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MULTIPLICITY RESULTS FOR NONLINEAR SCHRÖDINGER-POISSON SYSTEMS WITH SUBCRITICAL OR CRITICAL GROWTH

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Abstract. In this paper, we consider the following Schrödinger-Poisson system:

$$\begin{cases} -\triangle u + u + \lambda\phi u = \mu f(u) + |u|^{p-2}u, & \text{in } \Omega, \\ -\triangle\phi = u^2, & \text{in } \Omega, \\ \phi = u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$, $p \in (1,6]$, λ, μ are two parameters and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Using some critical point theorems and truncation technique, we obtain three multiplicity results for such a problem with subcritical or critical growth.

1. Introduction

The aim of this paper is to investigate the existence of multiple solutions for the following Schrödinger-Poisson system:

$$\text{(SP)} \qquad \begin{cases} -\triangle u + \omega u + \lambda\phi u = \mu f(u) + |u|^{p-2}u, & \text{in } \Omega, \\ -\triangle\phi = u^2, & \text{in } \Omega, \\ \phi = u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p \in (1,6]$, $\omega > 0$. This system appears in studying the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\triangle_x\psi + |\psi|^{q'-2}\psi, \tag{1.1}$$

which describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the motion. Here $\psi(x,t)$ is a complex function, $\hbar$, m express the normalized Plank constant and the mass of the particle, respectively, while the nonlinear term $|\psi|^{q'-2}\psi$: $\mathbb{R} \to \mathbb{R}$ describes either the interaction between the particles or an external nonlinear perturbation of the

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linearly charged fields in the presence of the electrostatic field (see, for example, [10, 29]). Here, for the sake of simplicity, we set $\hbar = 1$ and $m = 2$. In fact, the problem (SP) can be obtained via looking for stationary solutions

$$\psi(x,t) = u(x)e^{i\omega t}$$

for the Schrödinger equation (1.1) modelling a charged particle, in equilibrium with its own purely electrostatic field $\phi(x)$. The boundary conditions $u = \phi = 0$ on $\partial\Omega$ mean that the particle is constraint to live in Ω. In the following, referring to (SP) we will assume for simplicity $\omega = 1$.

Because of its importance in many different physical framework, Schrödinger-Poisson systems (sometimes called Schrödinger-Maxwell systems) have been extensively studied by using variational methods in the past years. In the case of bounded domain with $\mu = 0$, the linear version of Schrödinger-Poisson system (where $\mu f(u) + |u|^{p-2}u = 0$) has been approached as an eigenvalue problem in the pioneer paper of Benci and Fortunato [9]. The existence of infinitely many solutions of (SP) with $p \in (2, 5)$ has been discussed by Ruiz and Siciliano [25], and some analogous results have been established by Pisani and Siciliano [21] in the case where $p > 4$. Later, Siciliano [28] employed the abstract Lusternik-Schnirelmann theory to obtain multiplicity results for every $\lambda > 0$ and p sufficiently close to the critical exponent 2^*. More recently, Azzollini et al. [4] investigated a class of generalized Schrödinger-Poisson type system, and obtained some existence and multiplicity results in the case either of subcritical growth condition or of critical one. On the other hand, there are also many papers on unbounded domain which treat different aspects of the Schrödinger-Poisson system, even with an additional external and fixed potential $V(x)$. In particular, some important dynamical behaviors including ground states, radially and non-radially solutions, semiclassical limit and concentration of solutions, have been studied. See, for example, [1, 5, 6, 7, 8, 12, 13, 15, 18, 19, 31, 32, 33, 34].

Up to our knowledge, there are no result on the existence of multiple solutions for problem (SP) with the nonlinear term $f(u) + |u|^{p-2}u$. In particular, if f may be not odd, then the associated functional of (SP) may be nonsymmetric which leads to the significant difficulty in finding multiple solutions. Indeed problem (SP) with the nonsymmetric term f does possess a variational structure as well, problem (SP) can be attacked by means of variational methods. Namely, the weak solutions are characterized as critical points of a C^1 functional $I = I(u)$ defined on the Sobolev space $H_0^1(\Omega)$. Here, we obtain a sufficient result ensuring the existence of at least three weak solutions for problem (SP) whose the nonlinear term f may be nonsymmetric. The following is our result in the case of the subcritical exponent $p \in (1, 2)$:

Theorem 1.1. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$ is not an odd function and the following condition holds*:

- (f) *there exist three positive constants c_1, c_2, q such that $|f(u)| \le c_1 + c_2|u|^{q-1}$.*

If $q \in (1, 2)$ and $p \in (1, 2)$, then there exist $L > 0$ and an open interval J with $0 \in J$ such that, for every $\mu \in J$, problem (SP) *with $\lambda = \mu$ admits at least three weak solutions whose norms are less than or equal to L.*

