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Multiplication Graded Modules

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Abstract

Let G be a multiplicative group and R be a G-graded commutative ring and M a G-graded R-module. Various properties of multiplicative ideals in a graded ring are discussed and we extend this to graded modules over graded rings. We have also discussed the set of P-primary ideals and modules of R when P is a graded multiplication prime ideals and modules.

Keywords: graded rings, graded modules, graded multiplication Modules.

1. Introduction

Let G be a group. A ring R is called G-graded ring if there exist a family $\{R_g\}_{g\in G}$ of additive subgroup of R such that $R = \oplus_{g\in G} R_g$ that $R_g R_h \subset R_{gh}$ for each $g,h \in G$. A R- module M is called R-graded Module over G if $M = \oplus_{g\in G} M_g$ and $R_g M_h \subset M_{gh}$ for all $g,h \in G$. Thus each M_g submodule of M is $R = R_g$ - module. An element of a graded ring R is called homogeneous if it belongs to $\bigcup_{g\in G} M_g$. If an element $m \in M$ is belongs to $\bigcup_{g\in G} M_g$, then m is called homogeneous element and the set of all homogeneous elements of M is denoted by $H(M)$ (for a ring R is denoted by $H(R)$). A graded submodule N of a graded R -Module M (R is a graded ring) is a submodule such that $N = \oplus_{g\in G}(M_g \cap N) = \oplus_{g\in G} N_g$. Equivalently, N is graded in M if and only if

N has a homogeneous set of generators. If $R = \oplus_{g\in G} R_g$ and $R' = \oplus_{g\in G} R'_g$ are two graded ring, then the mapping $\phi : R \to R'$ whit $\phi(1_R) = 1_{R'}$ is called graded homomorphism if $\phi(R_g) \subset R'_g$, for all $g \in G$.

If $M = \oplus_{g\in G} M_g$ and $M' = \oplus_{g\in G} M'_g$ are two graded R-modules (R is a graded ring), the mapping $\lambda : M \to M'$ is called graded homomorphism if $\lambda(M_g) \subset M'_{g'}$ for all $g \in G$. A graded ideal P of a graded ring R is called gr-prime if whenever $x, y \in H(R)$ with $xy \in P$ then $x \in P$ or $y \in P$. And a graded submodule of a graded module M over a graded ring R is called gr-prime submodule if $rx \in N$, for $r \in H(R)$ and $x \in H(M)$, then $x \in N$ or $rM \subset N$. A graded ideal m of a graded ring R is called gr-maximal if it is maximal in the lattice of graded ideals of R. (similarly we have for R-modules). A graded ring R is called a gr-local ring if it has unique gr-maximal ideal. Let R be a graded ring and let $S \subset H(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called a gr-ring of fractions. Indeed, $S^{-1}R = \oplus_{g\in G} (S^{-1}R)_g$ where $(S^{-1}R)_g = \left\{ \frac{r}{s}, r \in R, s \in S, g = \frac{\deg(r)}{\deg(s)} \right\}$. And $S^{-1}M = \oplus_{g\in G} (S^{-1}M)_g$ where $(S^{-1}M)_g = \left\{ \frac{m}{s}, m \in M, s \in S, g = \frac{\deg(M)}{\deg(s)} \right\}$. Consider the ring gr-homomorphism $\pi : R \to S^{-1}R$ defined by $\pi(r) = \frac{r}{1}$. And $\pi : M \to S^{-1}M$ is called gr-homomorphism if $\pi(m) = \frac{m}{1}$. Let P be any gr-prime ideal of a graded ring R and consider the multiplicatively closed subset $S = H(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_p^g and we call it the gr-localization of R. This is a gr-local with the unique gr-maximal ideal $S^{-1}P$ which will be denoted by PR_p^g. Let I be a graded ideal in a graded ring R. The graded radical of I (gr-rad(I)) is defined the set of all $x_g \in R$ such that for each $g \in G$, there exists $n_g > 0$ such that $x_g^{n_g} \in I$. A graded radical submodule N of a graded R-module M (R is a graded ring) is the intersections of graded prime submodules of M such that containing N as a submodules. A submodule N of on R-module M is called multiplication if $N = IM$, for some gr-ideal I of R. If each sub module of M is gr-multiplication, M is called gr-multiplication R-module.

