MULTIPLICATIONS IN POSTNIKOV SYSTEMS AND THEIR APPLICATIONS

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Introduction. The present work concerns the theory of obstructions for Postnikov complexes of one-connected CW-complex to have multiplications and its application to Thom complexes. Multiplications are introduced stepwisely for dimensions of Postnikov complexes and the obstruction is a set of cohomology classes (See sections 1 and 2). The theory is applicable for $M(O(n))$ and results for a sum of a realizable classes with coefficients in Z_2 are obtained. We compute actually the obstructions for Postnikov complexes of $M(O(2))$ to be an H-space (See section 4). Parallel considerations are made for $M(SO(n))$ and sum of realizable classes with coefficients in Z or Z_p where p is an odd prime number (See section 5).

1. ***H*-spaces.** Let A be a topological space. Suppose that a continuous map

$$\mu : A \times A \longrightarrow A \tag{1}$$

is defined and there is the base point $e \in A$ such that

$$\mu(x, e) = x,\ \mu(e, y) = y \tag{2}$$

for any $x, y \in A$. Then A is called an *H-space* and the correspondence $\mu(x, y)$ is called a *multiplication*, which is occasionally denoted by $x \cdot y$. A *homotopy commutativity* and *homotopy associativity* are defined in usual ways. One can easily prove

PROPOSITION 1. *A product space of two H-spaces is again an H-space. If given H-spaces are homotopy commutative or homotopy associative, then the product space has the corresponding properties.*

PROOF. Let A_1 and A_2 be H-spaces with multiplication maps μ_1 and μ_2. Define a multiplication μ of $A_1 \times A_2$ by

$$\mu\{(a_1, a_2),\ (b_1, b_2)\} = \{\mu_1(a_1, b_1),\ \mu_2(a_2, b_2)\} \tag{3}$$

for any $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. Denoting by $e_1 \in A_1$ and $e_2 \in A_2$ respective identities, (e_1, e_2) is the identity for μ. The latter part of the proposition follows immediately from the definition of μ.

As usual we denote the subspace $A \times e \cup e \times A \subset A \times A$ by $A \vee A$. Let p_1 and p_2 be projections of $A \times A$ onto the first and the second factor. A cohomology class $\gamma \in H^{m+1}(A \times A,\ A \vee A\,;\ \pi_m(A))$ is said to be primitive with respect to μ, if we have

$$(4) \qquad (\mu^* - p_1{}^* - p_2{}^*)\gamma = 0.^{1)}$$

Let A' be another one-connected H-space with the multiplication map μ': $A' \times A' \to A'$. Given a homotopy multiplicative map $f\colon A \to A'$, we say f is *strictly homotopy multiplicative*[2], if there exists a homotopy $F(x, y, t)\,(0 \leqq t \leqq 1)$ such that

$$(5) \qquad \begin{aligned} F(x, y, 0) &= \mu'(f(x), f(y)), \\ F(x\ y, 1) &= f(\mu(x, y)), \end{aligned}$$

and

$$(6) \qquad F(x, e, t) = F(e, x, t) = f(x),$$

for all t. If f is exactly multiplicative, then we say f is a *homomorphism.*

PROPOSITION 2. *If f is strictly homotopy multiplicative, then the fiber space (E, p, A) induced by means of f from the fiber space of paths starting from the unity e' over A' admits an H-structure $\nu\colon E \times E \to E$.*

PROOF. Let (E', p', A') be the fiber space of paths over A'. An element of E' is a path $u\colon [0, r] \to A'$ for a real number r, such that $u(0) = e'$. Define a multiplication $\nu'(u, v)$ for $u\,(0 \leqq t \leqq r)$, $v\,(0 \leqq t \leqq s)$ E' by

$$(7) \qquad \begin{aligned} \nu'(u, v)(t) &= \mu'(u(t), v(st/r))\ \ 0 \leqq t \leqq s, \quad \text{if} \quad r \leqq s, \\ &= \mu'(u(rt/s), v(t))\ \ 0 \leqq t \leqq r, \quad \text{if} \quad r \geqq s. \end{aligned}$$

One obtains an H-structure of E'. The unity is the constant path $u(t) = e'(0 \leqq t \leqq 1)$.

