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MULTIPLICATIVE OPERATIONS IN BP^*BP

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One of the present computational difficulties in complex cobordism theory is the lack of a known algebra splitting of BP^*BP, the algebra of stable cohomology operations for the Brown-Peterson cohomology theory, analogous to the splitting isomorphism

$$MU^*MU \approx MU^*(pt) \otimes S$$

where S is the Landweber-Novikov algebra. S has the added advantage of being a cocommutative Hopf algebra over Z.

This paper does not remove this difficulty, but we will show that the monoid of multiplicative operations in BP^*BP, (i.e. those operations which induce ring endomorphisms on BP^*X for any space X), which we will denote by $\Gamma(BP)$, has a submonoid analogous to the monoid of multiplicative operations in S.

The latter monoid is known (see Morava [3]) to be isomorphic to the group of formal power series $f(x)$ over Z such that $f(0) = 0$ and $f'(0) = 1$ and where the group operation is composition of power series.

For the basic properties of MU^*MU and BP^*BP, see Adams [1], especially §§ 11 and 16.

The main construction of this paper was inspired by the work of Honda ([3]) although none of his results are needed here. I am grateful to Jack Morava for bringing Honda's work to my attention.

Before stating our main result we must review the description of $\Gamma(BP)$ implicit in [1] § 16. An operation $\alpha \in \Gamma(BP)$ is characterized by its action on the canonical generator $z \in BP^*(CP^\infty) \cong \pi_*BP[[z]]$. It is shown that $\alpha(z)$ is a power series $f(z)$ over π_*BP where

$$f^{-1}(z) = z +_\mu t_1 z^p +_\mu t_2 z^{p^2} +_\mu t_3 z^{p^3} +_\mu \cdots \tag{1}$$

where $+_\mu$ denotes the sum in the formal group defined over π_*BP and $t_i \in \pi_*BP$. (This formula appears on page 96 of [1].)

The action of α on π_*BP can be read off from (1). Let $l_n \in \pi_{2p^n-2}BP \otimes Q$ be defined by $\log^{BP} x = \sum_{n=0} l_n x^{p^n}$ (l_n is the m_{p^n-1} of [1]). Then we have

$$\alpha(l_n) = \sum_{0 \leqq i \leqq n} l_n t_{n-i}^{p^i} \tag{2}$$

and this formula also characterizes α.

In other words $\Gamma(BP)$ is an infinite dimensional affine space over

$\pi_* BP$ with coordinates t_i with a composition law which can be read off from (2). The difficulty mentioned above is that the subset of elements with integer coordinates is not a submonoid.

The main object of this paper is to construct new coordinates s_i of $\Gamma(BP)$ such that the subset of elements with coordinates in Q_p (the set of rational numbers with denominators prime to p) is a submonoid which will be denoted by $\gamma(BP)$. Moreover this submonoid is isomorphic to a direct product of countably many copies of Q_p, although this isomorphism is somewhat accidental in a sense to be described below. The s_i are not unique as they depend upon a choice of generators of $\pi_* BP$. What's worse, they are not algebraic functions of the t_i, so they do not lead to a splitting of BP^*BP.

Let $\{v_j \in \pi_* BP\}$ be a set of generators. Define a ring endomorphism σ of $\pi_* BP$ by $v_j^\sigma = v_j^p$, extending σ linearly over all of $\pi_* BP$. Then our main result is

THEOREM. *For every sequence of coordinates $\{t_n \in \pi_* BP\}$ there exists $\{s_n \in \pi_* BP\}$ such that*

$$\sum_{0 \leq i \leq n} l_i t_{n-i}^{p^i} = \sum_{0 \leq i \leq n} l_i s_{n-i}^{\sigma^i} \tag{3}$$

for every $n > 0$, and for every $\{s_n\}$ there exist $\{t_n\}$ satisfying the same conditions.

COROLLARY. *Let $\alpha', \alpha'' \in \Gamma(BP)$ have coordinates $s'_n, s''_n \in Q_p$ respectively. Then the coordinates s'''_n of $\alpha''' = \alpha' \cdot \alpha''$ are given by*

$$s'''_n = \sum_{i+j=n} s'_i s''_j$$

i.e. *if we define power series $s'(x)$, $s''(x)$, $s'''(x)$ by $s'(x) = 1 + \sum_{n>0} s'_n x^n$,* etc., *then $s'''(x) = s'(x)s''(x)$.*

The corollary follows from (3) by direct computation, remembering that σ fixes $Q_p \subset \pi_* BP$.

To prove the theorem we need an analogue of (1) involving the new coordinates s_n, and some elementary properties of formal groups beginning with

LEMMA A. *For every $r \in Q_p$ there is a power series $[r](z) \in \pi_* BP[[z]]$ such that*

(a) $[1](z) = z$

(b) $[r'](z) +_\mu [r''](z) = [r' + r''](z)$

(c) $[r']([r''](z)) = [r'r''](z)$

(d) $[r](z) \equiv rz$ *modulo*(z^2).