In this paper, we study some properties of gr-multiplication submodules in a graded multiplication R-module M, when M is gr-module over gr-ring R. And give a characterization of finitely generated gr-multiplication submodules of a gr-multiplication M over a gr-ring R.

Definition 1 Let R be a graded ring over the group G and M an R-graded module. A graded submodule N of M is called graded multiplication . If $K \underset{g}{<} N$ then there is an gr-ideal of R such that $K = NI$.

Definition 2 A graded R-module M is called gr-multiplication module if every gr-submodule of M is gr-multiplication .

Definition 3 A graded ideal Q of a graded ring R is called gr-primary if $Q \neq R$ and whenever, $a, b \in H(R)$ whit $ab \in Q$, then $a \in Q$ or $b^n \in Q$ .If Q is gr-primary ideal of R and gr-rad $(Q) = P$, we say that Q is gr-p-primary.

Definition 4 An gr-submodule N of graded R-module M is called gr-primary, if $a \in H(R)$, $b \in H(M)$ and $ab \in N$,then $b \in N$ or $a^n M \subset N$ for some integer $n \geq 0$.

Recall that if N, K are two gr-submodules of a graded R-module M , then $(N : K) = \left\{ r \in R \middle| rK \subset N \right\}$ is a graded ideal of R.

Lemma 1 Let I be a graded ideal in a graded ring R then I is multiplication if $I \cap J = I(J : I)$ for gr-ideal $J \subset I$.

Proof. Suppose that $J \subset I$ for some gr- ideal J of R.Then $J = I \cap J = I(J : I)$. Hence J is gr-multiplication ideal of I .

Conversely, Let I be a graded multiplication ideal in R, Let J be any graded ideal of R Then $I \cap J \subset I$, so there is a graded ideal K of R such that $I \cap J = IK$ .Therefore $K \subseteq ((I \cap J) : I) \subseteq (J : I)$, and then $I \cap J = IK \subset I(J : I)$. On the other hand, clearly $I(J : I) \subset I \cap J$. Hence $J = I \cap J = (J : I)I$.

Proposition 1 Let M be a graded R-module (R is a graded ring). Then M is gr-multiplication if for every gr-submodule N of M , $N = [N : M]M$.

Proof. Let M be gr-multiplication R-module, and N a gr-submodule of M , then there is an gr-ideal I of R such that $N = IM$, as $IM \subset N$ we have $I \subset [N : M]$ and $N = IM \subset M[N : M]$. Since $[N : M]M \subset N$, so $N = [N : M]M$. Conversely it is clearly. Recall that Graded R -module M is called graded cyclic if $M = Rx$, for some $x \in H(M)$.

Theorem 1 Let M be a gr-multiplication Module over a graded local ring R. Then M is gr-multiplication if M is a graded cyclic R-module .

Proof. If $M=<m>$ for some $m\in H(M)$ then clearly M is gr-multiplication R-module .Conversely, Let $M=<m_\alpha | \alpha\in A>$ where each M_α is a homogeneous element $(m_\alpha\in H(M))$. Since M is gr-multiplication we have $Rm_\alpha=[M_\alpha:M]M$,as $M=\sum_{\alpha\in A}Rm_\alpha=\sum_{\alpha\in A}[m_\alpha:M]M=M(\sum_{\alpha\in A}[m_\alpha:M])$. If $\sum_{\alpha\in A}[m_\alpha:M]=R$, then $[m_{\alpha_0}:M]=R$. Since otherwise if $\forall\alpha[m_\alpha:M]\neq R$, then $[m_\alpha:M]\subset J$, where J is the only maximal ideal of R, and hence $\sum_{\alpha\in A}[m_\alpha:M]=R\subset J$ that is a contradiction, so $[m_{\alpha_0}:M]=R$ for some $\alpha_\circ\in A$ therefore $<m_{\alpha_\circ}>=[m_{\alpha_\circ}:M]M=M$ Hence M is gr-principal. If $\sum_{\alpha\in A}[m_\alpha:M]\neq R$, then $\sum_{\alpha\in A}[m_\alpha:M]\subset J$, and then $M=\sum_{\alpha\in A}[m_\alpha:M]M\subseteq JM\subset M$ therefore $JM=M$, hence $M=<0>$.

Proposition 2 . If M is gr-multiplication R-module where R is a graded ring, and $S\subset H(R)$) is a multiplicatively closed subset of R. Then $S^{-1}M$ is a gr-multiplication $S^{-1}R$-module.