The proof of Theorem 1.1 is based on an abstract critical point theorem developed by Anello [2] in finding two local minimum points of the associated functional which is the two weak solutions of problem (SP). And then, we find the third weak solution different from the earlier two ones using a Mountain Pass Theorem coming from [22]. Now, we are interested in seeking for the existence conditions of infinity many solutions for problem (SP) with the symmetric term f (i.e., f is odd). For problem (SP) with $p \in (4, 6)$, the symmetric Mountain Pass Lemma can be employed to guarantee the existence of infinitely many high energy solutions because it is not difficult to analyze the Mountain Pass geometry and to prove the boundness of Palais-Smale ((PS) for short) sequence. In the case where $p \in (2, 4]$, however, the symmetric Mountain Pass Lemma doesn't work because the geometry of the associated functional cannot be deduced in a standard way and the boundness of (PS) sequence cannot be guaranteed either. In order to fill this gap, we make use of a truncation technique and the genus theory introduced by Krasnoselskii [16]. In [20], we employed the genus theory to obtain some sufficient conditions ensuring the existence of infinitely many negative energy solutions for problem (SP) in unbounded domain with sublinear term.

Theorem 1.2. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$ is an odd function,* (f) *hold and $f(t) \geq Dt^{p'}$ for all $t > 0$ with some $D > 0$ and $p' \in (0, 1)$. If $q \in (1, 2)$ and $p \in (2, 6)$, then there exists $\mu^* > 0$ such that problem* (SP) *has infinitely many negative energy solutions for every $\mu \in (0, \mu^*)$ and $\lambda > 0$.*

In particular, there is also no result on the existence of infinitely many nontrivial solutions for problem (SP) with critical growth, i.e., $p = 6$. Finally, we intend to extend the previous result to problem (SP) with critical growth. In this critical case, it is difficult to verify (PS) condition because of the lack of the compactness of the inclusion of $H_0^1(\Omega)$ in $L^6(\Omega)$. In order to recover the compactness, we introduce the concentration-compactness lemma due to [17]. Thus, we have the following result based on the proof of Theorem 1.2.

Theorem 1.3. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$ is an odd function,* (f) *hold and $f(t) \geq Dt^{p'}$ for all $t > 0$ with some $D > 0$ and $p' \in (0, 1)$. If $q \in (1, 2)$ and $p = 6$, then there exist $\mu^* > 0, \lambda^* > 0$, such that problem* (SP) *has infinitely many negative energy solutions for every $\mu \in (0, \mu^*)$ and $\lambda \in (0, \lambda^*)$.*

Throughout this paper, $C > 0$ denotes various positive generic constants. The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Sections 3 and 4, we shall give the proofs of Theorems 1.1 and 1.2, respectively. Finally the proof of Theorem 1.3 will be given in Section 5.

2. Preliminaries

We consider the Hilbert space $H_0^1(\Omega)$ with the standard norm

$$\|u\| := \left(\int_\Omega (|\nabla u|^2 + u^2)dx\right)^{\frac{1}{2}}.$$

For every $\rho > 0$ and every $z \in H_0^1(\Omega)$, set $B_\rho(z) := \{u \in H_0^1(\Omega) : \|u - z\| \leq \rho\}$. If $\Omega_0 \subset \Omega$, then $|\Omega_0|$ denotes its Lebesgue measure. Further, we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $H_0^1(\Omega)$ which induces the previous norm and recall that $H_0^1(\Omega)$ is compactly embedded in the space $L^s(\Omega)$ for all $s \in [1, 6)$ whose norm

$$\|u\|_s := \left(\int_\Omega |u|^s dx\right)^{\frac{1}{s}}.$$

Note that $D^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm

$$\|u\|_{D^{1,2}} := \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Recall that

$$S = \inf\{\|u\|^2_{D^{1,2}(\Omega)} : u \in H_0^1(\Omega), \|u\|_6^2 = 1\},$$

where S is the best constant in the Sobolev inclusion.

It is well known that (SP) can be reduced to a single equation with a non-local term (see [9, 24]). Now we recall this method. Indeed, let $\phi_u \in H_0^1(\Omega)$ be the unique solution of $-\Delta\phi = u^2$ and $\phi = 0$ on $\partial\Omega$ and the following properties for ϕ will be repeatedly used (see [24, 28]):

Lemma 2.1. *For any $u \in H_0^1(\Omega)$, we have*

(i) $\|\phi_u\|^2_{D^{1,2}} = \int_\Omega \phi_u u^2 dx \leq C\|u\|^4_{12/5} \leq C\|u\|^4$;
(ii) $\phi_u \geq 0$;
(iii) *if $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\Omega)$;*
(iv) *if $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then $\int_\Omega \phi_{u_n} u_n^2 dx \to \int_\Omega \phi_u u^2 dx$.*

