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MULTIPLICATION OPERATORS ON THE BERGMAN SPACE AND WEIGHTED SHIFTS

SHUNHUA SUN, DECHAO ZHENG and CHANGYONG ZHONG

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ABSTRACT. In this paper we show that the multiplication operator on the Bergman space is unitarily equivalent to a weighted unilateral shift operator of finite multiplicity if and only if its symbol is a constant multiple of the N-th power of a Möbius transform.

KEYWORDS: *Multiplication operators, Bergman space, weighted shifts.*

MSC (2000): Primary 47B37, 47B35; Secondary 47A15.

INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space L^2_a is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space $L^2(\mathbb{D}, dA)$ of square integrable functions on $\mathbb{D}$. Because the nonnegative powers $\{z^n\}$ span the Bergman space L^2_a, $\{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ form an orthonormal basis of L^2_a.

For a bounded analytic function ϕ on the unit disk, the multiplication operator M_ϕ is defined on the Bergman space L^2_a by

$$M_\phi h = \phi h$$

for $h \in L^2_a$.

Let $e_n = \sqrt{n+1}z^n$. Then $\{e_n\}_0^\infty$ form an orthonormal basis of the Bergman space L^2_a. On the basis $\{e_n\}$, the multiplication operator M_z by z is a weighted shift operator:

$$M_z e_n = \sqrt{\frac{n+1}{n+2}}e_{n+1}.$$

So it is usually called the Bergman shift.

A reducing subspace M for an operator T on a Hilbert space H is a subspace M of H such that $TM \subset M$ and $T^*M \subset M$. In [7] and [8] we have studied reducing subspaces of multiplication operators on the Bergman space via the Hardy space of the bidisk. The multiplication operator M_z is a weighted shift. The general multiplication operator M_ϕ is a holomorphic calculus of the weighted shift. Shift operators have been studied very extensively [2], [3]. In [4], Stessin and Zhu obtained a complete description of the reducing subspaces of weighted unilateral shift operators of finite multiplicity to shed a light on that M_{z^N} on the Bergman space has N nontrivial minimal reducing subspaces, but the multiplication operator by z^N on the Hardy space has infinitely many reducing subspaces.

A natural question is to characterize the multiplication operators on the Bergman space unitarily equivalent to a weighted unilateral shift operators of finite multiplicity. This paper continues our study on the multiplication operators M_ϕ on the Bergman space in [7], [8] by using the Hardy space of the bidisk to completely answer the question. Our main result of this paper almost says that only M_{z^N} up to unitary equivalence is a weighted unilateral shift operator of finite multiplicity.

THEOREM 0.1. *If the multiplication operator M_ϕ on the Bergman space is unitarily equivalent to a weighted unilateral shift operator of finite multiplicity, then $\phi = c\phi_\lambda^N$, for a constant c and some Möbius transform $\phi_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$.*

Let $\mathbb{T}$ denote the unit circle. The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$. Let $d\sigma$ be the rotation invariant Lebesgue measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{T}^2)$ is the subspace of $L^2(\mathbb{T}^2, d\sigma)$, each function in $H^2(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in the bidisk $\mathbb{D}^2$ with the square summable Fourier coefficients. The Toeplitz operator on $H^2(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2, d\sigma)$ is defined by

$$T_f(h) = P(fh),$$

for $h \in H^2(\mathbb{T}^2)$ where P is the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $H^2(\mathbb{T}^2)$.

For each integer $n \geqslant 0$, let

$$p_n(z,w) = \sum_{i=0}^{n} z^i w^{n-i}.$$

Let $\mathcal{H}$ be the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^{\infty}$. Thus

$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus \mathrm{cl}\{(z-w)H^2(\mathbb{T}^2)\}.$$

Let

$$\mathcal{B} = P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}$$

where $P_{\mathcal{H}}$ be the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $\mathcal{H}$. So $\mathcal{B}$ is unitarily equivalent to the *Bergman shift* M_z on the Bergman space L_a^2 via the following unitary operator $U : L^2_a(\mathbb{D}) \to \mathcal{H}$,

$$Uz^n = \frac{p_n(z,w)}{n+1}.$$

This implies that the Bergman shift is lifted up as the compression of an isometry on a nice subspace of $H^2(\mathbb{T}^2)$. Indeed, for each Blaschke product $\phi(z)$ with finite order, the multiplication operator M_ϕ on the Bergman space is unitarily equivalent to $\phi(\mathcal{B})$ on $\mathcal{H}$.

By Lemma 13 in [7], it is easy to see that for each Blaschke product ϕ with order N, $\mathcal{H}$ can be decomposed as a direct sum of at most N reducing subspaces of M_ϕ. We will show that if ϕ has more than two distinct roots and at least one root is repeated, then $\mathcal{H}$ can not be decomposed as a direct sum of N reducing subspaces of M_ϕ (Theorem 3.1).

1. PREMIMINARIES

We need some basic constructions from [7]. Let

$$\mathcal{K}_\phi = \operatorname{span}\{\phi^l(z)\phi^k(w)\mathcal{H}; l,k \geqslant 0\}.$$

Then $\mathcal{K}_\phi$ is a reducing subspace for both $T_{\phi(z)}$ and $T_{\phi(w)}$, and so $T_{\phi(z)}$ and $T_{\phi(w)}$ are also a pair of doubly commuting isometries on $\mathcal{K}_\phi$. Introduce the wandering space

$$\mathcal{L}_\phi = \ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{K}_\phi.$$

Let L_0 be $\ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{H}$. In [7], for each $e \in L_0$, we construct functions $\{d^k_e\}$ and d^0_e in $\mathcal{L}_\phi$ such that for each $l \geqslant 1$,

$$p_l(\phi(z),\phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z),\phi(w))d_e^{l-k} \in \mathcal{H}$$

and

$$p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d^0_e \in \mathcal{H}.$$

We have a precise formula of d^0_e but d^k_e is orthogonal to $\ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{H}$, and for a reducing subspace $\mathcal{M}$, and $e \in \mathcal{M}$,

$$p_l(\phi(z),\phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z),\phi(w))d_e^{l-k} \in \mathcal{M}.$$

The relation between d^1_e and d^0_e is given in [7] and stated as follows:

THEOREM 1.1. *If $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ orthogonal to the distinguished reducing subspace $\mathcal{M}_0$, for each $e \in \mathcal{M} \cap L_0$, then there is an element $\widetilde{e} \in \mathcal{M} \cap L_0$ and a number λ such that*

$$d^1_e = d^0_e + \widetilde{e} + \lambda e_0.$$

We will often use the above theorem and the following theorem from [7].

THEOREM 1.2. *If ϕ is a finite Blaschke product, then there is a unique reducing subspace $\mathcal{M}_0$ for $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{\mathcal{M}_0}$ is unitarily equivalent to the Bergman shift. In fact,*

$$\mathcal{M}_0 = \operatorname*{span}_{l \geqslant 0}\{p_l(\phi(z), \phi(w))e_0\},$$

and $\{\frac{p_l(\phi(z),\phi(w))e_0}{\sqrt{l+1}\|e_0\|}\}_0^\infty$ *form an orthonormal basis of* $\mathcal{M}_0$.

We call $\mathcal{M}_0$ the distinguished reducing subspace for $\phi(\mathcal{B})$. $\mathcal{M}_0$ is unitarily equivalent to a reducing subspace of M_ϕ contained in the Bergman space, denoted by $M_0(\phi)$. The space plays an important role in classifying the minimal reducing subspaces of M_ϕ [7], [8]. If 0 is a zero of ϕ, it was shown [5] that

$$M_0(\phi) = \operatorname{span}\{\phi'\phi^n : n = 0, 1, \ldots, m, \ldots\}.$$

The following lemmas give some properties for functions in $\mathcal{H}$ or $\mathcal{H}^\perp$.

