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MULTIPLICATION AND COMPOSITION OPERATORS ON LORENTZ-BOCHNER SPACES

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Abstract

In this paper we study the multiplication and composition operators induced by operator valued maps on Bochner spaces (Lorentz-Bochner and rearrangement invariant-Bochner) and discuss their closedness, compactness and spectrum.

Introduction

Let f be a complex-valued measurable function defined on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. For $s \geq 0$, define μ_f the *distribution function* of f as

$$\mu_f(s) = \mu(\{\omega \in \Omega : |f(\omega)| > s\}).$$

By f^* we mean the *non-increasing rearrangement* of f given as

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}, \quad t \geq 0.$$

For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \quad \text{and} \quad f^{**}(0) = f^*(0).$$

For $1 < p \leq \infty$, $1 \leq q \leq \infty$, and for a measurable function f on Ω define $\|f\|_{pq}$ as

$$\|f\|_{pq} = \begin{cases} \left\{\dfrac{q}{p}\displaystyle\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

The *Lorentz space* $L_{pq}(\Omega)$ consists of those measurable functions f on Ω such that $\|f\|_{pq} < \infty$. Also $\|\cdot\|_{pq}$ is a norm and $L_{pq}(\Omega)$ is a Banach space with respect to this norm. The L^p-spaces for $1 < p \leq \infty$ are equivalent to the spaces $L_{pp}(\Omega)$. Let us recall that simple functions are dense in $L_{pq}(\Omega)$ for $q \neq \infty$ and also the duality results $L_{p1}^* = L_{p'\infty}$ for $1 < p < \infty$ as well as $L_{pq}^* = L_{p'q'}$ for $1 < p, q < \infty$, where p', q'

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denote the conjugate exponent of p, q respectively, that is, $1/p + 1/p' = 1 = 1/q + 1/q'$. The reader is referred to ([3, 4, 8, 12, 14 and 17]) for more details on Lorentz spaces.

The *Banach function space* K is defined as the space of those complex-valued measurable functions on Ω for which the norm $\|\cdot\|_K$ on K has the following properties: For each measurable function f, g, f_n $(n \in \mathbb{N})$, we have

(1) $\|f\|_K = 0$ a.e. $\Leftrightarrow f = 0$ a.e.; $\|af\|_K = |a|\,\|f\|_K$; $\|f + g\|_K \le \|f\|_K + \|g\|_K$

(2) $|g| \le |f|$ a.e. $\Rightarrow \|g\|_K \le \|f\|_K$,

(3) $|f_n| \nearrow |f|$ a.e. $\Rightarrow \|f_n\|_K \nearrow \|f\|_K$, and

(4) $E \in \mathcal{A}$ with $\mu(E) < \infty \Rightarrow \|\chi_E\|_K < \infty$ and $\int_E |f|\,d\mu \le c_E \|f\|_K$ for some constant c_E, $0 < c_E < \infty$, depending on E and the norm $\|\cdot\|_K$ but independent of f.

A function f in a Banach function space K is said to have absolutely continuous norm if $\|f\chi_n\|_K \to 0$ for each sequence $\{E_n\}_n$ satisfying $E_n \to \varphi$ μ a.e. If each function in K has absolutely continuous norm then K is called Banach function space with absolutely continuous norm.

A *rearrangement invariant space* is a Banach function space K such that whenever $f \in K$ and g is equimeasurable function with f, then $g \in K$ and $\|g\|_K = \|f\|_K$. We recall a result from [3, p.59].

Theorem 0.1. *Let $(K(\Omega), \|\cdot\|_K)$ be a rearrangement invariant Banach function space on a resonant measure space $(\Omega, \mathcal{A}, \mu)$. Then the associate space $(K'(\Omega), \|\cdot\|_{K'})$ is also a rearrangement invariant space and these norms are given by*

$$\|g\|_{K'} = \sup\left\{\int_0^\infty f^*(s)g^*(s)\,ds : \|f\|_K \le 1\right\}, \quad g \in K'$$

and

$$\|f\|_K = \sup\left\{\int_0^\infty f^*(s)g^*(s)\,ds : \|g\|_{K'} \le 1\right\}, \quad f \in K.$$

For a Banach function space with absolutely continuous norm, the Banach dual coincide with its associate space. The Lorentz space $L_{pq}(\Omega)$, for $1 < p < \infty$, $1 \le q \le \infty$ is a rearrangement invariant Banach function space with upper and lower Boyd indices both equal to $1/p$ with the dual $L^*_{pq}(\Omega)$ coincides with the associate space $L'_{pq}(\Omega)$.

