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MULTIPLICITY OF SOLUTIONS FOR THE NONCOOPERATIVE p-LAPLACIAN OPERATOR ELLIPTIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this paper, we study the multiplicity of solutions for a class of noncooperative p-Laplacian operator elliptic system. Under suitable assumptions, we obtain a sequence of solutions by using the limit index theory.

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1. Introduction

In this paper we deal with the existence and multiplicity of solutions to the following p-Laplacian operator elliptic system with nonlinear boundary conditions.

$$\begin{cases} \Delta_p u - |u|^{p-2}u = F_u(x,u,v), & \text{in } \Omega, \\ -\Delta_p v + |v|^{p-2}v = F_v(x,u,v), & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = |u|^{p^*-2}u, \quad |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = |v|^{p^*-2}v, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < N$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is a p-Laplacian operator and $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $F = F(x,u,v)$, $F_u = \frac{\partial F}{\partial u}$, $F_v = \frac{\partial F}{\partial v}$, $p^* = Np/(N-p)$ is the critical exponent according to the Sobolev embedding.

In recent years, the existence and multiplicity of solutions for a noncooperative elliptic system have been obtained by many papers. In [1], Benci assumed X is a Hilbert space, f satisfies (PS)-condition and is the form

$$f(u) = \frac{1}{2}\langle Lu, u\rangle + \Phi(u),$$

where L is bounded self-adjoint operator and Φ' is compact.

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When $p = 2$ (a constant) with Dirichlet boundary condition, Lin and Li [9] considered the following system

$$\begin{cases} \Delta u = |u|^{2^*-2}u + F_u(x,u,v), & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + F_v(x,u,v), & \text{in } \Omega, \\ u = 0, \quad v = 0, & \text{on } \partial\Omega, \end{cases}$$

by applying the Limit Index Theory, they obtained the existence of multiple solutions under some assumptions on nonlinear part.

When $p \neq 2$, Huang and Li [6] considered the following the system of elliptic equations involving the p-Laplacian in the unbounded domain of $\mathbb{R}^N$ by applying the Limit Index Theory,

$$\begin{cases} \Delta_p u - |u|^{p-2}u = F_u(x,u,v), & \text{in } \mathbb{R}^N, \\ -\Delta_p v + |v|^{p-2}v = F_v(x,u,v), & \text{in } \mathbb{R}^N, \\ u, v \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

where $1 < p < N$ and they extended some results of [8].

We note that these papers deal with Dirichlet boundary condition [2,7]. However, nonlinear boundary conditions have only been considered in recent years. For the Laplace operator with nonlinear boundary conditions see for example [3, 14]. For elliptic systems with nonlinear boundary conditions see [5]. For previous work for the p-Laplacian with nonlinear boundary conditions of different type see [4, 13].

Motivated by papers above, a natural question arises whether the existence and multiplicity of solutions to the p-Laplacian operator elliptic system with nonlinear boundary conditions (1.1) can be obtained. In this paper we deal with the problem (1.1). Throughout this paper, we assume that $F(x,u,v)$ satisfies the following conditions:

(H_1) $F \in C(\overline{\Omega} \times R^2, R^+)$ and $F(x,s,t) = F(x,-s,-t)$ for all $(x,s,t) \in \Omega \times \in \mathbb{R}^2$;
(H_2) $\lim_{|t|\to\infty} \frac{F_t(x,s,t)}{|t|^{p-1}} = 0$ uniformly for $x \in \Omega$;
(H_3) $sF_s(x,s,t) \geq 0$ for all $(x,s,t) \in \overline{\Omega} \times R^2$.

Under assumptions (H_1) and (H_2), we have

$$F_v(x,u,v)v = o\left(|v|^p\right),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon),\ b(\varepsilon) > 0$ such that

$$|F(x,0,v)v| \leq a(\varepsilon) + \varepsilon|v|^p \tag{1.2}$$

and

$$|F_v(x,u,v)v| \leq b(\varepsilon) + \varepsilon|v|^p. \tag{1.3}$$

Hence, together with condition (1.2), (1.3) and the mean value theorem for any constants β and fixed u we have

$$|F(x,u,v) - \beta F_v(x,u,v)v| \leq c(\varepsilon) + \varepsilon|v|^p, \tag{1.4}$$

for some $c(\varepsilon) > 0$.