E is a subspace of $A \times E'$ consisting of points (x, u) such that $f(x) = p'(u) = u(r)$. We denote by $l_{x,y}$ a path from $\mu'(f(x), f(y))$ to $f(\mu(x, y))$ in A', given

1) See A. H. Copeland Jr., On H-spaces with two non-trivial homotopy groups, Proc. Amer. Math. Soc., Vol. 8 (1957), pp. 184–191.

2) If A is a sphere and if the suspension $E\colon \pi_m(A') \longrightarrow \pi_{m+1}(SA')$ is monomorph, then a homotopy multiplicative map $A \longrightarrow A'$ is strictly homotopy multiplicative. (See I. M. James, On sphere with multiplication II, Trans. Amer. Math. Soc., Vol. 84 (1957), pp. 545–558, Cor. (5. 5).

According to one of correspondences from Professor E. H. Spanier, Professor P. J. Hilton uses a similar notion called primitive.

by $F(x, y, t)$, $0 \leqq t \leqq 1$, for fixed points $x, y \in A$. Define a multiplication ν for two points (x, u) and (y, v) of E by

$$\nu\{(x, u), (y, v)\} = \{\mu(x, y), \nu'(u, v) \cdot l_{x,y}\}, \tag{8}$$

where the dot is the usual composition of paths.

From the choice of $l_{x,y}$, we have

$$p'(\nu'(u, v) \cdot l_{x,y}) = f(\mu(x, y)),$$

which means that $\nu\{(x, u), (y, v)\}$ is again a point of E. The continuity of ν follows immediately. The unity of E is (e, e').

From (8), one obtains

$$\begin{aligned} p \cdot \nu\{(x, u), (y, v)\} &= p\{\mu(x, y), \nu'(u, v) \cdot l_{x,y}\} \\ &= \mu(x, y) \\ &= \mu\{p(x, u), p(y, v)\}, \end{aligned}$$

that is the diagram

$$\begin{array}{ccc} E \times E & \xrightarrow{\nu} & E \\ \downarrow{\scriptstyle p \times p} & & \downarrow{\scriptstyle p} \\ A \times A & \xrightarrow{\mu} & A \end{array} \tag{9}$$

is exactly commutative and hence p is a homomorphism.

2. Multiplications in the Postnikov system. Let $K = \bigcup_{q \in Z^+} K_q$ be a semi-simplicial complex, wher Z^+ denotes the set of non-negative integers. K is called a *monoid* if K_q has an associative multiplication

$$K_q \times K_q \longrightarrow K_q \tag{10}$$

with a *unit element* for each q. If e_0 denotes the unit element of K_0 and s_0 denotes a degeneracy operator, then $(s_0)^q e_0$ gives that of K_q for each q. Now we shall prove the following,

LEMMA 1. *If A is an H-space, then the minimal subcomplex $M(A)$ of the singular complex $S(A)$ of A is a monoid.*

PROOF. Let $S_q(\mu) : S_q(A) \times S_q(A) \to S_q(A)$ be the monoid structure of $S(A)$ induced by the multiplication μ of A. Let λ be the natural chain map $S(A) \to M(A)$ and let i be the inclusion map $M(A) \subset S(A)$. Then we get the diagram

$$(11)\qquad \begin{array}{ccc} S_q(A)\times S_q(A) & \xrightarrow{S_q(\mu)} & S_p(A) \\ \uparrow{\scriptstyle i\times i} & & \downarrow{\scriptstyle \lambda} \\ M_q(A)\times M_q(A) & \longrightarrow & M_q(A). \end{array}$$