LEMMA 1.3. *If f is in $H^2(\mathbb{T}^2)$ and continuous on the closed bidisk and e is in $\mathcal{H}$, then*

$$\langle f(z,w), e(z,w)\rangle = \langle f(z,z), e(z,0)\rangle = \langle f(w,w), e(0,w)\rangle.$$

Proof. Since $f(z,w)$ is continuous on the closed bidisk, there are a sequence $\{P_n\}$ of polynomials of z and w converging uniformly to $f(z,w)$ on the closed bidisk. Thus it suffices to show

$$\langle P_n(z,w), e(z,w)\rangle = \langle P_n(z,z), e(z,0)\rangle = \langle P_n(w,w), e(0,w)\rangle.$$

Noting that $T^*_{z^l}|_{\mathcal{H}} = T^*_{w^l}|_{\mathcal{H}}$, we see that

$$T^*_{P_n(z,w)}e = T^*_{P_n(z,z)}e = T^*_{P_n(w,w)}e.$$

This gives

$$\begin{aligned}\langle P_n(z,w), e(z,w)\rangle &= \langle 1, \overline{P_n(z,w)}e(z,w)\rangle = \langle 1, T^*_{P_n(z,w)}e\rangle = \langle 1, T^*_{P_n(z,z)}e\rangle \\ &= \langle 1, \overline{P_n(z,z)}e(z,w)\rangle = \langle P_n(z,z), e(z,w)\rangle = \langle P_n(z,z), e(z,0)\rangle.\end{aligned}$$

Similarly we also obtain the following which completes the proof:

$$\langle P_n(z,w), e(z,w)\rangle = \langle P_n(w,w), e(0,w)\rangle. \quad ∎$$

The proofs of the following lemmas are easy and left for readers.

LEMMA 1.4. *For $h(z,w) \in H^2(\mathbb{T}^2)$, h is in $\mathcal{H}^\perp$ if and only if $h(z,z) = 0$, for $z \in \mathbb{D}$.*

LEMMA 1.5. *Suppose that $e(z,w)$ is in $\mathcal{H}$. If $e(z,z) = 0$ for each z in the unit disk, then $e(z,w) = 0$ for (z,w) on the torus.*

The above lemma tells us that a function in $\mathcal{H}$ is completely determined by its value on the diagonal. The following result says that $e(z,w)$ is symmetric with respect to z and w.

LEMMA 1.6. *If $e(z,w)$ is in $\mathcal{H}$, then*

$$e(z,w) = e(w,z).$$

LEMMA 1.7. *Suppose $f(z,w)$ is in $\mathcal{H}$. Let $F(z) = f(z,0)$. Then, for each $\lambda \in \mathbb{D}$,*

$$f(\lambda,\lambda) = \lambda F'(\lambda) + F(\lambda).$$

For $\alpha \in \mathbb{D}$, let k_α be the *reproducing kernel* of the Hardy space $H^2(\mathbb{T})$ at α. That is, for each function f in $H^2(\mathbb{T})$,

$$f(\alpha) = \langle f, k_\alpha \rangle.$$

For an integer $s \geqslant 0$, define

$$k_\alpha^s(z) = \frac{s! z^s}{(1-\overline{\alpha} z)^{s+1}}.$$

Let ϕ be a Blaschke product with zeros $\{\alpha_k\}_0^K$ and α_k repeats n_k+1 times. That is,

$$\phi(z) = \prod_{k=0}^{K} \left(\frac{z-\alpha_k}{1-\overline{\alpha}_k z} \right)^{n_k+1}.$$

The order of ϕ is given by

$$N = \sum_{i=0}^{K} (n_i + 1).$$

We assume that $\alpha_0 = 0$, and so $\phi(z) = z\phi_0(z)$ where ϕ_0 is the following Blaschke product:

$$\phi_0(z) = z^{n_0} \prod_{k=1}^{K} \left(\frac{z-\alpha_k}{1-\overline{\alpha}_k z} \right)^{n_k+1}.$$

For each $\alpha \in \mathbb{D}$ and integer $t \geqslant 0$, let

$$e_\alpha^t(z,w) = \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} k_\alpha^s(z) k_\alpha^{t-s}(w). \tag{1.1}$$

The Mittag-Leffler expansion of the finite Blaschke product ϕ_0 is

$$\phi_0(z) = \sum_{i=0}^{K} \sum_{t=0}^{n_i} c_i^t k_{\alpha_i}^t(z),$$

for some constants $\{c_i^t\}$. Define

$$e_0(z,w) = \sum_{i=0}^{K} \sum_{t=0}^{n_i} c_i^t e_{\alpha_i}^t(z,w).$$

Clearly,

$$e_0(z,0) = \phi_0(z).$$

Simple calculations give the following lemmas.

LEMMA 1.8. *For each $\alpha \in \mathbb{D}$ and $t \geqslant 0$, then*

$$e_\alpha^t(z,z) = \frac{(t+1)!z^t}{(1-\overline{\alpha}z)^{t+2}}.$$

LEMMA 1.9. *For each $F(z,w) \in H^2(\mathbb{T}^2)$,*

$$\langle F, e_\alpha^t \rangle = [(\partial_z + \partial_w)^t F(z,w)]|_{z=w=\alpha}.$$

Noting that the dimension of L_0 is N and $\{e_{\alpha_i}^{t_i}(z,w) : 0 \leqslant i \leqslant K,\ 0 \leqslant t_i \leqslant n_i\}$ are linearly independent, we immediately have the following lemma.

LEMMA 1.10. *We have*

$$L_0 = \operatorname{span}\{e_{\alpha_i}^{t_i}(z,w) : 0 \leqslant i \leqslant K,\ 0 \leqslant t_i \leqslant n_i\}.$$

Consequently, the above lemma gives the following lemma.

LEMMA 1.11. *For each function $F(z,w) \in \ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)}$, there is a function $E(z,w) \in L_0$ such that*

$$F(z,0) = E(z,0).$$

Theorem 18 in [7] only gives the existence of the family of functions $\{d_e^{(k)}\} \subset \mathcal{L}_\phi \ominus L_0$. It will be useful to know how those functions are constructed from e. Theorem 1.14 will give a recursive formula of $\{d_e^{(k)}\}$. First we need the following simple but useful lemma.

For two functions $x,\ y$ in $H^2(\mathbb{T}^2)$, the symbol $x \otimes y$ is the operator on $H^2(\mathbb{T}^2)$ defined, for $g \in H^2(\mathbb{T}^2)$, by

$$(x \otimes y)g = [\langle g, y\rangle_{H^2(\mathbb{T}^2)}]x.$$

LEMMA 1.12. *On the Hardy space $H^2(\mathbb{T}^2)$, the identity operator equals*

$$I = T_z T_z^* + \sum_{l \geqslant 0} w^l \otimes w^l = T_w T_w^* + \sum_{l \geqslant 0} z^l \otimes z^l.$$

LEMMA 1.13. *Suppose that $\phi(z) = z\phi_0(z)$ for some Blaschke product $\phi_0(z)$ with finite order. If f is a function in $H^2(\mathbb{T}^2)$, then for each $l \geqslant 1$,*

$$\begin{aligned} T^*_{z-w}(p_l(\phi(z),\phi(w))f) = p_l(\phi(z),\phi(w))T^*_{z-w}f + \phi_0(z)p_{l-1}(\phi(z),\phi(w))f(0,w) \\ - \phi_0(w)p_{l-1}(\phi(z),\phi(w))f(z,0). \end{aligned}$$