For details on Banach function spaces and rearrangement invariant spaces, one can refer to [3, 10] and references therein.

Let X be a Banach space and for a strongly measurable function $f : \Omega \to X$, where $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space, define a function $\|f\|$ as

$$\|f\|(\omega) = \|f(\omega)\| \quad \text{for all} \quad \omega \in \Omega.$$

All the notations make sense for f by replacing the modulus by norm. This leads to the natural definition of the rearrangement invariant-Bochner space $K(\Omega, X)$, with

norm $\|\cdot\|_K$ and the associate space $K'(\Omega, X)$ with norm $\|\cdot\|_{K'}$ given by

$$\|g\|_{K'} = \sup\left\{\int_0^\infty \|f\|^*(s)\|g\|^*(s)\,ds : \|f\|_K \le 1\right\}, \quad g \in K'$$

and

$$\|f\|_K = \sup\left\{\int_0^\infty \|f\|^*(s)\|g\|^*(s)\,ds : \|g\|_{K'} \le 1\right\}, \quad f \in K.$$

The Lorentz-Bochner space $L_{pq}(\Omega, X)$, is a rearrangement invariant-Bochner space, where the norm is defined by

$$\|f\|_{pq} = \begin{cases} \left\{\dfrac{q}{p}\displaystyle\int_0^\infty (t^{1/p}\|f\|^{**}(t))^q \frac{dt}{t}\right\}^{1/q}, & 1 < p < \infty,\ 1 \le q < \infty \\ \sup\limits_{t>0} t^{1/p}\|f\|^{**}(t), & 1 < p \le \infty, q = \infty. \end{cases}$$

The Lorentz space $L_{pq}(\Omega, X)$ is a Banach space and we still have the density of simple functions in it and its dual is

$$L^*_{pq}(\Omega, X) = L_{p'q'}(\Omega, X^*),$$

where X^* has the Radon-Nikodym property. The particular case when $p = q$ is studied in ([5]) whereas for more general case for certain Banach lattices including L_{pq} one can refer to ([6]). In ([4]) $L_{pq}(\Omega, X)$ is studied in terms of a space of vector measures.

For a strongly measurable function $u: \Omega \to \mathfrak{B}(X)$, the class of all bounded operators on Banach space X, the multiplication transformation $M_u: L_{pq}(\Omega, X) \to L(\Omega, X)$ is defined as

$$(M_u f)(\omega) = u(\omega)(f(\omega)), \quad \text{for all} \quad \omega \in \Omega,$$

where $L(\Omega, X)$ is the space of all strongly measurable functions. For a non-singular measurable transformation $T: \Omega \to \Omega$, the composition transformation $C_T: L_{pq}(\Omega, X) \to L(\Omega, X)$ is given by

$$(C_T f)(\omega) = f(T(\omega)), \quad \text{for all} \quad \omega \in \Omega.$$

These transformations are studied on various spaces by many researchers in ([1, 7, 9, 11, 13, 15, 16, 18 and 19]), in particular on L_p space in ([16]), in Orlicz space in ([9]) and on Lorentz space in ([11 and 13]). It is natural to extend the study to more general class. For the detail of these spaces one can refer to ([2, 3, 8, 10 and 17]) and the references therein.

This paper is divided into three sections. In the first section, the multiplication operators with some of its properties like invertibility, range and compactness are discussed. The next section is devoted to the study of composition operators and in the last section an attempt has been made to study the spectra of multiplication operators.

1. Characterizations

Let u be a strongly measurable operator valued map on Ω. Then the boundedness, invertibility and compactness of the multiplication operator M_u induced by u is characterized in terms of u.

