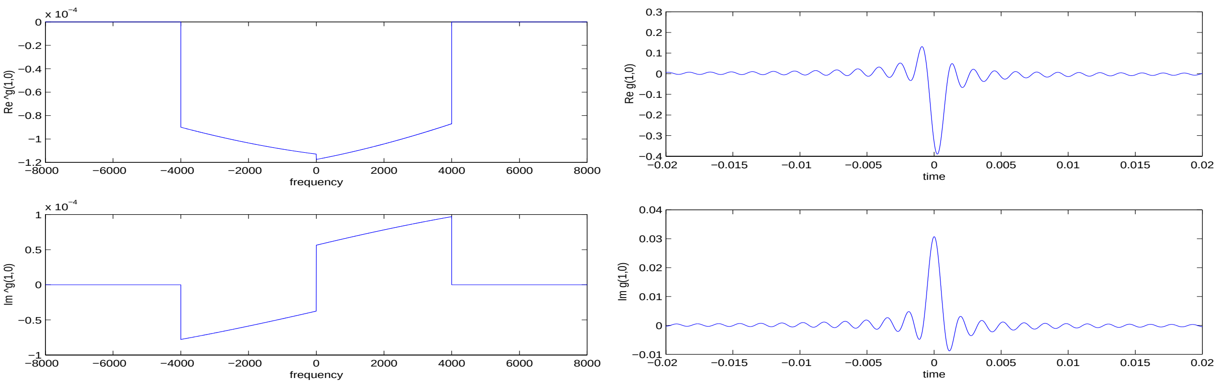

Figure 7. The decoding window g_1^0 (i.e. $n = 1, k = 0$) in frequency (left) and time (right) domain.

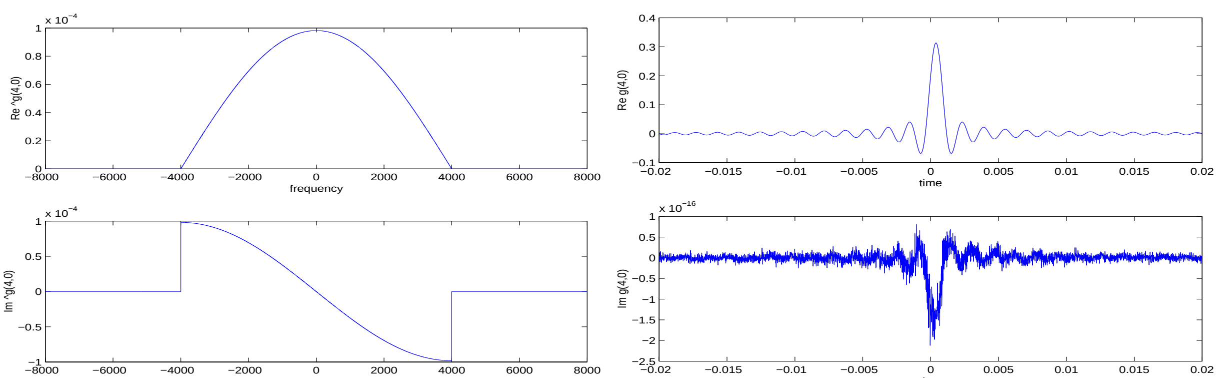

Figure 8. The decoding window g_4^0 (i.e. $n = 4, k = 0$) in frequency (left) and time (right) domain.

