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MULTIPLICATIVE η-QUOTIENTS

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ABSTRACT. Let $\eta(z)$ be the Dedekind η-function. In this work we exhibit all modular forms of integral weight $f(z) = \eta(t_1 z)^{r_1}\eta(t_2 z)^{r_2}\dots\eta(t_s z)^{r_s}$, for positive integers s and t_j and arbitrary integers r_j, such that both $f(z)$ and its image under the Fricke involution are eigenforms of all Hecke operators. We also relate most of these modular forms with the Conway group $2\mathrm{Co}_1$ via a generalized McKay-Thompson series.

1. Introduction

The Dedekind η-function is given by the infinite product

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

where $q = \exp(2\pi i z)$ and z lies in the complex upper half plane $\mathcal{H}$. We define an η-quotient to be a function $f(z)$ of the form

$$f(z) = \prod_{j=1}^{s} \eta(t_j z)^{r_j} \tag{1}$$

where $\{t_1, t_2, \dots, t_s\}$ is a finite set of positive integers and $r_1, r_2, \dots, r_s$ are arbitrary integers. In general this is a meromorphic modular form of weight $k = \frac{1}{2}\sum_j r_j$ and multiplier system for some congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$. In this paper we consider only η-quotients which are holomorphic modular forms of integral weight.

We denote the collection of integers $t_1, r_1, t_2, r_2, \dots, t_s, r_s$ defining (1) by the formal product $g = \prod t^{r_t} = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$, and write $\eta_g(z)$ for the corresponding η-quotient (1). If every integer in $r_1, r_2, \dots, r_s$ is non-negative, we refer to $\eta_g(z)$ as an η-product.

In [5] Dummit, Kisilevsky and McKay found the complete set of 30 η-products which are eigenforms for all Hecke operators (two of these have half-integral weight). Every element in this set is also a cusp form and an eigenform for the corresponding Fricke involution. By Theorem 9 in [14] this means that [5] exhibit all η-products which are primitive cusp forms (i.e. new forms which are eigenforms for all Hecke operators).

A second proof of the same classification is given in [10] by Koike. In [17] G. Mason gave yet another proof under the extra condition $k \equiv 0 \pmod 2$. Mason also showed that 21 of these η-products are part of a McKay-Thompson series associated to the Mathieu group M_{24}, i.e. they are traces of elements in M_{24} when this group

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is represented as an endomorphism group of certain graded, infinite dimensional, complex vector space. In fact, we know that every η-product in [5] appears in a particular McKay-Thompson series for the group $2^{24}M_{24}$.

In this work we study the more general η-quotients. We produce an explicit collection of modular forms of this type in Table I at the end of this paper, and prove the following

Theorem 1. *Let $g = t_1^{r_1}t_2^{r_2}\dots t_s^{r_s}$ where t_j, r_j, s are integers and $t_j, s > 0$. Assume that $\eta_g(z) = \prod_{j=1}^{s}\eta(t_jz)^{r_j}$ is a modular form of level N_g, weight k_g and character χ_g for some positive integers N_g and k_g. Denote by $\tilde{\eta}_g(z)$ the image of $\eta_g(z)$ under the Fricke involution $\begin{pmatrix} 0 & -1 \\ N_g & 0 \end{pmatrix}$.*

Then, both $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms for all Hecke operators if, and only if, g is one of the formal products listed in the second column of Table I.

In particular this proves the existence of only a finite number of such η-quotients. Of these, not all are cusp forms or are invariant under the corresponding Fricke involution, but by inspection and the theorem in [14] quoted above, it is easy to deduce the following

Corollary 2. *The formal product $g = t_1^{r_1}t_2^{r_2}\dots t_s^{r_s}$ determines a primitive cusp form $\eta_g(z)$ if, and only if, g is in the second column of Table I and $\eta_g(z)$ is a cusp form. This last property is indicated in the third column of the same table.*

The basic argument in the proof of Theorem 1 is the following: The Dedekind η-function, and therefore every η-quotient $\eta_g(z)$, is non-zero on the upper half plane. Hence $\eta_g(z)$ is completely determined by the order of its zeros at the cusps. We compare $\eta_g(z)$ with its image under the p-th Hecke operator $T_p\eta_g(z)$ at every cusp and every prime p (sections 3 and 4). If $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms for all T_p then the order of zero of $\eta_g(z)$ at any cusp is bounded (section 5, in particular Theorem 30). This puts a number of conditions on $g = t_1^{r_1}t_2^{r_2}\dots t_s^{r_s}$ and therefore limits the possible values for N_g and k_g. There are only a finite number of such pairs (N_g, k_g), and they can be computed (section 6). Each such pair (N_g, k_g) determines a finite number of systems of linear equations, whose solutions define explicit η-quotients (section 6). In this way we produce a list of formal products g which contains all of those such that both $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms of the Hecke algebra. In order to complete the proof of our main result we need only take every modular form $\eta_g(z)$ from the collection above and verify that it is an eigenform for all T_p. We show how to do this when p and N_g are relatively prime (Proposition 33). For the other cases we compute $T_p\eta_g(z)$ directly and compare it with $\eta_g(z)$.

In the last section of this paper (section 7), we indicate a connection between these multiplicative η-quotients and the Conway group (i.e. the automorphism group of the Leech lattice). Namely, we show that at least 72 of the 74 η-quotients characterized in Theorem 1 are elements of a generalized McKay-Thompson series defined by Mason for the Conway group.

This paper is a revised version of the author's doctoral thesis [15].

I would like to take this opportunity to thank Professor Geoffrey Mason for all his encouragement and support.

2. Preliminaries

If N and k are positive integers and χ is a Dirichlet character modulo N, we denote by $M_k(N,\chi)$ the space of modular forms of weight k and character χ on the group $\Gamma_0(N)$. If $g = \prod t^{r_t}$ defines an element $\eta_g(z)$ in $M_k(N,\chi)$ then χ is a real character. Hence, from now on we always assume $\chi(n) = \pm 1$ for $n \in \mathbb{Z}$, $\gcd(n, N) = 1$.

A complete set of representatives for the cusps of $\Gamma_0(N)$ is

$$\begin{aligned} \mathcal{C}_N = \{\frac{a_c}{c} \in \mathbb{Q};\ c \text{ divides } N,\ 1 \le a_c \le N,\ \gcd(a_c, N) = 1 \\ \text{and } a_c \equiv a'_c \pmod{\gcd(c, \frac{N}{c})} \text{ iff } a_c = a'_c\}. \end{aligned} \tag{2}$$

If $f(z)$ is in $M_k(N,\chi)$, the Fourier series of $f(z)$ at the cusp $\frac{a}{c} \in \mathcal{C}_N$ is

$$f(z)|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{n=0}^{\infty} c_n q_h^{n+\mu} \tag{3}$$

where $q_h = q^{\frac{1}{h}}$, b and d are integers such that $ad - bc = 1$, $h = h_{\frac{a}{c}}$ is the width of the cusp $\frac{a}{c}$ of $\Gamma_0(N)$, and $\mu = \mu_{\frac{a}{c}}$ is either 0 or $\frac{1}{2}$ depending upon $\frac{a}{c}$ being a regular or irregular cusp of $\Gamma_0(N)$ respectively.

The values for $h = h_{\frac{a}{c}}$ and $\mu = \mu_{\frac{a}{c}}$ are given by

$$h = \frac{N}{\gcd(c^2, N)}, \quad \chi(1 + ach) = \exp(2\pi i \mu). \tag{4}$$

We denote by $\nu_{\frac{a}{c}}$ the smallest integer n such that $c_n \neq 0$ in (3).

If $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$ defines $\eta_g(z) \in M_k(N,\chi)$ and $\frac{a}{c} \in \mathcal{C}_N$, the Fourier series of $\eta_g(z)$ at $\frac{a}{c}$ is of the form

$$\eta_g(z)|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = C \exp\left(\frac{2\pi i z}{24} \sum_{j=1}^{s} \frac{\gcd(t_j, c)^2}{t_j} r_j\right) G_{\frac{a}{c}}(z) \tag{5}$$

where C is a complex-valued constant depending on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $G_{\frac{a}{c}}(z)$ is a holomorphic function on some neighborhood of infinity with $\lim_{z\to\infty} g_{\frac{a}{c}}(z) \neq 0$ ([7], p. 49). In particular

$$\frac{\nu_{\frac{a}{c}} + \mu_{\frac{a}{c}}}{h_{\frac{a}{c}}} = \frac{1}{24} \sum_{j=1}^{s} \frac{\gcd(t_j, c)^2}{t_j} r_j. \tag{6}$$

Equations (4) and (6) show that $h_{\frac{a}{c}}$, $\nu_{\frac{a}{c}}$ and $\mu_{\frac{a}{c}}$ are independent of a if the modular form is an η-quotient. For the rest of this paper we assume that every modular form $f(z)$ that we consider has this property. Consequently, there is no ambiguity if for any divisor c of N we denote the previous values by h_c, ν_c and μ_c respectively.

Let p be a rational prime. If $f(z) \in M_k(N,\chi)$, its image under the p-th Hecke operator T_p is another element of $M_k(N,\chi)$. Most proofs in this paper are based on the study of Fourier series of $T_pf(z)$ at the cusps $\frac{1}{c}$ of $\Gamma_0(N)$, i.e.

$$T_pf(z)|_k\begin{pmatrix}1&0\\c&1\end{pmatrix}=p^{\frac{k}{2}-1}\Big(\sum_{b=0}^{p-1}f(z)|_k\begin{pmatrix}1&b\\0&p\end{pmatrix}\begin{pmatrix}1&0\\c&1\end{pmatrix}+\chi(p)f(z)|_k\begin{pmatrix}p&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\c&1\end{pmatrix}\Big) \tag{7}$$

From now on, and unless we say otherwise, we adopt the following notation; $f(z)\in M_k(N,\chi)$, p is a rational prime, χ_p is the p-part of χ, T_p is the p-th Hecke operator and λ_p denotes the eigenvalue of $f(z)$ under T_p whenever $f(z)$ is an eigenform of this operator. Moreover we assume that c is a factor of N, we write $a\|b$ if a is a divisor of b with $\gcd(a,\frac{b}{a})=1$, and let h_c, ν_c and μ_c be the real numbers defined by equation (3).

3. Some Fourier expansions of T_pf and some consequences for eigenforms of the Hecke algebra

First we relate the Fourier expansion of T_pf at the cusp $\frac{1}{c}$ with some Fourier series of $f(z)$. Basically there are three different situations, depending on the p-part of N being 1, p, or p^M with $M\geq 2$.

Lemma 3. (i) *If* $\gcd(p,N)=1$ *then*

(8)
$$T_pf(z)|_k\begin{pmatrix}1&0\\c&1\end{pmatrix}=p^{\frac{k}{2}-1}\left\{\sum_{l=0}^{p-1}f(z)|_k\begin{pmatrix}1&0\\pc&1\end{pmatrix}\begin{pmatrix}1&l(1-pp^*)\\0&p\end{pmatrix}+\chi(p)f(z)|_k\begin{pmatrix}1&0\\p^*c&1\end{pmatrix}\begin{pmatrix}p&-m\\0&1\end{pmatrix}\right\}$$

where p^* *satisfies* $pp^*\equiv 1 \pmod N$ *and* $m=\frac{N}{c}$.

(ii) *If* $p\|N$ *and* $\gcd(c,p)=1$ *then*

(9)
$$T_pf(z)|_k\begin{pmatrix}1&0\\c&1\end{pmatrix}=p^{\frac{k}{2}-1}\left\{\sum_{\substack{l=0\\lc\not\equiv1}}^{p-1}\chi_p(1-cl)f(z)|_k\begin{pmatrix}1&0\\pc&1\end{pmatrix}\begin{pmatrix}1&l-pn_l\\0&p\end{pmatrix}+f(z)|_k\begin{pmatrix}x'&l'\\c&p\end{pmatrix}\begin{pmatrix}p&0\\0&1\end{pmatrix}\right\}$$

where the congruence $lc\not\equiv 1$ *is modulo* p, l' *and* x' *are integers such that* $l'c+1=x'p$, *and for each* l *in* $\{0,1,\dots,p-1\}$ *with* $lc\not\equiv 1 \pmod p$ *the integer* n_l *is chosen such that* $l-pn_l\equiv 0 \pmod{\frac{N}{pc}}$.

(iii) *If* $p^M \| N$ *with* $M \geq 2$ *and* $\gcd(c,p) = 1$ *then*

(10)
$$T_p f(z)|_k \begin{pmatrix} 1 & 0 \\ p^{\alpha-1}c & 1 \end{pmatrix} = p^{\frac{k}{2}-1} \sum_{l=0}^{p-1} \chi_p(1 - p^{\alpha-1}c(l - pn_l)) f(z)|_k \begin{pmatrix} 1 & 0 \\ p^{\alpha}c & 1 \end{pmatrix} \begin{pmatrix} 1 & l - pn_l \\ 0 & p \end{pmatrix}$$

for all α *in* $\mathbb{Z}$ *with* $\frac{M+1}{2} \leq \alpha \leq M$. *The integers* n_l *are chosen such that* $l - pn_l \equiv 0 \pmod{\frac{N}{p^M}}$ *for each* $l = 0, 1, \ldots, p-1$.

Proof. (i) Let c^* be in $\{0, 1, \ldots, p-1\}$ such that $cc^* \equiv 1 \pmod p$. For any $l \in \{0, 1, \ldots, p-1\} - \{-c^*\}$ there is a unique l' in $\{0, 1, \ldots, p-1\} - \{c^*\}$ such that $(1 + lc)l' \equiv l \pmod p$, say $(1 + lc)l' + x_{l'}p = l$. Then

$$\begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 + lc & x_{l'} \\ pc & 1 - cl' \end{pmatrix} \begin{pmatrix} 1 & l' \\ 0 & p \end{pmatrix}. \tag{11}$$

Let p^* be in $\{0, 1, \ldots, N-1\}$ such that $pp^* \equiv 1 \pmod N$, and set $n_{l'} = p^*l'$ for all $l' = 0, 1, \ldots, N-1$. Then

$$\begin{pmatrix} 1 + lc & x_{l'} \\ pc & 1 - cl' \end{pmatrix} \begin{pmatrix} 1 & n_{l'} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ pc & 1 \end{pmatrix}^{-1} \in \Gamma_1(N). \tag{12}$$

Consequently,

(13)
$$f(z)|_k \begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = f(z)|_k \begin{pmatrix} 1 & 0 \\ pc & 1 \end{pmatrix} \begin{pmatrix} 1 & l'(1 - pp^*) \\ 0 & p \end{pmatrix}.$$

For $l = -c^*$ we have

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} p & l \\ c & \frac{1-cc^*}{p} \end{pmatrix} \begin{pmatrix} 1 & c^* \\ 0 & p \end{pmatrix}.$$

Moreover

$$\begin{pmatrix} p & -c^* \\ c & \frac{1-cc^*}{p} \end{pmatrix} \begin{pmatrix} 1 & p^*c^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ pc & 1 \end{pmatrix}^{-1} \in \Gamma_0(N).$$

Thus

(14)
$$\chi(p) f(z)|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = f(z)|_k \begin{pmatrix} 1 & 0 \\ pc & 1 \end{pmatrix} \begin{pmatrix} 1 & c^*(1 - pp^*) \\ 0 & p \end{pmatrix}.$$

Now we observe that

$$\begin{pmatrix} 1 & -c^* \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} p & -m \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ p^*c & 1 \end{pmatrix}^{-1} \in \Gamma_0(N). \tag{15}$$

Using equations (13), (14) and (15) in (7), we get the first identity of this lemma.