Therefore, the functional associated with (SP) reduces to be

$$(2.1)\quad I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)dx + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \mu\int_\Omega F(u)dx - \frac{1}{p}\int_\Omega |u|^p dx,$$

where $F(u) = \int_0^u f(t)dt$. This is a well-defined C^1 functional whose derivative is given by

(2.2)
$$I'(u)v = \int_\Omega \left(\nabla u \nabla v + uv + \lambda\phi_u uv - \mu f(u)v - |u|^{p-2}uv\right)dx, \quad \forall v \in H_0^1(\Omega),$$

whose critical points are the solutions of system (SP) (see e.g. [14]). Now we define the following integral momentums

$$(2.3)\qquad \Psi(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u|^p dx, \qquad \Phi(u) := \frac{1}{4}\int_\Omega \phi_u u^2 dx - \int_\Omega F(u)dx$$

for all $u \in H_0^1(\Omega)$. In order to obtain critical points of I, we will use the following abstract critical point theorems. Suppose that E is a Banach space and $I : E \to \mathbb{R}$ is a functional. A fundamental tool will be used in the sequel is the so-called Palais-Smale condition (PS for brevity): every sequence $\{u_n\}$ such that

$$\{I(u_n)\} \text{ is bounded and } I'(u_n) \to 0 \text{ in } E^{-1} \tag{2.4}$$

admits a convergent subsequence. Sequences which satisfy (2.4) are called Palais-Smale sequences.

Theorem 2.1 ([30]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the* (PS)*-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

Theorem 2.2 ([2]). *Let E be a reflexive Banach space and Φ, Ψ be two sequentially weakly lower semicontinuous real functionals defined on E. Suppose Ψ is* (*strongly*) *continuous. Moreover, assume that there exist $x_1, x_2, \ldots, x_n \in E$, $r_1, \ldots, r_n > 0$, with $r_i + r_j < \|x_i - x_j\|$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, such that for all $i \in \{1, \ldots, n\}$,*

(a) *the function $r \to \inf_{\|x\|=r} \Psi(x + x_i)$ is continuous in $\mathbb{R}^+$,*

(b) $\Psi(x_i) < \inf_{\|x\|=r_i} \Psi(x + x_i)$.

Then, there exists $\rho^* > 0$ such that for every $\rho > \rho^*$ the functional $\rho\Psi + \Phi$ admits at least n distinct local minimum points $y_1, \ldots, y_n$ such that $\|x_i - y_j\| < r_i$ for all $i = 1, \ldots, n$.

Theorem 2.3 ([22]). *Let X be a Banach space and assume that I satisfies the following conditions*:

(H_1) *there exist numbers a, r, R such that $0 < r < R$ and $I(x) \geq a$ for every $x \in A := \{x \in X : r < \|x\| < R\}$;*

(H_2) *$I(0) \leq a$ and $I(e) \leq a$ for some e with $\|e\| \geq R$.*

If I satisfies the Palais-Smale compactness condition, then there exists a critical point x, in X, different from 0 and e, with critical value $c \geq a$; moreover, $x \in A$ when $c = a$, where c is characterized by

$$\inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) = c, \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Let E be a Banach space, $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. Set

$$\Sigma = \{A \subset E - \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$

$$K_c = \{u \in E : I(u) = c, I'(u) = 0\},$$

$$I^c = \{u \in E : I(u) \leq c\}.$$

Definition 2.1 ([16, 23]). For $A \in \Sigma$, we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\varphi \in C(A, \mathbb{R}^n\backslash\{0\})$ and n is the smallest integer with this property.

As the last conclusion of this section, the following theorem is crucial to our arguments in Sections 4-5.

Theorem 2.4 ([16, 26])**.** *Let I be an even C^1 functional on E and satisfy the* (PS)*-condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geq n\}, \qquad c'_n = \inf_{A\in\Sigma_n} \sup_{u\in A} I(u).$$

(1) *If $\Sigma_n \neq \emptyset$ and $c'_n \in \mathbb{R}$, then c'_n is a critical value of I;*
(2) *If there exists $r \in \mathbb{N}$ such that*

$$c'_n = c'_{n+1} = \cdots = c'_{n+r} = c \in \mathbb{R},$$

and $c \neq I(0)$, then $\gamma(K_c) \geq r + 1$.

3. Proof of Theorem 1.1

In this section, under the condition that f may be not odd function, we give the result of the existence of at least three solutions for problem (SP) (i.e., Theorem 1.1). In what follows, we will give the proof of Theorem 1.1.