Proof. If r is a positive integer define $[r](z)$ inductively by $[r](z) = z +_\mu [r-1](z)$; the existence of $[-1](z)$ is one of the formal group axioms; and if r is an integer prime to p, define $[1/r](z)$ to be the formal inverse of $[r](z)$. Then (c) enables us to define $[r](z)$ for any $r \in Q_p$.

Now let $M = (m_1, m_2, m_3, \cdots)$ be a sequence of nonnegative integers, of which all but a finite number are zero, and let

$$v^M = \prod_{j>0} v_j^{m_j} \in \pi_* BP .$$

Then the v^M's form a Q_p-basis of $\pi_* BP$ and we can write

$$s_i = \sum s_{i,M} v^M$$

with $s_{i,M} \in Q_p$ and the sum ranging over all M. With this notation we have

LEMMA B. *Formula* (3) *is equivalent to*

$$(4) \qquad {}^\mu\sum_{n>0} t_n z^{p^n} = {}^\mu\sum_{n,M} [s_{n,M}](v^M z^{p^n})$$

where ${}^\mu\sum$ *denotes the formal group sum.*

Proof. Multiplying both sides of (3) by z^{p^n} and summing over all positive n gives

$$\sum_{n>0} \log t_n z^{p^n} = \sum_{n,M} s_{n,M} \log v^M z^{p^n}$$

which is equivalent to (4).

LEMMA C. *For* $r \in Q_p$, *there exist* $r_n \in \pi_* BP$ *with* $r_0 = r$ *such that for any* $u \in BP^* CP^\infty$

$$(5) \qquad [r](u) = {}^\mu\sum_{n\geqq 0} r_n u^{p^n} .$$

Proof. We will first show that there exist $r_{(k)} \in \pi_* BP$ such that

$$(6) \qquad [r](u) = {}^\mu\sum_{k>0} r_{(k)} u^k$$

and then show that $r_{(k)} = 0$ unless $k = p^n$. The proof of the former statement is by induction on k. Let $r_{(1)} = r$ and suppose we have found $r_{(1)}, r_{(2)}, \cdots, r_{(m)}$ such that

$$[r](u) = {}^\mu\sum_{k=1}^{m} r_{(k)} u^k \quad \text{modulo} (u^{m+1}) .$$

Then we can take $r_{(m+1)}$ to be the coefficient of u^{m+1} in

$$[r](u) - {}^{\mu}\sum_{k=1}^{m} r_{(k)}u^k$$

and the first statement follows. Taking the log of both sides of (6) we have

$$r \log u = \sum_{k>0} \log r_{(k)}u^k$$

i.e.

$$r \sum_{n\geqq 0} l_n u^{p^n} = \sum_{\substack{k>0\\ j\geqq 0}} l_j r_{(k)}^{p^j} u^{kp^j}$$

Equating coefficients of u^i yields the lemma.

LEMMA D. *For $a', a'' \in \pi_* BP$, there exist $a_n \in \pi_* BP$ for $n \geqq 0$ with $a_0 = a' + a''$ such that for any $u \in BP^* CP^\infty$*

$$a'u +_\mu a''u = \sum_{n\geqq 0}{}^{\mu} a_n u^{p^n} . \tag{7}$$

The proof of Lemma D is similar to that of Lemma C and is left to the reader.

We are now ready to prove the theorem via (4). Given $\{t_n\}$ we will construct $\{s_n\}$ by induction on n, beginning with $s_1 = t_1$. Suppose we have found $s_1, s_2, \cdots, s_m$ such that

$$\sum_{n>0}{}^{\mu} t_n z^{p^n} \equiv \sum_{\substack{0<n\leqq m\\ M}}{}^{\mu} [s_{n,M}](v^M z^{p^n} \quad \text{modulo}\,(z^{1+p^n})) \tag{8}$$

By repeated application of Lemmas C and D we can find $w_n \in \pi_* BP$ such that the right hand side of (8) is equal to $\sum_{n>0}^{\mu} w_n z^{p^n}$ and we know $w_n = t_n$ for $n \leqq m$. This allows us to set $s_{m+1} = t_{m+1} - w_{m+1}$, and the first half of the theorem is proved. Similarly if we are given $\{s_n\}$, let $t_1 = s_1$ and suppose we have found t_n for $n \leqq m$. Then we can set $t_{m+1} = w_{m+1} + s_{m+1}$, and the theorem is proved.

Now I will describe the sense in which the commutativity of $\gamma(BP)$ is accidental. If this paper were being written for number theorists rather than topologists, we would replace $\pi_* BP$ by $\pi_* BP \otimes_{Q_p} W(k)$, where $W(k)$ is the Witt ring of a finite field k of characteristic p. Lemma A would go through only for $r \in W(F_p)$, the p-adic integers. The ring endomorphism σ when restricted to $W(k)$ would be the lifting of the Frobenius automorphism on k. The corollary (with $s'_n, s''_n \in W(k)$) would then read

$$s'''_n = \sum_{i+j=n} s'_i (s''_j)^{\sigma^i}$$

The resulting group $\gamma_k(BP)$ would not be abelian if k has more than p elements.

References

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