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Multiplicity of positive solutions to second-order singular differential equations with a parameter

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Abstract

We study the existence and multiplicity of positive periodic solutions for second-order nonlinear damped differential equations by combing the analysis of positiveness of the Green function for a linear damped equation. Our nonlinearity may be singular in its dependent variable. The proof of the main result relies on the Guo-Krasnosel'skii fixed point theorem on compression and expansion of cones.

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1 Introduction

In this paper, we study the existence and multiplicity of positive T -periodic solutions for the following second-order singular differential equation:

$$-x'' + a(t)x' + b(t)x = \lambda f(t, x, x'), \quad 0 \leq t \leq T, \quad (1.1)$$

where $\lambda > 0$ is a positive parameter, $a, b \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and the nonlinearity $f \in ((\mathbb{R}/T\mathbb{Z}) \times (0, +\infty) \times \mathbb{R}, \mathbb{R})$. In particular, the nonlinearity may change sign and have a repulsive singularity at $x = 0$, which means that

$$\lim_{x \rightarrow 0^+} f(t, x, y) = +\infty \quad \text{uniformly in } (t, y) \in \mathbb{R}^2.$$

Equation (1.1) is a particular case of a more general class of Sturm equations of the type

$$-[p(t)x']' + q(t)x = f(t, x),$$

where p is a strictly positive absolutely continuous function. Such equations, even in the case $p \equiv 1$, where they are referred to as being of Schrödinger or Klein-Gordon type, appear in many scientific areas including quantum field theory, gas dynamics, fluid mechanics and chemistry.

Electrostatic or gravitational forces are the most important examples of singular interactions. During the last few decades, the study of the existence of positive solutions for singular differential equations has deserved the attention of many researchers [1–7]. Some

strong force conditions introduced by Gordon [8] are standard in the related earlier works [6, 7, 9]. Compared with the case of strong singularity, the study of the existence of periodic solutions under the presence of a weak singularity is more recent, but has also attracted many researchers [2, 3, 10, 11]. In particular, the degree theory [6, 7], the method of upper and lower solutions [11, 12], Schauder's fixed point theorem [2, 10], some fixed point theorems in cones for completely continuous operators [13–16] and a nonlinear Leray-Schauder alternative principle [17–19] are the most relevant tools.

However, singular differential equation (1.1), in which the nonlinearity is dependent on the derivative and does not require f to be nonnegative, has not attracted much attention in the literature. There are not so many existence results for (1.1) even when the nonlinearity is independent of the derivative. In this paper, we try to fill this gap and establish the existence of positive T -periodic solutions of (1.1) using the Guo-Krasnosel'skii fixed point theorem on compression and expansion of cones, which has been used to study positive solutions for systems of ordinary, functional differential equations [14–16]. We remark that it is sufficient to prove that $T : K \cap (\tilde{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is continuous and completely continuous in Lemma 3.2 (Section 3). This point is essential and advantageous.

As mentioned above, this paper is mainly motivated by the recent paper [14, 16]. The aim of this paper is to study the multiplicity of positive solutions to (1.1). It is proved that such a problem has at least two positive solutions under reasonable conditions (see Theorem 3.5). And the remaining part of this paper is organized as follows. In Section 2, we find the Green function of the linear damped equation

$$-x'' + a(t)x' + b(t)x = 0, \quad 0 \leq t \leq T, \tag{1.2}$$

subject to periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{1.3}$$

and prove its positiveness. The fact is very crucial to our arguments. Moreover, the one-signed property of the Green function implies that a maximum principle and an anti-maximum principle hold for the corresponding linear differential equations subject to various boundary conditions, which is an important topic in differential equations (see [20, 21]). In Section 3, by employing the Guo-Krasnosel'skii fixed point theorem, we prove the existence of twin positive solutions for (1.1) under the positiveness of the Green function associated with (1.2)-(1.3). To illustrate the new results, some applications are also given.