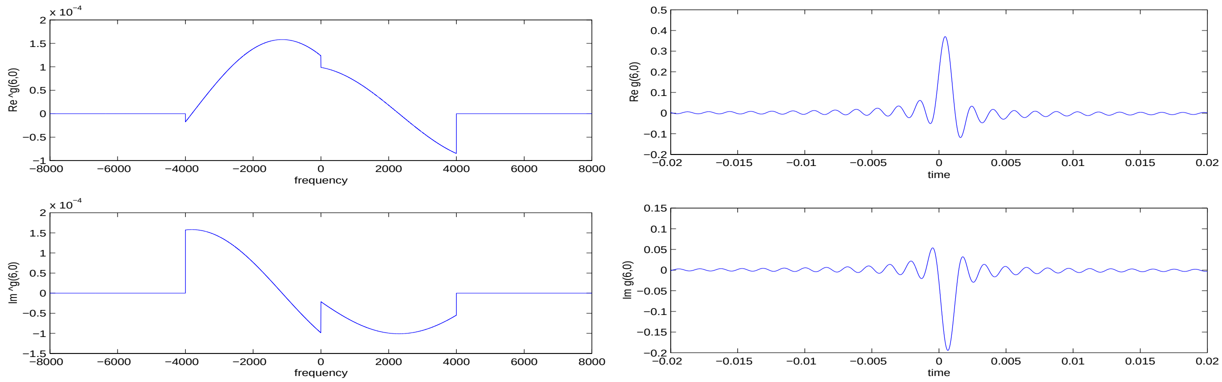

Figure 9. The decoding window g_6^0 (i.e. $n = 6, k = 0$) in frequency (left) and time (right) domain.

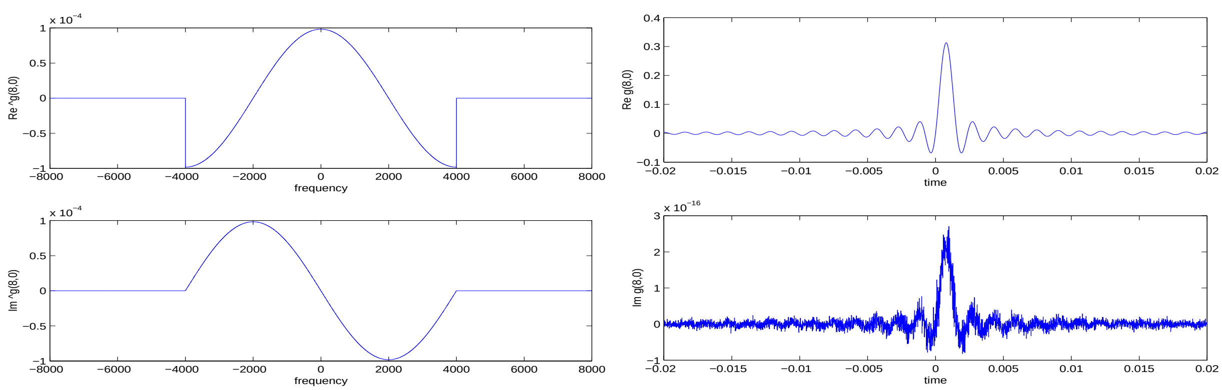

Figure 10. The decoding window g_8^0 (i.e. $n = 8, k = 0$) in frequency (left) and time (right) domain.

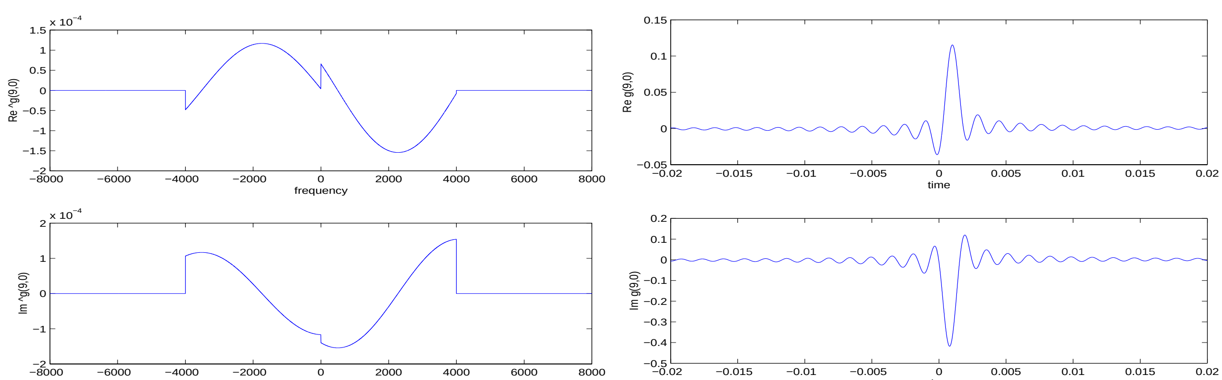

Figure 11. The decoding window g_9^0 (i.e. $n = 9, k = 0$) in frequency (left) and time (right) domain.

