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Multiplicity and Hilbert-Kunz Multiplicity of Monoid Rings

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In this paper, we will give a method to compute the multiplicity and the Hilbert-Kunz multiplicity of monoid rings. The multiplicity and the Hilbert-Kunz multiplicity are fundamental invariants of rings. For example, the multiplicity (resp. the Hilbert-Kunz multiplicity) of a regular local ring equals to one. Monoid rings are defined by lattice ideals, which are binomial ideals I in a polynomial ring R over a field such that any monomial is a non zero divisor on R/I. Affine semigroup rings are monoid rings. Hence we want to extend the thoery of affine semigroup rings to that of monoid rings.

1. Main Result.

Let $N > 0$ be an integer and $\mathbf{Z}$ the ring of integers. For $\alpha \in \mathbf{Z}^N$, we denote the i-th entry of α by α_i. We say $\alpha > 0$ if $\alpha \neq 0$ and $\alpha_i \geq 0$ for each i. And $\alpha > \alpha'$ if $\alpha - \alpha' > 0$. Let $R = k[X_1, \cdots, X_N]$ be a polynomial ring over a field k. For $\alpha > 0$, we simply write X^α in place of $\prod_{i=1}^N X_i^{\alpha_i}$.

For a positive submodule V of $\mathbf{Z}^N$ of rank r, we define an ideal $I(V)$ of R, which is generated by all binomials $X^\alpha - X^\beta$ with $\alpha - \beta \in V$ (we say that V is positive if it is contained in the kernel of a map $\mathbf{Z}^N \to \mathbf{Z}$ which is defined by positive integers). Put $d = N - r$. Then $R/I(V)$ is naturally a $\mathbf{Z}^d$-graded ring, which is called a monoid ring. Further, there is a positive submodule V' of $\mathbf{Z}^N$ of rank r containing V such that $\mathbf{Z}^N/V'$ is torsion free. That is, $\mathbf{Z}^N/V \cong \mathbf{Z}^N/V' \oplus T$, where $\mathbf{Z}^N/V' \cong \mathbf{Z}^d$ and T is a torsion module. Hence we can see an element of $\mathbf{Z}^N/V$ as a pair (α, β) where $\alpha \in \mathbf{Z}^d$ is a degree element and $\beta \in T$ is a torsion element. Put $t = |T|$(if $T = \{0\}$, put $t = 1$). Let $A = R/I(V)$ and $A' = R/I(V')$. For each $\alpha \in \mathbf{Z}^d$, we denote the degree α component of the $\mathbf{Z}^d$-graded ring A (resp. A') by A_α (resp. A'_α). It is clear $\dim_k A_\alpha \leq t$ and $\dim_k A'_\alpha \leq 1$ for $\alpha \in \mathbf{Z}^d$ and $\dim_k A_\alpha \geq \dim_k A_{\alpha'}$ if $\alpha > \alpha'$ and if there is a monomial of A of the degree $\alpha - \alpha'$.

EXAMPLE. Let V be a submodule of $\mathbf{Z}^3$ generated by $-e_1 + 2e_2 - e_3$, $-2e_1 - e_2 + 3e_3$ and $-3e_1 + e_2 + 2e_3$. Then $\mathbf{Z}^3/V \cong \mathbf{Z} \oplus \mathbf{Z}/5\mathbf{Z}$. And there is an isomorphism which corresponds e_1, e_2 and e_3 to $(1, 0)$, $(1, 1)$ and $(1, 2)$, respectively.

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LEMMA 1.1. *Let $A = R/I(V)$ be as above. Then there is $\alpha > 0$ with $\dim_k A_\alpha = t$.*

PROOF. Let $e_1, \cdots, e_N$ form a canonical basis of $\mathbf{Z}^N$. Since V is positive, $\alpha_i > 0$ for each i where (α_i, β_i) is the image of e_i. For each $(0, \beta) \in \mathbf{Z}^d \oplus T$, there is $\gamma \in \mathbf{Z}^N$ whose image is $(0, \beta)$. Let $T = \{\beta_1, \cdots, \beta_t\}$ and $\gamma_i \in \mathbf{Z}^N$ whose image in $\mathbf{Z}^N/V$ is $(0, \beta_i)$. Then there is $\delta_i \in \mathbf{Z}^N$ with $\delta_i > 0$ and $\gamma_i + \delta_i > 0$. Further, there is $\delta \in \mathbf{Z}^N$ with $\delta > 0$ and $\gamma_i + \delta > 0$ for each i. Then the degree part α of the image of δ in $\mathbf{Z}^N/V$ is also positive and $\dim_k A_\alpha = t$. Q.E.D.