Using the arguments given in ([9 and 11]) we can easily prove the following results:

Theorem 1.1. *The multiplication transformation $M_u: L_{pq}(\Omega, X) \to L_{pq}(\Omega, X)$ is bounded if and only if $u \in L^\infty(\Omega, \mathfrak{B}(X))$. Moreover*

$$\|M_u\| = \|u\|_\infty = \inf\{k > 0: \mu(\{\omega \in \Omega: \|u(\omega)\| > k\}) = 0\}.$$

For $X = \mathbb{C}^N$, a particular case of Theorem 1.1 is considered in ([7]).

Proof. In case $u \in L^\infty(\Omega, \mathfrak{B}(X))$, then simple computations give

$$\|M_u f\|^*(t) \leq \|u\|_\infty \|f\|^*(t), \quad \text{for all} \quad f \in L_{pq}(\Omega, X),$$

which implies

$$\|M_u f\|_{pq} \leq \|u\|_\infty \|f\|_{pq}.$$

Conversely, suppose M_u is a bounded operator on $L_{pq}(\Omega, X)$. In case u is not in $L^\infty(\Omega, \mathfrak{B}(X))$, then we can find a sequence $\langle E_{n_i} \rangle$ of disjoint measurable subsets with finite measure such that

$$\|u(\omega)\| > n_i \quad \text{for each} \quad \omega \in E_{n_i}, \text{ and } n_i \to \infty \text{ as } i \to \infty, \ n_1 = 1.$$

For each $\omega \in E_{n_i}$, let $x_{\omega_i} \in X$ be such that $\|x_{\omega_i}\| = 1$ and $\|u(\omega)x_{\omega_i}\| > n_i$. For each i, define f_i as

$$f_i(\omega) = \begin{cases} x_{\omega_i}, & \text{if} \quad \omega \in E_{n_i} \\ 0, & \text{otherwise.} \end{cases}$$

Then each $f_i \in L_{pq}(\Omega, X)$. Also we have

$$\|M_u f_i\|^*(t) \geq n_i \|f_i\|^*(t)$$

and hence

$$\|M_u f_i\|_{pq} \geq n_i \|f_i\|_{pq}.$$

This contradicts the boundedness of M_u. Moreover one can easily verify that $\|M_u\| = \|u\|_\infty$. □

Theorem 1.2. *If M_u is a linear transformation from $L_{pq}(\Omega, X)$ to itself, then M_u must be bounded.*

The proof is along the line of proof of Theorem 2.4 ([9]).

Theorem 1.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Then $M_u: L_{pq}(\Omega, X) \to L_{pq}(\Omega, X)$ is invertible with inverse M_v for some $v \in L^\infty(\Omega, \mathfrak{B}(X))$ if and only if*
(1) *$u(\omega)$ is invertible for μ-almost all $\omega \in \Omega$ and*
(2) *there exists $\epsilon > 0$ such that*

$$\|u(\omega)(x)\| \geq \epsilon \|x\|,$$

for all x in X and μ-almost all $\omega \in \Omega$.

Proof. Suppose M_u is invertible and $M_u^{-1} = M_v$ for some $v \in L^\infty(\Omega, \mathfrak{B}(X))$. For each $x \in X$, define $C_x: \Omega \to X$ as

$$C_x(\omega) = x, \quad \text{for all} \quad \omega \in \Omega.$$

Then

$$\|C_x\|^*(t) = \|x\| \chi_{[0, \mu(\Omega)]}(t)$$

and

$$\|C_x\|^{**}(t) = \begin{cases} \|x\|, & \text{if} \quad 0 \leq t < \mu(\Omega) \\ \dfrac{\|x\|}{t} \mu(\Omega), & \text{if} \quad t \geq \mu(\Omega). \end{cases}$$

This gives $\|C_x\|_{pq} = \|x\|(p')^{1/q} \mu(\Omega)$ and hence $C_x \in L_{pq}(\Omega, X)$. As $M_u^{-1} = M_v$ we have

$$u(\omega)v(\omega)x = v(\omega)u(\omega)x = x, \quad \text{for all} \quad x \in X \quad \text{and for all} \quad \omega \in \Omega.$$