2 The Green function and its positiveness

In this section, we consider the nonhomogeneous equation

$$-x'' + a(t)x' + b(t)x = h(t), \quad 0 \leq t \leq T. \tag{2.1}$$

We say that (1.2)-(1.3) is nonresonant if its unique T -periodic solution is the trivial one. When (1.2)-(1.3) is nonresonant, as a consequence of Fredholm's alternative, equation (2.1) admits a unique T -periodic solution which can be written as

$$x(t) = \int_0^T G(t,s)h(s) ds,$$

where $h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$, $G(t, s)$ is the Green function of (2.1), associated with (1.3), and we will prove its positiveness. Throughout this paper, we assume that the following condition is satisfied:

(H) $a(t), b(t)$ are continuous functions and $b(t) > 0$ for all $t \in [0, T]$.

Lemma 2.1 *Let $K(s, \tau) : [0, T] \times [0, T] \rightarrow [0, +\infty)$ be a continuous function. Then, for any nonnegative continuous function $\varphi(t)$ defined on $[0, T]$, the integral equation*

$$x(t) = \varphi(t) + \int_0^t \int_0^s K(s, \tau)x(\tau) d\tau ds, \quad t \in [0, T] \tag{2.2}$$

has a unique solution $x(t)$, which is continuous on $[0, T]$ and satisfies the following inequality:

$$x(t) \geq \varphi(t), \quad t \in [0, T]. \tag{2.3}$$

Proof We solve equation (2.2) by the method of successive approximations. Let

$$x_0(t) = \varphi(t), \quad x_n(t) = \int_0^t \int_0^s K(s, \tau)x_{n-1}(\tau) d\tau ds, \quad n = 1, 2, \dots$$

Take

$$\alpha_1 = \max_{0 \leq t \leq T} \varphi(t), \quad \beta_1 = \max_{0 \leq s, \tau \leq T} K(s, \tau).$$

One can easily verify that

$$0 \leq x_n(t) \leq \frac{\alpha_1 \beta_1^n}{(2n)!} t^{2n}, \quad n = 1, 2, \dots,$$

which implies that the series $\sum_{n=0}^\infty x_n(t)$ converges uniformly with respect to $t \in [0, T]$. Obviously, $x(t) = \sum_{n=0}^\infty x_n(t)$ is a continuous solution of (2.2). Moreover, inequality (2.3) holds because $\varphi(t)$ and $K(s, \tau)$ are nonnegative functions.

Next we prove the uniqueness. To do so, we first show that the solution of (2.2) is unique on $[t_1, t_2] \subset [0, T]$ with $t_2 < \sqrt{2/\beta_1}$. Then the uniqueness of the solution on $[0, T]$ is direct using the continuation property.

On the contrary, suppose that (2.2) has two solutions x_1 and x_2 on $[t_1, t_2]$. Then, for $t \in [t_1, t_2]$, we have

$$\begin{aligned} |x_1(t) - x_2(t)| &= \left| \int_0^t \int_0^s K(s, \tau)(x_1(\tau) - x_2(\tau)) d\tau ds \right| \\ &\leq \int_0^t \int_0^s K(s, \tau)|x_1(\tau) - x_2(\tau)| d\tau ds \\ &\leq \max_{t \in [t_1, t_2]} |x_1(t) - x_2(t)| \beta_1 t^2 / 2 \\ &< \max_{t \in [t_1, t_2]} |x_1(t) - x_2(t)|. \end{aligned}$$

Hence it follows that $x_1(t) = x_2(t)$ for all $t \in [t_1, t_2]$. □

Let us denote by $u(t)$ and $v(t)$ the solutions of (1.2) satisfying the initial conditions

$$u(0) = 1, \quad u'(0) = 0; \quad v(0) = 0, \quad v'(0) = 1.$$

Lemma 2.2 $u(t)$ and $v(t)$ satisfy the following integral equations:

$$u(t) = 1 + \int_0^t \int_0^s b(\tau) \exp\left(\int_\tau^s a(r) dr\right) u(\tau) d\tau ds, \tag{2.4}$$

$$v(t) = \int_0^t \exp\left(\int_0^s a(r) dr\right) ds + \int_0^t \int_0^s b(\tau) \exp\left(\int_\tau^s a(r) dr\right) v(\tau) d\tau ds. \tag{2.5}$$