DEFINITION. Let R be a graded k-algebra of dimension d, $\mathfrak{m}$ its maximal ideal, M a finite R-module and $\mathfrak{q} = (x_1, \cdots, x_s)$ a homogeneous $\mathfrak{m}$-primary ideal of R. We denote the multiplicity of $\mathfrak{q}$ by $e(\mathfrak{q}, M)$ i.e. $e(\mathfrak{q}, M) = \lim_{n\to\infty} d!\frac{l_R(M/\mathfrak{q}^n M)}{n^d}$ where l_R is the length. Similarly, Conca defined the generalized Hilbert-Kunz multiplicity $e_{\mathrm{HK}}(x_1, \cdots, x_s, M) = \lim_{n\to\infty} \frac{l_R(M/\mathfrak{q}^{[n]}M)}{n^d}$ where $\mathfrak{q}^{[n]} = (x_1^n, \cdots, x_s^n)$ ([3]). Generally, it is not clear that $e_{\mathrm{HK}}(x_1, \cdots, x_s, M)$ is always defined. But Monsky proved that $\lim_{e\to\infty} \frac{l_R(M/\mathfrak{q}^{[p^e]}M)}{p^{ed}}$ is well defined if *char* $R = p > 0$, which is called the Hilbert-Kunz multiplicity ([6]). In this case, it does not depend on a generating system of $\mathfrak{q}$. Note that the generalized Hilbert-Kunz multiplicity coincides with the Hilbert-Kunz multiplicity if *char* $R = p > 0$ and if it is defined. We also denote $e_{\mathrm{HK}}(x_1, \cdots, x_s, M)$ by $e_{\mathrm{HK}}(\mathfrak{q}, M)$. We write $e(R)$ (resp. $e_{\mathrm{HK}}(R)$) in place of $e(\mathfrak{m}, R)$ (resp. $e_{\mathrm{HK}}(\mathfrak{m}, R)$).

When R is a monoid ring over a field k and both $\mathfrak{q}$ and M are monomial ideal, the length $l_R(M/\mathfrak{q}^{[n]}M)$ is equal to the number of monomials in $M - \mathfrak{q}^{[n]}M$. Since $e_{\mathrm{HK}}(x_1, \cdots, x_s, M)$ is defined if *char* $k > 0$ and since $l_R(M/\mathfrak{q}^{[n]}M)$ does not depend on the base field k, $e_{\mathrm{HK}}(x_1, \cdots, x_s, M)$ is defined for any k.

NOTE. The following properties are known about the multiplicity;

- if $0 \to M' \to M \to M'' \to 0$ is a exact sequence,

$$e(\mathfrak{q}, M) = e(\mathfrak{q}, M') + e(\mathfrak{q}, M'') .$$

- $e(\mathfrak{q}, M) = \sum_{i=1}^t e(\overline{\mathfrak{q}_i}, R/\mathfrak{p}_i) l(M_{\mathfrak{p}_i})$ where $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_t\}$ is a set of a minimal prime ideals and $\bar{\mathfrak{q}}$ is the image of $\mathfrak{q}$ in $R/\mathfrak{p}_i$.
- $e(\mathfrak{q}, M) = e(\mathfrak{q}, R)\ \mathrm{rank}\, M$.

In turn, Monsky showed $e_{\mathrm{HK}}(\mathfrak{q}, M) = e_{\mathrm{HK}}(\mathfrak{q}, M') + e_{\mathrm{HK}}(\mathfrak{q}, M'')$, if $0 \to M' \to M \to M'' \longrightarrow 0$ is a exact sequence ([6, Theorem 1.8]). Hence another two formulas are also valid for the Hilbert-Kunz multiplicity.

THEOREM 1.2. *Let A, A' be monoid rings defined in this section and $\mathfrak{q}$ an $\mathfrak{m}$-primary monomial ideal of A. Then $e(\mathfrak{q}, A) = t \cdot e(\mathfrak{q}A', A')$ and $e_{\mathrm{HK}}(\mathfrak{q}, A) = t \cdot e_{\mathrm{HK}}(\mathfrak{q}A', A')$.*

NOTE. In general, A is not A'-module. In section two, we will prove that the generalized Hilbert-Kunz multiplicity for affine semigroup rings and $\mathfrak{m}$-primary monomial ideals is rational. Hence this theorem says that it is also rational in this case.