This implies $u(\omega)$ is invertible for all $\omega \in \Omega$. Also for each $x \in X$ and $\omega \in \Omega$,

$$\|x\| = \|v(\omega)u(\omega)x\| \leq \|v(\omega)\| \|u(\omega)x\| \leq \|v\|_\infty \|u(\omega)x\|.$$

Hence

$$\|u(\omega)x\| \geq \epsilon \|x\| \quad \text{for} \quad x \in X, \ \omega \in \Omega, \quad \text{where} \quad \epsilon = \frac{1}{\|v\|_\infty}.$$

Conversely if the conditions (1) and (2) are true, then if $v(\omega)$ is inverse of $u(\omega)$, we find for all $\omega \in \Omega$ and $f \in L_{pq}(\Omega, X)$,

$$u(\omega)v(\omega)(f(\omega)) = v(\omega)u(\omega)(f(\omega)).$$

Thus

$$M_u M_v f = f = M_v M_u f, \quad \text{for all} \quad f \in L_{pq}(\Omega, X).$$

Condition (2) implies

$$\epsilon\|v(\omega)x\| \le \|u(\omega)v(\omega)x\| = \|x\|,$$
$$\|v(\omega)x\| \le \epsilon^{-1}\|x\|, \quad \text{for all} \quad \omega \in \Omega \text{ and } x \in X,$$

and this gives $v \in L^{\infty}(\Omega, \mathfrak{B}(X))$. Therefore M_u is invertible and this proves the result. □

Theorem 1.4. *Let $M_u \in \mathfrak{B}(L_{pq}(\Omega, X))$. Then M_u has closed range if and only if there exists $\epsilon > 0$ such that for almost all $\omega \in S$, the support of u,*

$$\|u(\omega)x\| \ge \epsilon\|x\| \quad \textit{for all} \quad x \in X.$$

Proof. Suppose for some $\epsilon > 0$,

$$\|u(\omega)x\| \ge \epsilon\|x\|, \quad \text{for all} \quad x \in X \text{ and } \mu\text{-almost all } \omega \in S.$$

Let $L_{pq}(S)$ denote the subspace of all those f in $L_{pq}(\Omega, X)$ which vanish outside S, then $L_{pq}(S)$ is a closed invariant subspace of M_u. Also for each f in $L_{pq}(S)$, we have

$$\|M_u f\| \ge \epsilon\|f\|.$$

This gives $M_u|_{L_{pq}(S)}$ has closed range and so of M_u.

Conversely, suppose M_u has closed range. Then there exists an $\epsilon > 0$ such that

$$\|M_u f\|_{pq} \ge \epsilon\|f\|_{pq} \quad \text{for all} \quad f \text{ in } L_{pq}(\Omega, X).$$

Let $E = \{\omega \in S: \|u(\omega)x\| < (\epsilon/2)\|x\| \text{ for some } x \in X\}$.

If $\mu(E) > 0$, then we can find a measurable subset F of E such that $0 < \mu(F) < \infty$. For each $\omega \in F$, let $x_\omega \in X$ such that

$$\|u(\omega)x_\omega\| < \frac{\epsilon}{2}\|x_\omega\| \quad \text{with} \quad \|x_\omega\| = 1.$$

Define $f_F: \Omega \to X$ as

$$f_F(\omega) = \begin{cases} x_\omega, & \text{if} \quad \omega \in F \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_F \in L_{pq}(S)$ and $\|M_u f\|_{pq} < \epsilon\|f\|_{pq}$. Therefore $\mu(E) = 0$ and hence for μ-almost all $\omega \in S$,

$$\|u(\omega)x\| \ge \frac{\epsilon}{2}\|x\| \quad \text{for all} \quad x \in X.$$ □

EXAMPLE 1.5. Consider $\Omega = (0, 1]$ with Lebesgue measure. Let $X = l_2$. Define $u: \Omega \to \mathfrak{B}(l_2)$ as

$$u(\omega) = \begin{cases} I_2, & \text{if} \quad \omega \in \left(0, \frac{1}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

where $I_2(x_1, x_2, x_3, \dots) = (x_1, x_2, 0, \dots)$. Then $M_u \in \mathfrak{B}(L_{pq}(\Omega, X))$ as $u \in L^\infty(\Omega, \mathfrak{B}(X))$. But M_u is not invertible as $u(\omega)$ is not invertible for all ω in $(1/2, 1]$.