Proof Since $u(t)$ is a solution of (1.2), we have

$$\left(u'(t) \exp\left(-\int_0^t a(r) dr\right)\right)' = b(t) \exp\left(-\int_0^t a(r) dr\right) u(t). \tag{2.6}$$

Integrating (2.6) from 0 to t and noticing $u'(0) = 0$, we obtain

$$u'(t) = \int_0^t b(\tau) \exp\left(\int_\tau^t a(r) dr\right) u(\tau) d\tau.$$

To obtain (2.4), we only need to integrate the above equality from 0 to t and notice $u(0) = 1$. In a similar way, we can prove (2.5). □

Lemma 2.3 For the solution $x(t)$ of boundary value problem (2.1)-(1.3), the formula

$$x(t) = \int_0^T G(t,s)h(s) ds, \quad t \in [0, T], \tag{2.7}$$

holds, where

$$G(t,s) = \frac{v(T)}{D} u(t)u(s) - \frac{u'(T)}{D} v(t)v(s) + \begin{cases} \frac{v'(T)-1}{D} u(t)v(s) - \frac{u(T)-1}{D} u(s)v(t), & 0 \leq s \leq t \leq T, \\ \frac{v'(T)-1}{D} u(s)v(t) - \frac{u(T)-1}{D} u(t)v(s), & 0 \leq t \leq s \leq T, \end{cases} \tag{2.8}$$

is the Green function, the number D is defined by $D = u(T) + v'(T) - 2$.

Proof It is easy to see that the general solution of equation (2.1) has the form

$$x(t) = \alpha_2 u(t) + \beta_2 v(t) + \int_0^t (u(t)v(s) - u(s)v(t))h(s) ds,$$

where α_2 and β_2 are arbitrary constants. Substituting this expression for $x(t)$ in boundary condition (1.3), we can obtain

$$\alpha_2 = \alpha_2 u(T) + \beta_2 v(T) + u(T) \int_0^T v(s)h(s) ds - v(T) \int_0^T u(s)h(s) ds,$$

$$\beta_2 = \alpha_2 u'(T) + \beta_2 v'(T) + u'(T) \int_0^T v(s)h(s) ds - v'(T) \int_0^T u(s)h(s) ds.$$

After not very complicated calculations, we can get (2.7) and (2.8). □

Remark 2.4 As a direct application of Lemma 2.3, if $a(t) = 0$, $b(t) = k^2 > 0$, then the Green function $G(t, s)$ of boundary value problem (2.1)-(1.3) has the form

$$G(t, s) = \begin{cases} \frac{e^{k(t-s)} + e^{k(T-t+s)}}{2k(e^{kT}-1)}, & 0 \leq s \leq t \leq T, \\ \frac{e^{k(s-t)} + e^{k(T+t-s)}}{2k(e^{kT}-1)}, & 0 \leq t \leq s \leq T. \end{cases}$$

Lemma 2.5 Assume that (H) holds. Then the Green function $G(t, s)$ associated with (2.1)-(1.3) is positive for all $s, t \in [0, T]$.