Proof. Let $f \in H^2(T^2)$. By Lemma 1.12, we have

$$\begin{aligned} &T_z^*(p_l(\phi(z),\phi(w))f) \\ &\quad = T_z^*\Big[p_l(\phi(z),\phi(w))\Big(T_zT_z^* + \sum_{i\geqslant 0} w^i \otimes w^i\Big)f\Big] \\ &\quad = T_z^*[p_l(\phi(z),\phi(w))(T_zT_z^*f)] + T_z^*\Big[p_l(\phi(z),\phi(w))\Big(\sum_{i\geqslant 0} w^i \otimes w^i\Big)f\Big] \end{aligned}$$

$$= p_l(\phi(z),\phi(w))(T_z^* f) + T_z^*\Big[p_l(\phi(z),\phi(w))\Big(\sum_{i\geqslant 0} w^i \otimes w^i\Big) f\Big].$$

Noting

$$p_l(\phi(z),\phi(w)) = \sum_{k=0}^{l} \phi(z)^k \phi(w)^{l-k} = \phi(w)^l + \phi(z)\sum_{k=1}^{l}\phi(z)^{k-1}\phi(w)^{l-k}$$
$$= \phi(w)^l + z\phi_0(z)\sum_{k=1}^{l}\phi(z)^{k-1}\phi(w)^{l-k},$$

and

$$\Big(\sum_{i\geqslant 0} w^i \otimes w^i\Big) f = f(0,w),$$

we obtain

$$T_z^*\Big[p_l(\phi(z),\phi(w))\Big(\sum_{i\geqslant 0} w^i\otimes w^i\Big)f\Big]$$
$$= T_z^*[p_l(\phi(z),\phi(w))f(0,w)]$$
$$= T_z^*[\phi(w)^l f(0,w)] + T_z^*\Big[z\phi_0(z)\sum_{k=1}^{l}\phi(z)^{k-1}\phi(w)^{l-k}f(0,w)\Big]$$
$$= \phi_0(z)\Big[\sum_{k=1}^{l}\phi(z)^{k-1}\phi(w)^{l-k}\Big]f(0,w) = \phi_0(z)p_{l-1}(\phi(z),\phi(w))f(0,w).$$

This gives

$$T_z^*(p_l(\phi(z),\phi(w))f) = p_l(\phi(z),\phi(w))(T_z^* f) + \phi_0(z)p_{l-1}(\phi(z),\phi(w))f(0,w). \tag{1.2}$$

Similarly, we also have

$$T_w^*(p_l(\phi(z),\phi(w))f) = p_l(\phi(z),\phi(w))(T_w^* f) + \phi_0(w)p_{l-1}(\phi(z),\phi(w))f(z,0). \tag{1.3}$$

Combining (1.2) and (1.3) yields as desired

$$T_{z-w}^*(p_l(\phi(z),\phi(w))f) = p_l(\phi(z),\phi(w))T_{z-w}^* f + \phi_0(z)p_{l-1}(\phi(z),\phi(w))f(0,w)$$
$$- \phi_0(w)p_{l-1}(\phi(z),\phi(w))f(z,0). \quad \blacksquare$$

The following theorem gives a recursive formula for those functions $\{d_e^k\}$, which will be used in the construction of d_e.

THEOREM 1.14. *Suppose that e is in L_0 and $\{d_e^k\}$ are a family of functions in $H^2(\mathbb{T}^2)$. Then for a given integer $n \geqslant 1$,*

$$p_l(\phi(z),\phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z),\phi(w))d_e^{l-k} \in \mathcal{H},$$

for each $1 \leqslant l \leqslant n$, if and only if the following recursive formula holds

$$\phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T_{z-w}^* d_e^1(z,w) = 0;$$

and, for $1 \leqslant k \leqslant n-1$,

$$\phi_0(z)d_e^k(0,w) - \phi_0(w)d_e^k(z,0) + T^*_{z-w}(d_e^{k+1})(z,w) = 0.$$

Proof. For a given $e \in L_0$ and a family of functions $\{d_e^k\} \subset H^2(\mathbb{T}^2)$, for each integer $l \geqslant 1$, let

$$E_l = p_l(\phi(z),\phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z),\phi(w))d_e^{l-k}.$$

E_l is in $\mathcal{H}$ for each $1 \leqslant l \leqslant n$, if and only if $T^*_{z-w}E_l = 0$ for each $1 \leqslant l \leqslant n$. We need only show that for each $1 \leqslant l \leqslant n$, $T^*_{z-w}E_l = 0$ is equivalent to the recursive formula in the theorem.

By Lemma 1.13, we have

$$\begin{aligned}
&T^*_{z-w}E_l \\
&= T^*_{z-w}[p_l(\phi(z),\phi(w))e] + \sum_{k=0}^{l-1} T^*_{z-w}[p_k(\phi(z),\phi(w))d_e^{l-k}] \\
&= p_l(\phi(z),\phi(w))T^*_{z-w}e + \phi_0(z)p_{l-1}(\phi(z),\phi(w))e(0,w) \\
&\quad - \phi_0(w)p_{l-1}(\phi(z),\phi(w))e(z,0) + \sum_{k=1}^{l-1}[p_k(\phi(z),\phi(w))T^*_{z-w}d_e^{l-k} \\
&\quad + \phi_0(z)p_{k-1}(\phi(z),\phi(w))d_e^{l-k}(0,w) - \phi_0(w)p_{k-1}(\phi(z),\phi(w))d_e^{l-k}(z,0)] \\
&= p_{l-1}(\phi(z),\phi(w))[\phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T^*_{z-w}d_e^1] \\
&\quad + \sum_{k=0}^{l-2}[p_k(\phi(z),\phi(w))(T^*_{z-w}d_e^{l-k} + \phi_0(z)d_e^{l-k-1}(0,w) - \phi_0(w)d_e^{l-1-k}(z,0))]
\end{aligned}$$

since e is in L_0. Thus $T^*_{z-w}E_l = 0$ for each $1 \leqslant l \leqslant n$ if and only if

$$\phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T^*_{z-w}d_e^1 = 0,$$

and

$$T^*_{z-w}d_e^{l-k} + \phi_0(z)d_e^{l-k-1}(0,w) - \phi_0(w)d_e^{l-1-k}(z,0)) = 0,$$

for $1 \leqslant k < l \leqslant n$. This completes the proof. ∎

LEMMA 1.15. *If for a function* $f \in \mathcal{H}$, $p_l(\phi(z),\phi(w))f \in \mathcal{H}$, *for each* $l \geqslant 0$, *then* $f(z,0) = \lambda\phi_0(z)$, *for constant* λ.

Proof. Suppose that $p_l(\phi(z),\phi(w))f \in \mathcal{H}$, for each $l \geqslant 0$. Let $d_f^k = 0$. Then

$$p_l(\phi(z),\phi(w))f + \sum_{k=0}^{l-1} p_k(\phi(z),\phi(w))d_f^{l-k} \in \mathcal{H},$$

for each $l \geqslant 1$. By Theorem 1.14, we have

$$\phi_0(z)f(0,w) - \phi_0(w)f(z,0) = 0.$$

This gives

$$\frac{f(z,0)}{\phi_0(z)} = \frac{f(0,w)}{\phi_0(w)}$$

holds for all $(z,w) \in \mathbb{D} \times \mathbb{D}$ except for a finite vertical or horizontal lines. Thus the equality holds for an open subset of D^2, and so there is a constant λ such that $f(z,0) = \lambda\phi_0(z)$ on the unit disk. This completes the proof. ∎

The following theorem is proved in [7] and is used in the proof of Theorem 1.17.