Furthermore, we assume that $F(x,u,v)$ satisfies condition:

(H_4) There exist $L > 0$ (where L will be determined later) and

$$\xi < |\Omega|^{-1} \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right\}$$

such that $F(x,s,t)t \geq L|t|^p - \xi$, for every $(x,s,t) \in \overline{\Omega} \times \mathbb{R}^2$.

Notation. Weak (resp. strong) convergence is denoted by $\rightharpoonup$ (resp., $\rightarrow$). $|\cdot|_p$ is the usual norm in $L^p(\Omega)$. $L^p_2(\Omega) = L^p(\Omega) \times L^p(\Omega)$ with the norm $|(u,v)|_p := (|u|_p^p + |v|_p^p)^{\frac{1}{p}}$. $E := W^{1,p}(\Omega)$ with the norm $\|u\|_p := \int_\Omega (|\nabla u|^p + |u|^p)\mathrm{d}x$. $Y = E \times E$, $X_n = E \times E_n$. c_i denote a positive constant and can be determined in concrete conditions.

According to [15], there exists a Schauder basis $\{e_n\}_{n=1}^\infty$ for E. Furthermore, since E is reflexive, $\{e_n^*\}_{n=1}^\infty$ the biorthogonal functionals associated to the basis $\{e_n\}_{n=1}^\infty$ which are characterized by the relations

$$e_n^*(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases}$$

form a basis for E^* with the following properties (*cf.* [10] Prop. 1.b.1 and Thm. 1.b.5). Denote

$$E_n = \operatorname{span}\{e_1, \ldots, e_n\}, \quad E_n^\perp = \overline{\operatorname{span}\{e_{n+1}, \ldots\}}$$

and

$$E_n^* = \operatorname{span}\{e_1^*, \ldots, e_n^*\}.$$

Let $P_n : E \to E_n$ be the projector corresponding to decomposition $E = E_n \oplus E_n^\perp$ and $P_n^* : E^* \to E_n^*$ be the projector corresponding to the decomposition $E^* = E_n^* \oplus (E_n^*)^\perp$. Then $P_n u \to u$, $P_n^* v^* \to v^*$ for any $u \in E$, $v^* \in E^*$ as $n \to \infty$ and $\langle P_n^* v^*, u\rangle = \langle v^*, P_n u\rangle$. Let $\tau : E \to E^*$ be the mapping given by

$$\langle \tau u, \widetilde{u}\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \widetilde{u}\mathrm{d}x, \quad \text{for all } u, \widetilde{u} \in E.$$

It is easy to check that the operator τ is bounded, continuous. And if $u_n \to \widetilde{u}$ in E and $\langle \tau u_n - \tau\widetilde{u}, u_n - \widetilde{u}\rangle \to 0$, then $u_n \to \widetilde{u}$ in E (see [6,8])

The energy functional corresponding to problem (1.1) is defined as follows,

$$\begin{aligned} J(u,v) = &-\frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)\mathrm{d}x + \frac{1}{p}\int_\Omega (|\nabla v|^p + |v|^p)\mathrm{d}x \\ &-\frac{1}{p^*}\int_{\partial\Omega} |u|^{p^*}\mathrm{d}\sigma - \frac{1}{p^*}\int_{\partial\Omega} |v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,u,v)\mathrm{d}x. \end{aligned} \tag{1.5}$$

The main result of this paper is as follows.

Theorem 1.1. *Suppose that $F(x,u,v)$ satisfies conditions (H_1)–(H_4). Then there exists $k_0 > 1$ such that (1.1) possesses at least $k_0 - 1$ pairs weak nontrivial solutions.*

Remark 1.2. There are two difficulties in considering the elliptic problem (1.1). One is the functional $J(u,v)$ is strongly indefinite. Therefore one cannot apply the symmetric Mountain pass theorem of the functional $J(u,v)$. The other one in solving the problem is a lack of compactness which can be illustrated by the fact that the embedding of $W^{1,p}(\Omega)$ into $L^{p^*}(\partial\Omega)$ is no longer compact.

Remark 1.3. Theorem 1.1 is new as far as we know. We mainly follow the way in [8] to prove our main result.

2. Preliminaries and lemmas

First of all, we recall the limit index theory due to Li [8]. In order to do that, we introduce the following definitions.

Definition 2.1. [8,16] The action of a topological group G on a normed space Z is a continuous map

$$G \times Z \to Z : [g, z] \mapsto gz$$

such that

$$1 \cdot z = z, \quad (gh)z = g(hz) \quad z \mapsto gz \text{ is linear}, \quad \forall\, g, h \in G.$$

The action is isometric if

$$\|gz\| = \|z\|, \quad \forall\, g \in G, \quad z \in Z.$$

And in this case Z is called G-space.