Proof Since $G(t, s) = G(s, t)$, it is enough to prove that $G(t, s) > 0$ for $0 \leq s \leq t < T$. Recall that $u(t)$ and $v(t)$ satisfy integral equations (2.4) and (2.5). By condition (H) and Lemma 2.1, it follows that

$$\begin{aligned} u(t) &\geq 1, & v(t) &\geq \int_0^t \exp\left(\int_0^s a(r) dr\right) ds > 0, & t \in [0, T], \\ u(T) &> 1, & v'(T) &> 1. \end{aligned} \tag{2.9}$$

Now from (2.9) we get $D = u(T) + v'(T) - 2 > 0$. Setting

$$\begin{aligned} E_1(t, s) &= u(s)v(t) - u(t)v(s), \\ E_2(t, s) &= (v(T)u(t) - u(T)v(t))u(s) + (v'(T)u(t) - u'(T)v(t))v(s), \end{aligned}$$

for $s \leq t$, we have

$$G(t, s) = \frac{1}{D}(E_1(t, s) + E_2(t, s)).$$

Evidently, $E_2(T, 0) = 0$, $E_1(s, s) = 0$ for $s \in [0, T]$ holds. Let us now show that

$$E_1(t, s) > 0 \quad \text{for } s \in [0, T] \text{ and } t \in (s, T], \tag{2.10}$$

$$E_2(t, s) > 0 \quad \text{for } s \in [0, T], t \in [s, T] \text{ and } (t, s) \neq (T, 0). \tag{2.11}$$

To prove (2.10), we note that for fixed $s \in [0, T]$,

$$\frac{\partial^2 E_1(t, s)}{\partial t^2} = a(t) \frac{\partial E_1(t, s)}{\partial t} + b(t)E_1(t, s)$$

and

$$E_1(s, s) = 0, \quad \left. \frac{\partial E_1(t, s)}{\partial t} \right|_{t=s} = 1.$$

Hence it follows that for all $s \in (t, T]$, we have

$$\begin{aligned} E_1(t, s) &= \int_0^t \exp\left(\int_0^s a(r) dr\right) ds \\ &\quad + \int_0^t \int_0^\tau b(\omega) \exp\left(\int_\omega^\tau a(r) dr\right) E_1(\omega, t) d\omega d\tau. \end{aligned} \tag{2.12}$$

Using Lemma 2.1, we get from (2.12) that $E_1(t, s) > 0$ for all $t \in (s, T]$.

Next, we prove (2.11), note that for fixed $s \in [0, T]$,

$$\frac{\partial^2 E_2(t, s)}{\partial t^2} = a(t) \frac{\partial E_2(t, s)}{\partial t} + b(t) E_2(t, s)$$

and

$$E_2(T, s) = v(s), \quad \left. \frac{\partial E_2(t, s)}{\partial t} \right|_{t=T} = -u(s).$$

Hence it follows that, for all $t \in [s, T]$, we have

$$\begin{aligned} E_2(t, s) &= v(s) + u(s) \int_t^T \exp\left(\int_0^s a(r) dr\right) ds \\ &\quad + \int_t^T \int_0^\tau b(\omega) \exp\left(\int_\omega^\tau a(r) dr\right) E_2(\omega, t) d\omega d\tau. \end{aligned} \tag{2.13}$$

Again using Lemma 2.1, we get from (2.13) that $E_2(t, s) > 0$ if $(t, s) \neq (T, 0)$, and the proof is completed. \square

Under hypothesis (H), we always define

$$A = \min_{0 \leq s, t \leq T} G(t, s), \quad B = \max_{0 \leq s, t \leq T} G(t, s), \quad \sigma = A/B. \tag{2.14}$$

Thus $B > A > 0$ and $0 < \sigma < 1$. When $a(t) = 0$, $b(t) = k^2 > 0$, and a direct calculation shows that

$$A = \frac{e^{kT/2}}{k(e^{kT} - 1)}, \quad B = \frac{1 + e^{kT}}{2k(e^{kT} - 1)}, \quad \sigma = \frac{2e^{kT/2}}{1 + e^{kT}} < 1.$$

3 Main results

In this section, we state and prove the new existence results for (1.1). The proof is based on the following well-known fixed point theorem on compression and expansion of cones, which we state here for the convenience of the reader, after introducing the definition of a cone.

Definition 3.1 Let X be a Banach space and let K be a closed, nonempty subset of X . K is a cone if

- (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$,
- (ii) $u, -u \in K$ implies $u = 0$.