The composition map $\lambda\cdot S_q(\mu)\cdot(i\times i)$ is obviously a chain map and makes $M(A)$ a monoid complex. Let X be a one-connected CW-complex and suppose that its homotopy groups are countable in each demensions. Let the Postnikov system of X be

$$(12)\qquad \cdots\cdots \longrightarrow X^{(m)} \xrightarrow{p^{(m)}} X^{(m-1)} \xrightarrow{p^{(m-1)}} X^{(m-2)} \longrightarrow \cdots\cdots \xrightarrow{p^{(3)}} X^{(2)},$$

with k-invariants k_m, and let $p_{(m)}$ be the projection of X to $X^{(m)}$. $X^{(2)}$ is a CW-complex $K(\pi_2(X),2)$ which is a group complex. $X^{(m)}$ is a CW-complex of the same weak homotopy type with the fiber space $E^{(m)}$ induced from the fiber space of paths over $K(\pi_m(X),\ m+1)$ by means of the map $\varphi_m:\ X^{(m-1)}\to K(\pi_m(X),\ m+1)$, such that

$$(13)\qquad \varphi_m^*(b)=k_{m+1},$$

where b is the basic cohomology class of $K(\pi_m(X),\ m+1)$. From now on, we put $X^{(m)}=|M(E^{(m)})|$.

If $E^{(m)}$ is an H-space, then $M(E^{(m)})$ is a monoid complex, by Lemma 1. By the assumption for X, the geometric realization $|M(E^{(m)})|$ is a countable complex and hence it is an H-space by a theorem of J. Milnor[3]. The natural map $|M(E^{(m)})|\to E^{(m)}$ is a homomorphism.

Now we claim

LEMMA 2. *Suppose $X^{(m-1)}$ has an H-structure $\mu^{(m-1)}$. φ_m is strictly homotopy multiplicative if and only if k_{m+1} is primitive with respect to $\mu^{(m-1)}$.*

PROOF. Suppose φ_m is strictly homotopy multiplicative. $K(\pi_m(X), m+1)$ has a standard multiplication σ. The commutativity upto homotopy holds in the diagram

$$(14)\qquad \begin{array}{ccc} X^{(m-1)}\times X^{(m-1)} & \xrightarrow{\varphi_m\times\varphi_m} & K(\pi_m(X),\ m+1)\times K(\pi_m(X),\ m+1) \\ \downarrow{\scriptstyle \mu^{(m-1)}} & & \downarrow{\scriptstyle \sigma} \\ X^{(m-1)} & \xrightarrow{\varphi_m} & K(\pi_m(X),\ m+1). \end{array}$$

3) See J. Milnor, The geometric realization of a semi-simplicial complex, Lecture Note in Princeton Univeresity 1956.

Let $p_1^{(m-1)}$ and $p_2^{(m-1)}$ be projections of $X^{(m-1)} \times X^{(m-1)}$ onto the first and the second factor. Let p_1 and p_2 be projections of $K(\pi_m(X), m+1) \times K(\pi_m(X), m+1)$ onto the first and the second factor. One obtains, from the commutativity of (14),

$$\begin{aligned}\mu^{(m-1)*}k_{m+1} &= \mu^{(m-1)*}\cdot\varphi_m^*(b)\\ &= (\varphi_m \times \varphi_m)^*\cdot\sigma^*(b)\\ &= (\varphi_m \times \varphi_m)^*(p_1^*(b) + p_2^*(b))\\ &= p_1^{(m-1)*}\varphi_m^*(b) + p_2^{(m-1)*}\cdot\varphi_m^*(b)\\ &= p_1^{(m-1)*}k_{m+1} + p_2^{(m-1)*}k_{m+1},\end{aligned}$$

and hence we have

$$(\mu^{(m-1)*} - p_1^{(m-1)*} - p_2^{(m-1)*})k_{m+1} = 0,$$

which shows k_{m+1} is primitive with respect to $\mu^{(m-1)}$.