Proof. Let K be a graded $S^{-1}R$-submodule of $S^{-1}M$. Then $K=S^{-1}N$ for some graded submodule of N of M . Now since M is gr-multiplication R-module, then $N=[N:M]M$ so $S^{-1}N=(S^{-1}[N:M])(S^{-1}M)$ Hence $S^{-1}M$ is a gr-multiplication $S^{-1}R$-module.

Definition 5 A graded submodule N of graded R-module M is locally gr-principal if $N\cdot R_p^g$ is gr-principal for every gr-prime ideal P of R.

Proposition 3 Let R be a gr-local ring with graded maximal ideal J and M a graded R-module such that $M=<m_1,m_2,\cdots,m_k>$, where $m_1\in H(M)$ for every $1\le i\le k$, then $M=<m_i>$ for some $1\le i\le k$.

Proof. Suppose that $M=<a>$ for some $a\in H(M)$ and $M=<a_1,a_2,\cdots,a_k>$, then $a=\sum_{i=1}^{k}r_ia_i$ and each $a_i=s_ia$, so $a=\sum_{i=1}^{k}r_is_ia$

and $a(1-\sum_{i=1}^{k} r_i x_i)=0$ if $1-\sum_{i=1}^{k} r_i s_i$ is a unit in R then $a=0$ since $a_i = s_i a, a_i = 0$, for all $1 \le i \le k$. and $M =< 0 >=< a_i >$, for all $i = 1,2,\cdots,k$. If $1-\sum_{i=1}^{k} r_i s_i$ is not a unit, then $\sum_{i=1}^{k} r_i s_i \notin J$ and so $\sum_{i=1}^{k} r_i s_i$ is a unit. Therefore, there is an $i \in \{1,2,\cdots,k\}$ such that $r_i s_i$ is a unit. Otherwise since each $r_i s_i$ is not unit then $r_i x_i \in M$, for all $i = 1,2,\cdots,k$, hence $\sum_{i=1}^{k} r_i s_i \in M$. That is a contradiction. So $r_i s_i$ is a unit for some i, then s_i is a unit. Hence $a = a s_i s_i^{-1} = a_i^{-1} s \in < a_i >$ Then $M =< a_i >$.

Theorem 2 Let $M =< m_1, m_2, \cdots, m_k >$ be a finitely generated graded R-module over a graded ring R. Then the following are equivalent .

(1) M is gr-multiplication .

(2) M is locally gr-principal .

(3) $\sum_{i=1}^{k} [m_i : M_i] = R$, where $M_i =< a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k >$

Proof. (1) $\rightarrow$ (2) By Theorem 1 .

(2) $\rightarrow$ (3) Let M be a locally gr-principal. Then for graded prime ideal P of R, we have by Proposition 3 $MR_p^g =< \frac{m_1}{1}, \frac{m_2}{1}, \cdots, \frac{m_k}{1} >=< \frac{m_i}{1} >=< m_1 > R_p^g$, for some $i \in \{1,2,\cdots,k\}$}. Hence for any gr-prime ideal P of $R[(m_i)R_p^g : M_i R_p^g] = R_p^g$, where $M_i =< a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k >$ and then $(\sum_{i=1}^{k} [(m_i) : M_i] R_p^g = \sum_{i=1}^{k} [(m_i) R_p^g : M_i R_p^g] = R_p^g$. Since M_i is finitely generated for each i, There for $\sum_{i=1}^{k} ((m_i) : M_i) = R$.

(3) $\rightarrow$ (2) Suppose that $\sum_{i=1}^{k} ((m_i) : M_i) = R$. Then for any gr-prime P of R we have $(\sum_{i=1}^{k} [(m_i) R_p^g : M R_p^g] = \sum_{i=1}^{k} ([(m_i) : M]) R_p^g = (\sum_{i=1}^{k} [(m_i) : M_i]) R_p^g = R_p^g$. Therefore, there is $i \in \{1,2,\cdots,k\}$ such that $((m_i) R_p^g : M R_p^g) = R_p^g$ and then

$MR_p^g \subset (a_i)R_p^g = < \frac{a_i}{1} >$. It follows that $MR_p^g = < \frac{a_i}{1} >$ for each gr-prime ideal P of R. Hence M is locally gr-principal. If M is a graded module over the graded ring R we define the $\theta^g(M) = \sum_{x \in (M)} [(x):M]$. It is clear that $\theta^g(M)$ is a graded ideal of R.