THEOREM 1.16. *If for a function $f \in \mathcal{H}$, $p_l(\phi(z),\phi(w))f \in \mathcal{H}$, for each $l \geqslant 0$, then there exists a constant λ such that $f = \lambda e_0$.*

Next for a given $e \in L_0$, we will show that there is a unique function $d_e \in \mathcal{L}_\phi \ominus e_0$ such that, for each $l \geqslant 1$,

$$p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e \in \mathcal{H}.$$

THEOREM 1.17. *For a given $e \in L_0$, there is a unique function $d_e \in \mathcal{L}_\phi \ominus e_0$ such that*

$$p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e \in \mathcal{H}$$

for each $l \geqslant 1$. If e is linearly independent of e_0, then $d_e \neq 0$. Moreover, the mapping

$$e \to d_e$$

is a linear operator from L_0 into $\mathcal{L}_\phi \ominus e_0$.

Proof. First we show the existence of d_e. For the given e, by Theorem 18 in [7], there is a function $d_e^1 \in \mathcal{L}_\phi$ such that

$$p_1(\phi(z),\phi(w))e + d_e^1 \in \mathcal{H}.$$

By Theorem 1.14 we have

$$\phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T^*_{z-w}d_e^1(z,w) = 0. \tag{1.4}$$

Since $e(z,w)$ is in $\mathcal{H}$, by Lemma 1.6, $d_e^1(z,w)$ is symmetric with respect to z and w. In addition, $p_1(\phi(z),\phi(w))$ is also symmetric with respect to z and w. This gives $d_e^1(z,w) = d_e^1(w,z)$. Thus $d_e^1(z,0) = d_e^1(0,z)$. By Lemma 1.11, choose a function $\widetilde{e}(z,w) \in L_0$ such that $d_e^1(z,0) = \widetilde{e}(z,0)$. Hence $d_e^1(0,z) = \widetilde{e}(0,z)$, because $\widetilde{e}(z,w)$ is also symmetric with respect to z and w. Let $d_e = d_e^1 - \widetilde{e}$. Clearly,

$$p_1(\phi(z),\phi(w))e + d_e \in \mathcal{H}, \quad \text{and} \quad d_e(z,0) = d_e(0,z) = d_e^1(z,0) - \widetilde{e}(z,0) = 0.$$

Letting $\widetilde{d_e^1} = d_e$ and $\widetilde{d_e^k} = 0$, for $k > 1$, by (1.4), we have the following equations:

$$\begin{aligned}\phi_0(z)e(0,w) &- \phi_0(w)e(z,0) + T^*_{z-w}\widetilde{d_e^1}(z,w)\\ &= \phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T^*_{z-w}[d_e^1(z,w) - \widetilde{e}(z,w)] = 0,\\ \phi_0(z)\widetilde{d_e^k}(0,w) &- \phi_0(w)\widetilde{d_e^k}(z,0) + T^*_{z-w}(\widetilde{d_e^{k+1}})(z,w) = 0 - 0 - 0 = 0,\end{aligned}$$

for $1 \leqslant k \leqslant l-1$. The last equality in the first equation follows from $T^*_{z-w}\widetilde{e}(z,w) = 0$. By Theorem 1.14, we conclude that, as desired,

$$p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e \in \mathcal{H}.$$

Next we show that if there is another function $b_e \in \mathcal{L}_\phi$ such that

$$p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))b_e \in \mathcal{H},$$

for each $l \geqslant 1$, then $d_e - b_e = \mu e_0$ for some constant μ.

Since

$$\begin{aligned} p_{l-1}(\phi(z),\phi(w))[d_e - b_e] &= p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e \\ &\quad - (p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))b_e) \in \mathcal{H}, \end{aligned}$$

letting $f = d_e - b_e$, we have that $f \in \mathcal{H}$ and $p_l(\phi(z),\phi(w))f \in \mathcal{H}$. By Theorem 1.16, we obtain that $f = \lambda e_0$ to conclude

$$d_e = b_e + \lambda e_0.$$

If $d_e = 0$, i.e.,

$$p_l(\phi(z),\phi(w))e \in \mathcal{H},$$

then Theorem 1.16 again implies that $e = \lambda e_0$. This gives that if e is linearly independent of e_0, then $d_e \neq 0$.

As showed above, we know that the mapping $e \to d_e$ is well-defined from L_0 into $\mathcal{L}_\phi \ominus e_0$. To finish the proof we need to show that the mapping is linear. To do so, let e_1 and e_2 be in L_0. For given constants c_1 and c_2, we have

$$\begin{aligned} &p_l(\phi(z),\phi(w))e_1 + p_{l-1}(\phi(z),\phi(w))d_{e_1} \in \mathcal{H}, \\ &p_l(\phi(z),\phi(w))e_2 + p_{l-1}(\phi(z),\phi(w))d_{e_2} \in \mathcal{H}, \\ &p_l(\phi(z),\phi(w))[c_1e_1 + c_2e_2] + p_{l-1}(\phi(z),\phi(w))d_{c_1e_1+c_2e_2} \in \mathcal{H}. \end{aligned}$$

Thus $p_{l-1}(\phi(z),\phi(w))[c_1d_{e_1} + c_2d_{e_2} - d_{c_1e_1+c_2e_2}] \in \mathcal{H}$, for each $l \geqslant 1$. By Theorem 1.16,

$$c_1d_{e_1} + c_2d_{e_2} - d_{c_1e_1+c_2e_2} = c_3e_0,$$

for some constant c_3. But d_{e_1}, d_{e_2}, and $d_{c_1e_1+c_2e_2}$ are orthogonal to e_0. We conclude

$$c_1d_{e_1} + c_2d_{e_2} - d_{c_1e_1+c_2e_2} = 0. \quad \blacksquare$$

2. WEIGHTED SHIFTS

In this section we will characterize multiplication operators on the Bergman space which are unitarily equivalent to a weighted shift of finite multiplicity to prove our main result.

A weighted shift T of finite multiplicity n on Hilbert space H is an operator that maps each vector in some orthonormal basis $\{e_k\}_{k=0}^{\infty}$ into a scalar multiple of the next nth vector

$$Te_k = w_ke_{k+n},$$

for all k. The sequence $\{w_k\}$ is called the weight of the weighted shift T. In fact, T is unitarily equivalent to the multiplication operator by z^n on some Hilbert space of analytic functions on the unit disk. [2] and [3] contain many results on the shift operators, which will be used in this paper.

Indeed, a weighted shift of finite multiplicity is unitarily equivalent to a direct sum of finite weighted shifts. The following theorem tells us that if a multiplication operator on the Bergman space is unitarily equivalent to a weighted shift of finite multiplicity, then the first construction in [7] will become much simpler.

THEOREM 2.1. *Suppose that ϕ is a Blaschke product with order N. If there are N mutually orthogonal reducing subspaces $\{M_i\}$ of $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{M_i}$ is unitarily equivalent to a weighted shift, then for each $e_i \in M_i \cap L_0$ and each $l > 1$,*

$$d^l_{e_i} = 0.$$

Proof. By Theorem 1.2 we may assume that $\phi(\mathcal{B})|_{M_1}$ is unitarily equivalent to the Bergman shift. Let e_i be a nonzero vector in $M_i \cap L_0$. By Theorem 19 in [7], there are functions $d^l_{e_i} \in \mathcal{L}_\phi \ominus L_0$ such that

$$p_l(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-k}_{e_i} \in M_i.$$

Theorem 1.2 implies that $d^l_{e_1} = 0$ for $l \geqslant 1$ and $d^1_{e_i} \neq 0$, for $i > 1$. Let

$$E_{il} = p_l(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-k}_{e_i}.$$

Then E_{il} is in M_i and

$$\begin{aligned}\phi(\mathcal{B})^* E_{il} = T^*_{\phi(z)} E_{il} &= P\Big[\overline{\phi(z)}\Big(p_l(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-k}_{e_i}\Big)\Big] \\ &= p_{l-1}(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-2} p_k(\phi(z), \phi(w))d^{l-k}_{e_i} = E_{i(l-1)}.\end{aligned}$$