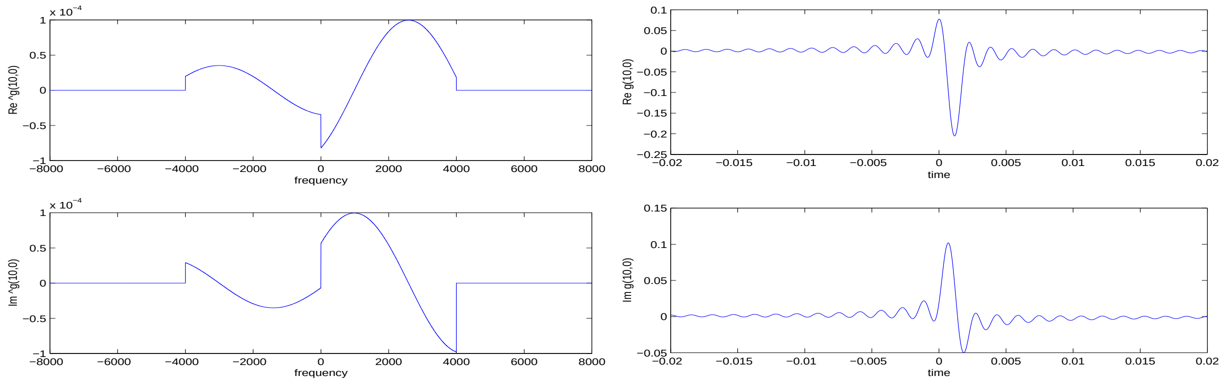

Figure 12. The decoding window g_{10}^{0} (i.e. $n = 10, k = 0$) in frequency (left) and time (right) domain.

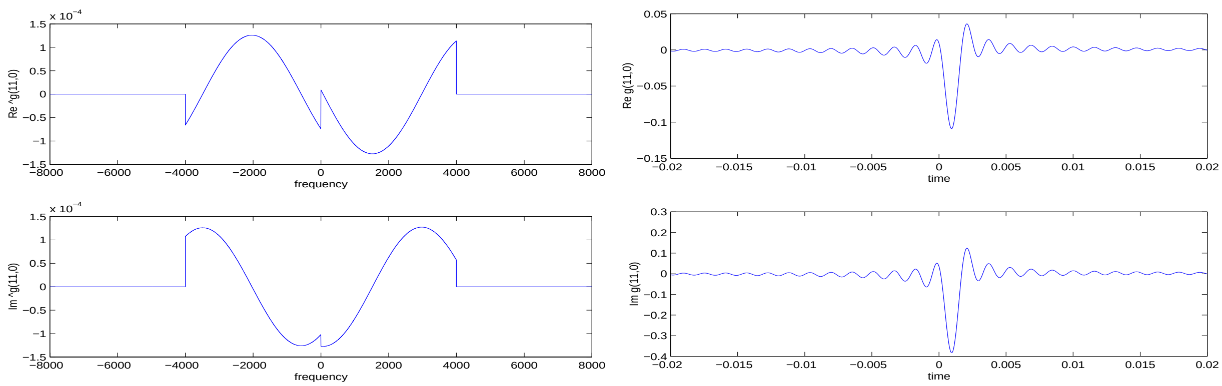

Figure 13. The decoding window g_{11}^{0} (i.e. $n = 11, k = 0$) in frequency (left) and time (right) domain.