EXAMPLE 1.6. Let $\Omega = (1, 2]$ and X any Banach space, and $u(\omega) = \omega I$ for all $\omega \in \Omega$. Then $u \in L^\infty(\Omega, \mathfrak{B}(X))$ and each $u(\omega)$ is invertible and $\|u(\omega)x\| \geq \|x\|$ for all $x \in X$ and M_u is invertible.

EXAMPLE 1.7. Let $\Omega = (0, 1]$. Define $u: \Omega \to \mathfrak{B}(l_2)$ as $u(\omega) = I_\omega$, where $I_\omega: l_2 \to l_2$ given by

$$I_\omega(x_1, x_2, x_3, \dots) = (\omega x_1, \omega x_2, 0, \dots).$$

Then for each $\omega \in (0, 1]$, $I_\omega \in \mathfrak{B}(l_2)$ and $\|I_\omega\| = \omega$. For $\epsilon > 0$,

$$\{\omega \in (0, 1]: \|u(\omega)x\| \geq \epsilon\|x\| \ \forall x \in l_2\} = \begin{cases} \phi, & \text{if} \quad \epsilon > 1 \\ [\epsilon, 1], & \text{if} \quad \epsilon \leq 1. \end{cases}$$

Thus there does not exist any $\epsilon > 0$ such that

$$\|u(\omega)x\| \geq \epsilon\|x\| \quad \text{for all} \quad x \in l_2 \text{ and a.e. on } \Omega.$$

Hence M_u does not have a closed range in view of Theorem 1.4.

For each $\epsilon > 0$, define u_ϵ as

$$u_\epsilon = \{\omega \in \Omega: \|u(\omega)x\| \geq \epsilon\|x\| \text{ for some } x \in X\}.$$

Theorem 1.8. *M_u in $\mathfrak{B}(L_{pq}(\Omega, X))$ is compact if $L_{pq}(u_\epsilon)$ is finite dimensional for each $\epsilon > 0$, where*

$$L_{pq}(u_\epsilon) = \{f \in L_{pq}(\Omega, X): f \text{ vanishes outside } u_\epsilon\}.$$

Proof. For each natural number n, define

$$u_n(\omega) = \begin{cases} u(\omega), & \text{if} \quad \omega \in u_{1/n} \\ 0, & \text{otherwise.} \end{cases}$$

Then each M_{u_n} is compact and for each f in $L_{pq}(\Omega, X)$,

$$\{\omega \in \Omega\colon \|(u_n - u)(\omega)(f(\omega))\| \geq s\} \subseteq \{\omega \in \Omega\colon \|f(\omega)\| > ns\},$$

$$\|(M_{u_n} - M_u) f\|^*(t) \leq \frac{1}{n}\|f\|^*(t).$$

Thus

$$\|(M_{u_n} - M_u) f\|_{pq} \leq \frac{1}{n}\|f\|_{pq}.$$

This gives that M_u is compact. □

EXAMPLE 1.9. $\Omega = (0, 1]$ and $X = l_2$. Define u as $u(\omega) = I_2$, where $I_2\colon l_2 \to l_2$ given by $I_2(x_1, x_2, x_3, \ldots) = (x_1, x_2, 0, \ldots)$. Then $\|u(\omega)\| = 1$, for all $\omega \in \Omega$. Also

$$u_\epsilon = \{\omega \in \Omega : \|u(\omega)\| \geq \epsilon\} = \begin{cases} \Omega, & \text{if } \epsilon > 1 \\ \phi, & \text{if } \epsilon \leq 1. \end{cases}$$

Thus

$$L_{pq}(u_\epsilon) = \begin{cases} 0, & \text{if } \epsilon > 1 \\ L_{pq}(\Omega, X), & \text{if } \epsilon \leq 1. \end{cases}$$

Hence for $\epsilon \leq 1$, $L_{pq}(u_\epsilon)$ is infinite dimensional for $q \geq p > 1$, as for each $i \geq 1$, $f_i(\omega) = (0, 0, \ldots, \omega, 0, \ldots)$ belongs to $L_{pq}(\Omega, X)$.