Proposition 4 Let M be a graded multiplication module over a graded ring R. Then

(1) $M = M\theta^g(M)$

(2) $N = N\theta^g(M)$ for any graded submodule N of M.

Proof. (1) Let $x \in M$ as M is graded multiplication R-module, then $< x >= [(x):M]M$ since

$$M = \sum_{x \in M} < x > = \sum_{x \in M} [(x):M]M = M \sum_{x \in M} [(x):M] = M\theta^g(M).$$

(2) suppose that N is a graded submodule of M. Then $N = [N:M]M$, where $[N:M]$ is a graded ideal of R. Hence $N = [N:M]M = [N:M]\theta^g(M)M = N\theta^g(M)$.

Proposition 5 Let N and K be graded submodules of graded multiplication R-module M and $S \subset H(R)$ be a multiplicatively closed subset of R. Then

(1) $\theta^g(N)\theta^g(K) \subset \theta^g(NK)$

(2) $S^{-1}(\theta^g(N)) \subseteq \theta^g(S^{-1}(N))$

Proof. (1) If M is a multiplication R-module and $N = IM$ and $K = JM$ we defined $NK = IJM$. If $x \in M$ and $y \in K$, then $xy = \sum_{i=1}^{n} r_i m_i$, where $r_i \in IJ$, for all $i = 1,2,\cdots,n$ and $n \geq 1$. See [2].

Let $a \in N \cap H(M)$ and $b \in K \cap H(M)$. It is enough to prove that $[(a):N][(b):K] \subseteq [(ab):NK]$. Let $\sum_{i=1}^{n} x_i y_i \in [(a):N][(b):K]$ where $x_i \in [(a):N]$ and $y_i \in [(b):K]$, for $i = 1,2,\cdots,n$. Then $x_i N \subset (a)$ and $y_i K \subset (b)$, for $i = 1,2,\cdots,n$. Hence, $x_i y_i NK \subset (ab)$ and then $x_i y_i \in [(ab):NK]$. Therefore $\sum_{i=1}^{n} x_i y_i \in [(ab):NK]$.

(2)

$$S^{-1}(\theta^g(N)) \subseteq S^{-1}(\sum_{x \in N \cap H(M)} [(x):N] = \sum_{x \in N \cap H(M)} s^{-1}[(x):N] \subseteq \sum_{x \in N \cap H(M)} [(\frac{x}{1}):s^{-1}N] \subseteq \theta^g(s^{-1}N).$$

Recall that a graded module M over graded ring R is called gr-finitely generated if M is generated by a finite set of homogeneous elements .

Theorem 3 Let M be a graded R-module where R is a graded ring. Then M is gr-finitely generated and locally gr-principal if $\theta^g(M) = R$.

Proof. Let J be a gr-maximal ideal in R. Then $MR_J^g = (x)R_J^g$ for some $x \in H(M)$. Hence, $R_J^g = [(x)R_J^g : MR_J^g] = [(x):M]R_J^g$ since M is gr-finitely generated. Therefore $R_J^g = \theta^g(M)R_J^g$ and they by local property $\theta^g(M) = R$.

Conversely, suppose $\theta^g(M) = R$. Then there exist, $m_1, m_2, \cdots, m_k \in H(N)$ such that $R = \theta^g(M) = [(m_1):M] + [(m_2):M] + \cdots + [(m_k):M]$. Thus

$$M = \theta^g(M)M = M[(m_1):M] + M[(m_2):M] + \cdots + M[(m_k):M] \subseteq (m_1) + (m_2) + \cdots + (m_n) \subset M$$

so $M = (m_1, m_2, \cdots m_k)$ is gr-finitely generated. Let J be a gr-maximal ideal of R. Since $\theta^g(M) = R$, there is $x \in H(M)$, with $[(x):M] \text{Ш} J$. Therefore, there exists $r \in R - J$ with $rM \subseteq (x)$ and then $rMR_J^g = <r> R_J^g \cdot MR_J^g = MR_J^g \subseteq (x)R_J^g$. Hence $MR_J^g = (x)R_J^g$, for any gr-maximal ideal J of R and so M is locally gr-principal .

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