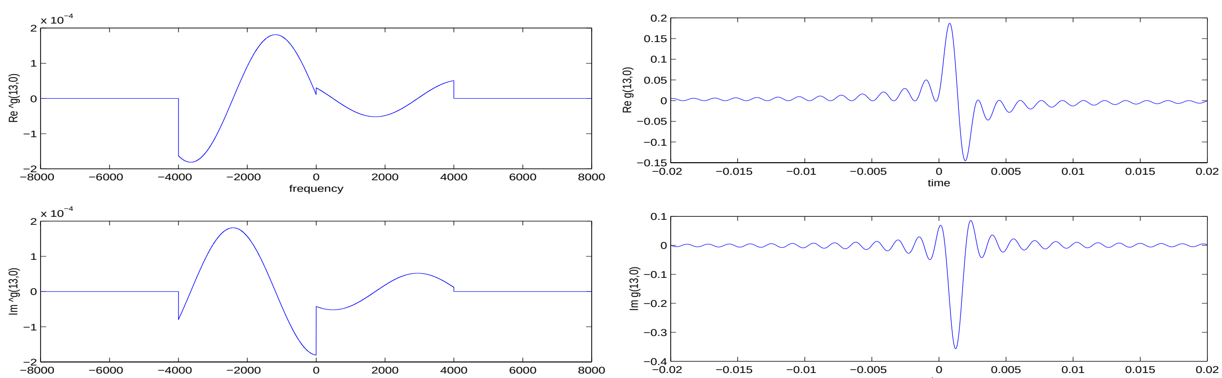

Figure 14. The decoding window g_{13}^{0} (i.e. $n = 13, k = 0$) in frequency (left) and time (right) domain.

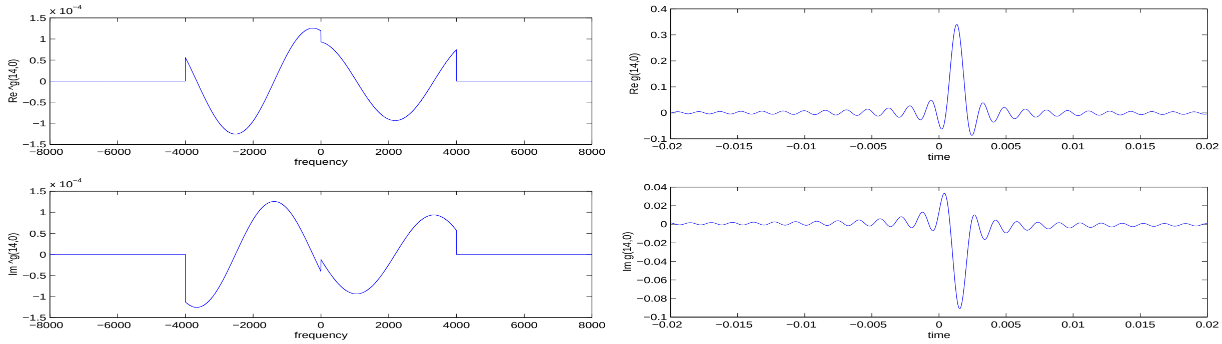

Figure 15. The decoding window g_{14}^0 (i.e. $n = 14, k = 0$) in frequency (left) and time (right) domain.

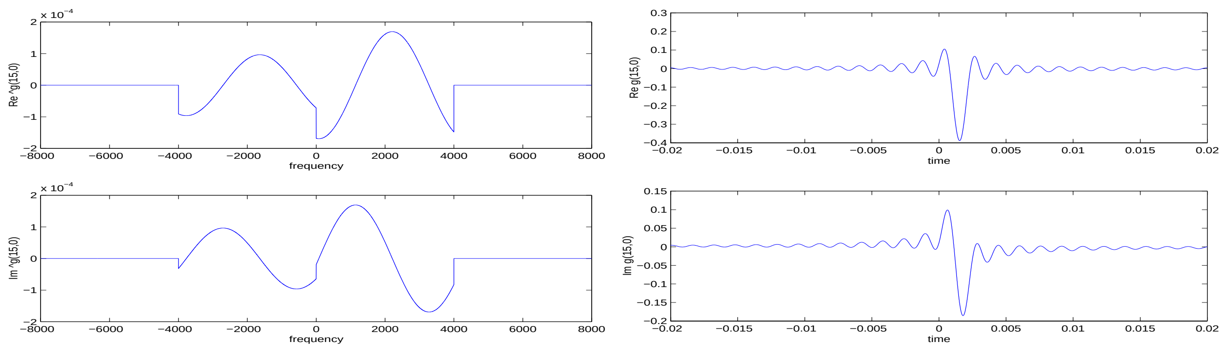

Figure 16. The decoding window g_{15}^0 (i.e. $n = 15, k = 0$) in frequency (left) and time (right) domain.

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