Theorem 1.10. *If u is a strongly measurable operator valued map such that for some $k > 0$, $\|u(\omega)x\| \geq k\|x\|$, $\forall x \in X$ whenever $\|u(\omega)\| \geq k$, then M_u in $\mathfrak{B}(L_{pq}(\Omega, X))$ is compact if and only if $L_{pq}(u_\epsilon)$ is finite dimensional for each $\epsilon > 0$.*

Proof. In case M_u is a compact operator then $M_u|_{L_{pq}(u_\epsilon)}$ is also a compact operator and also for each $f \in L_{pq}(\Omega, X)$,

$$\|M_u \chi_{u_\epsilon} f\| \geq \epsilon \|\chi_{u_\epsilon} f\|_{pq}.$$

Thus $M_u|_{L_{pq}(u_\epsilon)}$ is invertible and being compact we find that $L_{pq}(u_\epsilon)$ is finite dimensional. Converse follows by using the Theorem 1.8. □

EXAMPLE 1.11. Let X be a Banach space, $\Omega = (a, b)$, define $u\colon \Omega \to \mathfrak{B}(X)$ as $u(\omega) = \omega I$, $\forall \omega \in \Omega$. Then we find $u \in L^\infty(\Omega, \mathfrak{B}(X))$ so that $M_u \in \mathfrak{B}(L_{pq}(\Omega, X))$.

Also $\|u(\omega)\| \geq k$ if and only if $\|u(\omega)x\| \geq k\|x\|$, $\forall x \in X$.

Theorem 1.12. *Let $K(\Omega, X)$ be the rearrangement invariant spaces on resonant measure space $(\Omega, \mathcal{A}, \mu)$. Then the multiplication transformation M_u on $K(\Omega, X)$ is bounded if and only if $u \in L^\infty(\Omega, \mathfrak{B}(X))$. Moreover*

$$\|M_u\| = \|u\|_\infty = \inf\{k > 0 : \mu(\{\omega \in \Omega : \|u(\omega)\| > k\}) = 0\}.$$

Proof. In case $u \in L^\infty(\Omega, \mathfrak{B}(X))$, then simple computations give

$$\|M_u f\|^*(t) \leq \|u\|_\infty \|f\|^*(t), \quad \text{for all} \quad f \in K(\Omega, X).$$

Hence for $g \in K'(\Omega, X)$, the associate space,

$$\int_0^\infty \|M_u f\|^*(s)\|g\|^*(s)\, ds \leq \|u\|_\infty \int_0^\infty \|f\|^*(s)\|g\|^*(s)\, ds.$$

Taking supremum over all $g \in K'(\Omega, X)$ with $\|g\|_{K'} \leq 1$, we get

$$\|M_u f\|_K \leq \|u\|_\infty \|f\|_K, \quad \text{for all} \quad f \in K(\Omega, X).$$

The converse follows with the same computation as made in the Theorem 1.1. □

Similarly the Theorems 1.3, 1.4, 1.8 and 1.10 hold on rearrangement invariant-Bochner space $K(\Omega, X)$ along the same lines of proofs.

2. Composition operators

This section is devoted to a study of composition operators C_T on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ and $K(\Omega, X)$ induced by a non-singular measurable transformation $T : \Omega \to \Omega$, where $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space. The results presented here generalize the results of R.K. Singh ([16]), R. Kumar ([11]) and H. Takagi ([18 and 19]). Their proofs, being on the same lines can be easily formulated, therefore are omitted.