The last equality follows from $P(\overline{\phi(z)}e_i) = 0$ and $P(\overline{\phi(z)}d^l_{e_i}) = 0$. Thus $\{E_{il}\}_l$ are orthogonal to $\{E_{jl}\}_l$ for $i \neq j$ and so $\{d^l_{e_i}\}_l$ are orthogonal to $\{d^l_{e_j}\}_l$. Since $\dim[\mathcal{L}_\phi \ominus L_0]$ equals $N-1$ and $d^1_{e_i}$ does not equal zero for $i > 1$, $\{d^1_{e_i}\}$ form an orthogonal basis of $\mathcal{L}_\phi \ominus L_0$. This gives that there are constants β_{il} such that

$$d^l_{e_i} = \beta_{il} d^1_{e_i}.$$

Because $\phi(\mathcal{B})|_{M_i}$ is a weighted shift, there is an orthonormal basis $\{F_l\}$ of M_i such that

$$\phi(\mathcal{B})F_l = a_l F_{l+1}$$

where $\{a_l\}$ are weights of $\phi(\mathcal{B})$ on M_i. Thus F_0 is in the kernel of $[\phi(\mathcal{B})|_{M_i}]^*$, and so $F_0 = \lambda_0 e_i$ for some constant λ_0. Since $\phi(\mathcal{B})^* F_1 = a_0 F_0$, we have $\phi(\mathcal{B})^*[F_1 -$

$a_0\lambda_0 E_{i1}] = 0$. Thus $F_1 = a_0\lambda_0 E_{i1} + \mu_1 e_i$. But both F_1 and E_{i1} are orthogonal to e_i. So $\mu_1 = 0$. Hence there is a constant λ_1 such that

$$F_1 = \lambda_1 E_{i1}.$$

By induction, we obtain that there are constants λ_l such that

$$F_l = \lambda_l E_{il}.$$

This implies that $\{E_{il}\}$ form an orthogonal set. Note that

$$E_{il} = p_1(\phi(z), \phi(w))e_i + \Big[\sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))\beta_{i(l-k)}\Big] d^1_{e_i}.$$

We conclude that $\beta_{il} = 0$ for $l > 1$. This gives

$$E_{il} = p_1(\phi(z), \phi(w))e_i + p_{l-1}(\phi(z), \phi(w))d^1_{e_i} \in M_i$$

and $d^l_{e_i} = 0$ for $l > 1$. This completes the proof. ∎

THEOREM 2.2. *Suppose that ϕ is a finite Blaschke product and $\phi(0) = 0$. If ϕ has a nonzero root α, then there is a function $e \in L_0$ such that d^0_e is not orthogonal to L_0.*

Proof. Recall that L_0 equals $\ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{H}$. Assuming that for each $e \in L_0$, d^0_e is orthogonal to L_0, we will derive a contradiction.

Observe that $\{\{e^{s_k}_{\alpha_k}\}_{s_k=0,\dots,n_k}\}_{k=0,\dots,K}$ form a basis for L_0. So for each $e \in L_0$ there is a vector

$$(u^0_0, \dots, u^{n_0}_0, \dots, u^0_{\alpha_K}, \dots, u^{n_K}_{\alpha_K}) \in C^N$$

such that

$$e(z, w) = \sum_{i=0}^{K}\sum_{t=0}^{n_i} u^t_{\alpha_i} e^t_{\alpha_i}(z, w).$$

Noting that $\dim L_0 = N$, we see that $e \to (u^0_0, \dots, u^{n_0}_0, \dots, u^0_{\alpha_K}, \dots, u^{n_K}_{\alpha_K})$ is a linear invertible mapping from L_0 onto C^N.

Let α_j be a nonzero root of ϕ with multiplicity $n_j + 1$. Then

$$\phi^{(t)}(\alpha_j) = \langle \phi, k^t_{\alpha_j}\rangle = 0, \quad \text{for } 0 \leqslant t \leqslant n_j \quad \text{and} \quad \phi^{(n_j+1)}(\alpha_j) = \langle \phi, k^{n_j+1}_{\alpha_j}\rangle \neq 0.$$

Because d^0_e is orthogonal to L_0 and $\{e^t_{\alpha_j}\}^t_{t=0}$ is in L_0, we have

$$\begin{aligned} 0 = \langle d^0_e, e^t_{\alpha_j}\rangle &= \langle [w\phi_0(w)e(z, w) - we(0, w)e_0(z, w)], e^t_{\alpha_j}\rangle \\ &= \langle w\phi_0(w)e(z, w), e^t_{\alpha_j}\rangle - \langle we(0, w)e_0(z, w), e^t_{\alpha_j}\rangle. \end{aligned}$$

By Lemma 1.9,

$$\begin{aligned} \langle w\phi_0(w)e(z, w), e^t_{\alpha_j}\rangle &= \{[\partial_z + \partial_w]^t \phi(w)e(z, w)\}|_{z=w=\alpha_j} \\ &= \sum_{s=0}^{t} \frac{t!}{s!(t-s)!}\phi^{(s)}(\alpha_j)\{[\partial_z + \partial_w]^{t-s} e(z, w)\}|_{z=w=\alpha_j} = 0. \end{aligned}$$

Thus

$$\langle we(0,w)e_0(z,w), e^t_{\alpha_j}\rangle = 0$$

for $0 \leqslant t \leqslant n_j$. By Lemma 1.9 again, we have

$$0 = \langle we(0,w)e_0(z,w), e^t_{\alpha_j}\rangle = \{[\partial_z + \partial_w]^t we(0,w)e_0(z,w)\}|_{z=w=\alpha_j}$$

$$(2.1) \qquad = \sum_{s=0}^{t} \frac{t!}{s!(t-s)!}(we(0,w))^{(s)}(\alpha_i)\{[\partial_z + \partial_w]^{t-s} e_0(z,w)\}|_{z=w=\alpha_j}$$

for $0 \leqslant t \leqslant n_j$. When $t = 0$, the above equation gives $\alpha_j e(0,\alpha_j)e_0(\alpha_j,\alpha_j) = 0$. Noting that $\alpha_j e(0,\alpha_j) = 0$ is equivalent to $\sum\limits_{i=0}^{K} \sum_{t=0}^{n_i} u^t_i e^t_{\alpha_i}(0,\alpha_j) = 0$, we see that there is a function e in L_0 such that $\alpha_j e(0,\alpha_j) \neq 0$. Hence $e_0(\alpha_j,\alpha_j) = 0$. Letting $t = 1$, (2.1) gives

$$\alpha_j e(0,\alpha_j)\{[\partial_z + \partial_w]e_0(z,w)\}|_{z=w=\alpha_j} + (we(0,w))^{(1)}|_{w=\alpha_j} e_0(\alpha_j,\alpha_j) = 0,$$

Thus $\{[\partial_z + \partial_w]e_0(z,w)\}|_{z=w=\alpha_j} = 0$. By induction we obtain

$$\{[\partial_z + \partial_w]^t e_0(z,w)\}|_{z=w=\alpha_j} = 0,$$

for $0 \leqslant t \leqslant n_j$. In particular, $0 = \{[\partial_z + \partial_w]^{n_j} e_0(z,w)\}|_{z=w=\alpha_j}$. A simple calculation gives

$$\{[\partial_z+\partial_w]^{n_j} e_0(z,w)\}|_{z=w=\alpha_j} = \langle e_0, e^{n_j}_{\alpha_j}\rangle = \langle \overline{e^{n_j}_{\alpha_j}} e_0(z,w), 1\rangle = \langle P_{\mathcal{H}}[\overline{e^{n_j}_{\alpha_j}(z,w)} e_0(z,w)], 1\rangle.$$