(ii) For any l in $\{0, 1, \ldots, p-1\}$ with $1 + lc \not\equiv 0 \pmod p$ we take integers l' and $x_{l'}$ as above, so equation (11) holds. Since p^2 does not divide N there are integers $n_{l'}$ such that $l' - pn_{l'} \equiv 0 \pmod{\frac{N}{pc}}$. Hence the matrix (12) is in $\Gamma_0(N)$ and its lower right entry is $1 + c(pn_{l'} - l')$. Consequently

$$f(z)|_k \begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \chi_p(1 - cl') f(z)|_k \begin{pmatrix} 1 & 0 \\ pc & 1 \end{pmatrix} \begin{pmatrix} 1 & l' - pn_{l'} \\ 0 & p \end{pmatrix}.$$

If $l=-c^*$ we have

$$\begin{pmatrix} 1 & -c^* \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}=\begin{pmatrix} x' & -c^* \\ c & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

for some integer x'. Now we use these last two equalities in the left hand side of (7) and recall that $\chi(p)=0$ in order to obtain (9).

(iii) If $\alpha \geq \frac{M+1}{2}$ and $M \geq 2$ then $(1+lp^{\alpha-1}c)l+x_l p=l$ for some integer x_l. Hence

$$\begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p^{\alpha-1}c & 1 \end{pmatrix}=\begin{pmatrix} 1+lp^{\alpha-1}c & x_l \\ p^{\alpha}c & 1-lp^{\alpha-1}c \end{pmatrix}\begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix}.$$

Moreover, $\alpha \geq \frac{M+1}{2}$ implies that p^M divides $p^{2\alpha-1}$. Thus, if we choose an integer n_l for each l in $\{0,1,\ldots,p-1\}$ such that $l-pn_l \equiv 0 \pmod{\frac{N}{p^M}}$, then

$$\begin{pmatrix} 1+lp^{\alpha-1}c & x_l \\ p^{\alpha}c & 1-lp^{\alpha-1}c \end{pmatrix}\begin{pmatrix} 1 & n_l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p^{\alpha}c & 1 \end{pmatrix}^{-1} \in \Gamma_0(N)$$

and its lower right entry is congruent to 1 modulo $\frac{N}{p^M}$. Next we use equation (7) together with $\chi(p)=0$ and we get (10). □

Proposition 4. *Let* $\gcd(p,N)=1$ *and* $p^* \in \mathbb{Z}$ *such that* $pp^* \equiv 1 \pmod{N}$. *If*

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ pc & 1 \end{pmatrix}=\sum_{n=0}^{\infty} a_n q_{h_c}^n \text{ and } f(z)|_k \begin{pmatrix} 1 & 0 \\ p^*c & 1 \end{pmatrix}=\sum_{n=0}^{\infty} b_n q_{h_c}^n,$$

then

$$T_p f(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}=\sum_{n=0}^{\infty}(a_{np}+\chi(p)p^{k-1}b_{\frac{n}{p}})q_{h_c}^n \tag{16}$$

As usual, $b_{\frac{n}{p}}=0$ *whenever* p *does not divide* n.

Proof. From equation (8)

$$T_p f(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}=p^{-1}\sum_{n=0}^{\infty}\sum_{l=0}^{p-1}\exp\left(2\pi i\frac{n}{h_c}\frac{l(1-pp^*)}{p}\right)a_n q_{h_c}^{\frac{n}{p}}$$
$$+p^{k-1}\chi(p)\sum_{n=0}^{\infty}\exp\left(-2\pi i\frac{n}{h_c}m\right)b_n q_{h_c}^{np}.$$

Since $m=\frac{N}{c}$, $\frac{m}{h_c}$ is an integer. Furthermore $\gcd(\frac{1-pp^*}{h_c},p)=1$. Hence the equation above yields (16). □

Proposition 5. *Let* $p\|N$ *and* $\gcd(c,p)=1$. *Assume that*

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ pc & 1 \end{pmatrix}=\sum_{n=0}^{\infty} a_n q_{h_{pc}}^{n+\mu_{pc}} \text{ and } f(z)|_k \begin{pmatrix} x' & l \\ c & p \end{pmatrix}=\sum_{n=0}^{\infty} b_n q_{h_c}^n$$

where l is in $\{0,1,\ldots,p-1\}$ with $lc+1=x'p$ for some integer x'. Then

(17)

$$T_pf(z)|_k\begin{pmatrix}1&0\\c&1\end{pmatrix}=p^{-1}\sum_{n=0}^{\infty}\sum_{l''=1}^{p-1}\chi_p(l'')\exp\left(2\pi ina'c^*\frac{1-l''}{p}\right)a_nq_{h_c}^{n}+p^{k-1}\sum_{n=0}^{\infty}b_nq_{h_c}^{pn}$$

for some a', c^* in $\mathbb{Z}$ with $\gcd(a',p)=\gcd(c^*,p)=1$.

Proof. From equation (4) we have $h_{pc}=\frac{h_c}{p}$. Hence $\exp(2\pi i\mu_{pc})=\exp(2\pi i\mu_c)=1$ and therefore $\mu_{pc}=\mu_c=0$. Now, from (9) we get

(18)

$$T_pf(z)|_k\begin{pmatrix}1&0\\c&1\end{pmatrix}=p^{-1}\sum_{n=0}^{\infty}\left(\sum_{\substack{l=0\\lc\not\equiv1\pmod p}}^{p-1}\chi_p(1-cl)\exp(2\pi in\frac{l-pn_l}{ph_{pc}})\right)a_nq_{h_{pc}}^{\frac{n}{p}}+p^{k-1}\sum_{n=0}^{\infty}b_nq_{h_c}^{pn}.$$

Let c^* and l'' be in $\{0,1,\ldots,p-1\}$ such that $c^*c\equiv1\pmod p$ and $l''\equiv1-lc\pmod p$ for each l in $\{0,1,\ldots,p-1\}-\{c^*\}$. Since p^2 does not divide N there is an integer a' such that $a'h_{pc}\equiv1\pmod p$. Using these new variables in (18), equation (17) follows. □

Observe that $\chi_p(1+lp)=1$ for any odd prime p and integer l since χ is a real character. Similarly, $\chi_2(1+2^3l)=1$. The following Dirichlet characters will play a distinguished role in the arguments ahead, so we write

$$\psi_0(x)=1,\quad\psi_1(x)=(-1)^{\frac{x^2-1}{8}},$$
$$\psi_2(x)=(-1)^{\frac{x-1}{2}},\quad\psi_3(x)=\psi_1(x)\psi_2(x)$$

for every odd integer x.

Notice that every p-factor χ_p of χ can be considered as a real Dirichlet character modulo p whenever p is odd or $\chi_2=\psi_0,\psi_2$.

Proposition 6. *Let $p^M\|N$ for some $M\geq2$ and $\gcd(c,p)=1$. Assume*

$$f(z)|_k\begin{pmatrix}1&0\\p^{\alpha}c&1\end{pmatrix}=\sum_{n=0}^{\infty}a_nq_{h_{p^{\alpha}c}}^{n}$$

where α is any integer such that $\frac{M+1}{2}\leq\alpha\leq M$. Then

$$T_pf(z)|_k\begin{pmatrix}1&0\\p^{\alpha-1}c&1\end{pmatrix}=\sum_{\substack{n=0\\p|n}}^{\infty}a_nq_{h_{p^{\alpha}c}}^{\frac{n}{p}}\tag{19}$$

in any of the following cases:

(i) *p is odd,*
(ii) *$p=2$ and $\alpha\geq4$,*
(iii) *$(p,\chi_p)=(2,\psi_0)$,*
(iv) *$(p,\chi_p)=(2,\psi_2)$ and $\alpha\geq3$.*

Proof. For p and χ_p as in the proposition, $\chi_p(1-p^{\alpha-1}c(l-pn_l))=1$ (see previous remark). Hence, from equation (10) we get

$$T_pf(z)|_k\begin{pmatrix}1&0\\p^{\alpha-1}c&1\end{pmatrix}=p^{-1}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{p-1}\exp\left(2\pi i\frac{n}{p}\frac{l-pn_l}{h_{p^\alpha c}}\right)\right)a_nq_{h_{p^\alpha c}}^{\frac{n}{p}}.$$

As $h_{p^\alpha c}$ divides $\frac{N}{p^M}$ the rational number $\frac{l-pn_l}{h_{p^\alpha c}}$ is an integer and the proposition follows. □

Now we apply the previous results to modular forms $f(z)$ which are eigenforms for some or all Hecke operators.

Proposition 7. *Let $f(z)$ be an eigenform of T_p for all p such that* $\gcd(p,\frac{N}{c})=1$. *If $\mu_c=0$ then, either $\nu_cf\in\{0,1\}$ or any prime divisor of ν_cf is a divisor of $\frac{N}{c}$.*

Proof. Consider the Fourier expansion of $f(z)$ at $\frac{1}{c}$ given by (3). If p is any prime with $\gcd(p,N)=1$ we use (16) and get

$$\lambda_pc_n=a_{np}+\chi(p)p^{k-1}b_{\frac{n}{p}}$$

for all $n\geq 0$, where a_m and b_m are defined as in Proposition 4.

Since p and p^* do not divide N (p^* as in Proposition 4), then $\frac{p}{c}$ and $\frac{1}{pc}$ (resp. $\frac{p^*}{c}$ and $\frac{1}{p^*c}$) represent the same cusp of $\Gamma_0(N)$. Therefore $\nu_{\frac{1}{pc}}f=\nu_{\frac{1}{p^*c}}f=\nu_cf$, say $\nu_cf=\nu$.

Suppose that p divides $\nu\neq 0$. Then

$$\lambda_pc_{\frac{\nu}{p}}=a_\nu+\chi(p)p^{k-1}b_{\frac{\nu}{p^2}}.$$

Hence $a_\nu=0$, a contradiction.

Next, let p be a prime factor of N which is relatively prime to $\frac{N}{c}$.

Since p divides c, there is an integer x_l for each l in $\{0,1,\dots,p-1\}$ such that $(1+cl)l+x_pp=l$. Hence

$$T_pf(z)|_k\begin{pmatrix}1&0\\c&1\end{pmatrix}=p^{\frac{k}{2}-1}\sum_{l=0}^{p-1}f(z)|_k\begin{pmatrix}1+cl&x_l\\pc&1-cl\end{pmatrix}\begin{pmatrix}1&l\\0&p\end{pmatrix}.$$

For each $l=0,1,\dots,p-1$ there is an integer m_l such that

$$\begin{pmatrix}1+cl&x_l\\pc&1-cl\end{pmatrix}\begin{pmatrix}1&m_l\\0&1\end{pmatrix}\begin{pmatrix}1&0\\pc&1\end{pmatrix}^{-1}\in\Gamma_1(N).$$

Therefore

(20)
$$T_pf(z)|_k\begin{pmatrix}1&0\\c&1\end{pmatrix}=p^{\frac{k}{2}-1}\sum_{l=0}^{p-1}f(z)|_k\begin{pmatrix}1&0\\pc&1\end{pmatrix}\begin{pmatrix}1&l-pm_l\\0&p\end{pmatrix}.$$

One can show that $\frac{1}{pc}$ and $\frac{t}{c}$ represent the same cusp of $\Gamma_0(N)$ for some integer t with $\gcd(N,t)=1$. Hence

$$f(z)|_k\begin{pmatrix}1&0\\pc&1\end{pmatrix}=\sum_{n=0}^{\infty}d_nq_{h_c}^n.$$

Now use that $\frac{l-pm_l}{h_c}$ is an integer, and deduce from equation (20) the following

$$\lambda_p f(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \sum_{\substack{n=0 \\ p|n}}^{\infty} d_n q_{h_c}^{\frac{n}{p}}.$$

If p divides ν then the coefficient of $q_{h_c}^{\frac{\nu}{p}}$ in the right hand side of the previous equation is non-zero, thus ν must be zero. □

Proposition 8. *Let $f(z)$ be an eigenform for all Hecke operators. Assume that $p\|N$. Then*

(i) $\lambda_p \neq 0$.

(ii) *For any divisor c of N with $\gcd(p,c)=1$ and $\mu_c = 0$, either $\nu_c f = 0$ or $\gcd(p, \nu_c f) = 1$. Moreover, $\nu_c f \neq 0$ and $\nu_{pc} f \neq 0$ imply $\nu_c f = \nu_{pc} f$.*

Proof. Let c be a factor of N as in (ii). In Proposition 5 we showed $h_{pc} = \frac{h_c}{p}$ and $\mu_{pc} = \mu_c = 0$. Hence, from equations (3) and (17) we get

(21)

$$\lambda_p c_n = p^{-1} a_n \exp\left(2\pi i n \frac{a'c^*}{p}\right)\left(\sum_{l''=1}^{p-1} \chi_p(l'') \exp\left(-2\pi i n \frac{a'c^*l''}{p}\right)\right) + p^{k-1} b_{\frac{n}{p}}$$

for all non-negative integers n (here we are using the notation introduced in Proposition 5). If $c = \frac{N}{p}$ then c satisfies the conditions in (ii) and $a_1 \neq 0$ (see [9], p. 163). Therefore $n=1$ in (21) implies $\lambda_p c_1 \neq 0$, as the sum of $p-1$ terms involving χ_p in (21) is a Gauss sum. This proves (i).

For (ii) set $\nu_c f = \nu$ and $\nu_{pc} f = \omega$. By Proposition 7, $\omega = 0$ or $\gcd(p,\omega) = 1$.

If $\omega \neq 0$ we put $n = \omega$ in (21) and conclude $c_\omega \neq 0$. Hence $\nu \leq \omega$. Suppose $\nu < \omega$. Then $\lambda_p c_\nu = p^{k-1} b_{\frac{\nu}{p}}$ from (21), and $b_{\frac{\nu}{p}} \neq 0$. Consequently $\nu = \nu_{\frac{x'}{c}} f \leq \frac{\nu}{p}$, so we must have $\nu = 0$.

Suppose next that $\omega = 0$. If $\nu = 0$ then there is nothing else to prove. Otherwise $\nu \neq 0$ and from (21) we get $\lambda_p c_0 = p^{-1} a_0 \sum_{l''=1}^{p-1} \chi_p(l'') + p^{k-1} b_0$. Since $b_0 = c_0 = 0$ and $a_0 \neq 0$ we conclude $\sum_{l''}^{p-1} \chi_p(l'') = 0$. Thus $\lambda_p c_n = p^{k-1} b_{\frac{n}{p}}$ for all non-negative integers n divisible by p. Hence, $\lambda_p \neq 0$ and $c_\nu \neq 0$ imply that p does not divide ν. □

In the previous proposition we considered the case $p\|N$. For those cases in which a higher power of p divides N we have to look at two distinct situations, namely $\lambda_p = 0$ and $\lambda_p \neq 0$.

Proposition 9. *Let $\gcd(c,p) = 1$ and $p^M\|N$ with $M \geq 2$. Assume that $f(z)$ is an eigenform of T_p with eigenvalue $\lambda_p \neq 0$. Let the Fourier expansion of $f(z)$ at $\frac{1}{p^M c}$ be*

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ p^M c & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n q_h^n$$

where $h = h_{p^M c}$.