PROOF. By Lemma 1.1, there is $\alpha > 0$ with $\dim_k A_\alpha = t$. Put $M = \bigoplus_{\alpha' \geq \alpha} A_{\alpha'}$. Then it is not only an A-module but also an A'-module. Note that $\operatorname{rank}_A M = 1$ and $\operatorname{rank}_{A'} M = t$. Hence

$$e(\mathfrak{q}, A) = e(\mathfrak{q}, M) = e(\mathfrak{q}A', M) = t \cdot e(\mathfrak{q}A', A'),$$
$$e_{\mathrm{HK}}(\mathfrak{q}, A) = e_{\mathrm{HK}}(\mathfrak{q}, M) = e_{\mathrm{HK}}(\mathfrak{q}A', M) = t \cdot e_{\mathrm{HK}}(\mathfrak{q}A', A'). \qquad \text{Q.E.D.}$$

2. Hilbert-Kunz multiplicity of semigroup rings.

In this section, we treat the case of affine semigroup rings. Watanabe gives a method to compute the Hilbert-Kunz multiplicity for normal semigroup rings([10]). We will extend them to the generalized Hilbert-Kunz multiplicity for affine semigroup rings. We always assume that all semigroups are finitely generated.

Let S be a semigroup contained in $\mathbf{Z}^N$, $\mathfrak{q} = (x^{a_1}, x^{a_2}, \cdots, x^{a_v}) \subset k[S]$ such that $k[S]/\mathfrak{q}$ is finite length and $\bar{S} = \{p \in \mathbf{Z}^N \mid ap \in S \text{ for } \exists a > 0\}$. Then $k[\bar{S}]$ is finite as $k[S]$-module. By applying [11, Theorem 2.7] to them, we have

$$e_{\mathrm{HK}}(\mathfrak{q}, k[S]) = e_{\mathrm{HK}}(\mathfrak{q}k[\bar{S}], k[\bar{S}]).$$

Note that $k[\bar{S}]$ is normal. By the above lemma, we can extend Watanabe's result ([10]) of rationality of the Hilbert-Kunz multiplicity for normal semigroup rings to that for general semigroup rings;

COROLLARY 2.1. *The Hilbert-Kunz multiplicity for semigroup rings is always rational.*

We will give a way to compute the Hilbert-Kunz multiplicity for semigroup rings.

THEOREM 2.2. *Let S be a semigroup and $a_1, \cdots, a_v \in S(\subset \mathbf{Z}^N)$ elements such that $k[S]/\mathfrak{q}$ is finite length where $\mathfrak{q} = (x^{a_1}, \cdots, x^{a_v})$. Let C denote the convex rational polyhedral cone spanned by S in $\mathbf{R}^N$ and $\mathcal{P} = \{p \in C \mid p \notin a_j + C$ for each $j\}$. Then*

$$e_{\mathrm{HK}}(\mathfrak{q}, k[S]) = vol\,\bar{\mathcal{P}},$$

where $\bar{\mathcal{P}}$ is the closure of $\mathcal{P}$ and vol denote the relative volume ([8, p. 239]).

PROOF. We may assume S is normal. Let $d = \dim k[S]$ and $n\mathcal{P} = \{p \in C \mid p \notin na_j + C$ for each $j\}$. Since $C \cap \mathbf{Z}^N = S$,

$$n\mathcal{P} \cap \mathbf{Z}^N = \{p \in S \mid p \text{ in not of the form } na_j + b \text{ for } \exists b \in S\}.$$

Thus $l_{k[S]}(k[S]/\mathfrak{q}^{[n]}) = |n\mathcal{P} \cap \mathbf{Z}^N|$. If $\lim_{n\to\infty} \frac{|n\mathcal{P}\cap\mathbf{Z}^N|}{n^d} = vol\,\bar{\mathcal{P}}$, we finish the proof.

We will prove the above equality. Let P be a rational polytope of dimension d containing the origin as a vertex. Put $i(P, n) = |nP \cap \mathbf{Z}^N|$. Then $\lim_{n\to\infty} \frac{i(P,n)}{n^d}$ exists ([7, Theorem 2.8], [8, p. 273, Ex. 33]). Further, there is $c > 0$ such that cP is integral. Since cP is integral,

$\lim_{n\to\infty}\frac{i(cP,n)}{n^d}$ is equal to $vol\,cP$ ([8, Proposition 4.6.30]). Hence

$$\begin{aligned} vol\ P &= \frac{1}{c^d} vol\,cP = \lim_{n\to\infty}\frac{|ncP\cap \mathbf{Z}^N|}{(nc)^d} \\ &= \lim_{n\to\infty}\frac{|nP\cap \mathbf{Z}^N|}{n^d} = \lim_{n\to\infty}\frac{i(P,n)}{n^d}\,. \end{aligned}$$