The set of invariant points is defined by

$$\mathrm{Fix}G := \{z \in Z : gz = z,\ \forall\, g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G$, $z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Suppose Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma := \{A \subset Z : A \text{ is closed and } gA = A, \forall\, g \in G\}$$

be a family of all G-invariant closed subset of Z, and let

$$\Gamma := \left\{h \in C^0(Z, Z) : h(gu) = g(hu), \quad \forall\, g \in G\right\}$$

be the class of all G-equivariant mapping of Z. Finally, we call the set

$$O(u) := \{gu : g \in G\}$$

G-orbit of u.

Definition 2.2. [8] An index for (G, Σ, Γ) is a mapping $i : \Sigma \to \mathcal{Z}_+ \cup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, $h \in \Gamma$, the following conditions are satisfied:

(1) $i(A) = 0 \Leftrightarrow A = \emptyset$;
(2) (monotonicity) $A \subset B \Rightarrow i(A) \leq i(B)$;
(3) (subadditivity) $i(A \cup B) \leq i(A) + i(B)$;
(4) (supervariance) $i(A) \leq i(\overline{h(A)}), \forall\, h \in \Gamma$;
(5) (continuity) If A is compact and $A \cap \mathrm{Fix}G = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighbourhood N of A such that $i(\overline{N}) = i(A)$;
(6) (normalization) If $x \notin \mathrm{Fix}G$, then $i(O(x)) = 1$.

Definition 2.3. [1] An index theory is said to satisfy the d-dimension property if there is a positive integer d such that

$$i(V^{dk} \cap S_1) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}G = \{0\}$, where S_1 is the unit sphere in Z.

Suppose U and V are G-invariant closed subspaces of Z such that

$$Z = U \oplus V,$$

where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \ldots$, and $V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$ Let

$$Z_j = U \bigoplus V_j,$$

and $\forall\ A \in \Sigma$, let

$$A_j = A \bigoplus Z_j.$$

Definition 2.4. [8] Let i be an index theory satisfying the d-dimension property. A limit index with respect to (Z_j) induced by i is a mapping

$$i^\infty : \Sigma \to \mathcal{Z} \cup \{-\infty, +\infty\}$$

given by

$$i^\infty(A) = \lim_{j\to\infty} \sup\,(i(A_j) - n_j).$$

Proposition 2.5. [8] *Let $A, B \in \Sigma$. Then i^∞ satisfies:*
(1) $A = \emptyset \Rightarrow i^\infty = -\infty$;
(2) *(monotonicity)* $A \subset B \Rightarrow i^\infty(A) \le i^\infty(B)$;
(3) *(subadditivity)* $i^\infty(A \cup B) \le i^\infty(A) + i^\infty(B)$;
(4) *If* $V \cap \mathrm{Fix}G = \{0\}$, *then* $i^\infty(S_\rho \cap V) = 0$, *where* $S_\rho = \{z \in Z : \|z\| = \rho\}$;
(5) *If* Y_0 *and* $\widetilde{Y_0}$ *are* G*-invariant closed subspaces of* V *such that* $V = Y_0 \oplus \widetilde{Y_0}$, $\widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim\widetilde{Y_0} = dm$, *then* $i^\infty(S_\rho \cap Y_0) \ge -m$.

Definition 2.6. [16] A functional $J \in C^1(Z, R)$ is said to satisfy the condition $(PS)^*_c$ if any sequence $\{u_{n_k}\}$, $u_{n_k} \in Z_{n_k}$ such that

$$J(u_{n_k}) \to c, \quad \mathrm{d}J_{n_k}(u_{nk}) \to 0, \quad \text{as } k \to \infty$$

possesses a convergent subsequence, where Z_{n_k} is the n_k-dimension subspace of Z, $J_{n_k} = J|_{Z_{n_k}}$.