Then, for any integer t with $\gcd(t,N) = 1$ and $t \equiv 1 \pmod{\frac{N}{p^M c}}$

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tp^\alpha c & 1 \end{pmatrix} = \lambda_p^{\alpha - M} \sum_{\substack{n=0 \\ p^{M-\alpha}|n}}^{\infty} a_n q_h^{\frac{n}{p^{M-\alpha}}} \tag{22}$$

for $\alpha = 3, 4, \dots, M$*. This equation also holds for*

(a) $\alpha = 2$ *if* p *is either odd or* $(p, \chi_p) = (2, \psi_0),\ (2, \psi_2)$.
(b) $\alpha = 1$ *if* p *is either odd or* $(p, \chi_p) = (2, \psi_0)$.

Moreover, in all the other cases the corresponding Fourier expansions are the following:
If $p = 2$ *and* $\chi_2 = \psi_2$*, then*

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tpc & 1 \end{pmatrix} = \lambda_p^{1-M} \sum_{m=0}^{\infty} a_{(2m+1)p^{M-2}} q_h^{m+\frac{1}{2}}. \tag{23}$$

If $p = 2$ *and* $\chi_2 = \psi_1, \psi_3$*, then*

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tp^2c & 1 \end{pmatrix} = \lambda_p^{2-M} \sum_{m=0}^{\infty} a_{(2m+1)p^{M-3}} q_h^{m+\frac{1}{2}} \tag{24}$$

and

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tpc & 1 \end{pmatrix} = \lambda_p^{1-M} \sum_{m=0}^{\infty} p^{-1} a_{(2m+1)p^{M-3}} \xi_m q_{h_{pc}}^{(m+\frac{1}{2})p^{M-3}} \tag{25}$$

for some $\xi_m \neq 0$ *in* $\mathbb{C}$.
The same identities hold if c *is replaced by* $-c$.

Proof. We prove (22) by induction on α. Since

$$\begin{pmatrix} 1 & 0 \\ p^M c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{N}{p^M c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ tp^M c & 1 \end{pmatrix}^{-1} \in \Gamma_1(N)$$

we have

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tp^M c & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n \exp\left(2\pi i n \frac{N/p^M c}{h}\right) q_h^n.$$

This establishes (22) for $\alpha = M$.

For the general case we argue as in the proof of Lemma 3 (iii) and get the following equation

(26)

$$T_p f(z)|_k \begin{pmatrix} 1 & 0 \\ tp^\alpha c & 1 \end{pmatrix} = p^{\frac{k}{2}-1} \sum_{l=0}^{p-1} \chi_p(1 - tp^\alpha c(l - pn_l)) f(z)|_k \begin{pmatrix} 1 & 0 \\ l' tp^{\alpha+1} c & 1 \end{pmatrix} \begin{pmatrix} 1 & l - pn_l \\ 0 & p \end{pmatrix}$$

where n_l and l' are integers such that $pn_l - l \equiv 0 \pmod{\frac{N}{p^M}}$, $l' \equiv 1 \pmod{\frac{N}{p^M}}$ and $l'(1 + tp^\alpha c(pn_l - l)) \equiv 1 \pmod{p^M}$.

Assuming that (22) holds for $\alpha+1$ this equation becomes

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tp^\alpha c & 1 \end{pmatrix} = \lambda_p^{\alpha-M} p^{-1} \sum_{\substack{n=0 \\ p^{M-\alpha-1}|n}}^{\infty} \left(\sum_{l=0}^{p-1} \exp\left(2\pi i \frac{n}{p^{M-\alpha-1}} \frac{l-pn_l}{hp}\right)\right) a_n q_h^{\frac{n}{p^{M-\alpha}}} = \lambda_p^{\alpha-M} \sum_{\substack{n=0 \\ p^{M-\alpha}|n}}^{\infty} a_n q_h^{\frac{n}{p^{M-\alpha}}}$$

whenever $\chi_p(1-tp^\alpha c(l-pn_l)) = 1$ for all $l = 0, 1, \dots, p-1$.

Next, in order to show (23), we take $n_0 = 0$ and $l_0' = 1$ in (26) and observe that $\psi_2(1-tpc(1-pn_1)) = -1$. Thus

$$\lambda_p f(z)|_k \begin{pmatrix} 1 & 0 \\ tpc & 1 \end{pmatrix} = \lambda_p^{2-M} p^{-1} \sum_{\substack{n=0 \\ p^{M-2}|n}}^{\infty} \left(1 - \exp\left(2\pi i \frac{1-pn_1}{h} \frac{n}{p^{M-1}}\right)\right) a_n q_h^{\frac{n}{p^{M-1}}}.$$

Since $\frac{1-pn_1}{h}$ is an integer we obtain (23).

The identities (24) and (25) can be deduced similarly from equation (26). □

Proposition 10. *Let* $\gcd(c,p) = 1$ *and* $p^M \| N$ *with* $M \geq 2$. *Assume that* $f(z)$ *is an eigenform of* T_p *with eigenvalue* $\lambda_p = 0$. *Furthermore, assume any one of the following cases;*

(i) p *is odd,*
(ii) $p = 2$ *and* $\alpha \geq 4$,
(iii) $(p, \chi_p) = (2, \psi_0)$,
(iv) $(p, \chi_p) = (2, \psi_2)$ *and* $\alpha \geq 3$.

Then p *does not divide* $\nu_{p^\alpha c} f$ *whenever* α *is an integer such that* $\frac{M+1}{2} \leq \alpha \leq M$ *and* $\mu_{p^\alpha c} = 0$.

Proof. It follows from Proposition 6. □

All previous results on eigenforms of Hecke operators are about $\nu_c f$ at regular cusps of $\Gamma_0(N)$. In order to have information about these eigenforms at the others cusps as well, we first determine for which levels N, characters χ and divisors c of N we get irregular cusps $\frac{1}{c}$.

Let $N = \prod_i p_i^{e_i}$ be the prime factorization of N. Then $\exp(2\pi i \mu_c) = \chi(1+ch_c) = \prod_i \chi_{p_i}(1-ch_c)$ where χ_{p_i} denotes the p_i-part of χ. From equation (4) and remark previous to Proposition 6 is easy to conclude that $\frac{1}{c}$ is an irregular cusp of $\Gamma_0(N)$ with respect to χ if, and only if, one of the following situations holds:

(27)

2-part of N	2-part of c	χ_2
2^2	2	Ψ_2
2^3	2	$\Psi_1,\ \Psi_3$
2^3	2^2	$\Psi_1,\ \Psi_3$
2^4	2^2	$\Psi_1,\ \Psi_3$

Consider $N' = 2^\sigma N$ in each one of the cases listed above, with $\sigma = 2$ for the case in the third row and $\sigma = 1$ for the rest. Denote by χ' the Dirichlet character modulo N' induced from χ. Then, the irregular cusp $\frac{1}{c}$ of $\Gamma_0(N)$ is a regular cusp of $\Gamma_0(N')$. Let $f'(z)$ be the image of $f(z)$ under the canonical injection of $M_k(N,\chi)$ into $M_k(N',\chi')$. If $\frac{1}{c}$ is a irregular cusp of $\Gamma_0(N)$, the Fourier series of $f(z)$ at $\frac{1}{c}$ is of the form

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n q_{h_c}^{n+\frac{1}{2}}.$$

Consequently, the Fourier series of $f'(z)$ at $\frac{1}{c}$ is

$$f'(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n q_{h'_c}^{2n+1}$$

where h'_c denotes the width of $\frac{1}{c}$ as a cusp of $\Gamma_0(N')$.

If p is any prime, let T'_p be the p-th Hecke operator defined on $M_k(N',\chi')$. The restriction of T'_p to the subspace $M_k(N,\chi)$ coincide with p-th Hecke operator T_p acting on $M_k(N,\chi)$ because $N' = 2^\sigma N$ and 2 is a divisor of N. Therefore, if $f(z) \in M_k(N,\chi)$ is an eigenform of T_p with eigenvalue λ_p then $T'_p f'(z) = \lambda_p f'(z)$.

Proposition 11. *Let $f(z)$ be an eigenform for all Hecke operators.*

(i) *If $\mu_c \neq 0$ then any prime divisor of $2\nu_c f + 1$ is a divisor of $\frac{N}{c}$.*
Furthermore, assume that $p^M \| N$ and $\gcd(c,p) = 1$.

(ii) *If $\mu_c \neq 0$ and $M = 1$ then $\gcd(p, 2\nu_c f + 1) = 1$, $\mu_{pc} \neq 0$ and $\nu_c f = \nu_{pc} f$.*

(iii) *If $\mu_{p^M c} \neq 0$ and $\lambda_p \neq 0$ then $p \neq 2$. Moreover, $\mu_{p^\alpha c} \neq 0$ implies $p^{M-\min\{2\alpha,M\}}$ divides $2\nu_{p^\alpha c} f + 1$, and $\mu_{p^\alpha c} = 0$ implies $p^{M-\min\{2\alpha,M\}}$ divides $\nu_{p^\alpha c} f$, for all $\alpha = 1, 2, \ldots, M$.*

(iv) *If $M \geq 2$, $\lambda_p = 0$ and $\mu_{p^\alpha c} \neq 0$, then p does not divide $2\nu_{p^\alpha c} f + 1$ for any α with $\frac{M+1}{2} \leq \alpha \leq M$.*

Proof. Consider the modular form $f'(z) \in M_k(N',\chi')$ defined by $f(z)$ as above. Then use Propositions 7, 8, 9 and 10 in order to get (i), (ii), (iii) and (iv) respectively. ☐

4. The Fricke involution

In the previous section we deduced some properties of $\nu_c f$ when $f(z)$ is an eigenform of the Hecke algebra at several cusps $\frac{1}{c}$ of $\Gamma_0(N)$. But clearly the argument used in Proposition 6 does not give similar information for cusps of the form $\frac{1}{p^\alpha c}$ when $\frac{N}{p^{2\alpha}}$ is still divisible by p. In order to deal with this problem we restrict ourselves to the study of modular forms satisfying an additional property involving the Fricke involution. Namely, from now on we only consider modular forms $f(z)$ in $M_k(N,\chi)$ such that:

(A) The smallest power of q with a non-zero coefficient in the Fourier series of $f(z)$ at the cusp $\frac{a}{c}$ (where $\gcd(a,c) = 1$ and c is a divisor of N) is independent of a.

(B) $f(z)$ is an eigenform for all Hecke operators.

(C) $\tilde{f}(z) = f(z)|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is an eigenform for all Hecke operators.

We denote by $\tilde{\lambda}_p$ the eigenvalue of $\tilde{f}(z)$ under the Hecke operator T_p.

Lemma 12. *If $f(z)$ satisfies* (A), (B) *and* (C), *then so it does $\tilde{f}(z)$. Moreover*

$$\tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ \frac{N}{c} & 1 \end{pmatrix} = f(z)|_k \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix} \begin{pmatrix} \frac{N}{c} & 1 \\ 0 & c \end{pmatrix} \tag{28}$$

and $\mu_{\frac{N}{c}} = \mu_c$, $\nu_{\frac{N}{c}} \tilde{f} = \nu_c f$, for any divisor c of N.

Proof. Clearly $\tilde{f}(z)$ satisfies (B) and (C).

Since χ is a real character $\tilde{f}(z)$ is in $M_k(N, \chi)$. If c is any factor of N and a is any integer such that $\gcd(\frac{N}{c}, a) = 1$ then $ad - b\frac{N}{c} = 1$ for some b and d in $\mathbb{Z}$. Therefore

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} a & b \\ \frac{N}{c} & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ ac & -1 \end{pmatrix} \begin{pmatrix} \frac{N}{c} & d \\ 0 & c \end{pmatrix}$$

and so

$$\tilde{f}(z)|_k \begin{pmatrix} a & b \\ \frac{N}{c} & d \end{pmatrix} = f(z)|_k \begin{pmatrix} -1 & 0 \\ ac & -1 \end{pmatrix} \begin{pmatrix} \frac{N}{c} & d \\ 0 & c \end{pmatrix}. \tag{29}$$

If $a = d = 1$ and $b = 0$ in the equation above, we get (28).

As $h_{\frac{a}{N/c}} = \frac{c}{N/c} h_{\frac{-1}{ac}}$, equation (29) yields $\nu_{\frac{a}{N/c}} \tilde{f} = \nu_{\frac{-1}{ac}} f$ and $\mu_{\frac{a}{N/c}} = \mu_{\frac{-1}{ac}}$. Now we use that both $\frac{-1}{ac}$ and $\frac{-a}{c}$ represent the same cusp of $\Gamma_0(N)$ and conclude that $\tilde{f}(z)$ satisfies (A). □

Proposition 13. (i) *If $\mu_c = 0$ then either $\nu_c f \in \{0, 1\}$ or any prime divisor of $\nu_c f$ is a divisor of* $\gcd(c, \frac{N}{c})$.

(ii) *If $\mu_c \neq 0$ then any prime divisor of $2\nu_c f + 1$ divides* $\gcd(c, \frac{N}{c})$.

Proof. It follows from the previous lemma, Proposition 7 and Proposition 11 (i). □

Lemma 14. *Let $p^M \| N$ with $M \geq 3$. Assume that (p^M, χ_p) is none of the following: $(2^3, \psi_1), (2^3, \psi_2), (2^3, \psi_3), (2^4, \psi_1)$ or $(2^4, \psi_3)$.*

Then $\lambda_p = 0$ if, and only if, $\tilde{\lambda}_p = 0$.

Proof. By Lemma 12 it suffices to show that $\tilde{\lambda}_p \neq 0$ implies $\lambda_p \neq 0$.

First assume that p is odd or $(p, \chi_p) = (2, \psi_0)$. If $c_0 = \frac{N}{p^M}$ then $\mu_{p^M c_0} = 0$ and $\nu_{p^M c_0} \tilde{f} \in \{0, 1\}$. If $\nu_{p^M c_0} \tilde{f} = 0$ then $\tilde{\lambda}_p \neq 0$ implies $\mu_{pc_0} = 0$ and $\nu_{pc_0} \tilde{f} = 0$, by Proposition 9. Consequently, $\mu_{p^{M-1}} = 0$ and $\nu_{p^{M-1}} f = 0$ by Lemma 12. Since $M \geq 3$, Proposition 10 yields $\lambda_p \neq 0$.

If $\nu_{p^M c_0} \tilde{f} = 1$ we consider the Fourier series $\tilde{f}(z) = \sum_{n=1}^{\infty} a_n q^n$. Since $\tilde{f}(z)$ is an eigenform of T_p we have $a_{p^{M-1}} = \tilde{\lambda}_p^{M-1} a_1 \neq 0$. If we apply Proposition 9 to $\tilde{f}(z)$ then equation (22) with $tp^\alpha c = pc_0$ shows that $\mu_{pc_0} = 0$ and $\nu_{pc_0} \tilde{f} = p^{M-2}$. Hence $\mu_{p^{M-1}} = 0$ and p divides $\nu_{p^{M-1}} f$. Then, as above, $\lambda_p \neq 0$.

In those cases where $p = 2$ and $\chi_2 \neq \psi_0$ we must have $M \geq 5$ or $(p^M, \chi_2) = (2^4, \psi_2)$. As $\tilde{\lambda}_p \neq 0$, we get from equations (23) and (25) that $\mu_{pc_0} = 0$ and p is a factor of $\nu_{pc_0} \tilde{f}$. Thus $\mu_{p^{M-1}} = 0$ and p divides $\nu_{p^{M-1}} f$. Then $\lambda_p \neq 0$ follows from Proposition 10. □

Lemma 15. *Let $p^2 \| N$,* $\gcd(p, c) = 1$ *and $\mu_{p^2 c} = 0$. If $p = 2$ assume also $\chi_2 = \psi_0$. Then $\lambda_p = 0$ if, and only if, $\tilde{\lambda}_p = 0$.*

Moreover, $\lambda_p \neq 0$ implies $\nu_c f = 0$ and $\nu_{p^2 c} f = 0$.