Because $e^{n_j}_{\alpha_j}$ is in $H^\infty(\mathbb{T}^2)$ and $e_0(z,w)$ is in $\mathcal{H}$, we have

$$P_{\mathcal{H}}[\overline{e^{n_j}_{\alpha_j}(z,w)} e_0(z,w)] = P_{\mathcal{H}}[\overline{e^{n_j}_{\alpha_j}(z,z)} e_0(z,w)].$$

Thus

$$\begin{aligned}\{[\partial_z + \partial_w]^{n_j} & e_0(z,w)\}|_{z=w=\alpha_j} \\ &= \langle P_{\mathcal{H}}[\overline{e^{n_j}_{\alpha_j}(z,z)} e_0(z,w)], 1\rangle = \langle \overline{e^{n_j}_{\alpha_j}(z,z)} e_0(z,w), 1\rangle = \langle e_0(z,w), e^{n_j}_{\alpha_j}(z,z)\rangle \\ &= \langle e_0(z,0), e^{n_j}_{\alpha_j}(z,z)\rangle = \Big\langle \phi_0(z), \frac{(n_j+1)!z^{n_j}}{(1-\overline{\alpha}_j z)^{n_j+2}}\Big\rangle.\end{aligned}$$

On the other hand, we also have

$$0 = \phi_0^{(n_j)}(\alpha_j) = \langle \phi_0, k^{n_j}_{\alpha_j}\rangle = \Big\langle \phi_0, \frac{n_j! z^{n_j}}{(1-\overline{\alpha}_j z)^{n_j+1}}\Big\rangle.$$

Combining the above two equalities gives

$$0 = \Big\langle \phi_0(z), \Big[\frac{z^{n_j}}{(1-\overline{\alpha}_j z)^{n_j+2}} - \frac{z^{n_j}}{(1-\overline{\alpha}_j z)^{n_j+1}}\Big]\Big\rangle = \Big\langle \phi_0(z), \frac{\overline{\alpha}_j z^{n_j+1}}{(1-\overline{\alpha}_j z)^{n_j+2}}\Big\rangle.$$

Hence

$$\phi_0^{(n_j+1)}(\alpha_j) = \langle \phi_0(z), k_{\alpha_j}^{n_j+1}(z) \rangle = \frac{(n_j+1)!}{\bar{\alpha}_j} \Big\langle \phi_0(z), \frac{\bar{\alpha}_j z^{n_j+1}}{(1-\bar{\alpha}_j z)^{n_j+2}} \Big\rangle = 0.$$

This contradicts the fact that α_j is a nonzero root of ϕ_0 with multiplicity $n_j + 1$. ∎

We are ready to prove our main result.

Proof of Theorem 0.1. We may assume that $\|M_\phi\| = 1$. Suppose that M_ϕ is unitarily equivalent to the direct sum $\bigoplus_{i=1}^{N} W_i$ where W_i is a weighted shift. Then

$$\dim\ker M_\phi^* = \sum_i \dim\ker W_i^*$$

and the essential spectrum of M_ϕ is

$$\sigma_e(M_\phi) = \bigcup_{i=1}^{N} \sigma_e(W_i).$$

Noting that W_i is subnormal, we see that the essential spectrum of W_i is a circle with center at origin. So $\bigcup_{i=1}^{N} \sigma_e(W_i)$ is a union of circles with the same center at origin. On the other hand, by Corollary 20 of [6], the essential spectrum of M_ϕ is connected. Thus $\bigcup_{i=1}^{N} \sigma_e(W_i)$ is the unit circle and $|\phi(z)| = 1$ on $\mathbb{T}$. So ϕ is an inner function.

We claim that ϕ is a Blaschke product with N zeros in the unit disk. If ϕ is not so, there is a singularity $z_0 \in \mathbb{T}$ of $\phi(z)$ (that is a point that $\phi(z)$ does not extend analytically), by Theorem 6.6 in [1], the cluster set of $\phi(z)$ is the closed unit disk. Note that a point η in the cluster set of $\phi(z)$ at z_0 if and only if there are points z_n in $\mathbb{D}$ tending to z_0 such that $\phi(z_n)$ converges to η. This implies that the cluster set of $\phi(z)$ at every point z_0 on the unit circle is contained in the essential spectrum of M_ϕ, which is a contradiction.

By Theorem 1.17, there are N linearly independent functions $\{e_i\}$ of L_0 such that $\{d_{e_i}\}$ are orthogonal to e_0 and

$$p_l(\phi(z), \phi(w))e_i + p_{l-1}(\phi(z), \phi(w))d_{e_i} \in \mathcal{H}.$$

Also we have $p_l(\phi(z), \phi(w))e_i + p_{l-1}(\phi(z), \phi(w))d_{e_i}^0 \in \mathcal{H}$, for $l \geqslant 0$. Thus $p_l(\phi(z), \phi(w))(d_{e_i} - d_{e_i}^0) \in \mathcal{H}$. So $d_{e_i} - d_{e_i}^0$ is in L_0 and hence Theorem 1.16 gives that there are constants λ_i such that $d_{e_i} = d_{e_i}^0 + \lambda_i e_0$. Since $e_0^{n_0}$ is in L_0 and d_{e_i} is orthogonal to L_0, we have

$$0 = \langle d_{e_i}, e_0^{n_0} \rangle = \langle d_{e_i}^0, e_0^{n_0} \rangle + \lambda_i \langle e_0, e_0^{n_0} \rangle.$$

On the other hand, Lemma 1.9 gives

$$\begin{aligned}\langle e_0, e_0^{n_0}\rangle &= \langle e_0(z,w), e_0^{n_0}(z,z)\rangle = \langle e_0(z,0), e_0^{n_0}(z,z)\rangle = (n_0+1)!\langle \phi_0(z), z^{n_0}\rangle \\ &= (n_0+1)!\phi_0^{(n_0)}(0) \neq 0, \\ \langle d_{e_i}^0, e_0^{n_0}\rangle &= \langle w\phi_0(w)e_i(z,w) - we_i(0,w)e_0(z,w), e_0^{n_0}(z,w)\rangle \\ &= \langle \phi(w)e_i(z,w), e_0^{n_0}(z,w)\rangle - \langle we_i(0,w)e_0(z,w), e_0^{n_0}(z,w)\rangle.\end{aligned}$$

The Leibniz rule and Lemma 1.9 give

$$\begin{aligned}\langle \phi(w)e_i(z,w), e_0^{n_0}(z,w)\rangle &= [(\partial_z+\partial_w)^{n_0}(\phi(w)e_i(z,w))]|_{z=w=0} \\ &= \sum_{s=0}^{n_0}\frac{n_0!}{s!(n_0-s)!}\phi^{(s)}(0)[(\partial_z+\partial_w)^{n_0-s}e_i](0,0) = 0.\end{aligned}$$

The last equality follows from the fact that 0 is a root of ϕ with multiplicity n_0+1. Similarly, we have

$$\begin{aligned}\langle we_i(0,w)e_0(z,w), e_0^{n_0}(z,w)\rangle &= [(\partial_z+\partial_w)^{n_0}(we_i(0,w)e_0(z,w))]|_{z=w=0} \\ &= \sum_{s=0}^{n_0}\frac{n_0!}{s!(n_0-s)!}(we_i(0,w))^{(s)}(0)[(\partial_z+\partial_w)^{n_0-s}e_0](0,0).\end{aligned}$$