Theorem 2.7. [8] *Assume that*

(B_1) $J \in C^1(Z, R)$ *is* G*-invariant;*
(B_2) *there are* G*-invariant closed subspaces* U *and* V *such that* V *is infinite dimensional and* $Z = U \oplus V$*;*
(B_3) *there is a sequence of* G*-invariant finite dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots, \quad \dim V_j = dn_j,$$

such that $V = \overline{\cup_{j=1}^\infty V_j}$*;*
(B_4) *there is an index theory* i *on* Z *satisfying the* d*-dimension property;*
(B_5) *there are* G*-invariant subspaces* Y_0, $\widetilde{Y_0}$, Y_1 *of* V *such that* $V = Y_0 \oplus \widetilde{Y_0}$, $Y_1, \widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim\widetilde{Y_0} = \mathrm{d}m < \mathrm{d}k = \dim Y_1$*;*
(B_6) *there are* α *and* β, $\alpha < \beta$ *such that* f *satisfies* $(PS)^*_c$, $\forall\ c \in [\alpha, \beta]$*;*
(B_7)

$$\begin{cases} (a)\ \textit{either}\ \mathrm{Fix}G \subset U \oplus Y_1, \quad \textit{or} \quad \mathrm{Fix}G \cap V = \{0\}, \\ (b)\ \textit{there is}\ \rho > 0\ \textit{such that}\ \forall\ u \in Y_0 \cap S_\rho, f(z) \ge \alpha, \\ (c)\ \forall\ z \in U \oplus Y_1,\ f(z) \le \beta, \end{cases}$$

if i^∞ *is the limit index corresponding to* i, *then the numbers*

$$c_j = \inf_{i^\infty(A)\ge j} \sup_{z\in A} f(u), \quad -k+1 \le j \le -m,$$

are critical values of f, *and* $\alpha \le c_{-k+1} \le \cdots \le c_{-m} \le \beta$. *Moreover, if* $c = c_l = \cdots = c_{l+r}$, $r \ge 0$, *then* $i(\mathbb{K}_c) \ge r + 1$, *where* $\mathbb{K}_c = \{z \in Z : \mathrm{d}f(z) = 0, f(z) = c\}$.

3. Local Palais-Smale condition

To prove Theorem 1.1, noting the lack of compactness, in the inclusion $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, we can no longer expect the Palais-Smale condition to hold. Anyway we can prove a local Palais-Smale condition that will hold for $J(u, v)$ below a certain value of energy. Let u_n be a bounded sequence in $W^{1,p}(\Omega)$ then there exists a subsequence that we still denote u_n such that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),\\
&u_n \to u \quad \text{strongly in } L^r(\Omega),\ 1 \le r < p^*,\\
&|\nabla u_n|^p \rightharpoonup \mathrm{d}\mu, \quad |u_n|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta,
\end{aligned}$$

weakly-* in the sense of measures. Observe that $\mathrm{d}\eta$ is a measure supported on $\partial\Omega$.

If we consider $\phi \in C^\infty(\overline{\Omega})$, from the Sobolev trace inequality we obtain, passing to the limit

$$\left(\int_{\partial\Omega} |\phi|^{p^*}\mathrm{d}\eta\right)^{\frac{1}{p^*}} S^{\frac{1}{p}} \le \left(\int_\Omega |\phi|^p \mathrm{d}\mu + \int_\Omega |u|^p|\nabla\phi|^p \mathrm{d}x + \int_\Omega |\phi u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.1}$$

where S is the best constant in the Sobolev trace embedding theorem. From (3.1) we observe that if $u = 0$ we get a reverse Hölder-type inequality (but it involves one integral over Ω) between the two measures μ and η.

Similar to the proof of [11,12], we have the following lemma.

Lemma 3.1. [4] *Let u_j be a weakly convergent sequence in $W^{1,p}(\Omega)$ with weak limit u such that*

$$|\nabla u_j|^p \rightharpoonup \mathrm{d}\mu, \quad \textit{and} \quad |u_j|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta,$$

weakly- in the sense of measures. Then there exists $x_1, \dots, x_l \in \partial\Omega$ such that*

(i) $\mathrm{d}\eta = |u|^{p^*} + \sum_{j=1}^{l} \eta_j\delta_{x_j}$, $\eta_j > 0$;

(ii) $\mathrm{d}\mu \ge |\nabla u|^p + \sum_{j=1}^{l} \mu_j\delta_{x_j}$, $\mu_j > 0$;

(iii) $(\eta_j)^{\frac{p}{p^*}} \le \frac{\mu_j}{S}$.

Similar to [6,16], it is easy to obtain the following lemma:

Lemma 3.2. *Assume $1 \le \theta_1, \theta_2, \theta < \infty$, $I \in C(\overline{\Omega} \times R^2, R)$ and*

$$I(x, u, v) \le C\left(|u|^{\frac{\theta_1}{\theta}} + |v|^{\frac{\theta_2}{\theta}}\right).$$

Then for every $(u, v) \in L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$, $I(\cdot, u, v) \in L^\theta(\Omega)$ and the operator

$$T : (u, v) \mapsto I(x, u, v)$$

is a continuous map from $L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$ to $L^\theta(\Omega)$.