We also recall that a compact operator means an operator which transforms every bounded set into a relatively compact set. Let us define the function $\omega(t) = \lambda \int_0^T G(t, s) ds$ and use $\|\cdot\|_1$ to denote the usual L^1 -norm over $(0, T)$, by $\|\cdot\|$ we denote the supremum norm of $C[0, T]$.

Lemma 3.2 [22] *Let X be a Banach space and $K (\subset X)$ be a cone. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$\mathcal{A} : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

be a completely continuous operator such that either

- (i) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$; or
- (ii) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then \mathcal{A} has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Let $X = C[0, T]$ and define

$$K = \left\{ x \in X : x(t) \geq 0 \text{ and } \min_{0 \leq t \leq T} x(t) \geq \sigma \|x\| \right\},$$

where σ is as in (2.14).

One may readily verify that K is a cone in X . Now, suppose that $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ is a continuous function. Define an operator

$$(\mathcal{A}x)(t) = \int_0^T G(t, s)F(s, x(s), x'(s)) ds$$

for $x \in X$ and $t \in [0, T]$.

Lemma 3.3 $\mathcal{A} : X \rightarrow K$ is well defined.

Proof Let $x \in X$, then we have

$$\begin{aligned} \min_{0 \leq t \leq T} (\mathcal{A}x)(t) &= \min_{0 \leq t \leq T} \int_0^T G(t, s)F(s, x(s), x'(s)) ds \\ &\geq A \int_0^T F(s, x(s), x'(s)) ds \\ &= \sigma B \int_0^T F(s, x(s), x'(s)) ds \\ &\geq \sigma \max_{0 \leq t \leq T} \int_0^T G(t, s)F(s, x(s), x'(s)) ds \\ &= \sigma \|\mathcal{A}x\|. \end{aligned}$$

This implies that $\mathcal{A}(X) \subset K$ and the proof is completed. □

It is easy to prove.

Lemma 3.4 \mathcal{A} is continuous and completely continuous.

Now we present our main result.

Theorem 3.5 Suppose that (1.1) satisfies (H). Furthermore, assume that

(H₁) $f : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a constant $M > 0$ such that

$$F(t, x, y) = f(t, x, y) + M \geq 0 \quad \text{for all } (t, x, y) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}.$$

(H₂) $\lim_{x \rightarrow 0^+} f(t, x, y) = +\infty$ and $\lim_{x \rightarrow +\infty} f(t, x, y)/x = +\infty$ uniformly $(t, y) \in \mathbb{R}^2$.

Then (1.1) has at least two positive T -periodic solutions for sufficiently small λ .

Proof To show that (1.1) has a positive solution, we should only show that

$$-x'' + a(t)x' + b(t)x = \lambda F(t, x(t) - M\omega(t), x'(t) - M\omega'(t)) \tag{3.1}$$

has a positive solution x satisfying (1.3) and $x(t) > M\omega(t)$ for $t \in [0, T]$. If it is right, then $\phi(t) = x(t) - M\omega(t)$ is a solution of (1.1) since

$$\begin{aligned} & -\phi''(t) + a(t)\phi'(t) + b(t)\phi(t) \\ &= -(x(t) - M\omega(t))'' + a(t)(x(t) - M\omega(t))' + b(t)(x(t) - M\omega(t)) \\ &= \lambda F(t, x(t) - M\omega(t), x'(t) - M\omega'(t)) - \lambda M \\ &= \lambda f(t, x(t) - M\omega(t), x'(t) - M\omega'(t)) \\ &= \lambda f(t, \phi(t), \phi'(t)), \end{aligned}$$

where $-\omega''(t) + a(t)\omega'(t) + b(t)\omega(t) = \lambda$ is used.