Proof. It is enough to show that $\lambda_p \neq 0$ implies $\tilde{\lambda}_p \neq 0$, $\nu_c f = 0$ and $\nu_{p^2c} f = 0$. Let

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ p^2c & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n q_h^n$$

where $h = h_{p^2c}$. If $\lambda_p \neq 0$ then

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tpc & 1 \end{pmatrix} = \lambda_p^{-1} \sum_{\substack{n=0 \\ p|n}}^{\infty} a_n q_h^{\frac{n}{p}} \tag{30}$$

for any t with $\gcd(t, N) = 1$ and $t \equiv 1 \pmod{\frac{N}{p^2c}}$, by Proposition 9.

Let c^* be in $\{0, 1, \ldots, p-1\}$ such that $cc^* \equiv 1 \pmod p$. For each $l \in \{0, 1, \ldots, p-1\} - \{-c^*\}$ take the unique solution $l' \in \{0, 1, \ldots, p-1\} - \{c^*\}$ of $(1+lc)l' + x_{l'}p = l$ for some integer $x_{l'}$. Then

$$\begin{aligned} &\lambda_p f(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \\ &= p^{\frac{k}{2}-1} \left(\sum_{\substack{l=0 \\ l \neq -c^*}}^{p-1} f(z)|_k \begin{pmatrix} 1+lc & x_{l'} \\ pc & 1-l'c \end{pmatrix} \begin{pmatrix} 1 & l' \\ 0 & p \end{pmatrix} \right. \\ &\qquad \left. + f(z)|_k \begin{pmatrix} x' & -c^* \\ c & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) \end{aligned}$$

where $x' \in \mathbb{Z}$ is chosen in such a way that $1 - cc^* = x'p$. For each l' consider integers $t_{l'}$ and $n_{l'}$ satisfying $l' - pn_{l'} \equiv 0 \pmod{\frac{N}{p^2}}$, $t_{l'}(1 - cl') \equiv 1 \pmod p$ and $t_{l'} \equiv 1 \pmod{\frac{N}{p^2}}$. Then $\gcd(t_{l'}, N) = 1$ and

$$\begin{pmatrix} 1+lc & x_{l'} \\ pc & 1-l'c \end{pmatrix} \begin{pmatrix} 1 & n_{l'} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_{l'}pc & 1 \end{pmatrix}^{-1} \in \Gamma_0(N).$$

As h divides $l' - pn_{l'}$, we get from (30) and the last equation the following identity

(31)

$$\begin{aligned} &\lambda_p f(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} - p^{\frac{k}{2}-1} f(z)|_k \begin{pmatrix} x' & -c^* \\ c & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad = p^{-1}\lambda_p^{-1} \sum_{\substack{n=0 \\ p|n}}^{\infty} a_n \exp\left(2\pi i \frac{n}{p} \frac{c^* h^*}{p}\right) \left(\sum_{l''=1}^{p-1} \chi_p(l'') \exp\left(-2\pi i \frac{n}{p} c^* h^* \frac{l''}{p}\right)\right) q_{hp}^{\frac{n}{p}} \end{aligned}$$

where h^* and l'' are integers such that $hh^* \equiv 1$ and $l'' = 1 - c(l' - pn_{l'}) \pmod p$.

Since $h = h_{p^2c} = \frac{1}{p^2} h_c$, the series above is in integral powers of $q_{h_c}^p$.

Notice that $\mu_{p^2c} = 0$ implies $\mu_c = 0$. As $\nu_{\frac{x'}{c}} f = \nu_c f$, the assumption $\nu_c f \neq 0$ yields $\gcd(p, \nu_c f) \neq 1$ from equation (31). But this impossible by Proposition 13, thus $\nu_c f = 0$.

Next we assume $\tilde{\lambda}_p = 0$. Then

(32)
$$0 = T_p \tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ \frac{N}{pc} & 1 \end{pmatrix} = p^{\frac{k}{2}-1} \sum_{l=0}^{p-1} \tilde{f}(z)|_k \begin{pmatrix} 1 + l\frac{N}{pc} & x_l \\ \frac{N}{c} & 1 - l\frac{N}{pc} \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix}$$

where $\left(1 + l\frac{N}{pc}\right) l + x_l p = l$. Let n_l be an integer such that $pn_l \equiv l \pmod{c}$. Then (32) becomes

$$0 = p^{\frac{k}{2}-1} \sum_{l=0}^{p-1} \tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ \frac{N}{c} & 1 \end{pmatrix} \begin{pmatrix} 1 & l - pn_l \\ 0 & p \end{pmatrix}.$$

If we use Lemma 12 we get

$$0 = p^{\frac{k}{2}-1} \sum_{l=0}^{p-1} f(z)|_k \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix} \begin{pmatrix} \frac{N}{c} & \frac{N}{c}(l - pn_l) + p \\ 0 & pc \end{pmatrix}.$$

Therefore the constant term of $f(z)|_k \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$ must be zero, i.e. $\nu_{\frac{-1}{c}} f > 0$, a contradiction. Thus $\tilde{\lambda}_p \neq 0$.

Finally, the argument after equation (31) applied to $\tilde{f}$ shows that $\nu_{\frac{N}{p^2 c}} \tilde{f} = 0$. Hence $\nu_{p^2 c} f = 0$ by Lemma 12. □

Proposition 16. *Let $p^M \| N$ with $M \geq 3$. Assume that (p^M, χ_p) is none of the following; $(2^3, \psi_1), (2^3, \psi_2), (2^3, \psi_3), (2^4, \psi_1)$ or $(2^4, \psi_3)$.*
Then $\lambda_p = 0$.

Proof. Suppose $\lambda_p \neq 0$. Then $\tilde{\lambda}_p \neq 0$ by Lemma 14. From (27) $\mu_{p^M} = 0$, thus

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ -p^M & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n q_h^n$$

where $h = h_{p^M}$. By Lemma 12 and Proposition 9 we may write the following identities;

(33)
$$\tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ \frac{N}{p} & 1 \end{pmatrix} = \chi(-1) N^{\frac{k}{2}} p^{-k} \lambda_p^{1-M} \sum_{\substack{n=0 \\ p^{M-1}|n}}^{\infty} a_n \exp\left(2\pi i \frac{n}{p^{M-1}} \frac{1}{ph}\right) q_h^{\frac{n}{p^{M-1}} \frac{N}{p^2}}$$

if p is odd or if $(p, \chi_p) = (2, \psi_0)$.

(34)
$$\tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ \frac{N}{p} & 1 \end{pmatrix}$$
$$= \chi(-1) N^{\frac{k}{2}} p^{-k} \lambda_p^{1-M} \sum_{m=0}^{\infty} a_{(2m+1)p^{M-2}} \exp\left(2\pi i \frac{m + \frac{1}{2}}{ph}\right) q_h^{(m+\frac{1}{2})\frac{N}{p^2}}$$

if $(p, \chi_p) = (2, \psi_2)$.

(35)
$$\tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ \frac{N}{p} & 1 \end{pmatrix} = \chi(-1)N^{\frac{k}{2}}p^{-k-1}\lambda_p^{1-M}\sum_{m=0}^{\infty} a_{(2m+1)p^{M-3}}\xi_m \times \exp\left(2\pi i\frac{(m+\frac{1}{2})p^{M-3}}{ph_p}\right)q_{h_p}^{(m+\frac{1}{2})p^{M-3}\frac{N}{p^2}}$$

if $(p,\chi_p) = (2,\psi_1),\ (2,\psi_3)$.

Observe that the right hand side of (33) (resp. (34), (35)) is a power series in $q^{p^{M-2}}$ (resp. $q^{p^{M-3}},\ q^{p^{M-4}}$).

By Proposition 9 the coefficient of q in the Fourier series of $\tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ \frac{N}{p} & 1 \end{pmatrix}$ is $\tilde{\lambda}_p^{-1}b_p$, where $\tilde{f}(z) = \sum_{n=0}^{\infty} b_n q^n$. As $\tilde{f}(z)$ is an eigenform of T_p, we know $\tilde{\lambda}_p^{-1}b_p = b_1 \neq 0$. Thus $M-2=0$ (resp. $M-3=0,\ M-4=0$), a contradiction. □

Proposition 17. *Let* $\gcd(c,p)=1$ *and* $p^M\|N$ *with* $M \geq 3$. *Assume that* (p^M,χ_p) *is none of the following:* $(2^3,\psi_1),(2^3,\psi_2),(2^3,\psi_3),(2^4,\psi_1),(2^4,\psi_2),(2^4,\psi_3)$, $(2^5,\psi_1)$ *or* $(2^5,\psi_3)$. *Then, for every* α *in* $\{0,1,\dots,M\}-\{\frac{M}{2}\}$,

(i) $\mu_{p^\alpha c}=0$ *implies* p *is not a factor of* $\nu_{p^\alpha c}f$,
(ii) $\mu_{p^\alpha c}\neq 0$ *implies* p *is not a factor of* $2\nu_{p^\alpha c}f+1$.

Proof. By Lemma 14 and the previous proposition, $\lambda_p = \tilde{\lambda}_p = 0$. Then (i) and (ii) above follow from Propositions 10, 11 (iv) and the identity in Lemma 12. □

Remarks 1. (i) If $(p^M,\chi_2)=(2^5,\psi_1),\ (2^5,\psi_3)$ then $\mu_{p^\alpha c}=0$ for all $\alpha=0,1,\dots,5$ (see table (27)). Moreover $\gcd(p,\nu_{p^\alpha c}f)=1$ for $\alpha=0,1,4,5$ by Propositions 10, 16 and Lemma 12.
(ii) If $(p^M,\chi_p)=(2^4,\psi_2)$ then $\mu_{p^\alpha c}=0$ for all $\alpha=0,1,\dots,4$ and $\gcd(p,\nu_{p^\alpha c}f)=1$ for $\alpha=0,1,3,4$ by the same argument.

Lemma 18. *Let* p *be an odd prime with* $p^2\|N$. *Then* $\lambda_p=0$.

Proof. Let c be a factor of N such that $\gcd(p,c)=1$. We argue as in the proof of Lemma 15 and obtain (31).

Now, for every $l'\neq c^*$ we choose $n_{l'}\in\mathbb{Z}$ such that $pn_{l'}+l'\equiv 0 \pmod{\frac{N}{p^2}}$. Then there exist an integer $t_{l'}$ such that $t_{l'}(1-l'c-pcn_{l'})\equiv 1 \pmod N$. In particular $\gcd(t_{l'},N)=1$. Moreover, $f(z)|_k \begin{pmatrix} 1 & 0 \\ t_{l'}pc & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n^{(l')} q_{h_{pc}}^{n+\mu_{pc}}$ implies that the right hand side of the equation (31) is equal to

(36)
$$p^{-1}\sum_{n=0}^{\infty}\left(\sum_{\substack{l'=0\\ l'\neq c^*}}^{p-1}\chi_p(1-l'c)a_n^{(l')}\exp\left(2\pi i\frac{n+\mu_{pc}}{h_c}(l'+pn_{l'})\right)\right)q_{h_c}^{(n+\mu_{pc})p}.$$

As $\mu_p=0$, the expression above for $c=1$ is a series in integral powers of $q_{h_1}^p$. If $\tilde{f}(z)=\sum_{n=0}^{\infty} b_n q^n$, the identity in Lemma 12 yields

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = N^{-\frac{k}{2}}\sum_{n=0}^{\infty} b_n \exp\left(2\pi i\frac{n}{N}\right)q_N^n \tag{37}$$

and this shows that the coefficient of q_{h_1} in (37) is non-zero (since $\tilde{f}(z)$ is an eigenform of the Hecke algebra).

In particular $\lambda_p \neq 0$ implies that the left hand side of (31) with $c = 1$ is a power series where q_{h_1} has a non-zero coefficient, a contradiction. □

Proposition 19. *Let p be as in the previous lemma and* $\gcd(c, p) = 1$. *Then* $\mu_{pc} = \mu_c$ *and* $\nu_{pc}f \leq \nu_c f$.

Proof. By Lemma 18 $\lambda_p = 0$. Thus, the Fourier expansion of $f(z)|_k \begin{pmatrix} x' & -c^* \\ c & p \end{pmatrix}$ is a series in integral powers of q_{h_c} if, and only if, (36) is a series in integral powers of q_{h_c}. Hence $\mu_{pc} = \mu_c$.

Furthermore, the smallest power of q_{h_c} with a non-zero coefficient in the left hand side of (31) has exponent $(\nu_c f + \mu_c)p$. Therefore (36) and $\nu_{\frac{1}{t_{l'}pc}} f = \nu_{pc} f$ imply $\nu_{pc} f + \mu_{pc} \leq \nu_c f + \mu_c$. □

We finish this section with a technical result that will allow us to get information about $\nu_c f$ at those cusps $\frac{1}{c}$ not considered in Proposition 17, i.e. whenever $p^{\frac{M}{2}} \| c$ for some prime p with $p^M \| N$. But before we need to observe the following:

Remarks 2. (i) Lemma 3, and Propositions 4, 5 and 6 also hold if c is replaced by tc, where t is any integer with $\gcd(t, N) = 1$.

(ii) For t as above and $t' \in \{0, 1, \ldots, p-1\}$ such that $t' \equiv t \pmod N$

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tc & 1 \end{pmatrix} = f(z)|_k \begin{pmatrix} 1 & 0 \\ t'c & 1 \end{pmatrix}.$$

So, from now on we take t to be in $\{0, 1, \ldots, p-1\}$.

Proposition 20. *Assume there is at most one prime p such that $p^M \| N$, $M > 0$ even, and $p^{\frac{M}{2}} \| c$.*

Assume also that $(2^{M_1}, \chi_2)$ is none of the following: $(2^2, \psi_2)$, $(2^3, \psi_1)$, $(2^3, \psi_2)$, $(2^3, \psi_3)$, $(2^4, \psi_1)$, $(2^4, \psi_2)$, $(2^4, \psi_3)$, $(2^5, \psi_1)$ or $(2^5, \psi_3)$, where 2^{M_1} denotes the 2-part of N. Then $\mu_c = 0$.

Furthermore, if the Fourier series of $f(z)$ at $\frac{1}{c}$ is given by (3) and $\nu_c f \neq 0$ then $c_n = 0$ whenever $\nu_c f$ is not a factor of n.

Proof. Since $(2^{M_1}, \chi_2) = (2^{M_1}, \psi_0)$ for $M_1 = 2, 3, 4$, we have $\mu_c = 0$. If $\nu_c f = 1$ there is nothing else to prove, hence we assume $\nu_c f > 1$ for the rest of this proof. By Propositions 13 and 17 we know that $\nu_c f = p^\nu$ for some positive integer ν.

Suppose there exist some $c_n \neq 0$ such that n is not divisible by $\nu_c f$. If we write

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ tc & 1 \end{pmatrix} = \sum_{n=0}^{\infty} c_n^{(t)} q_{h_c}^n$$

for every $0 \leq t \leq N$, $\gcd(t, N) = 1$, our assumption implies the existence of

$$n_0 = \min\{n \in \mathbb{Z} : c_n^{(t)} \neq 0,\ n \text{ is not divisible by } p^\nu\}$$

where $1 \leq t \leq N$ with $\gcd(t, N) = 1$. Fix t_0 such that $c_{n_0}^{(t_0)} \neq 0$. Since $\nu_{\frac{1}{t_0 c}} f = p^\nu$, there is a prime $r \neq p$ such that r is a factor of n_0.