Lemma 3.3. *Suppose that $F(x, u, v)$ satisfies conditions (H_1)–(H_3). Then*

(i) $J \in C^1(X, R)$;

(ii)

$$\begin{aligned}
\langle \mathrm{d}J(u, v), (\widehat{u}, \widehat{v})\rangle = &-\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\widehat{u} + |u|^{p-2}u\widehat{u}\mathrm{d}x - \int_{\partial\Omega} |u|^{p^*-2}u\widehat{u}\mathrm{d}\sigma\\
&+ \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\widehat{v} + \ll |v|^{p-2}v\widehat{v}\mathrm{d}x - \int_{\partial\Omega} |v|^{p^*-2}v\widehat{v}\mathrm{d}\sigma\\
&- \int_\Omega F_u(x, u, v)\widehat{u}\mathrm{d}x - \int_\Omega F_v(x, u, v)\widehat{v}\mathrm{d}x;
\end{aligned}$$

(iii) A critical point of J is a weak solution of system (1.1).

Now set

$$X = U \oplus V, \quad U = E \times \{0\}, \quad V = \{0\} \times E,$$
$$Y_0 = \{0\} \times E_1^{\perp}, \quad V = Y_0 \oplus \widetilde{Y_0},$$
$$Y_1 = \{0\} \times E_{k_0}, \quad E_{k_0} = \mathrm{span}\{e_1, \dots, e_{k_0}\},$$

then $\dim \widetilde{Y_0} = 1$, $\dim Y_1 = k_0$.

Define a group action $G_2 = \{1, \tau\} \cong \mathcal{Z}_2$ by setting $\tau(u, v) = (-u, -v)$, then $\mathrm{Fix}G = \{0\} \times \{0\}$ (also denote $\{0\}$). It is clear that U and V are G-invariant closed subspaces of X, and Y_0, $\widetilde{Y_0}$ and Y_1 are G-invariant subspace of V. Set

$$\Sigma := \{A \subset X \setminus \{0\} : \ A \text{ is closed in } X \text{ and } (u, v) \in A \Rightarrow (-u, -v) \in A\}.$$

Define an index γ on Σ by:

$$\gamma(A) = \begin{cases} \min\{N \in \mathcal{Z} : \exists\ h \in C(A, \mathbb{R}^N \backslash \{0\} \text{ such that } h(-u, -v) = h(u, v))\}, \\ 0, \quad \text{if } A = \emptyset, \\ +\infty, \quad \text{if such } h \text{ does not exist.} \end{cases}$$

Then we have the following proposition from [6]: γ is an index satisfying the properties given in Definition 2.2. Moreover, γ satisfies the one-dimension property. According to Definition 2.4 we can obtain a limit index γ^{∞} with respect to (X_n) from γ.

Now we turn to prove local Palais-Smale condition.

Lemma 3.4. *Assume condition (H_1)–(H_3) hold, Then the functional J satisfies the local $(PS)_c$ condition in*

$$c \in \left(-\infty, \frac{1}{N} S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right),$$

in the following sense: if

$$J(u_{n_k}, v_{n_k}) \to c \in \left(-\infty, \frac{1}{N} S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right), \quad \mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}) \to 0, \quad \text{as } k \to \infty,$$

where $J_{n_k} = J|_{X_{n_k}}$. Then $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in X.

Proof. First, we show that $\{(u_{n_k}, v_{n_k})\}$ is bounded in X.

We note that by condition (H_3),

$$\begin{aligned} o(1)\|u_{n_k}\|_p &\geq \langle -\mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \\ &= \int_{\Omega} |\nabla u_{n_k}|^p + |u_{n_k}|^p \mathrm{d}x + \int_{\partial\Omega} |u_{n_k}|^{p^*} \mathrm{d}\sigma + \int_{\Omega} F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \mathrm{d}x \\ &\geq \int_{\Omega} |\nabla u_{n_k}|^p + |u_{n_k}|^p \mathrm{d}x + \int_{\partial\Omega} |u_{n_k}|^{p^*} \mathrm{d}x \\ &\geq \|u_{n_k}\|_p^p, \end{aligned} \tag{3.2}$$

since $p > 1$, from (3.2), we know that $\|u_{n_k}\|_p$ is bounded. On the one hand, we have

$$\begin{aligned} &J_{n_k}(0, v_{n_k}) - \frac{1}{p^*}\langle \mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\Omega} (|\nabla v_{n_k}|^p + |v_{n_k}|^p)\, \mathrm{d}x - \int_{\Omega} \left[F(x, u_{n_k}, v_{n_k}) - \frac{1}{p^*} F_v(x, u_{n_k}, v_{n_k}) v_{n_k}\right] \mathrm{d}x \\ &= c + o(1)\|v_n\|_p, \end{aligned}$$

i.e.