Problem (3.1)-(1.3) is equivalent to the following fixed point of the operator equation:

$$x(t) = (\mathcal{A}x)(t),$$

where \mathcal{A} is a completely continuous operator defined by

$$(\mathcal{A}x)(t) = \lambda \int_0^T G(t, s)F(s, x(s) - M\omega(s), x'(s) - M\omega'(s)) ds.$$

Since $\lim_{x \rightarrow +\infty} \frac{f(t, x, y)}{x} = +\infty$, there exists $r_1 \geq MT$ such that

$$x \geq \sigma r_1 \Rightarrow \frac{f(t, x, y)}{x} \geq \frac{1}{\sigma}.$$

For $r > 0$, let $\Omega_r = \{x \in K : \|x\| < r\}$ and note that $\partial\Omega_r = \{x \in K : \|x\| = r\}$.

First we show

$$\|\mathcal{A}x\| \leq \|x\| \quad \text{for } x \in K \cap \partial\Omega_{r_1}.$$

Let $h(t) = \max\{f(t, x, x') : \frac{\sigma}{2}r_1 \leq x \leq r_1\}$ and $\lambda^* = \min\{\sigma^2/2A, MT/B\|h\|_1\}$. For any $x \in \partial\Omega_{r_1}$ and $0 < \lambda < \lambda^*$, we can verify that

$$\begin{aligned} x(t) - M\omega(t) &\geq \sigma \|x\| - M\omega(t) = \sigma r_1 - M\omega(t) \\ &\geq \sigma r_1 - \lambda MBT \geq \sigma r_1 - \frac{\sigma r_1}{2} = \frac{\sigma r_1}{2}. \end{aligned}$$

Then we have

$$\begin{aligned} (\mathcal{A}x)(t) &= \lambda \int_0^T G(t, s)F(s, x(s) - M\omega(s), x'(s) - M\omega'(s)) ds \\ &\leq \lambda B \int_0^T F(s, x(s) - M\omega(s), x'(s) - M\omega'(s)) ds \end{aligned}$$

$$\begin{aligned} &\leq \lambda B \|h\|_1 \\ &\leq r_1 = \|x\|. \end{aligned}$$

This implies $\|\mathcal{A}x\| \leq \|x\|$.

In view of the assumption

$$\lim_{x \rightarrow +\infty} \frac{f(t, x, x')}{x} = \lim_{x \rightarrow +\infty} \frac{F(t, x - M\omega, x' - M\omega')}{x - M\omega'} = +\infty,$$

then there is $r_2 > \sigma r_2 > r_1$ such that

$$\frac{F(t, x - M\omega, x' - M\omega')}{x - M\omega} \geq \lambda^{-1} \sigma^{-1} A^{-1} T^{-1}, \quad x \geq \sigma r_2.$$

Hence, we have

$$F(t, x - M\omega, x' - M\omega') \geq \lambda^{-1} A^{-1} T^{-1} r_2, \quad x \geq \sigma r_2.$$

Next, we show that

$$\|\mathcal{A}x\| \geq \|x\| \quad \text{for } x \in K \cap \partial\Omega_{r_2}.$$

To see this, let $x \in K \cap \partial\Omega_{r_2}$, then

$$\begin{aligned} \|\mathcal{A}x\| &= \max_{0 \leq t \leq T} \left| \lambda \int_0^T G(t, s) F(s, x(s) - M\omega(s), x'(s) - M\omega'(s)) ds \right| \\ &\geq \lambda A \int_0^T F(s, x(s) - M\omega(s), x'(s) - M\omega'(s)) ds \\ &\geq \lambda A \int_0^T \lambda^{-1} A^{-1} T^{-1} r_2 ds \\ &\geq r_2 = \|x\|. \end{aligned}$$

It follows from Lemma 3.2 that \mathcal{A} has a fixed point $\tilde{x}_1(t)$ such that $\tilde{x}_1(t) \in \bar{\Omega}_{r_2} \setminus \Omega_{r_1}$, which is a positive periodic solution of (3.1) for $\lambda < \lambda^*$ satisfying

$$r_1 < \|\tilde{x}_1\| < r_2.$$

So, equation (1.1) has a positive solution $x_1(t) = \tilde{x}_1(t) - M\omega(t) \geq \sigma r_1 - \frac{\sigma MT}{2} \geq \frac{\sigma MT}{2}$.