In the following we divide this proof into four cases and show that each of them yields a contradiction.

Case 1. $\gcd(r, N) = 1$.

Let $r^* \in \mathbb{Z}$ such that $r^* r \equiv 1 \pmod{N}$. Let $t_1 = r^* t_0$. Then

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ rt_1 c & 1 \end{pmatrix} = \sum_{n=0}^{\infty} c_n^{(t_0)} q_{h_c}^n$$

By equation (16) the coefficient of $q_{h_c}^{\frac{n_0}{r}}$ in the Fourier series of $\lambda_r f(z)|_k \begin{pmatrix} 1 & 0 \\ t_1 c & 1 \end{pmatrix}$ is

$$\lambda_r c_{\frac{n_0}{r}}^{(t_1)} = c_{n_0}^{(t_0)} + \chi(r) r^{k-1} c_{\frac{n_0}{r^2}}^{(r^* t_1)} = c_{n_0}^{(t_0)} \neq 0$$

This is a contradiction to the minimality of n_0.

Case 2. $r \| N$ and $\gcd(r, c) \neq 1$.

Let $r^* \in \mathbb{Z}$ such that $r^* r \equiv 1 \pmod{\frac{N}{c}}$ and $\gcd(r^*, c) = 1$. Let t_1 be as above. It is possible to show, as in the proof of Lemma 3 (ii), the following identity

$$T_r f(z)|_k \begin{pmatrix} 1 & 0 \\ t_1 c & 1 \end{pmatrix} = r^{\frac{k}{2}-1} \sum_{l=0}^{r-1} f(z)|_k \begin{pmatrix} 1 & 0 \\ rt_1 c & 1 \end{pmatrix} \begin{pmatrix} 1 & l - rn_l \\ 0 & r \end{pmatrix}$$

where $n_l \in \mathbb{Z}$ with $l - rn_l \equiv 0 \pmod{\frac{N}{c}}$. This equation yields

$$\lambda_r \sum_{n=0}^{\infty} c_n^{(t_1)} q_{h_c}^n = r^{-1} \sum_{n=0}^{\infty} c_n^{(t_0)} \left(\sum_{l=0}^{r-1} \exp\left(2\pi i \frac{n}{h_c} \frac{l - rn_l}{r} \right) \right) q_{h_c}^{\frac{n}{r}}.$$

In particular we get $\lambda_r c_{\frac{n_0}{r}}^{(t_1)} = c_{n_0}^{(t_0)} \neq 0$. But again, this is impossible by the minimality of n_0.

Case 3. $r \| N$ and $\gcd(r, c) = 1$.

Let $0 \leq l \leq r-1$ such that $t_0 c l + 1 = x' r$ for some integer x'. There is $t_1 \in \mathbb{Z}$ such that $t_0 \equiv \left(t_0 \frac{N}{r} + r\right) t_1 \pmod{\frac{N}{c}}$ and $\gcd(t_1, N) = 1$. Hence

$$\begin{aligned} f(z)|_k \begin{pmatrix} x' & l \\ t_0 c & r \end{pmatrix} &= \chi\left(t_0 \frac{N}{r} + r\right) f(z)|_k \begin{pmatrix} 1 & 0 \\ t_1 c & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{N}{rc} \\ 0 & 1 \end{pmatrix} \\ &= \chi\left(t_0 \frac{N}{r} + r\right) \sum_{n=0}^{\infty} c_n^{(t_1)} \exp\left(-2\pi i n \frac{\gcd(\frac{N}{c}, c)}{r} \right) q_{h_c}^n. \end{aligned}$$

If we put $f(z)|_k \begin{pmatrix} 1 & 0 \\ t_0 rc & 1 \end{pmatrix} = \sum_{n=0}^{\infty} a_n q_{h_{rc}}^n$ (recall $\mu_{rc} = \mu_c = 0$ by an argument in the proof of Proposition 5), equation (17) implies

(38)

$$\lambda_r c_{n_0}^{(t_0)} = r^{-1} a_{n_0} \sum_{l''=1}^{r-1} \chi_r(l'') + r^{k-1} \chi\left(t_0 \frac{N}{r} + r\right) c_{\frac{n_0}{r}}^{(t_1)} \exp\left(-2\pi i n_0 \frac{\gcd(\frac{N}{c}, c)}{r^2} \right).$$

By minimality of n_0 we must have

$$\lambda_r c_{n_0}^{(t_0)} = r^{-1} a_{n_0} \sum_{l''=1}^{r-1} \chi_r(l'')$$

Since $\lambda_r \neq 0$ (see Proposition 8) we conclude that χ_r is the trivial character. Then $\nu_{\frac{1}{t_0 rc}} f = \nu_{rc} f \neq 0$ by equation (17). Moreover $\nu_{\frac{1}{t_1 c}} f = \nu_{\frac{1}{t_0 c}} f = \nu_c f > 0$, hence $\nu_c f = \nu_{rc} f$ by Proposition 8. This give us a factor of N, namely rc, satisfying the same conditions than c in the statement of this proposition, with $a_{n_0} \neq 0$ and $\nu_{rc} f = p^\nu$. Furthermore, the identity (38) above and the minimality of n_0 (defined in terms of c) imply

$$n_0 = \min\{n \in \mathbb{Z}; b_n^{(t)} \neq 0, n \not\equiv 0 \pmod{p^\nu}\}$$

where the $b_n^{(t)}$'s are defined by $f(z)|_k \begin{pmatrix} 1 & 0 \\ trc & 1 \end{pmatrix} = \sum_{n=0}^\infty b_n^{(t)} q_{h_{rc}}^n$.

Now we just apply the arguments from the previous case to rc and obtain the desired contradiction.

Case 4. r^2 divides N.

Let $r^{M_r} \| N$. Since $r \neq p$, then $r^\alpha \| c$ for some $\alpha \in \{0, 1, \dots, M_r\} - \{\frac{M_r}{2}\}$.

Suppose $M_r \geq 3$. Then $\lambda_r = 0$ by Proposition 16. Consequently

$$0 = \sum_{\substack{n=0 \\ r|n}}^\infty c_n^{(t_0)} q_{h_c}^{\frac{n}{r}}$$

if $\frac{M_r+1}{2} \leq \alpha \leq M_r$ (see equation (19)). Thus $c_{n_0}^{(t_0)} \neq 0$ implies $\alpha \leq \frac{M_r-1}{2}$.

From Lemma 14 we also have $\tilde{\lambda}_r = 0$, thus the previous argument shows $\tilde{c}_{n_0}^{(t_0^*)} = 0$ where

$$\tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ t_0^* \frac{N}{c} & 1 \end{pmatrix} = \sum_{n=0}^\infty \tilde{c}_n^{(t_0^*)} q_{h_{\frac{N}{c}}}^n$$

for any $0 \leq \alpha \leq \frac{M_r-1}{2}$ and any integer t_0^* satisfying $\gcd(t_0^*, N) = 1$.

On the other hand, $\gcd(t_0, N) = 1$ implies $t_0 y + \frac{N}{c} x = 1$ for some $x, y \in \mathbb{Z}$. Then

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} t_0 & x \\ -\frac{N}{c} & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t_0 c & 1 \end{pmatrix} \begin{pmatrix} \frac{N}{c} & -y \\ 0 & c \end{pmatrix}.$$

Since there is $m \in \mathbb{Z}$ such that $\gcd(-\frac{N}{c}m + y, c) = 1$, we can take $t_0^* \in \mathbb{Z}$ satisfying $t_0^*(-\frac{N}{c}m + y) \equiv 1 \pmod{c}$ and $\gcd(t_0^*, N) = 1$. Then, we use the previous identity and get

$$f(z)|_k \begin{pmatrix} 1 & 0 \\ t_0 c & 1 \end{pmatrix} \begin{pmatrix} \frac{N}{c} & -y \\ 0 & c \end{pmatrix} = \chi(y - \frac{N}{c}m) \tilde{f}(z)|_k \begin{pmatrix} 1 & 0 \\ t_0^* \frac{N}{c} & 1 \end{pmatrix} \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}.$$

As $h_{\frac{N}{c}} = \frac{c^2}{N} h_c$, one obtains

$$\frac{N^{\frac{k}{2}}}{c^k} \sum_{n=0}^\infty c_n^{(t_0)} \exp\left(-2\pi i n \frac{y}{h_c c}\right) q_{h_c}^{n\frac{N}{c^2}} = \chi(y - \frac{N}{c}m) \sum_{n=0}^\infty \tilde{c}_n^{(t_0^*)} \exp\left(-2\pi i n \frac{m\frac{N}{c}}{h_c c}\right) q_{h_c c}^{n\frac{N}{c}}.$$

Thus $\tilde{c}_{n_0}^{(t_0^*)} = 0$ if, and only if, $c_{n_0}^{(t_0)} = 0$.

Consequently $\tilde{c}_{n_0}^{(t_0^*)} \neq 0$, which implies $\tilde{\lambda}_r \neq 0$ and therefore $M_r = 2$.

If $\lambda_r = 0$ then $\tilde{\lambda}_r = 0$ by Lemma 15. Hence the previous argument yields a contradiction. Finally, if $\lambda_r \neq 0$ then $\nu_c f = 0$ by Lemma 15 again. But this is impossible by assumption. □

Remark 1. One can also use the proof above in order to show the following: Let $(2^M, \chi_2) = (2^5, \psi_1)$, $(2^5, \psi_3)$ (resp. $(2^4, \psi_2)$) and $c = 2^3 c'$ (resp. $c = 2^2 c'$) for some odd integer c'. Assume $\nu_c f = 2^\nu$ and $r \| N$ for any prime factor r of c'. Then, if the Fourier series of $f(z)$ at $\frac{1}{c}$ is given by (3), the equation $c_n = 0$ holds whenever $\nu_c f$ is not a factor of n.

Before we close this section we observe that it is also possible to get information about $\nu_c f$ in those exceptional cases not covered by Proposition 17. Nevertheless, we prefer to postpone the analysis of these cases until we are able to show some explicit upper bounds for $\nu_c f$.

5. On η-quotients which are eigenforms of Hecke operators

From now on we only work with modular forms satisfying conditions (B) and (C) of section 3 and which are of the form $\eta_g(z) \in M_{k_g}(N_g, \chi_g)$ for some $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$. Since $\eta_g(z)$ also satisfies (A), we can make use of any result from the previous section.

Proposition 21. *Let $g = \prod_{j=1}^{s} t_j^{r_j}$ such that $\eta_g(z) \in M_{k_g}(N_g, \chi_g)$. Let $p^M \| N_g$. If $M \geq 2$ then $p = 2$ or 3. Furthermore, $M \leq 8$ if $p = 2$ and $M \leq 3$ if $p = 3$.*

Proof. Let $c_0 = \frac{N_g}{p^M}$. If p is odd then $\mu_{p^\alpha c_0} \eta_g = 0$ for all $0 \leq \alpha \leq M$ (see (27)). Moreover $\nu_{p^\alpha c_0} \eta_g = 1$ for all $0 \leq \alpha \leq M$, $\alpha \neq \frac{M}{2}$. This follows from Propositions 13 and 17 when $M \geq 3$, and from Lemmas 12, 15, 18 and Propositions 10 and 13 if $M = 2$. The above also holds for $p = 2$ and $M \geq 6$ by the same argument.

Thus, from equation (6) we get

$$h_{p^\alpha c_0} \sum_{d \in D} \Big(\sum_{l=0}^{\alpha} p^l d r_{p^l d} + \sum_{l=\alpha+1}^{M} p^{2\alpha - l} d r_{p^l d} \Big) = 24 \tag{39}$$

for all α in $\{0, 1, \dots, M\} - \{\frac{M}{2}\}$, where $D = \{d \in \mathbb{Z}; d > 0,\ d \text{ divides } N_g,\ \gcd(p, d) = 1\}$.

One deduces from this system of equations that $\sum_{d \in D} d r_{p^l d} = 0$ for all $l \in \{0, 1, \dots, M\} - \{\frac{M-1}{2}, \frac{M+1}{2}\}$ if M is odd, and $\sum_{d \in D} d r_{p^l d} = 0$ for all $l \in \{0, 1, \dots, M\} - \{\frac{M}{2} - 1, \frac{M}{2}, \frac{M}{2} + 1\}$ if M is even. Consequently, the system (39) reduces to

$$\sum_{d \in D} \big(p^{\frac{M-1}{2}} d r_{p^{\frac{M-1}{2}} d} + p^{\frac{M+1}{2}} d r_{p^{\frac{M+1}{2}} d} \big) = 24, \tag{40}$$

$$\sum_{d \in D} \big(p^{\frac{M+1}{2}} d r_{p^{\frac{M-1}{2}} d} + p^{\frac{M-1}{2}} d r_{p^{\frac{M+1}{2}} d} \big) = 24 \tag{41}$$

if M is odd, and

$$\sum_{d \in D} \big(p^{\frac{M}{2} - 1} d r_{p^{\frac{M}{2} - 1} d} + p^{\frac{M}{2}} d r_{p^{\frac{M}{2}} d} + p^{\frac{M}{2} + 1} d r_{p^{\frac{M}{2} + 1} d} \big) = 24, \tag{42}$$

$$\sum_{d \in D} \big(p^{\frac{M}{2} - 1} d r_{p^{\frac{M}{2} - 1} d} + p^{\frac{M}{2}} d r_{p^{\frac{M}{2}} d} + p^{\frac{M}{2} - 1} d r_{p^{\frac{M}{2} + 1} d} \big) = \begin{cases} 0 \\ 24 p^u, \end{cases} \tag{43}$$

$$\sum_{d \in D} \big(p^{\frac{M}{2} + 1} d r_{p^{\frac{M}{2} - 1} d} + p^{\frac{M}{2}} d r_{p^{\frac{M}{2}} d} + p^{\frac{M}{2} - 1} d r_{p^{\frac{M}{2} + 1} d} \big) = 24 \tag{44}$$

if M is even, where u is some non-negative integer.

Assume $M \geq 3$. From the equations above we know that either $p^{\frac{M}{2} - 1}$ or $p^{\frac{M-1}{2}}$ is a factor of 24, hence $p = 2$ or 3. Moreover, $M \leq 8$ if $p = 2$ and $M \leq 4$ if $p = 3$. In order to rule out the case $p = 3$ and $M = 4$ we observe that for these values

$$\sum_{d \in D} d r_{3^3 d} = \begin{cases} 1 \\ 1 - 3^u \end{cases}$$

by subtracting (44) from (42) and then subtracting (43) from (42). Then, equation (43) is equal to

$$6+9\sum_{d\in D} dr_{3^2d}=0 \text{ or } 6(1-3^u)+9\sum_{d\in D} dr_{3^2d}=24\cdot 3^u$$

both of which yield a contradiction.