Conversely, suppose k_{m+1} is primitive with respect to $\mu^{(m-1)}$. We have to prove the existence of a homotopy $F(x, y, t)$ of maps from $X^{(m-1)} \times X^{(m-1)}$ to $K(\pi_m(X), m+1)$, for $0 \leqq t \leqq 1$ and for any $x, y \in X^{(m-1)}$, such that

$$\text{(15)} \qquad \begin{cases} F(x, y, 0) = \sigma(\varphi_m(x), \varphi_m(y)),\\ F(x, y, 1) = \varphi_m\cdot\mu^{(m-1)}(x, y),\\ F(x, e, t) = F(e, x, t) = \varphi_m(x).\end{cases}$$

The obstruction to construct the homotopy is obviously given by

$$\begin{aligned}(\varphi_m\cdot\mu^{(m-1)})^*(b) &- (\varphi \quad \times \varphi_m)^*\sigma^*(b)\\ &= \mu^{(m-1)*}\cdot\varphi_m^*(b) - (\varphi_m \times \varphi_m)^*\sigma^*(b)\\ &= \mu^{(m-1)*}k_{m+1} - p_1^*k_{m+1} - p_2^*k_{m+1},\end{aligned}$$

which is zero by the assumption. Thus the map of $X^{(m-1)} \times X^{(m-1)} \times \dot{I} \cup (X^{(m-1)} \vee X^{(m-1)}) \times I\,(I = [0, 1])$ to $K(\pi_m(X), m+1)$ defined by (15) can be extended to the map $F(x, y, t)$ over the whole complex $X^{(m-1)} \times X^{(m-1)} \times I$. φ_m is, therefore, strictly homotopy multiplicative. Lemma 2 is proved.

Immediately we obtain, from Proposition 2,

PROPOSITION 3. *Let X be a one-connected CW-complex with countable homotopy groups in each dimensions. Suppose that the complex $X^{(m-1)}$ of the Postnikov system of X has an H-structure $\mu^{(m-1)}$. If k_{n+1} is primitive with respect to $\mu^{(m-1)}$, the complex $X^{(m)}$ has again an H-structure and the projection $p^{(m)}: X^{(m)} \to X^{(m-1)}$ is a homomorphism.*

REMARK. Several multiplications $\mu_\alpha^{(m)}$ of $X^{(m)}$ in the above proposition may exist. The complex $X^{(m+1)}$ has an H-structure if k_{m+2} is primitive with respect to one of $\mu_\alpha^{(m)}$.

Now we consider to construct multiplications of $X^{(m)}$ stepwisely. $X^{(2)} = K(\pi_2(X), 2)$ has a standard multiplication $\mu^{(2)}$. If k_4 is primitive with respect to $\mu^{(2)}$, then $X^{(3)}$ has multiplications $\mu_\alpha^{(3)}$. The obstructions for $X^{(4)}$ to be an H-space is given by a set of cohomology classes $(\mu_\alpha^{(3)*} - p_1{}^{(3)*} - p_2{}^{(3)*})k_5$, and so on. In general, assuming $X^{(m-1)}$ has H-structures $\mu_\alpha^{(m-1)}$, we put

$$O_\alpha^{(m)} = (\mu_\alpha^{(m-1)*} - p_1{}^{(m-1)*} - p_2{}^{(m-1)*})k_{m+1},$$

which is an element of $H^{m+1}(X^{(m-1)} \times X^{(m-1)}, X^{(m-1)} \vee X^{(m-1)}; \pi_m(X))$. The obstruction for $X^{(m)}$ to be an H-space is a set of cohomology classes $\mathbf{O}^{(m)} = \{O_\alpha^{(m)}\}$. Let $m_0 + 1$ be the least integer of m such that $\mathbf{O}^{(m)}$ does not contain the zero element. We call m_0 an *index of multiplicativity* of X. Proposition 3 leads easily,