Proof. Firstly, with the help of Theorem 2.2, we show that problem (SP) has at least two weak solutions. It follows from the conditions of Theorem 1.1 and Lemma 2.1 that Φ, Ψ are two well-defined differentiable and sequentially weakly lower semicontinuous functionals. Moreover, Ψ is strongly continuous and coercive. Hence, it is clear that Ψ is bounded from below in $H^1_0(\Omega)$. Next we show Ψ satisfies the (PS)- condition. Assume that $\{w_n\} \subset H^1_0(\Omega)$ such that $\{\Psi(w_n)\}$ is bounded and $\Psi'(w_n) \to 0$ as $n \to \infty$. Then there exist positive constants $C, C_p > 0$ such that

$$C > \Psi(w_n) \geq \frac{1}{2}\|w_n\|^2 - \frac{1}{p}C_p\|w_n\|^p.$$

In view of the above inequality, we know that $\|w_n\|$ is bounded. Hence, there exists $w_0 \in H^1_0(\Omega)$ such that $w_n \rightharpoonup w_0$. It follows from the definition of Ψ that

$$\langle\Psi'(w_n)-\Psi'(w_0), w_n-w_0\rangle = \|w_n-w_0\|^2 - \int_\Omega (|w_n|^{p-2}w_n - |w_0|^{p-2}w_0)(w_n-w_0). \tag{3.1}$$

Noting that, by Sobolev embedding theorem and hölder inequality, we have

$$\begin{aligned} &|\int_\Omega (|w_n|^{p-2}w_n - |w_0|^{p-2}w_0)(w_n - w_0)| \\ &\leq \|w_n\|_6^{\frac{p-1}{6}} \|w_n - w_0\|_{\frac{6}{7-p}}^{\frac{7-p}{6}} + |\int_\Omega |w_0|^{p-2}w_0(w_n - w_0)| \to 0 \end{aligned} \tag{3.2}$$

as $n \to \infty$. Moreover, it is easy to see that $\langle\Psi'(w_n) - \Psi'(w_0), w_n - w_0\rangle \to 0$ as $n \to \infty$. Combining (3.1) with (3.2), we deduce that $w_n \to w_0$ as $n \to \infty$. Therefore, Ψ satisfies the (PS)-condition. Theorem 2.1 implies that there exists w satisfying $\Psi(w) = \inf_{H^1_0(\Omega)} \Psi$ and $\Psi'(w) = 0$. We claim that $w \neq 0$. Let $\bar{w} \in H^1_0(\Omega)/\{0\}$, then $\Psi(s\bar{w}) < 0$ for sufficiently small positive number s.

Therefore $\Psi(w) = \inf_{H_0^1(\Omega)} \Psi < 0$. Our claim is true. Moreover, the standard elliptic estimates imply that $w \in L^\infty(\Omega)$ and $v > 0$ follows from the strong maximum principle. We can easily deduce from [11] and the definition of Ψ that w and $-w$ are the unique two global minimum points of the functional Ψ over $H_0^1(\Omega)$. It is shown in [27] that, if $u_0 \in H_0^1(\Omega)$, then the real function

$$r \to \inf_{\|u\|=r} \Psi(u-u_0) = \frac{r^2}{2} + \frac{\|u_0\|^2}{2} - \sup_{\|u\|=r}\left(\langle u, u_0\rangle + \frac{1}{p}\int_\Omega |u-u_0|^p dx\right)$$

is continuous in $\mathbb{R}^+$. Now we claim that, for each fixed $r \in (0, 2\|w\|)$,

$$\inf_{\|u\|=r} \Psi(u \pm w) > \inf_{H_0^1(\Omega)} \Psi = \Psi(\pm w). \tag{3.3}$$

If not, without loss of generality, suppose that there exists $r_0 \in (0, 2\|w\|)$ such that

$$\inf_{\|u\|=r_0} \Psi(u+w) = \inf_{H_0^1(\Omega)} \Psi. \tag{3.4}$$

The argument is same as the case where

$$\inf_{\|u\|=r_0} \Psi(u-w) = \inf_{H_0^1(\Omega)} \Psi.$$

So we only consider the case (3.4). It follows from (3.4) that there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ with $\|u_n\| = r_0$ such that

$$\lim_{n\to+\infty} \Psi(u_n+w) = \inf_{H_0^1(\Omega)} \Psi.$$

From this, up to subsequence, again denoted by $\{u_n\}$, we have $u_n \rightharpoonup u^*$ in $H_0^1(\Omega)$. Therefore, by using Sobolev embedding theorem we have

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \Psi(u_n+w) - \Psi(w)\\
&= \lim_{n\to+\infty}\left(\frac{r_0^2}{2} + \frac{\|w\|^2}{2} + \langle u_n, w\rangle - \frac{1}{p}\int_\Omega |u_n+w|^p dx\right) - \Psi(w)\\
&= \frac{r_0^2}{2} + \langle u^*, w\rangle + \frac{1}{p}\int_\Omega (|w|^p - |u^*+w|^p) dx.
\end{aligned} \tag{3.5}$$