Lemmas 1.3 and 1.9 give

$$\begin{aligned}[(\partial_z+\partial_w)^{n_0-s}e_0](0,0) &= \langle e_0(z,w), e_0^{n_0-s}(z,w)\rangle = \langle e_0(z,w), e_0^{n_0-s}(z,z)\rangle \\ &= \langle e_0(z,0), e_0^{n_0-s}(z,z)\rangle = \langle \phi_0(z), (n_0-s+1)!z^{n_0-s}\rangle = 0\end{aligned}$$

for $0 < s \leqslant n_0$. The second equality follows from $P_{\mathcal{H}}[\overline{e_0^{n_0-s}(z,w)}e_0(z,w)] = P_{\mathcal{H}}[\overline{e_0^{n_0-s}(z,z)}e_0(z,w)]$. Thus

$$\sum_{s=0}^{n_0}\frac{n_0!}{s!(n_0-s)!}(we_i(0,w))^{(s)}(0)[(\partial_z+\partial_w)^{n_0-s}e_0](0,0) = 0,$$

and so

$$\langle we_i(0,w)e_0(z,w), e_0^{n_0}(z,w)\rangle = 0.$$

Hence we have that the constant $\lambda_i = 0$. Therefore $d_{e_i}^0$ is orthogonal to L_0 for each i. Noting that $\{e_i\}$ form a basis for L_0 we see that d_e^0 is orthogonal to L_0 for each $e \in L_0$. By Theorem 2.2, we conclude that $\phi = \phi_\lambda^N$, to complete the proof. ∎

3. DECOMPOSITION OF $\mathcal{H}$

The proof of Theorem 0.1 in the previous section suggests a more general result stating that if ϕ has more than two distinct roots and at least one root is repeated, then $\mathcal{H}$ can not be decomposed as a direct sum of N reducing subspaces of M_ϕ. In this section we will prove the result.

THEOREM 3.1. *Suppose that ϕ is a Blaschke product of order N. If 0 is a zero and a critical point of ϕ and the zero set of ϕ contains at least one nonzero point in the unit disk, then $\mathcal{H}$ cannot be decomposed as a direct sum $\bigoplus_{i=0}^{N-1} M_i$ of N mutually orthogonal nontrivial reducing subspaces $\{M_i\}_{i=0}^{N-1}$ of $\phi(\mathcal{B})$.*

Proof. By the assumption, we may write

$$\phi = z\phi_0 = z^{n_0+1}\phi_1,$$

where

$$\phi_0 = z^{n_0}\phi_{\alpha_1}^{n_1+1}\cdots\phi_{\alpha_K}^{n_K+1} \quad \text{and} \quad \phi_1 = \phi_{\alpha_1}^{n_1+1}\cdots\phi_{\alpha_K}^{n_K+1}$$

for some nonzero points $\alpha_1,\ldots,\alpha_K$ in the unit disk and nonegative integers $n_0,\ldots,n_K$.

Recall that L_0 is equal to $\ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{H}$. Then

$$L_0 = \text{span}\{1, p_1, \ldots, p_{n_0}, e^0_{\alpha_1}, \ldots, e^{n_1}_{\alpha_1}, \ldots, e^0_{\alpha_K}, \ldots, e^{n_K}_{\alpha_K}\}.$$

Assume that $\phi(\mathcal{B})$ has N mutually orthogonal nontrivial reducing subspaces $\{M_i\}_{i=0}^{N-1}$ such that

$$\mathcal{H} = \bigoplus_{i=0}^{N-1} M_i$$

where M_0 is the distinguished reducing subspace $\mathcal{M}_0$ in Theorem 1.2.

By Lemma 1.10, for each i, there is an $e_i \neq 0$ such that $e_i \in M_i \cap L_0$, and

$$L_0 = \text{span}\{e_0, e_1, \ldots, e_{N-1}\}.$$

By Theorems 19 in [7], there are functions $\{d^1_{e_i}\} \subset \mathcal{L}_\phi \ominus L_0$ such that

$$p_1(\phi(z),\phi(w))e_i + d^1_{e_i} \in M_i.$$

Since M_i is orthognal to M_j for distinct i and j, we have

$$\langle p_1(\phi(z),\phi(w))e_i + d^1_{e_i}, p_1(\phi(z),\phi(w))e_j + d^1_{e_j}\rangle = 0.$$

On the other hand, a simple calculation gives

$$\begin{aligned}
&\langle p_1(\phi(z),\phi(w))e_i + d^1_{e_i}, p_1(\phi(z),\phi(w))e_j + d^1_{e_j}\rangle \\
&\quad = \langle p_1(\phi(z),\phi(w))e_i + d^1_{e_i}, p_1(\phi(z),\phi(w))e_j\rangle + \langle p_1(\phi(z),\phi(w))e_i + d^1_{e_i}, d^1_{e_j}\rangle \\
&\quad = \langle p_1(\phi(z),\phi(w))e_i, p_1(\phi(z),\phi(w))e_j\rangle + \langle d^1_{e_i}, d^1_{e_j}\rangle = \langle d^1_{e_i}, d^1_{e_j}\rangle.
\end{aligned}$$

The second equality follows from the fact that d_{e_i} and d_{e_j} are in $\mathcal{L}_\phi \ominus L_0$. The equality follows since e_i and e_j are in L_0. Thus,

$$\langle d^1_{e_i}, d^1_{e_j}\rangle = 0.$$

By Theorems 19 in [7], each $d^1_{e_i} \neq 0$ for $i > 0$ and

$$\{d^1_{e_i}\}_{i=1}^{N-1} \subset \mathcal{L}_\phi \ominus L_0$$

are linearly independent.

By Theorem 1.1, there are numbers β_i, λ_i such that

$$d^1_{e_i} = d^0_{e_i} + \beta_i e_i + \lambda_i e_0 \quad i = 1, \dots, N-1. \tag{3.1}$$

We will show that $d^0_{e_i}$ and e_0 are in

$$\{1, p_1, \dots, p_{n_0-1}, e^0_{\alpha_1}, \dots, e^{n_1-1}_{\alpha_1}, \dots, e^0_{\alpha_K}, \dots, e^{n_K-1}_{\alpha_K}\}^\perp.$$

To do this, observe that for $0 \leqslant k \leqslant n_0$,

$$\begin{aligned}
&-\langle d^0_{e_i}, p_k\rangle \\
&\quad=\langle \phi(w)e_i - we_i(0,w)e_0, p_k\rangle \\
&\quad=\langle \phi(w)e_i(w,w), p_k(0,w)\rangle - \langle we_i(0,w)e_0(w,w), p_k(0,w)\rangle \\
&\quad=\langle \phi(w)e_i(w,w), w^k\rangle - \langle we_i(0,w)(w\phi_0'(w) + \phi_0(w)), w^k\rangle \\
&\quad=\langle w^{n_0+1-k}\phi_1(w)e_i(w,w), 1\rangle - \langle w^{n_0+1-k}[w\phi_1'(w) + (n_0+1)\phi_1(w)]e_i(0,w), 1\rangle = 0.
\end{aligned}$$

The second equality follows from Lemma 1.3 and the third equality follows from Lemma 1.7.