Finally, assume $M=2$. Again, if we subtract (44) from (42) and use this equation when we subtract (43) from (42) we obtain

$$(p^2-1)\sum_{d\in D} dr_{p^2d}=\begin{cases} 24 \\ 24(1-p^u). \end{cases}$$

If $u=0$ then there is nothing to prove, so we assume $u\geq 1$. Consequently $p=2,\ 3,\ 5$ or equation (43) becomes $24p^u=2\sum_d dr_d+p\sum_d dr_{pd}$. In the former case $p=5$ implies $\sum_d dr_{p^2d}=1$ or $1-p^u$, which yields a contradiction if we consider the identity (43). In the latter case we conclude that p is a factor of $\sum_d dr_d$, and therefore it is also a factor of 24 by (42). ☐

Corollary 22. *Let $g=\prod_{j=1}^s t_j^{r_j}$ such that $\eta_g(z)\in M_{k_g}(N_g,\chi_g)$. If $\mu_c\neq 0$ then $\nu_c\eta_g=0$.*

Proof. By Proposition 13 any prime divisor of $2\nu_c\eta_g+1$ is a divisor of $\gcd(c,\frac{N_g}{c})$. Then, by the last proposition, $2\nu_c\eta_g+1=3^{m_c}$ for some non-negative integer m_c. Moreover, $m_c\neq 0$ implies $3^2\|N_g$ and $3\|c$ (see Propositions 13, 17 and 21).

From Proposition 19 $\mu_{\frac{c}{3}}\eta_g=\mu_c\eta_g\neq 0$ and $\nu_c\eta_g\leq\nu_{\frac{c}{3}}\eta_g$. Hence, the previous argument shows that $2\nu_{\frac{c}{3}}\eta_g+1=1$. Thus $\nu_c\eta_g=0$. ☐

As we mentioned before, the Fourier series of $\eta_g(z)\in M_{k_g}(N_g,\chi_g)$ at the cusp $\frac{1}{c}$ is given by (5). The holomorphic function $G_{\frac{1}{c}}(z)$ in that equation can be written explicitly as the infinite product

$$G_{\frac{1}{c}}(z)=\prod_{j=1}^{s}\prod_{n=1}^{\infty}(1-\exp(2\pi in\frac{\gcd(t_j,c)}{t_j}v_j)q^{n\frac{\gcd(t_j,c)^2}{t_j}})^{r_j} \tag{45}$$

where each $v_j\in\mathbb{Z}$ satisfies $cv_j\equiv\gcd(t_j,c)\pmod{t_j}$ (see [7]).

Next we study the function $G_{\frac{1}{c}}(z)$ in order to find some upper bounds for $\nu_c\eta_g$ at those cusps $\frac{1}{c}$ of $\Gamma_0(N_g)$ where there is an even integer $M>0$ such that $2^M\|N_g$ and $2^{\frac{M}{2}}\|c$. For the following six results we assume these conditions on N_g and c, plus the hypothesis $\gcd(3,c)=1$. Moreover, we order the set $g=t_1^{r_1}t_2^{r_2}\dots t_s^{r_s}$ in such a way that $1\leq j_1\leq j_2\leq s$ implies $\frac{\gcd(t_{j_1},c)^2}{t_{j_1}}\leq\frac{\gcd(t_{j_2},c)^2}{t_{j_2}}$.

Lemmas 23 and 25 below are stated without proof since they follow from straightforward algebraic manipulations.

Lemma 23. *Let u and N be some positive integers such that $2^u\|N$. Let $a_1,\ a_2\in\mathbb{Z}$ not both zero, $\xi\in\mathbb{C}$ a primitive N-th root of unity, and $l\in\mathbb{Z}$ with $\gcd(l,\frac{N}{2^u})=1$.*

If ξ is a root of $a_1X^{2^ul}+a_2X\in\mathbb{Z}[X]$ then $u=1$ and $a_1=a_2$.

Proposition 24. *Let $c=2^{\frac{M}{2}}c'$ with c' a positive integer relatively prime to 6. If the coefficient of $q^{\frac{\gcd(t_1,c)^2}{t_1}}$ in the product $G_{\frac{1}{c}}(z)$ is zero, then $t_1=2^{\frac{M}{2}-1}3^{\beta_1}t_1'$ for some $t_1'\geq 1$, $\gcd(t_1',6)=1$ and $\beta_1\geq 0$. Moreover $t_2=2^{\frac{M}{2}+1}3^{\beta_1}t_1'$, $r_1=r_2$ and $\frac{\gcd(t_1,c)^2}{t_1}<\frac{\gcd(t_3,c)^2}{t_3}$.*

Proof. Assume that r_1 is non-zero. Let $t_1 = 2^{\alpha_1}3^{\beta_1}t_1'$ for some integers $\alpha_1,\ \beta_1$ and t_1' with $\gcd(t_1', 6) = 1$.

If t_2 is a divisor of N_g with $\frac{\gcd(t_1,c)^2}{t_1} = \frac{\gcd(t_2,c)^2}{t_2}$ then $t_2 = 2^{M-\alpha_1}3^{\beta_1}t_1'$. As the coefficient of $q^{\frac{\gcd(t_1,c)^2}{t_1}}$ in (45) is zero and $\frac{\gcd(t_1,c)^2}{t_1}$ is minimal by the ordering in g, we get $\alpha_1 \neq M - \alpha_1$, say $\alpha_1 < \frac{M}{2}$. One can choose integers $l,\ v$ such that

$$\frac{c}{\gcd(t_1,c)}lv \equiv 1 \pmod{\frac{t_1}{\gcd(t_1,c)}} \textit{ and } \frac{c}{\gcd(t_2,c)}v \equiv 1 \pmod{\frac{t_2}{\gcd(t_2,c)}}.$$

Then, the coefficient of $q^{\frac{\gcd(t_1,c)^2}{t_1}}$ in (45) is given by

$$-r_1 \exp(2\pi i\frac{\gcd(t_1',c')}{3^{\beta_1}t_1'}lv) - r_2 \exp(2\pi i\frac{\gcd(t_1',c')}{2^{\frac{M}{2}-\alpha_1}3^{\beta_1}t_1'}v).$$

Therefore $\alpha_1 = \frac{M}{2} - 1$ and $r_1 = r_2$ by Lemma 23. □

Lemma 25. *Let N be an odd positive integer and $\theta \in \mathbb{C}$ an N-th root of unity.*

(i) *If $\alpha \in \mathbb{Z}$, $\alpha > 0$ satisfies $2\alpha \equiv 1 \pmod N$, then $\theta^{2\alpha+1} + \theta^{4\alpha} + \theta^2 \neq 0$.*

(ii) *If $a_1,\ a_2 \in \mathbb{Z}$, not both zero and $a_1(\theta^{2\alpha+1} + \theta^{4\alpha} + \theta^2) + a_2\theta^2 = 0$ then $\theta = 1$ and $a_2 = -3a_1$.*

Lemma 26. *Let c be as in Proposition 24. Assume that the coefficients of $q^{\frac{\gcd(t_1,c)^2}{t_1}}$ and $q^{2\frac{\gcd(t_1,c)^2}{t_1}}$ in the product $G_{\frac{1}{c}}(z)$ are zero. Then $2\frac{\gcd(t_1,c)^2}{t_1} = \frac{\gcd(t_3,c)^2}{t_3}$.*

Furthermore, either $G_{\frac{1}{c}}(z) = 1$ or $s \geq 4,\ r_4 \neq 0$ and

$$G_{\frac{1}{c}}(z) = \prod_{j=4}^{s}\prod_{n=1}^{\infty}(1 - \exp(2\pi i n\frac{\gcd(t_j,c)}{t_j}v_j)q^{n\frac{\gcd(t_j,c)^2}{t_j}})^{r_j}. \tag{46}$$

Proof. By the previous proposition there are some non-negative integers β_1 and t_1' with $\gcd(t_1', 6) = 1$, such that $t_1 = 2^{\frac{M}{2}-1}3^{\beta_1}t_1'$, and $t_2 = 2^{\frac{M}{2}+1}3^{\beta_1}t_1'$. Moreover $r_1 = r_2$. Thus

$$\begin{aligned} G_{\frac{1}{c}}(z) = &\prod_{n=1}^{\infty}(1 - \xi^{2ln}q^{n\frac{\gcd(t_1,c)^2}{t_1}})^{r_1}(1 - \xi^n q^{n\frac{\gcd(t_1,c)^2}{t_1}})^{r_1} \\ &\times \prod_{j=3}^{s}\prod_{n=1}^{\infty}(1 - \exp(2\pi i n\frac{\gcd(t_j,c)}{t_j}v_j)q^{n\frac{\gcd(t_j,c)^2}{t_j}})^{r_j} \end{aligned} \tag{47}$$

where $\xi = \exp(2\pi i\frac{\gcd(t_1',c')}{2\cdot 3^{\beta_1}t_1'}v_2)$ and l is defined in Proposition 24.

Since the coefficient of $q^{2\frac{\gcd(t_1,c)^2}{t_1}}$ in this infinite product is zero, we must have $2\frac{\gcd(t_1,c)^2}{t_1} = \frac{\gcd(t_3,c)^2}{t_3}$ and $r_3 \neq 0$ by Lemma 25 (i). This implies $t_3 = 2^{\frac{M}{2}}3^{\beta_1}t_1'$ and therefore the coefficient of $q^{2\frac{\gcd(t_1,c)^2}{t_1}}$ in $G_{\frac{1}{c}}(z)$ is $r_1\xi^{2l+1} - r_1\xi^{4l} - r_1\xi^2 - r_3\xi^2$. From Lemma 25 (ii) we get $r_3 \neq 0$, $\xi = -1$ and $r_3 = -3r_1$. Thus $\beta_1 = 0$ and t_1' is a factor of c'. Finally, equation (46) can be obtained from (47) after some algebraic manipulations. □

Proposition 27. *Let c be as in Proposition 24. Then there is a power of q in the infinite product $G_{\frac{1}{c}}(z)$, say q^f for some $f \in \mathbb{Q}$, which has a non-zero coefficient and satisfies $0 \leq$ 2-part of $f \leq 2^{\frac{M}{2}}$.*

Proof. Obviously, we must have one of the following cases:

a) the coefficient of $q^{\frac{\gcd(t_1,c)^2}{t_1}}$ in $G_{\frac{1}{c}}(z)$ is not zero,

b) the coefficients of $q^{\frac{\gcd(t_1,c)^2}{t_1}}$ and $q^{2\frac{\gcd(t_1,c)^2}{t_1}}$ in $G_{\frac{1}{c}}(z)$ are zero and non-zero respectively,

c) the coefficients of $q^{\frac{\gcd(t_1,c)^2}{t_1}}$ and $q^{2\frac{\gcd(t_1,c)^2}{t_1}}$ are both zero.

If a) occurs we take $f = \frac{\gcd(t_1,c)^2}{t_1}$. If b) occurs let $f = 2\frac{\gcd(t_1,c)^2}{t_1}$. If c) occurs, either $G_{\frac{1}{c}}(z) = 1$ and we take $f = 0$, or $G_{\frac{1}{c}}(z)$ is given by (46), in which case we repeat the previous argument on this product. □

Corollary 28. *Let $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$ and c as in Proposition 24. Assume that $(2^M, \chi_2)$ is none of the following: $(2^2, \psi_2)$, $(2^4, \psi_1)$ or $(2^4, \psi_3)$. Then $\mu_c = 0$ and $\nu_c \eta_g \le 2^{\frac{M}{2}}$.*

Proof. From Propositions 20 and 21 we know that $\mu_c = 0$. Moreover $\nu_c \eta_g = 0$ or $\nu_c \eta_g = 2^u$ for some non-negative integer u. In the latter case $\nu_c \eta_g = 2^u$ must divide the exponent of every power of q_{h_c} in the series (5) with a non-zero coefficient. Hence 2^u must divide the exponent of every power of q_{h_c} in the product (45). Thus, $2^u \le 2^{\frac{M}{2}}$ by Proposition 27. □

Remark 2. Clearly, all four Lemmas 23, 24, 25 and 26 also hold if $2^M \| N_g$ with M odd and $2^{\frac{M+1}{2}} \| c$. We only have to enlarge the original formal product $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$ with factors of the form t^0 where t is any factor of $2N_g$ which does not divide N_g, and then take $G_{\frac{1}{c}}(z)$ as the infinite product defined by this bigger set.

Consequently, we can have a statement analogous to Proposition 27 for this case. Namely, if M is odd and $2^{\frac{M+1}{2}} \| c$, there is a rational power of q, say q^f, in the infinite product $G_{\frac{1}{c}}(z)$ such that it has a non-zero coefficient and satisfies $0 \le$ 2-part of $f \le 2^{\frac{M+1}{2}}$.

Now we put together several previous results in the following.

Proposition 29. *Let $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$ such that $\eta_g(z) \in M_{k_g}(N_g, \chi_g)$. Let $2^M \| N_g$ and assume that $(2^M, \chi_2)$ is none of the following: $(2^2, \psi_2), (2^3, \psi_1), (2^3, \psi_2), (2^3, \psi_3), (2^4, \psi_1), (2^4, \psi_2), (2^4, \psi_3), (2^5, \psi_1)$ or $(2^5, \psi_3)$.*

If c is any divisor of N_g then $0 \le \nu_c \eta_g \le 2^{\frac{M}{2}}$. Furthermore, either $\nu_c \eta_g \in \{0, 1\}$ or M is an even positive integer and $2^{\frac{M}{2}} \| c$.

Proof. Assume $\nu_c \eta_g \notin \{0, 1\}$. By Corollary 22 we have $\mu_c = 0$. Then, from Propositions 13, 17 and 21, M is some even positive integer and $2^{\frac{M}{2}} \| c$ or $3^2 \| N_g$ and $3 \| c$. In the latter case we consider $\frac{c}{3}$ and by Proposition 19 $\nu_{\frac{c}{3}} \eta_g \notin \{0, 1\}$. Hence M must be positive, even and $2^{\frac{M}{2}} \| c$. Finally, the inequalities follow from the corollary above. □

For those cases not covered by Proposition 29 we also have an upper bound on $\nu_c \eta_g$, but for this we need to study each case by itself.

First we note that for $p=2$, equation (7) yields

$$\lambda_2\eta_g(z)|k\begin{pmatrix}1&0\\2^{\alpha-1}c&1\end{pmatrix}=2^{\frac{k}{2}-1}(\eta_g(z)|_k\begin{pmatrix}1&0\\2^{\alpha}c&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}$$

(48)

$$+\chi_2(1-2^{\alpha-1}c(1-2n))\eta_g(z)|_k\begin{pmatrix}1&0\\2^{\alpha}c&1\end{pmatrix}\begin{pmatrix}1&1-2n\\0&2\end{pmatrix})$$

where $1-2n\equiv 0 \pmod{\frac{N}{2^M}}$ and $\gcd(2,c)=1$, for all integers α with $\frac{M+1}{2}\leq\alpha\leq M$.

Assume that c is a factor of N_g relatively prime to 6. Then, in each of the exceptional cases we argue as follows:

(i) $(2^M,\chi_2)=(2^5,\Psi_1),\ (2^5,\Psi_3)$.

By Propositions 10 and 16 $\gcd(2,\nu_{2^4c}\eta_g)=\gcd(2,\nu_{2^5c}\eta_g)=1$. Hence $\nu_{2^4c}\eta_g=\nu_{2^5c}\eta_g=1$ from Proposition 13. Next we use Lemma 14 and by a similar argument conclude $\nu_c\eta_g=\nu_{2c}\eta_g=1$.

(Here, and for the rest of this section we are making constant use of Lemma 12 and Proposition 21).

If we put together the remarks given after Proposition 20 and Corollary 28 we get the inequality $\nu_{2^3c}\eta_g\leq 2^3$. Then, we also get $\nu_{2^2c}\eta_g\leq 2^3$.

(ii) $(2^M,\chi_2)=(2^4,\Psi_2)$.

By the same reasons $\nu_{2^\alpha c}\eta_g=1$ for $\alpha=0,1,3,4$ and $\nu_{2^2c}\eta_g\leq 2^2$.

(iii) $(2^M,\chi_2)=(2^4,\Psi_1),\ (2^4,\Psi_3)$.