If we put $B_1 = \{x \in \Omega : u^* \neq 0\}$, then $|B_1| > 0$. Or else, by (3.5), it would be $r_0 = 0$, against the choice of r_0. On the other hand, if we put $B_2 = \{x \in \Omega : u^* \neq -2w\}$, then $|B_2| > 0$. Otherwise, again by (3.5), it would be $r_0 = 2\|w\|$, which contradicts with the choice of r_0. Consequently, the function $u^* + w$ is different from w and $-w$. So it follows from Fatou lemma, Sobolev embedding theorem and $\Psi(w) = \inf_{H_0^1(\Omega)} \Psi$ that

$$\begin{aligned}
\Psi(w) &= \lim_{n\to+\infty} \Psi(u_n+w)\\
&= \lim_{n\to+\infty} \int_\Omega [\frac{1}{2}(|\nabla(u_n+w)|^2 + (u_n+w)^2) - \frac{1}{p}|u_n+w|^p] dx\\
&\geq \int_\Omega [\frac{1}{2}(|\nabla(u^*+w)|^2 + (u^*+w)^2) - \frac{1}{p}|u^*+w|^p] dx > \Psi(w),
\end{aligned} \tag{3.6}$$

which is impossible. Hence, (3.3) holds. Fix $r \in (0, \|w\|)$, then it is easy to check that all the hypotheses of Theorem 2.2 are fulfilled if we take $n = 2$, $r_1 = r_2 = r$ and $x_1 = -x_2 = w$. Hence there exists $\rho^* > 0$ such that for all $\rho > \rho^*$ the functional $\rho\Psi + \Phi$ contains at least two distinct local minimum points u_1^ρ, u_2^ρ satisfying

$$\max\{\|u_1^\rho - w\|, \|u_2^\rho + w\|\} < r.$$

Indeed, such minimum points are critical points of the same functional. Hence, if we put $\mu_1^* = \frac{1}{\rho^*}$, we have that, for all $\mu \in (0, \mu_1^*)$, functional $\Psi + \mu\Phi$ admits at least two critical points u_1^*, u_2^* such that

$$\max\{\|u_1^*\|, \|u_2^*\|\} \le 2[\|w\| + r].$$

That is, problem (SP) admits at least two weak solutions for $\lambda = \mu \in (0, \mu_1^*)$. It is easy to see that, for $\mu = 0$, the same conclusion holds and the two weak solutions are exactly w and $-w$. Now, consider the functional

$$\Phi_1(u) = -\Phi(u) = \int_\Omega F(u)dx - \frac{1}{4}\int_\Omega \phi_u u^2 dx$$

defined for all $u \in H_0^1(\Omega)$. Repeating the same above argument applied to Φ_1 and Ψ, we easily deduce that there exists $\mu_2^* > 0$ such that the previous conclusion holds for all $\mu \in (-\mu_2^*, 0)$. Put $\sigma = 2[\|w\| + r]$ and $J = (-\mu_2^*, \mu_1^*)$, then problem (SP) contains at least two weak solutions whose norms are less or equal than σ.

Secondly, using a Mountain Pass Theorem (see Theorem 2.3), we will find the third weak solution for problem (SP). We show that $\Psi + \mu\Phi$ satisfies the (PS)-condition. Assume that $\{u_n\}$ is a sequence in $H_0^1(\Omega)$ such that $\{\Psi(u_n) + \mu\Phi(u_n)\}$ is bounded and $\Psi'(u_n) + \mu\Phi'(u_n) \to 0$ as $n \to +\infty$.

By (2.1), (2.2) and (2.3), there exist two positive constant a_1, a_2 such that

$$\begin{aligned} a_1 + a_2\|u_n\| &\ge \Psi(u_n) + \mu\Phi(u_n) - \frac{1}{4}[\Psi'(u_n)u_n + \mu\Phi'(u_n)u_n] \\ &\ge \frac{1}{4}\|u_n\|^2 - (\frac{1}{p} - \frac{1}{4})\int_\Omega |u_n|^p dx + \mu\int_\Omega (\frac{1}{4}f(u_n)u_n - F(u_n))dx \\ &\ge \frac{1}{4}\|u_n\|^2 - (\frac{1}{p} - \frac{1}{4})C\|u\|^p + \mu(c_1C\|u\| + c_2C\|u\|^q) \end{aligned}$$

for $\mu \in J$. In view of the earlier inequality, using the fact that $p, q < 2$ we conclude that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ for $\mu \in J$ and, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in} \quad H_0^1(\Omega), \\ &u_n \to u \quad \text{in} \quad L^s(\Omega) \quad \text{for} \quad 1 \le s < 6, \\ &u_n(x) \to u(x) \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{3.7}$$

Hence, $\Psi'(u_n)(u_n - u) + \mu\Phi'(u_n)(u_n - u) = o_n(1)$, that is,

$$\langle u_n, u_n - u\rangle + \int_\Omega \big(\mu\phi_{u_n}u_n - \mu f(u_n) - |u_n|^{p-2}u_n)(u_n - u)\big)\, dx = o_n(1). \tag{3.8}$$