On the other hand, since

$$\lim_{x \rightarrow 0^+} f(t, x, y) = +\infty,$$

hence, there exists a positive number $0 < r_3 < r_1$ such that

$$f(t, x, y) > 0, \quad x \in \mathbb{R}^+ \text{ with } 0 < x \leq r_3 < \frac{\sigma MT}{2},$$

problem (1.1)-(1.3) is equivalent to the following fixed point of the operator equation:

$$x(t) = (\mathcal{A}'x)(t),$$

where \mathcal{A}' is a continuous and completely continuous operator defined by

$$(\mathcal{A}'x)(t) = \lambda \int_0^T G(t,s)f(s,x(s),x'(s)) ds.$$

And for any $\rho > 0$, define

$$\Delta(\rho) = \max\{f(t,x,y) : x \in \mathbb{R}^+, \sigma\rho \leq x \leq \rho, (t,y) \in [0, T] \times \mathbb{R}\}.$$

Furthermore, for any $x \in K \cap \partial\Omega_{r_3}$, we have

$$\begin{aligned} (\mathcal{A}'x)(t) &= \lambda \int_0^T G(t,s)f(s,x(s),x'(s)) ds \\ &\leq \lambda B \int_0^T f(s,x(s),x'(s)) ds \\ &\leq \lambda B \Delta(r_3)T. \end{aligned}$$

Thus, from the above inequality, there exists λ^{**} such that

$$\|\mathcal{A}'x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_3}, 0 < \lambda < \lambda^{**}.$$

Since $\lim_{x \rightarrow 0^+} f(t,x,x') = +\infty$, then there is a positive number $0 < r_4 < \sigma r_3 < r_3$ such that

$$f(t,x,x') > \gamma x, \quad x \in \mathbb{R}^+ \text{ with } 0 < x \leq r_4,$$

where γ satisfies $\lambda\gamma\sigma AT > 1$.

If $x \in K \cap \partial\Omega_{r_4}$, then

$$\begin{aligned} \|\mathcal{A}'x\| &= \max_{0 \leq t \leq T} \left| \lambda \int_0^T G(t,s)f(s,x(s),x'(s)) ds \right| \\ &\geq \lambda A \int_0^T f(s,x(s),x'(s)) ds \\ &\geq \lambda A \int_0^T \gamma x ds \\ &\geq \lambda A \int_0^T \gamma \sigma \|x\| ds \\ &\geq \|x\|. \end{aligned}$$

It follows from Lemma 3.2 that \mathcal{A}' has a fixed point $x_2(t)$ such that $x_2(t) \in \bar{\Omega}_{r_3} \setminus \Omega_{r_4}$, which is a positive periodic solution of (1.1) for $\lambda < \lambda^{**}$ satisfying

$$r_4 < \|x_2\| < r_3.$$

Noting that

$$r_4 < \|x_2\| < r_3 < \frac{\sigma r_1}{2} < \|x_1\| < r_2,$$

we can conclude that x_1 and x_2 are the desired distinct positive periodic solutions of (1.1) for $\lambda < \min\{\lambda^*, \lambda^{**}\}$. \square

Example Let the nonlinearity in (1.1) be

$$f(t, x, y) = (1 + |y|^\gamma)(c(t)x^{-\alpha} + d(t)x^\beta + e(t)), \quad 0 \leq t \leq T,$$

where $\alpha > 0$, $\beta > 1$, $\gamma \geq 0$, $c(t), d(t), e(t) \in \mathbb{C}[0, T]$. It is clear that $f(t, x, y)$ satisfies conditions (H_1) , (H_2) . Then (1.1) has at least two positive T -periodic solutions for sufficiently small λ .

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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