COROLLARY 4. *Each complex $X^{(m)}$ $(m \leqq m_0)$ has a multiplication and $p^{(m)}$ is a homomorphism.*

3. Sums of realizable classes with coefficients in Z_2. Now we turn to the study of the Thom complex $M(O(n))$. Let the Postnikov system of the complex $M(O(n))$ be

$$\cdots\cdots \longrightarrow X^{(m)} \xrightarrow{p^{(m)}} X^{(m-1)} \longrightarrow \cdots\cdots \longrightarrow X^{(n)} = K(Z_2, n)$$

and $p_{(m)}$ be the projection of $M(O(n))$ to $X^{(m)}$. Let $m_0(n)$ denote the index of multiplicativity of $M(O(n))$. We obtain the

THEOREM. *In a compact differentiable manifold M of dimension $\leqq m_0(n)$, a sum of two realizable classes of dimension n with coeffcients in the group Z_2 of integers modulo 2 is again realizable.*

PROOF. Let u and v be two realizable cohomology classes with coefficients in Z_2. We have maps $f, g : M \rightarrow M(O(n))$ such that

$$\begin{aligned} u &= f^*U_n, \\ v &= g^*U_n, \end{aligned} \tag{16}$$

where U_n is the fundamental class of $M(O(n))$. The matter is to construct a map $h: M \rightarrow M(O(n))$ such that

$$u + v = h^*U_n. \tag{17}$$

Since dim $M \leqq m_0(n)$ and $M(O(n))$ have the same $(m_0 + 1)$-type with $X^{(m_0)}$, the problem is reduced to construct maps of M to $X^{(m_0)}$. We put $p_{(m)}f = f_m$,

$p_{(m)}g = g_m$ and put $p^{*-1}_{(m)} U_n = U_{n,(m)}$. We obtain easily

$$u = f^*_m U_{n,(m)},$$
$$v = g^*_m U_{n,(m)}. \tag{18}$$

By the assumption for m_0, $X^{(m_0)}$ has a multiplcation $\mu^{(m_0)}: X^{(m_0)} \times X^{(m_0)} \to X^{(m_0)}$. Now define a map $f_m \circ g_m : M \to X^{(m_0)} \times X^{(m_0)}$ by the equation

$$f_{m_0} \circ g_{m_0}(x) = (f_{m_0}(x),\ g_{m_0}(x))$$

$$M \xrightarrow{f_{m_0} \circ g_{m_0}} X^{(m_0)} \times X^{(m_0)} \xrightarrow{\mu^{(m_0)}} X^{(m_0)}.$$
$$\xrightarrow[\quad h_{m_0} \quad]{}$$

It induces a homomorphisim $h^*_{m_0}: H^*(X^{(m_0)}; Z_2) \to H^*(M; Z_2)$ satisfying the relation,

$$\begin{aligned} h^*_{m_0} U_{n,(m_0)} &= (f_{m_0} \circ g_{m_0})^* \mu^{(m_0)*}(U_{n,(m_0)}) \\ &= (f_{m_0} \circ g_{m_0})^*(U_{n,(m_0)} \otimes \omega + \omega \otimes U_{n,(m_0)}) \\ &\quad \text{(where } \omega \text{ is the unit class of } H^*(X^{(m_0)}; Z_2),) \\ &= (f_{m_0} \circ g_{m_0})^*(U_{n,(m_0)}) \otimes \omega) + (f_{m_0} \circ g_{m_0})^*(\omega \otimes U_{n,(m_0)}) \\ &= f_{m_0}(U_{n,(m_0)}) \cdot \omega + \omega \cdot g_{m_0}(U_{n,(m_0)}) \\ &\quad \text{(where } \omega \text{ is the unit class of } H^*(M; Z_2),) \\ &= u + v. \end{aligned}$$