From Lemma 2.1, we have

$$\int_\Omega \mu\phi_{u_n} u_n(u_n - u)dx = o_n(1). \tag{3.9}$$

Using (f), (3.7) and $1 < p < 2$, we obtain

$$\int_\Omega \mu f(u_n)(u_n - u)dx = o_n(1), \quad \text{and} \quad \int_\Omega |u_n|^{p-1}u_n(u_n - u)dx = o_n(1). \tag{3.10}$$

Therefore, combining (3.9), (3.10) with (3.8), we get

$$\langle u_n, u_n - u\rangle = o_n(1). \tag{3.11}$$

With the help of the fact that $\langle u, u_n - u\rangle = o_n(1)$, we deduce that $u_n \to u$ in $H_0^1(\Omega)$. In fact, the two weak solutions u_1^*, u_2^* turn out to be the global minimum points for the restriction of the functional $\Psi + \mu\Phi$ to the set $B_r(w)$ and $B_r(-w)$, respectively. Therefore, by applying Theorem 2.3, for every $\mu \in J$, we can get a critical point u_3^* of $\Psi + \mu\Phi$ different from u_1^* and u_2^* such that

$$\Psi(u_3^*) + \mu\Phi(u_3^*) = c(\mu), \tag{3.12}$$

where

$$c(\mu) = \inf_{\gamma\in\Gamma_\mu} \sup_{t\in[0,1]} (\Psi(\gamma(t)) + \mu\Phi(\gamma(t))),$$

and

$$\Gamma_\mu = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = u_1^* \,\text{and}\, \gamma(1) = u_2^*\}.$$

Note that, for every $\mu \in J$ and $t \in [0,1]$, if we take

$$\gamma_0(t) = tu_2^* + (1-t)u_1^*,$$

then $\gamma_0 \in \Gamma_\mu$ and $\|\gamma_0(t)\| \le \sigma$. Consequently, for $\mu \in J$ we have

$$\begin{aligned} c(\mu) &= \inf_{\gamma\in\Gamma_\mu} \sup_{t\in[0,1]} (\Psi(\gamma(t)) + \mu\Phi(\gamma(t))) \\ &\le \sup_{t\in[0,1]} (\Psi(\gamma_0(t)) + \mu\Phi(\gamma_0(t))) \\ &\le \frac{\sigma^2}{2} + |\mu|(\frac{1}{4}C\|\gamma_0(t)\|^4 + \sup_{t\in[0,1]} \int_\Omega |F(\gamma_0(t))|dx) \\ &\le \frac{\sigma^2}{2} + (\mu_1^* + \mu_2^*)(\frac{1}{4}C\sigma^4 + c_1C\sigma + c_2C\sigma^q) := C^*. \end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\begin{aligned} C^* &\ge c(\mu) = \Psi(u_3^*) + \mu\Phi(u_3^*) - \frac{1}{4}[\Psi'(u_3^*)u_3^* + \mu\Phi'(u_3^*)u_3^*] \\ &\ge \frac{1}{4}\|u_3^*\|^2 - (\frac{1}{p} - \frac{1}{4})\int_\Omega |u_3^*|^p dx + \mu\int_\Omega (\frac{1}{4}f(u_3^*)u_3^* - F(u_3^*))dx \\ &\ge \frac{1}{4}\|u_3^*\|^2 - (\frac{1}{p} - \frac{1}{4})C\|u_3^*\|^p + \mu(c_1C\|u_3^*\| + c_2C\|u_3^*\|^q). \end{aligned} \tag{3.14}$$

Since $p, q < 2$, for $\mu \in J$, there exists a positive constant C_2 such that $\|u_3^*\| \leq C_2$. Therefore, let $L = \max\{\sigma, C_2\}$, then we have $\max\{\|u_1^*\|, \|u_2^*\|, \|u_3^*\|\} \leq L$ for all $\mu \in J$. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In the proof of Theorem 1.2, we need that I is bounded from below, which contradicts with $p \in (2, 6)$. To overcome this difficulty, motivated by the idea in [3], we introduce a truncation in the functional I. It follows from (f), (2.1) and Sobolev's embedding that, there exist three positive constants C_1, C_q, C_p such that

$$I(u) \geq \frac{1}{2}\|u\|^2 - \mu(c_1 C_1 \|u\| + \frac{c_2 C_q}{q}\|u\|^q) - \frac{C_p}{p}\|u\|^p. \tag{4.1}$$

Set $g(t) = \frac{1}{2}t^2 - \mu(c_1 C_1 t + \frac{c_2 C_q}{q}t^q) - \frac{C_p}{p}t^p$, then $I(u) \geq g(\|u\|)$. Note that there exists $\mu^* > 0$ such that, if $\mu \in (0, \mu^*)$, then $g(t) \geq 0$ over some interval $J' = [R_1, R_2]$, where R_1, R_2 depend on the choice of μ and $R_2 > R_1 > 0$, $g(R_1) = g(R_2) = 0$. Moreover, $g(t) < 0$ over interval $(0, R_1)$ or $(R_2, +\infty)$.