If $\lambda_2=0$ one can prove $\nu_c\eta_g=\nu_{2c}\eta_g$ from equation (7). On the other hand, $\tilde{\lambda}_2=0$ implies $\gcd(2,\nu_{\frac{N}{2c}}\tilde{\eta}_g)=1$ by equation (48). But this yields that $\nu_c\eta_g$ is even, which is impossible by Proposition 10 (iv) for $\tilde{\eta}_g(z)$. Hence $\tilde{\lambda}_2\neq 0$. This in turn implies that $\nu_{2^3c}\eta_g$ is odd by equation (25) on $\tilde{\eta}_g(z)$. But $\nu_{2^3c}\eta_g$ is even according to the identity (48), hence we have a contradiction which shows $\lambda_2\neq 0$. By a symmetric argument we also have $\tilde{\lambda}_2\neq 0$. Next we use equation (25) again and conclude that $\nu_{2c}\eta_g$ and $\nu_{2^3c}\eta_g$ are odd. Hence $\nu_c\eta_g$ and $\nu_{2^4c}\eta_g$ are also odd by Proposition 7 and equation (22). Consequently, $\nu_{2^\alpha c}\eta_g=1$ for $\alpha=0,1,3,4$. By Corollary 22 $\nu_{2^2c}\eta_g=0$.

(iv) $(2^M,\chi_2)=(2^3,\Psi_2)$.

If $\lambda_2\neq 0$ then $\nu_{2c}\eta_g$ is odd by equation (23). Hence, from (7) and Proposition 13 we obtain $\nu_c\eta_g=0$. In particular $\nu_{\frac{N_g}{c}}\tilde{\eta}_g=0$. Now, if we suppose that $\tilde{\lambda}_2\neq 0$ then $\nu_{\frac{N_g}{2c}}\tilde{\eta}_g=0$ by equation (22), i.e. $\nu_{2c}\eta_g=0$, a contradiction. Therefore we must have $\tilde{\lambda}_2=0$. But in this case we get $\gcd(2,\nu_{\frac{N_g}{c}}\tilde{\eta}_g)=1$ by Proposition 10. Thus $\nu_c\eta_g$ is odd, which again is a contradiction. Hence $\lambda_2=0$. Similarly, $\tilde{\lambda}_2=0$. Under these conditions $\nu_c\eta_g$ must be odd and therefore $\nu_c\eta_g=1$ (see Propositions 13 and 21). From equation (48) we get $\nu_{2c}\eta_g=\nu_c\eta_g$. By the same arguments on $\tilde{\eta}_g$ we conclude $\nu_c\eta_g=\nu_{2c}\eta_g=\nu_{2^2c}\eta_g=\nu_{2^3c}\eta_g=1$.

(v) $(2^M,\chi_2)=(2^3,\Psi_1),\ (2^3,\Psi_3)$.

By Corollary 22 $\nu_{2c}\eta_g=\nu_{2^2c}\eta_g=0$, and from Proposition 13 $\nu_c\eta_g,\ \nu_{2^3c}\eta_g\in\{0,1\}$.

(vi) $(2^M,\chi_2)=(2^2,\Psi_2)$.

By the argument above, $\nu_{2c}\eta_g=0$ and $\nu_c\eta_g,\ \nu_{2^2c}\eta_g\in\{0,1\}$.

We end this section with a summary of all the information that we have about the integers $\nu_c\eta_g$.

Theorem 30. *Let $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$ such that $\eta_g(z) \in M_{k_g}(N_g, \chi_g)$. Assume that both $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms for all Hecke operators.*

Let $N_g = 2^M 3^{M_1} N'$ for some non-negative integers $M,\ M_1,\ N'$ with $\gcd(N', 6) = 1$, and denote by χ_2 the 2-part of χ_g.

Then, for any factor c of N_g, the non-negative integer $\nu_c\eta_g$ satisfies one the properties below:

(i) *If $M \in \{0, 1\}$ then $\nu_c\eta_g \in \{0, 1\}$.*
(ii) *If $(2^M, \chi_2) \in \{(2^3, \psi_0), (2^5, \psi_0), (2^5, \psi_2)\}$ or $M = 7$ then $\nu_c\eta_g = 1$.*
(iii) *If $M \in \{4, 6, 8\}$ then $0 \le \nu_c\eta_g \le 2^{\frac{M}{2}}$. Moreover $\nu_c\eta_g = 1$ for $\gcd(2^M, c) \neq 2^{\frac{M}{2}}$.*
(iv) *If $(2^M, \chi_2) \in \{(2^5, \psi_1), (2^5, \psi_3)\}$ then $0 \le \nu_c\eta_g \le 2^3$. Moreover $\nu_c\eta_g = 1$ for c such that $\gcd(2^5, c) \neq 2^2$ and $\gcd(2^5, c) \neq 2^3$.*
(v) *If $(2^M, \chi_2) \in \{(2^3, \psi_1), (2^3, \psi_3)\}$ then $\nu_c\eta_g \in \{0, 1\}$. Moreover $\nu_c\eta_g = 0$ whenever $\gcd(2^3, c) = 2$ or $\gcd(2^3, c) = 2^2$.*
(vi) *If $(2^M, \chi_2) = (2^3, \psi_2)$ then $\nu_c\eta_g = 1$.*
(vii) *If $M = 2$ then $0 \le \nu_c\eta_g \le 2$. Moreover $\nu_c\eta_g \in \{0, 1\}$ for c such that $\gcd(2^2, c) \neq 2$. In fact, $\nu_c\eta_g = 0$ for $\gcd(2^2, c) = 2$ if $\chi_2 \neq \psi_0$.*

Furthermore in every case, $\nu_c\eta_g \ge 2$ implies that the only prime divisors of $\nu_c\eta_g$ are 2 or 3, with 3 dividing it only if $3^2 \| N_g$ and $3 \| c$.

Proof. If $\nu_c\eta_g \neq 0$ then $\mu_c = 0$ by Corollary 22. Hence any prime divisor of $\nu_c\eta_g \ge 2$ is 2 or 3 (by 13 and 21).

Suppose that 3 divides $\nu_c\eta_g$. Then $3^2 \| N_g$ and $3 \| c$ by Propositions 13, 17 and 21. Moreover $\nu_c\eta_g \le \nu_{\frac{c}{3}}\eta_g$, by Proposition 19. Consequently, if we want an upper bound for $\nu_c\eta_g$ we may assume that 3 does not divide c.

Now, the proof of each statement above is the following:

(i) If $M \in \{0, 1\}$ then 2 does not divide $\gcd(c, \frac{N}{c})$ and therefore $0 \le \nu_c\eta_g \le 1$.
(ii) In this case $\mu_c = 0$ by the table (27) and so 2 is not a factor of $\nu_c\eta_g$ by Proposition 17 (this is true when 3 is a factor of c too). Thus $\nu_c\eta_g = 1$.
(iii) For $M = 6, 8$ and $(2^M, \chi_2) = (2^4, \psi_0)$ we use the previous argument and get $\nu_c\eta_g = 1$ whenever $\gcd(2^{\frac{M}{2}}, c) \neq 2^{\frac{M}{2}}$. The inequality follows from Corollary 28. If $(2^M, \chi_2)$ is $(2^4, \psi_1), (2^4, \psi_2)$ or $(2^4, \psi_3)$ we obtain the same conclusion from the analysis of the exceptional cases done above.

Similarly, (iv), (v) and (vi) follow from the remarks preceding this theorem and the fact that these also hold if c is a multiple of 3.

Finally, we get (vi) by Proposition 13, Corollary 28 and the last exceptional case studied above. □

6. The computation of η-quotients which are eigenforms of the Hecke algebra

Here we show that is possible to determine explicitly all $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$ characterized in Theorem 30.

First, let's recall that a complete set of representatives for the cusps of $\Gamma_0(N)$ is given in section 2 by the set $\mathcal{C}_N$. If $Ker\chi$ denotes the kernel of the real Dirichlet character χ and A is in $\Gamma_0(N) - Ker\chi$ then

$$\mathcal{C}_{N,A} = \mathcal{C}_N \cup \{A(\frac{a}{c});\ \frac{a}{c} \in \mathcal{C}_N,\ \mu_{\frac{a}{c}} = 0\}$$

is a complete set of representatives for all cusps of $Ker\chi$.

Let $f(z)$ be in $M_k(N,\chi)$. Consider $f(z)$ as a modular form on $Ker\chi$ and denote its order of zero at the cusp $x \in \mathcal{C}_{N,A}$ by $\nu_{x,\chi} f$. Then it is easy to prove that $\mu_{\frac{a}{c}} = 0$ implies $\nu_{\frac{a}{c},\chi} f = \nu_{\frac{a}{c}} f$ and $\nu_{A(\frac{a}{c}),\chi} f = \nu_{\frac{a}{c}} f$, as well as $\mu_{\frac{a}{c}} \neq 0$ implies $\nu_{\frac{a}{c},\chi} f = 2\nu_{\frac{a}{c}} f + 1$.

Since $f(z)$ is a modular form invariant under $Ker\chi$, if we assume $f(z) \neq 0$ for all $z \in \mathcal{H}$ we get

$$\sum_{\frac{a}{c} \in \mathcal{C}} \nu_{\frac{a}{c},\chi} f + \sum_{\substack{\frac{a}{c} \in \mathcal{C} \\ \mu_{\frac{a}{c}} = 0}} \nu_{A(\frac{a}{c}),\chi} f = \frac{k}{12} [\mathrm{SL}_2(\mathbb{Z}) : Ker\chi].$$

Consequently, from the previous identities and Corollary 22, we have the following

Lemma 31. *Let $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$ such that $\eta_g(z) \in M_{k_g}(N_g, \chi_g)$. Then*

$$2 \sum_{\substack{\frac{a}{c} \in \mathcal{C} \\ \mu_{\frac{a}{c}} = 0}} \nu_{\frac{a}{c}} \eta_g + \sum_{\substack{\frac{a}{c} \in \mathcal{C} \\ \mu_{\frac{a}{c}} \neq 0}} 1 = \frac{k_g}{6} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_g)]. \tag{49}$$

The number of elements $\frac{a_c}{c}$ in $\mathcal{C}$ for a fixed divisor c of N_g is $\phi(\gcd(\frac{N_g}{c}, c))$, where ϕ denotes the Euler function. If we use the upper bounds for $\nu_{\frac{a_c}{c}} \eta_g = \nu_c \eta_g$ given in Theorem 30, we obtain from the previous equation an upper bound for $\frac{k_g}{6}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_g)]$. For example

Lemma 32. *Let $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$, as in theorem 30. Let $2^M \| N_g$ and assume that $(2^M, \chi_2)$ is none of the following: $(2^2, \psi_2)$, $(2^3, \psi_1)$, $(2^3, \psi_2)$, $(2^3, \psi_3)$, $(2^4, \psi_1)$, $(2^4, \psi_2)$, $(2^4, \psi_3)$, $(2^5, \psi_1)$ or $(2^5, \psi_3)$. Then*

$$\frac{k_g}{6} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_g)] \leq 2 \sum_{c | N_g} \phi(\gcd(\frac{N_g}{c}, c))$$

if $M = 0, 1, 3, 5, 7$, and

$$\frac{k_g}{6} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_g)] \leq 2 (\sum_{c | N_g} \phi(\gcd(\frac{N_g}{c}, c)) + (2^{\frac{M}{2}} - 1) \sum_{\substack{c | N_g \\ 2^{\frac{M}{2}} \| c}} \phi(\gcd(\frac{N_g}{c}, c)))$$

if $M = 2, 4, 6, 8$.

Analogous inequalities can be obtained for each one of the exceptional cases.

Since the mapping on $\mathbb{Z}$ defined by $N \mapsto \sum_{c|N} \phi(\frac{N}{c}, c))$ is multiplicative, the right hand side of the inequalities in the lemma above can be written in terms of $\sum_{c|2^M} \phi(\frac{2^M}{c}, c))$, $\sum_{c|3^{M_1}} \phi(\frac{3^{M_1}}{c}, c))$ and $\sum_{c|N'} \phi(\frac{N'}{c}, c))$, where $N_g = 2^M 3^{M_1} N'$ and $\gcd(N', 6) = 1$. Consequently, one obtains

$$\begin{array}{ll}
2^{\frac{M-1}{2}} k_g \prod_{p|N'} (p+1) \leq 2^3 \prod_{p|N'} 2 & \text{if } M > 0 \text{ odd},\ M_1 = 0, \\
2^{\frac{M-1}{2}} 3^{\frac{M_1-1}{2}} k_g \prod_{p|N'} (p+1) \leq 2^2 \prod_{p|N'} 2 & \text{if } M > 0 \text{ odd},\ M_1 = 1, 3, \\
2^{\frac{M-1}{2}} 3 k_g \prod_{p|N'} (p+1) \leq 2^3 \prod_{p|N'} 2 & \text{if } M > 0 \text{ odd}, M_1 = 2,
\end{array}$$

$$\begin{array}{ll}
2^{\frac{M}{2}} k_g \prod_{p|N'} (p+1) \leq (2^{\frac{M}{2}} + 2) 2^2 \prod_{p|N'} 2 & \text{if } M > 0 \text{ even},\ M_1 = 0, \\
2^{\frac{M}{2}} 3^{\frac{M_1-1}{2}} k_g \prod_{p|N'} (p+1) \leq (2^{\frac{M}{2}+1} + 2^2) \prod_{p|N'} 2 & \text{if } M > 0 \text{ even},\ M_1 = 1, 3, \\
2^{\frac{M}{2}} 3 k_g \prod_{p|N'} (p+1) \leq (2^{\frac{M}{2}} + 2) 2^2 \prod_{p|N'} 2 & \text{if } M > 0 \text{ even},\ M_1 = 2.
\end{array}$$

We get similar inequalities for $M = 0$ and for all exceptional cases.

The left hand side of these inequalities always growth faster than the right hand side. Hence, there exist only a finite number of pairs (N_g, k_g) satisfying at least one of these. This means there are only a finite number of modular forms $\eta_g(z) \in M_k(N_g, \chi_g)$ such that both $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms for all Hecke operators.

It is not hard to compute all pairs (N_g, k_g) satisfying any of the inequalities obtained above. For each N_g in one of these pairs we consider the system of linear equations in the variables r_t

$$\frac{1}{24}\frac{N_g}{\gcd(N_g, c^2)}\sum_{t|N_g}\frac{\gcd(t,c)^2}{t}r_t = a_c \tag{50}$$

where c and t are running in the set of positive divisors of N_g. The values $a_c = \nu_c\eta_g + \mu_c \in \frac{1}{2}\mathbb{Z}$ are subject to the conditions and bounds given in the table (27) and Theorem 30.

This defines a finite number of square systems of linear equations for N_g, and it is a fact that each of them has a unique solution.

Collecting in one set the integral solutions for all systems of equations (50), we define a set L' of formal products $g = t_1^{r_1}t_2^{r_2}\ldots t_s^{r_s}$. By Theorem 30 and the previous computations of (N_g, k_g) we know that L' must contain all those g such that $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms of the Hecke algebra.

Consequently, we only have to decide which elements in L' are indeed eigenforms for all T_p in order to get the complete list of η-quotients with this property.

Let $g = t_1^{r_1}t_2^{r_2}\ldots t_s^{r_s} \in L'$. If p is a prime divisor of N_g we compute the coefficients of $\eta_g(z)$ and $T_p\eta_g(z)$ at the cusp at infinity up to a certain power of q. If this power is large enough we can decide whether or not $\eta_g(z)$ is an eigenform of T_p just by comparing these two set of coefficients. Hence, with a computer, we can easily find all those elements g in L' such that $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms of the Hecke operators T_p with p a factor of N_g. Call this new set L.