The last formula follows from (15). Let $q_m: X^{(m)} \to M(O(n))$ be a homotopy inverse of $p_{(m)}$ for the m-skeleton, which induces an isomorphism of cohomology rings $H^*(X^{(m)}; Z_2)$ and $H^*(M(O(n)); Z_2)$ upto the dimension m. Let h be the composed map $q_{m_0} \cdot h_{m_0}$. One can easily see from (17) that

$$\begin{aligned} h^* U_n &= h^*_{m_0} \cdot q_{m_0} U_n \\ &= h^*_{m_0} U_{n,(m_0)} \\ &= u + v, \end{aligned}$$

which is the required relation (14). Thus our theorem is proved.

We know the following[4]. One takes an integer j and let $d(j)$ be the number of non-dyadique subdivisions,

$$\lambda = \{a_1, a_2, \ldots\ldots, a_r \mid a_i \text{ integers} \neq 2^m - 1,\ \Sigma\, a_i = j\}.$$

4) R. Thom, Quelques propriétés globales des variétés différentiables, Comm. Math. Helv., Vol. 28 (1954), 17-86, Chap. II.

We put a CW-complex

$$Y = K(Z_2, n) \times K(Z_2, n+2) \times \cdots\cdots \times (K(Z_2, n+2))^{d(l)} \times \cdots\cdots \times (K(Z_2, 2n))^{d(n)}.$$

There exists a natural map $F: M(O(n)) \to Y$, which induces isomorphisms of cohomology groups with coefficient group Z_2 for dimension $< 2n$. Since we have $H^m(Y; Z_p) = 0$ and $H^m(M(O(n)); Z_p) = 0$ for an odd prime number p and for $m < 2n$, Y and $M(O(n))$ are of the same $2n$-type. And hence there is a map g of the $2n$-skeleton of Y into $M(O(n))$ such that composed maps $g \cdot F$ and $F \cdot g$ are homotopic to the identity maps of $(2n-1)$-skeletons in $M(O(n))$ and in Y respectively. g is a so-called homotopy inverse of F for the $2n$-skeleton of Y. The sequence of complexes

$$Y, \ldots\ldots, K(Z_2, n) \times K(Z_2, n+2) \times \cdots\cdots \times (K(Z_2, n+l))^{d(l)},$$
$$K(Z_2, n) \times K(Z_2, n+2) \times \cdots\cdots \times (K(Z_2, n+l-1))^{d(l-1)},$$
$$\ldots\ldots, K(Z_2, n) \times K(Z_2, n+2),\ K(Z_2, n)$$

gives the Postnikov system of $M(O(n))$ for dimension $< 2n$. Its k-invariants

$$k_{n+1},\ k_{n+2}, \ldots\ldots, k_{2n}$$

are all zero.

Obviously, Y is an H-space from Proposition 1, Theorem leads immediately

COROLLARY 5. *In a compact differentiable manifold of dimension $< 2n$, a sum of two realizable cohomology classes of dimension n with coefficients in Z_2 is again realizable*[6].

k_{2n} is, however, not trivial in general and it should be computed in respective cases for n. We consider the case $n = 2$ in the following section.

4. 2-dimensional realizable classes with coefficients in Z_2. As for the system of $M(O(n))$, following results are known[4),5)]. Homotopy groups of $M(O(2))$ in lower dimensions are

$$\pi_2 = Z_2,\quad \pi_3 = 0,\quad \pi_4 = Z,\quad \pi_5 = Z_2. \tag{19}$$

5) H. Suzuki, On the realization of the Stiefel-Whitney characteristic classes by submanifolds, Tôhoku Math. Journ., Vol. 10 (1958), pp. 91-115, Chap. II.

6) Professor Thom wrote to the author that the result can be proved directly by a geometrical method.