$$\frac{1}{N}\int_{\Omega}(|\nabla v_{n_k}|^p+|v_{n_k}|^p)\,\mathrm{d}x=\int_{\Omega}\left[F(x,u_{n_k},v_{n_k})-\frac{1}{p^*}F_v(x,u_{n_k},v_{n_k})v_{n_k}\right]\mathrm{d}x$$
$$+c+o(1)\|v_{n_k}\|_p.$$

Then by (1.4), we have

$$\left(\frac{1}{N}-\varepsilon\right)\|v_{n_k}\|_p^p\leq c(\varepsilon)|\Omega|+c+o(1)\|v_{n_k}\|_p,$$

where $|\cdot|$ denote by Lebesgue measure. Setting $\varepsilon=1/2N$, we get

$$\|v_{n_k}\|_p^p\leq M+o(1)\|v_{n_k}\|_p, \tag{3.3}$$

where $o(1)\to 0$ and M is a some positive number. Thus (3.3) implies that $\{v_{n_k}\}$ is bounded in $W^{1,p}(\Omega)$. This implies $\|u_{n_k}\|_p+\|v_{n_k}\|_p$ is bounded in X.

Next, we prove that $\{(u_{n_k},v_{n_k})\}$ contains a subsequence converging strongly in X.

We note that $\{u_{n_k}\}$ is bounded in E. Hence, up to a subsequence, $u_{n_k}\rightharpoonup u$ weakly in E and $u_{n_k}(x)\to u(x)$, a.e. in $\mathbb{R}^N$. We claim that $u_{n_k}\to u$ strongly in E. In fact, note that

$$\int_{\Omega}|\nabla u_{n_k}-\nabla u|^p+|u_{n_k}-u|^p\mathrm{d}x+\int_{\partial\Omega}|u_{n_k}-u|^{p^*}\mathrm{d}\sigma+\int_{\Omega}F_u(x,u_{n_k}-u,v_{n_k})(u_{n_k}-u)\mathrm{d}x$$
$$=\langle -\mathrm{d}J_{n_k}(u_{n_k}-u,v_{n_k}),(u_{n_k}-u,0)\rangle\to 0\quad\text{as } n\to\infty,$$

and condition (H_3) imply that

$$u_{n_k}\to u\quad\text{strongly in } E. \tag{3.4}$$

In the following we will prove that there exists $v\in E$ such that

$$v_{n_k}\to v\quad\text{strongly in } E. \tag{3.5}$$

By Lemma 3.1 and (3.3) there exists a subsequence, there exists a subsequence, that we still denote v_{n_k} such that

$$\begin{aligned}&v_{n_k}\rightharpoonup v\quad\text{weakly in } W^{1,p}(\Omega),\\&v_{n_k}\to v\quad\text{strongly in } L^r(\Omega),\ 1\leq r<p^*,\ \text{and a.e. in } \Omega\\&|\nabla v_{n_k}|^p\rightharpoonup \mathrm{d}\mu\geq|\nabla v|^p+\sum_{k=1}^{l}\mu_k\delta_{x_k},\\&|v_{n_k}|_{\partial\Omega}|^{p^*}\rightharpoonup \mathrm{d}\eta=|v|_{\partial\Omega}|^{p^*}+\sum_{k=1}^{l}\eta_k\delta_{x_k}.\end{aligned}$$