Let $b_1, b_2, \cdots, b_u$ be generators of C i.e. $C = \{\sum d_j b_j \mid d_j \geq 0 \text{ for each } j\}$. Let $F_1, F_2, \cdots, F_w$ are cones divided by all hyperplanes spanned by a_i and b_j with $a_i \neq b_j$. Then the closure of the complement of $a_i + C$ is a finite union of convex sets $\overline{F_j - (a_i + C)}$. Hence $\mathcal{P} = C \cap (\bigcap_i \overline{(a_i + C)^c})$ is a finite union of rational polytopes $P_1, P_2, \cdots, P_s$ containing the origin such that $P_j \cap P_{j'}$ is rational polytope of dimension $< d$ if $j \neq j'$. We will prove $\lim_{n\to\infty}\frac{i(\bar{\mathcal{P}},n)}{n^d} = \sum_j \lim_{n\to\infty}\frac{i(P_j,n)}{n^d}$. Let $P_{\leq j} = P_1 \cup \cdots \cup P_j$ for each j. Since

$$i(P_{\leq j_0}, n) = i(P_{\leq j_0 - 1}, n) + i(P_{j_0}, n) - i(P_{\leq j_0 - 1} \cap P_{j_0}, n)$$

and $\lim_{n\to\infty}\frac{i(P_{\leq j_0-1}\cap P_{j_0},n)}{n^{d-1}}$ is finite, we have

$$\lim_{n\to\infty}\frac{i(P_{\leq j_0},n)}{n^d} = \lim_{n\to\infty}\frac{i(P_{\leq j_0-1},n)}{n^d} + \lim_{n\to\infty}\frac{i(P_{j_0},n)}{n^d}\,.$$

The claim follows from this. Further, we have $vol\,\bar{\mathcal{P}} = \sum_j vol\,P_j$. Therefore, we conclude $vol\,\bar{\mathcal{P}} = \lim_{n\to\infty}\frac{i(P,n)}{n^d}$. Q.E.D.

EXAMPLE. Let $A = k[X,Y,Z,W]/(XW - YZ)$. Then $A \cong k[X,Y,Z] \oplus W\,k[Y,Z,W]$ as k-vector space. Hence $e(A) = 2$. There is a grading with

$$\deg X = (1,0,0)\,,\quad \deg Y = (0,1,0)\,,\quad \deg Z = (0,0,1) \quad\text{and}\quad \deg W = (-1,1,1)\,.$$

Thus, by considering $\mathcal{P}$, we have $e_{\mathrm{HK}}(A) = 4/3$.

By the following corollary, we can easily compute the Hilbert-Kunz multiplicity of the semigroup rings in special case. It also directly follows from [11, Theorem 2.7]. And it also can follow from the above theorem, by putting $C = \mathbf{N}_0^d$ and by noting $vol\,\bar{\mathcal{P}} = \frac{1}{\delta} l_R(R/J)$.

COROLLARY 2.3. *Let $S \subset \mathbf{N}_0^d = \bigoplus_{i=1}^d \mathbf{N}_0 e_i$ be an affine semigroup such that there is $c_i > 0$ with $c_i e_i \in S$ for each i and $\delta = |\mathbf{Z}^d/\mathbf{Z}S| < \infty$ and $J = (x^\alpha \mid \alpha \in S)$ the ideal of $R = k[X_1, \cdots, X_d]$, where $\mathbf{N}_0 = \{0,1,2,\cdots\}$. Then $e_{\mathrm{HK}}(k[S]) = \frac{1}{\delta} l_R(R/J)$.*

EXAMPLE. Let $A = k[s^4, s^3t, st^3, t^4]$. Then

$$e_{\mathrm{HK}}(A) = \frac{1}{4} l_{k[s,t]}(k[s,t]/(s^4, s^3t, st^3, t^4)) = \frac{11}{4}\,.$$

EXAMPLE (Veronese subrings cf [11, Example 2.8]). Let $R = k[X_1, \cdots, X_d]$, $\mathfrak{m}_R = (X_1, \cdots, X_d)$, and $A = R^{(c)}$ the Veronese subring of R, which is generated by all monomials of degree $c > 0$. Put $A_0 = A$ and $A_i \subset R$ be an A-module generated by all monomials of degree i for $i = 1, 2, \cdots, c-1$. Further, let $W \subset \mathbf{Z}^d \oplus \mathbf{Z}/c\mathbf{Z}$ generated by $(e_i, 1)$ for

$i = 1, 2, \cdots, d$. Then $W \cong \mathbf{Z}^d$ and the monoid ring defined by W is isomorphic to R and equal to $\bigoplus_{i=0}^{c-1} A_i$ as A_0-module. Hence, for the maximal ideal $\mathfrak{m}_A$ of A, we have

$$e(\mathfrak{m}_A^a, A) = \frac{1}{c} e(\mathfrak{m}_R^{ac}, R) = a^d c^{d-1},$$

$$e_{\mathrm{HK}}(\mathfrak{m}_A^a, A) = \frac{1}{c} e_{\mathrm{HK}}(\mathfrak{m}_R^{ac}, R) = \frac{1}{c} l_R(R/\mathfrak{m}_R^{ac}) = \frac{1}{c}\binom{d+ac-1}{d}.$$

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