Let x_0 be a fixed element of X with $\|x_0\| = 1$. Then for each measurable subset E of Ω, define the characteristic function χ_E as

$$\chi_E(\omega) = \begin{cases} x_0, & \text{if} \quad \omega \in E \\ 0, & \text{otherwise.} \end{cases}$$

Then we find

$$\|\chi_E\|^*(t) = \chi_{[0,\mu(E))}(t) \quad \text{and} \quad \|\chi_E\|_{pq} = (p')^{1/q}(\mu(E))^{q/p}.$$

Theorem 2.1. *A non-singular measurable transformation $T: \Omega \to \Omega$ induces the composition operator $C_T: L_{pq}(\Omega, X) \to L_{pq}(\Omega, X)$ $(f \mapsto f \circ T)$ if and only if for some $b > 0$,*

$$\mu T^{-1}(E) \leq b\mu(E) \quad \text{for all} \quad E \in \mathcal{A}.$$

Moreover

$$\|C_T\| = (\inf\{k > 0: \mu T^{-1}(E) \leq k\mu(E),\ \forall E \in \mathcal{A}\})^{1/p}.$$

Corollary 2.2. *T induces C_T on $L_{pq}(\Omega, X)$ if and only if μT^{-1} is absolutely continuous with respect to μ and f_T, the Radon-Nikodym derivative of μT^{-1} with respect to μ, belong to $L^{\infty}(\Omega)$.*

Corollary 2.3. *T is measure preserving if and only if C_T is an isometry.*

Theorem 2.4. *Let $C_T \in \mathfrak{B}(L_{pq}(\Omega, X))$, $1 < p \leq \infty$, $1 \leq q < \infty$. Then C_T has closed range if and only if there exists $\epsilon > 0$ such that $f_T(\omega) \geq \epsilon$ for almost all $\omega \in S$, the support of f_T, where f_T is the Radon-Nikodym derivative of μT^{-1} with respect to μ.*

Theorem 2.5. *$C_T \in \mathfrak{B}(L_{pq}(\Omega, X))$, $q \neq \infty$ has dense range if and only if $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

Theorem 2.6. *Let $K(\Omega, X)$ be a rearrangement invariant space on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. A non-singular measurable transformation $T: \Omega \to \Omega$ induces the composition operator C_T on $K(\Omega, X)$ if and only if for some $b > 0$,*

$$\mu T^{-1}(E) \leq b\mu(E) \quad \text{for all} \quad E \in \mathcal{A}.$$

3. Spectra

In this section an attempt is made to find the spectrum of multiplication operator M_u on $L_{pq}(\Omega, X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ or $K(\Omega, X)$ induced by the measurable function $u: \Omega \to \mathfrak{B}(X)$. Throughout this section we assume $(\Omega, \mathcal{A}, \mu)$ to be a finite measure space. The symbols P_T, P_T^a and $\sigma(T)$ refer to the point spectrum, the approximate point spectrum (the set of complex numbers λ such that $T - \lambda I$ is not bounded below) and the spectrum of the operator T respectively.

Theorem 3.1. $P_{M_u} = \bigcup_{\Omega': \mu(\Omega')>0} \left(\bigcap_{\omega \in \Omega'} P_{u(\omega)} \right)$.

Proof. If $\lambda \in \bigcap_{\omega\in\Omega'} P_{u(\omega)}$, then for each $\omega \in \Omega'$ assume x_ω to be a unital vector (i.e. $\|x_\omega\| = 1$) satisfying

$$u(\omega)x_\omega = \lambda x_\omega.$$

Define f as

$$f(\omega) = \begin{cases} x_\omega, & \text{if} \quad \omega \in \Omega' \\ 0, & \text{otherwise.} \end{cases}$$

Then f is the eigenvector corresponding to λ that is $M_u f = \lambda f$.