Let $\phi(x)\in C^\infty(\Omega)$ such that $\phi(x)\equiv 1$ in $B(x_k,\varepsilon)$, $\phi(x)\equiv 0$ in $\Omega\setminus(x_k,2\varepsilon)$ and $|\nabla\phi|\leq 2/\varepsilon$, where x_k belongs to the support of $\mathrm{d}\eta$. Consider Then $\{\phi v_{n_k}\}$ is bounded in E, Obviously, $\langle \mathrm{d}J_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k}\phi)\rangle\to 0$, *i.e.*

$$\lim_{n\to\infty}\left[\int_{\Omega}(|\nabla v_{n_k}|^p+|v_{n_k}|^p)\,\phi\mathrm{d}x-\int_{\partial\Omega}|v_{n_k}|^{p^*}\phi\mathrm{d}\sigma-\int_{\Omega}F_v(x,u_{n_k},v_{n_k})v_{n_k}\phi\mathrm{d}x\right]$$
$$=-\lim_{n\to\infty}\int_{\Omega}\left(v_{n_k}|\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\nabla\phi\right)\mathrm{d}x. \tag{3.6}$$

On the other hand, by Hölder inequality and weak convergence, we obtain

$$\begin{aligned}
0 &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega v_{n_k}|\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\nabla\phi \mathrm{d}x\right| \\
&\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega |v_{n_k}|^p|\nabla\phi|^p \mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla v_{n_k}|^q \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_\Omega |v|^p|\nabla\phi|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)} |\nabla\phi|^N \mathrm{d}x\right)^{\frac{1}{N}}\left(\int_{B(x_j,\varepsilon)} |v|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)} |v|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} = 0.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$0=\lim_{\varepsilon\to 0}\left[\int_{\partial\Omega}\phi \mathrm{d}\eta-\int_\Omega \phi \mathrm{d}\mu-\int_\Omega |v|^p\phi \mathrm{d}x-\int_\Omega F_v(x,u,v)v\phi \mathrm{d}x\right]=\eta_k-\mu_k. \tag{3.8}$$

Combing this with Lemma 3.1, we obtain $(\mu_k)^{p/p^*}S\le\mu_k$. This result implies that

$$\mu_k=0 \qquad\text{or}\qquad \mu_k\ge S^{p^*/(p^*-p)}.$$

If the second case $\mu_k\ge S^{p^*/(p^*-p)}$ holds, for some $k\in J$, then by using Lemma 3.1 and the Hölder inequality, we have that

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J_{n_k}(0,v_{n_k})-\frac{1}{p^*}\langle \mathrm{d}J_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k})\rangle\right) \\
&= \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega\left(|\nabla v_{n_k}|^p+|v_{n_k}|^p\right)\mathrm{d}x-\int_\Omega\left[F(x,u_{n_k},v_{n_k})-\frac{1}{p^*}F_v(x,u_{n_k},v_{n_k})v_{n_k}\right]\mathrm{d}x \\
&\ge \frac{1}{N}\int_\Omega \mathrm{d}\mu-c\left(\frac{1}{2N}\right)|\Omega| \\
&\ge \frac{1}{N}\int_\Omega |\nabla v_{n_k}|^p \mathrm{d}x+\frac{1}{N}S^{p^*/(p^*-p)}-c\left(\frac{1}{2N}\right)|\Omega| \\
&\ge \frac{1}{N}S^{p^*/(p^*-p)}-c\left(\frac{1}{2N}\right)|\Omega|,
\end{aligned}$$

where $\varepsilon=1/2N$. This is impossible. Consequently, $\mu_k=0$ for all $k\in J$. From (3.8) we know that $\eta_k=0$ for all $k\in J$ and hence

$$\int_{\partial\Omega}|v_{n_k}|^{p^*}\mathrm{d}\sigma\to\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma.$$

Now $v_{n_k}\rightharpoonup v$ in E and Brezis-Lieb lemma [2] implies that

$$\lim_{n\to\infty}\int_{\partial\Omega}|v_{n_k}-v|^{q^*}\mathrm{d}\sigma=0.$$

Thus, we have

$$\begin{aligned} o(1)\|v_{n_k}\|_p &= \|v_{n_k}\|_p^p - \int_\Omega |v_{n_k}|^{p^*}\mathrm{d}\sigma - \int_\Omega F_v(x,u_{n_k},v_{n_k})v_{n_k}\mathrm{d}x \\ &= \|v_{n_k}-v\|_p^p + \|v\|_p^p - \int_\Omega |v|^{p^*}\mathrm{d}\sigma - \int_\Omega F_v(x,u,v)v\mathrm{d}x \\ &= \|v_{n_k}-v\|_p^p + o(1)\|v\|_p, \end{aligned}$$