Assume $\mu \in (0, \mu^*)$, and introduce the following truncation of I: Take $\xi \in C_0^\infty(\mathbb{R}^+, [0, 1])$, such that $\xi(t) = 1$ if $t \in [0, R_1]$ and $\xi(t) = 0$ if $t \in [R_2, +\infty)$. Now, we consider the truncated functional

$$I_1(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \mu \int_\Omega F(u) dx - \xi(\|u\|)\frac{1}{p}\int_\Omega |u|^p dx.$$

As in (4.1), $I_1(u) \geq g_1(\|u\|)$, where $g_1 : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$g_1(t) = \frac{1}{2}t^2 - \mu(c_1 C_1 t + \frac{c_2 C_q}{q}t^q) - \xi(t)\frac{C_p}{p}t^p.$$

Note that $g_1(t) < 0$ over interval $(0, R_1)$ and $g_1(t) > 0$ over $(R_1, +\infty)$.

The following lemma ensures that I_1 has infinitely many critical points.

Lemma 4.1. *Assume that $\mu \in (0, \mu^*)$, and $\lambda > 0$ is fixed, then I_1 has infinitely many critical points.*

Proof. Using assumption (f) and $q \in (1, 2)$, it is easy to see that I_1 is coercive. So I_1 is bounded from below. Noticing that I_1 is even and using the similar argument as that in the proof of Theorem 1.1, we see that I_1 satisfies the (PS) condition. We now claim that for every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that

$$\gamma(I_1^{-\varepsilon}) \geq n. \tag{4.2}$$

Let us consider $\{e_1, e_2, \ldots\}$ an orthogonal basis of $H_0^1(\Omega)$ and for each $n \in \mathbb{N}$ consider $E_n = \{e_1, \ldots, e_n\}$, the subspace of $H_0^1(\Omega)$ generated by n vectors $e_1, \ldots, e_n$. Set

$$S_n = \{u \in E_n : \|u\| = 1\}.$$

So for every $u \in E_n$, there exist $\pi_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ such that

$$u(x) = \sum_{i=1}^{n} \pi_i e_i. \tag{4.3}$$

Therefore,

$$\|u\|_{p'+1} = \left(\int_\Omega |u|^{p'+1} dx\right)^{\frac{1}{p'+1}} = \left(\sum_{i=1}^{n} |\pi_i|^{p'+1} \int_\Omega |e_i(x)|^{p'+1} dx\right)^{\frac{1}{p'+1}}. \tag{4.4}$$

Since all norms of a finite dimensional normed space are equivalent, there is a constant $c' > 0$ such that

$$c'\|u\| \le \|u\|_{p'+1} \quad \text{for} \ \ u \in E_n. \tag{4.5}$$

By (2.1), (4.4) and (4.5), we obtain

$$\begin{aligned} I_1(su) &\le \frac{s^2}{2}\|u\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{su}(su)^2 dx - \mu \int_\Omega F(su) dx \\ &\le \frac{s^2}{2}\|u\|^2 + \lambda C\|su\|^4 - \frac{\mu}{p'+1}\sum_{i=1}^{n}\int_\Omega D|s\pi_i e_i|^{p'+1} dx \\ &\le \frac{s^2}{2}\|u\|^2 + \lambda C\|su\|^4 - \frac{\mu}{p'+1} D s^{p'+1} \sum_{i=1}^{n} |\pi_i|^{p'+1} \int_\Omega |e_i|^{p'+1} dx \\ &= \frac{s^2}{2}\|u\|^2 + \lambda C\|su\|^4 - C s^{p'+1}\|u\|_{p'+1}^{p'+1} \\ &\le \frac{s^2}{2}\|u\|^2 + \lambda C\|su\|^4 - C s^{p'+1} c'^{p'+1}\|u\|^{p'+1} \\ &= \frac{s^2}{2} + \lambda s^4 C - C s^{p'+1} c'^{p'+1}, \quad \forall u \in S_n, \ 0 < s < 1. \end{aligned} \tag{4.6}$$

Hence, $0 < p' < 1$ and (4.6) imply that there exist $\varepsilon > 0$ and $s_1 > 0$ such that

$$I_1(s_1 u) < -\varepsilon \quad \text{for } u \in S_n. \tag{4.7}$$

Let

$$S_n^{s_1} = \{s_1 u : u \in S_n\}, \quad \Pi = \{(\pi_1, \pi_2, \dots, \pi_n) : \sum_{i=1}^{n} \pi_i^2 < s_1^2\}.$$