Since $e^t_{\alpha_j}$ is in the kernel of $T^*_{\phi(w)}$ and $\phi^{(s)}(\alpha_j) = 0$ for $0 \leqslant s \leqslant n_j$, we have that for $0 \leqslant t \leqslant n_j - 1$ and $j = 1, \dots, K$,

$$\begin{aligned}
\langle d^0_{e_i}, e^t_{\alpha_j}\rangle &= \langle we_i(0,w)e_0(w,w) - \phi(w)e_i, e^t_{\alpha_j}\rangle = \langle we_i(0,w)e_0(w,w), e^t_{\alpha_j}(0,w)\rangle \\
&= \langle we_i(0,w)[w\phi_0'(w) + \phi_0(w)], e^t_{\alpha_j}(0,w)\rangle \\
&= \langle we_i(0,w)\phi', k^t_{\alpha_j}\rangle = (we_i(0,w)\phi')^{(t)}|_{w=\alpha_j} = 0; \\
\langle d^0_{e_i}, e^{n_j}_{\alpha_j}\rangle &= [we_i(0,w)\phi'(w)]^{(n_j)}|_{\alpha_j} = \alpha_j e_i(0,\alpha_j)\phi^{(n_j+1)}(\alpha_j).
\end{aligned}$$

These give that

$$d^0_{e_i} \perp \{1, p_1, \dots, p_{n_0-1}, e^0_{\alpha_1}, \dots, e^{n_1-1}_{\alpha_1}, \dots, e^0_{\alpha_K}, \dots, e^{n_K-1}_{\alpha_K}\}. \tag{3.2}$$

We also have that for $0 \leqslant k \leqslant n_0 - 1$

$$\begin{aligned}
\langle e_0, p_k\rangle &= \langle e_0(w,w), p_k(0,w)\rangle = \langle \phi'(w), w^k\rangle = 0, \\
\langle e_0, p_{n_0}\rangle &= \frac{1}{n_0!}\phi^{(n_0+1)}(0) \neq 0.
\end{aligned}$$

A simple calculation shows that for $j = 1, \dots, K$, $0 \leqslant t \leqslant n_j - 1$

$$\begin{aligned}
\langle e_0, e^t_{\alpha_j}\rangle &= [e_0(w,w)]^{(t)}|_{\alpha_j} = \phi^{(t+1)}(\alpha_j) = 0, \\
\langle e_0, e^{n_j}_{\alpha_j}\rangle &= \phi^{(n_j+1)}(\alpha_j) \neq 0.
\end{aligned}$$

These give

$$e_0 \perp \{1, p_1, \dots, p_{n_0-1}, e^0_{\alpha_1}, \dots, e^{n_1-1}_{\alpha_1}, \dots, e^0_{\alpha_K}, \dots, e^{n_K-1}_{\alpha_K}\}. \tag{3.3}$$

We claim that there are at most K nonzero β_i's. If β_{i_0} does not equal 0 for some i_0, (3.1) yields

$$e_{i_0} = \frac{1}{\beta_{i_0}}[d^1_{e_{i_0}} - d^0_{e_{i_0}} - \lambda_{i_0} e_0].$$

Noting that $d^1_{e_i}$ is orthogonal to L_0, by (3.2) and (3.3) we have

$$e_{i_0} \perp \{1, p_1, \ldots, p_{n_0-1}, e^0_{\alpha_1}, \ldots, e^{n_1-1}_{\alpha_1}, \ldots, e^0_{\alpha_K}, \ldots, e^{n_K-1}_{\alpha_K}\}.$$

Thus

$$e_{i_0} \perp \{1, p_1, \ldots, p_{n_0-1}, e^0_{\alpha_1}, \ldots, e^{n_1-1}_{\alpha_1}, \ldots, e^0_{\alpha_K}, \ldots, e^{n_K-1}_{\alpha_K}, e_0\}. \tag{3.4}$$

So there are at most K nonzero β_i's and hence our claim holds.

On the other hand if $\beta_i = 0$, then (3.1) gives

$$d^1_{e_i} = d^0_{e_i} + \lambda_i e_0.$$

Since p_{n_0} is in L_0 and $d^1_{e_i} \perp L_0$, we have that $d^0_{e_i} \perp p_{n_0}$, and

$$\langle e_0, p_{n_0} \rangle \neq 0,$$

to obtain that $\lambda_i = 0$ and $d^0_{e_i} = d^1_{e_i}$ is orthogonal to L_0. By Theorem 2.2, there is at least one nonzero β_i.

Without loss of generality, assume that for some m, $\beta_{N-j} \neq 0$ for $1 \leqslant j \leqslant m$ and $\beta_j = 0$ for $1 \leqslant j \leqslant N-m-1$, (3.4) gives

$$e_{N-j} \perp \{1, p_1, \ldots, p_{n_0-1}, e^0_{\alpha_1}, \ldots, e^{n_1-1}_{\alpha_1}, \ldots, e^0_{\alpha_K}, \ldots, e^{n_K-1}_{\alpha_K}, e_0\}$$

for $1 \leqslant j \leqslant m$. Now we extend

$$\{1, p_1, \ldots, p_{n_0-1}, e^0_{\alpha_1}, \ldots, e^{n_1-1}_{\alpha_1}, \ldots, e^0_{\alpha_K}, \ldots, e^{n_K-1}_{\alpha_K}, e_0, e_{N-1}, \ldots, e_{N-m}\}$$

to a basis of L_0

$$\{1, p_1, \ldots, p_{n_0-1}, e^0_{\alpha_1}, \ldots, e^{n_1-1}_{\alpha_1}, \ldots, e^0_{\alpha_K}, \ldots, e^{n_K-1}_{\alpha_K}, e_0, e_{N-1}, \ldots, e_{N-m}, f_1, \ldots, f_{K-m}\}$$

by adding some elements $f_1, \ldots, f_{K-m}$ in L_0. Let $\{g_j\}_{j=1}^{N-m-1}$ denote

$$\{1, p_1, \ldots, p_{n_0-1}, e^0_{\alpha_1}, \ldots, e^{n_1-1}_{\alpha_1}, \ldots, e^0_{\alpha_K}, \ldots, e^{n_K-1}_{\alpha_K}, f_1, \ldots, f_{K-m}\}.$$

Since for $1 \leqslant j \leqslant N-m-1$, e_j is in L_0 and

$$e_j \perp \{e_0, e_{N-1}, \ldots, e_{N-m}\}$$

we have that e_j is in the subspace span$\{1, g_2, \ldots, g_{N-m-1}\}$ of L_0. This implies that there are numbers $\{c_{jl}\}_{j,l=1}^{N-m-1}$ such that for $1 \leqslant j \leqslant N-m-1$

$$e_j = c_{j1} + c_{j2} g_2 + \cdots + c_{jN-m-1} g_{N-m-1}. \tag{3.5}$$

On the other hand, because $\beta_j = 0$ for $1 \leqslant j \leqslant N-m-1$, we have that $d^0_{e_j} = d^1_{e_j}$ is orthogonal to L_0, and

$$\langle d^0_{e_j}, e^{n_1}_{\alpha_1} \rangle = \alpha_1 e_j(0, \alpha_1) \phi^{(n_1+1)}(\alpha_1) = 0.$$

This implies that $e_j(0,\alpha_1) = 0$. Hence (3.5) gives

$$e_j(0,\alpha_1) = c_{j1}1 + c_{j2}g_2(0,\alpha_1) + \cdots + c_{jN-m-1}g_{N-m-1}(0,\alpha_1) = 0$$

for $1 \leqslant j \leqslant N-m-1$. Thus the determinant $\det[c_{jk}]$ of the coefficient matrix of the above system must be zero. So there is a nonzero vector $(x_1,\ldots,x_{N-m-1})$ such that

$$c_{1l}x_1 + c_{2l}x_2 + \cdots + c_{N-m-1l}x_{N-m-1} = 0$$

for $1 \leqslant l \leqslant N-m-1$. This implies

$$x_1e_1 + x_2e_2 + \cdots + x_{N-m-1}e_{N-m-1} = 0.$$

We obtain a contradiction that $e_1,\ldots,e_{N-m-1}$ are linearly independent to complete the proof. ∎

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SHUNHUA SUN, INSTITUTE OF MATHEMATICS, JIAXING UNIVERSITY, JIAXING, ZHEJIANG, 314001, P. R. CHINA

E-mail address: shsun@mail.zjxu.edu.cn

DECHAO ZHENG, DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TN 37240, USA
E-mail address: dechao.zheng@vanderbilt.edu

CHANGYONG ZHONG, DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGIA STATE UNIVERSITY, ATLANTA, GA 30303, USA
E-mail address: matcyz@langate.gsu.edu

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