since $\mathrm{d}J_{n_k}(0,v)=0$. Thus we prove that $\{v_{n_k}\}$ strongly converges to v in E. Thus (3.5) holds. (3.4) and (3.5) imply the conclusion of Lemma 3.4 follows. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Now we shall verify the conditions of Theorem 2.7. Obviously, (B_1), (B_2), (B_4) in Theorem 2.7 are satisfied. Set $V_j = E_j = \mathrm{span}\{e_1, e_2, \ldots, e_j\}$, then (B_3) is also satisfied. Since $1 = \dim \widetilde{Y_0} < k_0 = \dim Y_1$, (B_5) is satisfied. In the following we verify the conditions in (B_7). Since $\mathrm{Fix}G \cap V = 0$, that is (a) of (B_7) holds. It remains to verify (b), (c) of (B_7). Choose a number α such that

$$\alpha < \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|, \frac{1}{N}2^{\frac{p^*}{p-p^*}}S^{\frac{pp^*}{p-p^*}} - b\left(\frac{1}{2p}\right)|\Omega|\right\}. \tag{4.1}$$

(i) If $(0,v)\in Y_0\cap S_\rho$ (where ρ is to be determined) then by (H_2),

$$\begin{aligned} J(0,v) &= \frac{1}{p}\int_\Omega |\nabla v|^p + |v|^p\mathrm{d}x - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,0,v)\mathrm{d}x \\ &\ge \left(\frac{1}{p}-\varepsilon\right)\cdot\int_\Omega |\nabla v|^p + |v|^p\mathrm{d}x - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - b(\varepsilon)|\Omega| \\ &\ge \frac{1}{2p}\|v\|_p^p - \frac{1}{p^*}S^{p^*}\|v\|_p^{p^*} - b\left(\frac{1}{2p}\right)|\Omega|, \end{aligned} \tag{4.2}$$

where $\varepsilon = \frac{1}{2p}$. Since

$$\max_{t\in\mathbb{R}}\left(\frac{1}{2p}t^p - \frac{1}{p^*}S^{p^*}t^{p^*} - b\left(\frac{1}{2p}\right)|\Omega|\right) = \frac{1}{N}2^{\frac{p^*}{p-p^*}}S^{\frac{pp^*}{p-p^*}} - b\left(\frac{1}{2p}\right)|\Omega|,$$

Therefore, there exists $\rho>0$ such that $J(0,v)\ge\alpha$ for every $\|v\|_p=\rho$, that is (b) of (B_7) holds.

(ii) For each $(u,v)\in U\oplus Y_1$, by condition (H_4), we have

$$\begin{aligned} J(u,v) &= -\frac{1}{p}\int_\Omega(|\nabla u|^p+|u|^p)\mathrm{d}x + \frac{1}{p}\int_\Omega(|\nabla v|^p+|v|^p)\mathrm{d}x \\ &\quad -\frac{1}{p^*}\int_{\partial\Omega}|u|^{p^*}\mathrm{d}\sigma - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,u,v)\mathrm{d}x \\ &\le \frac{1}{p}\|v\|_p^p - L|v|_p^p + \xi|\Omega| \\ &\le \max_{v\in E_{k_0}}\left(\frac{1}{p}\|v\|_p^p - L|v|_p^p\right) + \xi|\Omega| \\ &= \max_{\{t\ge0, v\in\partial B_1(0)\cap E_{k_0}\}}\left[t^p\left(\frac{1}{p} - L|v|_p^p\right)\right] + \xi|\Omega|. \end{aligned} \tag{4.3}$$

Let $r = \min\{\int_\Omega |v|^p \mathrm{d}x : v \in \partial B_1(0) \cap E_{k_0}\}$. By taking $L \geq \frac{1}{pr}$, we have

$$\frac{1}{p} - L|v|_p^p \leq \frac{1}{p} - Lr \leq 0. \tag{4.4}$$

It follows from (4.3), (4.4) and (H_4) that

$$J(u,v) \leq \xi|\Omega| \leq \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right\}.$$

Let $\beta = \xi|\Omega|$, so we get (c) in (B_7). By Lemma 3.4, for any $c \in [\alpha, \beta]$, $J(u,v)$ satisfies the condition of $(PS)_c^*$, then (B_6) in Theorem 2.7 holds. So according to Theorem 2.7,

$$c_j = \inf_{i^\infty(A)\geq j} \sup_{z\in A} f(u), \quad -k_0+1 \leq j \leq -1,$$

are critical values of J, $\alpha \leq c_{-k_0+1} \leq \cdots \leq c_{-1} \leq \beta < 0$ and J has at least $k_0 - 1$ pairs critical points. □

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