We recall that $p_{(m)}^{*-1}U_2 = U_{2,(m)}$. The complex $X^{(2)} = K(Z_2, 2)$ has a standard multiplication $\mu^{(2)}$ and one can take $X^{(3)} = X^{(2)}$, because of $\pi_3 = 0$ in (18). The k-invariant of dimension 5 is

$$k_5 = (1/2)\delta\mathfrak{p}(U_{2,(3)}),$$

where $\mathfrak{p}$ denotes the Pontryagin square operation and $(1/2)\delta$ is the Bockstein operator for the coboundary homomorphism δ. The obstruction[7] for $X^{(4)}$ to be an H-space is

$$(\mu^{(3)*} - p_1^{(3)*} - p_2^{(3)*})k_5 = 0. \tag{20}$$

The above relation is proved as follows:

$$\begin{aligned}
&(\mu^{(3)*} - p_1^{(3)*} - p_2^{(3)*})k_5 \\
&\quad = (\mu^{(3)*} - p_1^{(3)*} - p_2^{(3)*})(1/2)\delta\mathfrak{p}(U_{2,(3)}) \\
&\quad = \mu^{(3)*}(1/2)\delta\mathfrak{p}(U_{2,(3)}) - (1/2)\delta(p_1^{(3)*} + p_2^{(3)*})\mathfrak{p}(U_{2,(3)}).
\end{aligned}$$

Computing the first term in the right side, we have

$$\begin{aligned}
&\mu^{(3)*}(1/2)\delta\mathfrak{p}(U_{2,(3)}) \\
&\quad = (1/2)\delta\mathfrak{p}(U_{2,(3)} \otimes \omega + \omega \otimes U_{2,(3)}) \\
&\qquad \text{(where } \omega \in H^*(Z_2, 2; Z_2) \text{ is the unit element,)} \\
&\quad = (U_{2,(3)} \otimes \omega + \omega \otimes U_{2,(3)}) \cup \delta(U_{2,(3)} \otimes \omega + \omega \otimes U_{2,(3)}) \\
&+ (1/2)[\delta(U_{2,(3)} \otimes \omega + \omega \otimes U_{2,(3)}) \cup_1 \delta(U_{2,(3)} \otimes \omega + \omega \otimes U_{2,(3)})] \\
&\quad = U_{2,(3)}\delta U_{2,(3)} \otimes \omega + \omega \otimes U_{2,(3)}\delta U_{2,(3)} \\
&\qquad + U_{2,(3)} \otimes \delta U_{2,(3)} + \delta U_{2,(3)} \otimes U_{2,(3)} \\
&\qquad + (1/2)[\delta(U_{2,(3)} \otimes \omega) \cup_1 \delta(U_{2,(3)} \otimes \omega)] \\
&\qquad + (1/2)[\delta(\omega \otimes U_{2,(3)}) \cup_1 \delta(\omega \otimes U_{2,(3)})] \\
&\qquad + (1/2)(\delta U_{2,(3)} \otimes \omega) \cup_1 (\omega \otimes \delta U_{2,(3)}) \\
&\qquad + (1/2)(\omega \otimes \delta U_{2,(3)}) \cup_1 (\delta U_{2,(3)} \otimes \omega).
\end{aligned}$$

The second term is given by the formula

$$\begin{aligned}
&(1/2)\delta(p_1^{(3)*} + p_2^{(3)*})\mathfrak{p}(U_{2,(3)}) \\
&\quad = (1/2)\delta\mathfrak{p}(p_1^{(3)*}U_{2,(3)}) + (1/2)\delta\mathfrak{p}(p_2^{(3)*}U_{2,(3)}) \\
&\quad = (1/2)\delta\mathfrak{p}(U_{2,(3)} \otimes \omega) + (1/2)\delta\mathfrak{p}(\omega \otimes U_{2,(3)})
\end{aligned}$$

7) This obstruction is unique, because if k_5 is primitive, then the induced H-structure of $X^{(3)}$ by that of $X^{(4)}$ is the standard one. See A. H. Copeland Jr., On H-spaces with two non-trivial homotopy groups (Foot-note 1).