Converse is easy to prove. □

Theorem 3.2. $P^a_{M_u} = \bigcup_{A\in\mathcal{C}} \left(\bigcap_{\omega\in A} P^a_{u(\omega)}\right)$, *where*

$$\mathcal{C} = \{A \subseteq \Omega\colon A \text{ is measurable and } \mu(\Omega \setminus A) \geq 0\}.$$

Proof. In case $\lambda \in P^a_{M_u}$ then we can find a sequence $\langle f_n\rangle$ such that

$$\|M_u f_n - \lambda f_n\|_{pq} \to 0 \quad \text{as} \quad n \to \infty.$$

This gives $\lim_{n\to\infty}(u(\omega)(f_n(\omega)) - \lambda f_n(\omega)) = 0$ for a.e. ω in Ω. Thus we can find some $A \in \mathcal{C}$ such that

$$\lim_{n\to\infty} (u(\omega) - \lambda I) f_n(\omega) = 0 \quad \text{for all} \quad \omega \in A.$$

Thus $\lambda \in P^a_{u(\omega)}$ for all $\omega \in A$ and hence $\lambda \in \bigcap_{\omega\in A} P^a_{u(\omega)}$.

Conversely if $\lambda \in \bigcup_{A\in\mathcal{C}} \left(\bigcap_{\omega\in A} P^a_{u(\omega)}\right)$, then for $\epsilon > 0$ and for every $\omega \in A$, let x_ω in X be such that

$$\|x_\omega\| = 1 \quad \text{and} \quad \|(u(\omega) - \lambda)x_\omega\| < \epsilon.$$

Define $f\colon \Omega \to X$ as

$$f(\omega) = \begin{cases} x_\omega, & \text{if} \quad \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|M_u f - \lambda f\|_{pq} < \epsilon \|f\|_{pq}.$$

This gives $\lambda \in P^a_{M_u}$ and hence the result. □

For a strongly measurable function $u\colon \Omega \to \mathfrak{B}(X)$, the set

$$\text{ess}_u = \{\lambda \in \mathbb{C}\colon \mu(\{\omega \in \Omega\colon \|u(\omega) - \lambda\| < \epsilon\}) \neq 0 \ \forall \epsilon > 0\}$$

is called the essential range of u.

DEFINITION 3.3. For a strongly measurable function $u\colon \Omega \to \mathfrak{B}(X)$, we define ess_u^s, the strong essential range of u as the set

$$\{\lambda \in \mathbb{C}\colon \mu(\{\omega \in \Omega\colon \|(u(\omega) - \lambda)x\| < \epsilon\|x\| \text{ for some } x \in X\}) \neq 0,\ \forall \epsilon > 0\}.$$

It is obvious to check that $\operatorname{ess}_u \subseteq \operatorname{ess}_u^s$.

Theorem 3.4. $\sigma(M_u) \subseteq \left(\bigcup_{\omega\in\Omega} \sigma(u(\omega))\right) \cup \operatorname{ess}_u^s$.

The proof follows using Theorem 1.3.

Finally we present some examples to verify the proper inclusion $\operatorname{ess}_u \subset \operatorname{ess}_u^s$ and to ensure that there is no relation between ess_u^s and $\bigcup_{\omega\in\Omega} \sigma(u(\omega))$.

EXAMPLE 3.5. For a projection operator P (or any operator that is not one-one) on a Banach space X, define $u\colon \Omega \to \mathfrak{B}(X)$ as $u(\omega) = P$, $\forall \omega \in \Omega$. Then 0 does not belong to ess_u whereas $0 \in \operatorname{ess}_u^s$.

EXAMPLE 3.6. Let $\Omega = (0, 1]$, X any Banach space, define $u\colon \Omega \to \mathfrak{B}(X)$ as $u(\omega) = \omega I$ for $\omega \in \Omega$. Then each $u(\omega)$ is invertible so that 0 does not belong to $\bigcup_{\omega\in\Omega} \sigma(u(\omega))$. But for any $\epsilon > 0$,

$$\mu(\{\omega \in \Omega\colon \|u(\omega)x\| < \epsilon\|x\|,\ \forall x \in X\}) = \epsilon.$$

Thus $0 \in \operatorname{ess}_u^s$.

EXAMPLE 3.7. For any operator T on a Banach space X that is bounded below but not invertible if we define $u\colon \Omega \to \mathfrak{B}(X)$ as $u(\omega) = T$, $\forall \omega \in \Omega$. Then 0 does not belong to ess_u^s but $0 \in \bigcup_{\omega\in\Omega} \sigma(u(\omega))$.

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