We exhibit the elements of L in the second column of our next table. The first column gives the level N_g for the corresponding modular form, and the third column says if $\eta_g(z)$ is a cusp form. The meaning of the last column will be explained in the final section.

Now we deal with the action of T_p on $\eta_g(z)$ for $g \in L$ and $\gcd(p, N_g) = 1$.

Proposition 33. *Let $g = t_1^{r_1}t_2^{r_2}\ldots t_s^{r_s} \in L$ and $\gcd(p, N_g) = 1$. Then $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms of T_p.*

Proof. Let c be any divisor of N_g. If $\nu_c\eta_g \in \{0, 1\}$ then

$$\nu_c T_p\eta_g \geq \nu_c\eta_g \tag{51}$$

by Proposition 4.

If $\nu_c\eta_g = \nu \geq 2$ we check by direct computation that every power of q_{h_c} with a non-zero coefficient in the product $G_{\frac{1}{c}}(z)$ has ν as a factor of its exponent. Thus, every power of q_{h_c} with a non-zero coefficient in $\eta_g(z)|_k \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ has an exponent divisible by ν. Hence, inequality (51) holds by Proposition 4.

As $\eta_g(z)$ is non-zero in $\mathcal{H}$, (51) implies that the quotient $\frac{T_p\eta_g(z)}{\eta_g(z)}$ is a constant in $\mathbb{C}$, therefore $\eta_g(z)$ is an eigenform of T_p. Since $\gcd(p, N_g) = 1$, the above implies that $\tilde{\eta}_g(z)$ is also an eigenform of T_p. $\square$

The main consequence of Proposition 33 is that $g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s} \in L$ if, and only if, $\eta_g(z) \in M_{k_g}(N_g, \chi_g)$ and both $\eta_g(z)$ and $\tilde{\eta}_g(z)$ are eigenforms for all Hecke operators. This is precisely the statement of Theorem 1.

7. Multiplicative η-quotients and finite groups

As we mentioned in the introduction, most of the η-products which are eigenforms for the Hecke algebra can be related to the largest Mathieu group M_{24}. Namely, one can show the existence of some graded, infinite-dimensional, complex vector space $V = \bigoplus_{n=1}^{\infty} V_n q^n$ such that

(i) for every $n \geq 1$ the subspace V_n is a finite-dimensional $\mathbb{C}M_{24}$-module, and
(ii) for every $g \in M_{24}$ its graded trace in V, $tr_V(g) = \sum_{n=1}^{\infty} tr_{V_n}(g) q^n$, is precisely one of the multiplicative η-products listed in [5].

An explicit construction of V is given in [17]. The above is an example of a McKay-Thompson series for the finite group M_{24}.

In [18] and [20] the concept of a McKay-Thompson series is generalized to what is called elliptic system (see also [21]). Basically, an elliptic system of a finite group G is a mapping that associate to every element h in G some graded, infinite-dimensional, complex vector space V_h, such that

(i) every homogeneous component of V_h affords a finite-dimensional, complex representation for the centralizer of h in G, and
(ii) if $g \in G$ commutes with h then its graded trace in V_h, $tr_{V_h}(g)$, is a modular function or modular form.

Moreover, there is a particular functional equation relating $tr_{V_h}(g)$ and $tr_{V_{h'}}(g')$ whenever the commuting pairs (h, g) and (h', g') generate the same subgroup of G.

In [20] G. Mason constructs an explicit elliptic system for a large class of finite groups. In [21] Mason study this elliptic system for the group M_{24}, and exhibits the graded traces $tr_{V_h}(g)$ for all commuting pairs (h, g) in $M_{24} \times M_{24}$ for which the action of g in V_h is rational. This list of modular forms contains all η-products and some of the η-quotients from the second column of Table I.

Table I: Multiplicative η-quotients

N_g	$g = t_1^{r_1} t_2^{r_2} \dots t_s^{r_s}$	**cusp form**	**Conway group**
1	1^{24}	yes	$1A$
2	$1^8 \cdot 2^8$	yes	$2A$
3	$1^{-3} \cdot 3^9$	no	$3C$
3	$1^9 \cdot 3^{-3}$	no	$(3C)\|W_9$
3	$1^6 \cdot 3^6$	yes	$3B$
4	$1^{-4} \cdot 2^{10} \cdot 4^{-4}$	no	$(8D)\|W_8T$
4	$1^{-4} \cdot 2^6 \cdot 4^4$	no	$-4C$
4	$1^4 \cdot 2^6 \cdot 4^{-4}$	no	$(-4C)\|W_4$
4	2^{12}	yes	$2C$
4	$1^4 \cdot 2^2 \cdot 4^4$	yes	$4C$
5	$1^{-1} \cdot 5^5$	no	$5C$
5	$1^5 \cdot 5^{-1}$	no	$(5C)\|W_5$
5	$1^4 \cdot 5^4$	yes	$5B$
6	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	yes	$6E$
7	$1^3 \cdot 7^3$	yes	$7B$

Table I (continued)

8	$2^4 \cdot 4^4$	yes	$4D$
8	$1^{-2} \cdot 2^3 \cdot 4^3 \cdot 8^{-2}$	no	$(16A)\|TW_{16}T$
8	$1^{-2} \cdot 2^3 \cdot 4 \cdot 8^2$	no	$-8E$
8	$1^2 \cdot 2 \cdot 4^3 \cdot 8^{-2}$	no	$(-8E)\|W_8$
8	$1^2 \cdot 2 \cdot 4 \cdot 8^2$	yes	$8E$
9	$1^3 \cdot 3^{-2} \cdot 9^3$	no	$9C$
9	3^8	yes	$3D$
11	$1^2 \cdot 11^2$	yes	$11A$
12	$1^{-2} \cdot 2^2 \cdot 3^2 \cdot 4 \cdot 12$	no	$-12I$
12	$1 \cdot 3 \cdot 4^2 \cdot 6^2 \cdot 12^{-2}$	no	$(-12I)\|W_{12}$
12	$1^2 \cdot 3^{-2} \cdot 4 \cdot 6^2 \cdot 12$	no	$12I$
12	$1 \cdot 2^2 \cdot 3 \cdot 4^{-2} \cdot 12^2$	no	$12H$
12	$2^3 \cdot 6^3$	yes	$6G$
14	$1 \cdot 2 \cdot 7 \cdot 14$	yes	$14B$
15	$1^2 \cdot 3^{-1} \cdot 5^{-1} \cdot 15^2$	no	$15E$
15	$1^{-1} \cdot 3^2 \cdot 5^2 \cdot 15^{-1}$	no	$(15E)\|W_3$
15	$1 \cdot 3 \cdot 5 \cdot 15$	yes	$15D$
16	$2^4 \cdot 4^{-4} \cdot 8^4$	no	$8B$
16	$2^{-4} \cdot 4^{16} \cdot 8^{-4}$	yes	$(4D)\|T$
16	$2^{-12} \cdot 4^{36} \cdot 8^{-12}$	yes	$(2C)\|T$
16	4^6	yes	$4F$
20	$1 \cdot 2 \cdot 4^{-1} \cdot 5^{-1} \cdot 10 \cdot 20$	no	$-20C$
20	$1^{-1} \cdot 2 \cdot 4 \cdot 5 \cdot 10 \cdot 20^{-1}$	no	$(-20C)\|W_5$
20	$2^2 \cdot 10^2$	yes	$10F$
23	$1 \cdot 23$	yes	$23A$
24	$1 \cdot 3^{-1} \cdot 4 \cdot 6 \cdot 8^{-1} \cdot 24$	no	$-24F$
24	$1^{-1} \cdot 2 \cdot 3 \cdot 8 \cdot 12 \cdot 24^{-1}$	no	$(-24F)\|W_3$
24	$2 \cdot 4 \cdot 6 \cdot 12$	yes	$12J$
27	$3^2 \cdot 9^2$	yes	$(3D, 3B)$
32	$2^2 \cdot 4^{-1} \cdot 8^{-1} \cdot 16^2$	no	$16A$
32	$2^{-2} \cdot 4^9 \cdot 8^{-5} \cdot 16^2$	no	$(-8E)\|TW_{16}T$
32	$2^2 \cdot 4^{-5} \cdot 8^9 \cdot 16^{-2}$	no	$(-8E)\|TW_{16}TW_{32}$
32	$2^{-2} \cdot 4^5 \cdot 8^5 \cdot 16^{-2}$	yes	$(2C, 4D)$
32	$4^2 \cdot 8^2$	yes	$8F$
36	$1 \cdot 2^{-1} \cdot 3^{-2} \cdot 4 \cdot 6^4 \cdot 9 \cdot 12^{-2} \cdot 18^{-1} \cdot 36$	no	?
36	6^4	yes	$6I$
44	$2 \cdot 22$	yes	$22A$
48	$2 \cdot 4^{-1} \cdot 6 \cdot 8 \cdot 12^{-1} \cdot 24$	no	$24E$
48	$2^{-1} \cdot 4^4 \cdot 6^{-1} \cdot 8^{-1} \cdot 12^4 \cdot 24$	yes	$(12J)\|T$
48	$2^{-3} \cdot 4^9 \cdot 6^{-3} \cdot 8^{-3} \cdot 12^9 \cdot 24^{-3}$	yes	$(6G)\|T$
63	$3 \cdot 21$	yes	$21C$
64	$4^2 \cdot 8^{-2} \cdot 16^2$	no	$(8D)\|T$
64	$4^{-2} \cdot 8^8 \cdot 16^{-2}$	yes	$(8F)\|T$
64	$4^{-6} \cdot 8^{18} \cdot 16^{-6}$	yes	$(4F)\|T$
64	$4^{-14} \cdot 8^{38} \cdot 16^{-14}$	yes	$(2C, 2A)\|T$
80	$2^{-2} \cdot 4^6 \cdot 8^{-2} \cdot 10^{-2} \cdot 20^6 \cdot 40^{-2}$	yes	$(10F)\|T$
80	$4 \cdot 20$	yes	$20B$
96	$2 \cdot 4^{-3} \cdot 6^{-1} \cdot 8^4 \cdot 12^4 \cdot 16^{-1} \cdot 24^{-3} \cdot 48$	no	$(-24F)\|TW_{48}TW_3$

Table I (continued)

96	$2^{-1}\cdot 4^4\cdot 6\cdot 8^{-3}\cdot 12^{-3}\cdot 16\cdot 24^4\cdot 48^{-1}$	no	$(-24F)\vert TW_{48}T$
108	$6\cdot 18$	yes	$(6G,3D)$
128	$8\cdot 16$	yes	$(4F,4D)$
144	$6^{-4}\cdot 12^{12}\cdot 24^{-4}$	yes	$(6I)\vert T$
144	12^2	yes	$12M$
176	$2^{-1}\cdot 4^3\cdot 8^{-1}\cdot 22^{-1}\cdot 44^3\cdot 88^{-1}$	yes	$(22A)\vert T$
256	$8^{-1}\cdot 16^4\cdot 32^{-1}$	yes	$(4F,4D)\vert T$
320	$4^{-1}\cdot 8^3\cdot 16^{-1}\cdot 20^{-1}\cdot 40^3\cdot 80^{-1}$	yes	$(20B)\vert T$
432	$6^{-1}\cdot 12^3\cdot 18^{-1}\cdot 24^{-1}\cdot 36^3\cdot 72^{-1}$	yes	$(6G,3D)\vert T$
576	$12^{-2}\cdot 24^6\cdot 48^{-2}$	yes	$(12M)\vert T$
576	$4^{-2}\ 8^5\ 12^2\ 16^{-2}\ 24^{-4}\ 36^{-2}\ 48^2\ 72^5\ 144^{-2}$	no	?

In order to find a similar conection between the set of η-quotients classified by Theorem 1 and some finite group we consider the elliptic system defined in [20] for the group G of automorphisms of the Leech lattice, i.e. the Conway group.

The non-trivial 24-dimensional permutation representation of G associates to any g in G a formal product $t_1^{r_1}t_2^{r_2}\dots t_s^{r_s}$, where t_j, r_j, s are integers, $t_j, s > 0$, called the Frame shape of g (see [16]). These Frame shapes are listed in [13]. The elliptic system for G that we are considering is such that

$$tr_{V_{\mathbf{1}}}(g) = \prod_{t_j=1}^{s} \eta(t_j z)^{r_j} \tag{52}$$

where $V_{\mathbf{1}}$ is the vector space corresponding to the identity element $\mathbf{1}$ of G. We should mention that if the Frame shape of g defines an η-product, i.e. all $r_1, r_2, \dots, r_s$ are non-negatives, then the level N_g of the form $tr_{V_{\mathbf{1}}}(g) = \eta_g(z)$ is the product of the smallest and the largest of $t_1, t_2, \dots, t_s$ for which the corresponding exponents $r_1, r_2, \dots, r_s$ are non-zero.

In any elliptic system, two modular forms associated to different pairs of commuting elements (h, g) and (h', g') are related provide that both pairs generate the same group. For example, if we write $f(h, g, z)$ for the modular form $tr_{V_h}(g)$ we must have

$$f(h,g,z)|_k \begin{pmatrix} Qa & b \\ Nc & Qd \end{pmatrix} = Cf(h^{Qd}g^{-\frac{N}{Q}c}, h^{-b}g^{a}, Qz) \tag{53}$$

where C is some complex number, k is the weight of $f(h, g, z)$, $Q\|N$, $N = N_hN_g$, N_h (resp. N_g) is the level of the modular form $f(\mathbf{1}, h, z)$ (resp. $f(\mathbf{1}, g, z)$) and

$$W_Q = \begin{pmatrix} Qa & b \\ Nc & Qd \end{pmatrix}$$

is the corresponding Atkin-Lehner involution.

Similarly, if $g \in G$ and 2^e divides N_g

$$f(\mathbf{1},g,z)|_k \begin{pmatrix} 0 & -1 \\ N_g & 2^{-e}N_g \end{pmatrix} = C'f(g^{-1}, g^{2^{-e}N_g}, N_g z). \tag{54}$$

From the Frame shapes of the elements g in G we compute all modular forms in this elliptic system of type $f(\mathbf{1}, g, z)$. Since M_{24} is a subset of G, all forms $f(h, g, z)$ computed in [21] are also part of it. Then, we apply to these modular forms the transformations defined by equation (53) above or, in those cases where $f(\mathbf{1}, g, z)$ is an η-product, the transformation $f(\mathbf{1}, g, z)|_k T$ given by equation (54), with 2^e being the largest power of 2 dividing every t_j that has a non-zero r_j in the Frame shape of g. We get in this way a few more elements of this particular elliptic system.

After the computations described above we observe that at least 72 of the 74 η-quotients satisfying the conditions of Theorem 1 can be realized as modular forms $f(h, g, z)$ in this elliptic system for G. We summarize the explicit correspondence in the last column of Table I. The notation that we use take the names for $g \in G$ from the Atlas of finite simple groups [4]. We write a pair $(h, g) \in G \times G$ for the form $f(h, g, z)$, a single element g for $f(\mathbf{1}, g, z)$, and put $(h, g)|W_Q$ and $(h, g)|T$ for $f(h, g, z)|_k W_Q$ and $f(h, g, z)|_k T$ respectively. Notice that our computations do not exhaust all modular forms of this elliptic system for G, hence it is tempting to think that all η-quotients in Table I are related to the Conway group via this construction.

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