$$= (U_{2,(3)} \otimes \omega) \cup \delta(U_{2,(3)} \otimes \omega)$$
$$+ (1/2)[\delta(U_{2,(3)} \otimes \omega) \cup_1 \delta(U_{2,(3)} \otimes \omega)]$$
$$+ (\omega \otimes U_{2,(3)}) \cup \delta(\omega \otimes U_{2,(3)})$$
$$+ (1/2)[\delta(\omega \otimes U_{2,(3)}) \cup_1 \delta(\omega \otimes U_{2,(3)})]$$
$$= U_{2,(3)}\delta U_{2,(3)} \otimes \omega + \omega \otimes U_{2,(3)}\delta U_{2,(3)}$$
$$+ (1/2)(\delta U_{2,(3)} \otimes \omega) \cup_1 (\delta U_{2,(3)} \otimes \omega)$$
$$+ (1/2)(\omega \otimes \delta U_{2,(3)}) \cup_1 (\omega \otimes \delta U_{2,(3)}).$$

Since we have $\frac{1}{2}(\delta U_{2,(3)} \otimes \omega) \cup_1 (\omega \otimes \delta U_{2,(3)}) \sim 0$, $\frac{1}{2}(\omega \otimes \delta U_{2,(3)}) \cup_1 (\delta U_{2,(3)} \otimes \omega) \sim 0$ and

$$U_{2,(3)} \otimes \delta U_{2,(3)} + \delta U_{2,(3)} \otimes U_{2,(3)} \sim 0,$$

we obtain

$$(\mu^{(3)*} - p_1^{(3)*} - p_2^{(3)*})k_5 = 0.$$

Thus the CW-complex $X^{(4)}$ of the Postnikov system of $M(O(2))$ has an H-structure.

Now we proceed to a further step. The 6-dimensional k-invariant of $M(O(2))$ is given by

$$k_6 = (Sq^1(U_{2,(4)}))^2 + U_{2,(4)} \cdot V_{(4)},$$

where V_4 is the class of $H^4(X^{(4)}; Z_2)$ which goes to the basic class of $H^4(Z, 4; Z_2)$ under the homomorphism i^*: $H^4(X^{(4)}; Z_2) \to H^4(Z, 4; Z_2)$ induced by the inclusion i: $K(Z, 4) \subset X^{(4)}$. More precisely, the projection $p_{(4)}$: $M(O(2)) \to X^{(4)}$ which is the equivalence of 5-type induces an isomorphism $H^4(X^{(4)}; Z_2) \approx H^4(M(O(2)); Z_2)$. V_4 is determined by

$$p^*_{(4)}(V_4) = U_2(W_1)^2,$$

where W_1 is the 1-dimensional Stiefel-Whitney class of the Grassmann manifold of non-oriented 2-planes and $U_2(W_1)^2$ is the notation by Thom.

An obstruction for $X^{(5)}$ to be an H-space is

$$(\mu^{(4)*} - p_1^{(4)*} - p_2^{(4)*})k_6,$$

which is not zero. This fact does not lead the CW-complex $X^{(5)}$ of the Postnikov system of $M(O(2))$ has an H-structure.

Summarizing the above results, we see $m_0(M(O(2))) = 5$ and obtain

COROLLARY 6. *In a compact differentiable manifold of dimension* $\leqq 5$, *a sum of two realizable classes of dimension* 2 *with coefficients in* Z_2 *is again realizable.*

5. Sum of realizable classes with coefficients in Z **or** Z_p. We briefly touch on the case of $M(SO(n))$. Let $m_0(M(SO(n)))$ be the multiplicative index fo the Postnikov system of $M(SO(n))$. By arguments being similar to section 3, we see formally that *in a compact orientable differentiable manifold of dimension* $\leqq m_0(M(SO(n)))$, *a sum of realizable classes of dimension* n *with coefficients in* Z *or* Z_p *is again realizable.*

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