It follows from (4.7) that $I_1(u) < -\varepsilon$ for $u \in S_n^{s_1}$, which, together with the fact that $I_1 \in C^1(E, R)$ is even, implies that

$$S_n^{s_1} \subset I_1^{-\varepsilon} \in \Sigma.$$

Since $S_n^{s_1}$ and $\partial\Pi$ are homeomorphic, by some properties of the genus (see 3° of Propositions 7.5 and 7.7 in [23]), we deduce

$$\gamma(I^{-\varepsilon}) \ge \gamma(S_n^{s_1}) = \gamma(\partial\Pi) = n. \tag{4.8}$$

Therefore, (4.2) holds. Take

$$c'_n = \inf_{A\in\Sigma_n} \sup_{u\in A} I(u). \tag{4.9}$$

Using (4.8) and the fact that I_1 is bounded from below , $-\infty < c'_n \leq -\varepsilon < 0$ for $n \in \mathbb{N}$. By Theorem 2.4, I_1 has infinitely many nontrivial critical points. □

Lemma 4.2. *The critical points of I_1 are critical points of I.*

Proof. In view of Lemma 4.1, we know that each critical point u of I_1 satisfies $g_1(\|u\|) \leq I_1(u) < 0$. Hence, $\|u\| < R_1$ and $I_1(u) = I(u)$. Furthermore, since I_1 is a continuous functional, there exists a neighborhood U of u in $H_0^1(\Omega)$ such that $I_1(u) < 0$ for all $u \in U$. Therefore, $I_1(u) \equiv I(u) < 0$ for all $u \in U$, where we deduce that $I'_1(u) = I'(u) = 0$, that is, u is a critical point of I. □

Lemmas 4.1-4.2 give the proof Theorem 1.2.

5. Proof of Theorem 1.3

In this section, $p = 6$ is one of the main difficulty in solving problem (SP), because of the lack of compactness in the inclusion of $H_0^1(\Omega)$ in $L^6(\Omega)$. Generally speaking, this means that the (PS)-condition is not satisfied. To overcome the lack of compactness, we will use the concentration-compactness principle due to Lions [17]. Moreover, as in the case $p \in (2, 6)$ in Section 4, the functional I can not be bounded below. Here we repeat the same truncation appeared in Section 4. That is, we also consider the functional I_1 which is bounded from below. Now we will show that I_1 satisfies the local (PS)-condition. For this, we give the following technical result about the functional I.

Lemma 5.1. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a bounded sequence satisfying $I(u_n) \to c$ and $I'(u_n) \to 0$. If $c < \frac{1}{24}S^{3/2}$ and the conditions of Theorem 1.3 hold, then $\{u_n\}$ contains a convergent subsequence in $H_0^1(\Omega)$.*

Proof. Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, taking a subsequence, we may obtain that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ such that $|\nabla u_n|^2$ converges in the weak*-sence of measures to a measure μ' and $|u_n|^6$ converges in the weak*-sence of measures to a measure ν. Using the concentration-compactness principle due to Lions [17], we obtain an at-most countable index set J_1 and sequences $\{x_i\} \subset \bar{\Omega}$ and $\{\mu'_i\}, \{\nu_i\} \subset (0, +\infty)$, such that

$$\mu' \geq |\nabla u|^2 + \sum_{i\in J_1} \mu'_i \delta_{x_i}, \quad \nu = |u|^6 + \sum_{i\in J_1} \nu_i \delta_{x_i} \text{ and } S\nu_i^{1/3} \leq \mu'_i \tag{5.1}$$

for all $i \in J_1$, where δ_{x_i} is the Dirac mass at $x_i \in \Omega$. Take $x_i \in \bar{\Omega}$ in the support of the singular part of μ', ν. Now for fixed $\epsilon > 0$, we consider $\chi(x) := \chi(x - x_i)$, where $\chi \in C_0^\infty(\Omega, [0, 1])$ is such that $\chi \equiv 1$ on $B_\epsilon(0)$, $\chi \equiv 0$ on $\Omega \setminus B_{2\epsilon}(0)$ and

$|\nabla\chi| \in [0, \frac{2}{\epsilon}]$. It is clear that the sequence $\{\chi u_n\}$ is bounded, then we have $I'(u_n)(\chi u_n) \to 0$. By (2.2), one has

$$\int_\Omega u_n \nabla u_n \nabla \chi dx = \int_\Omega \left(-|\nabla u_n|^2 - u_n^2 - \lambda\phi_{u_n} u_n^2 + \mu f(u_n)u_n + |u_n|^6\right)\chi_\epsilon dx + o_n(1). \tag{5.2}$$

Using the Hölder inequality and the right sides of (5.2), we obtain the following limit expression:

$$\begin{aligned} |\int_\Omega u_n \nabla u_n \nabla \chi_\epsilon dx| &\le (\int_{B_{2\epsilon}(x_i)} (u_n \nabla u_n)^{\frac{3}{2}} dx)^{\frac{2}{3}} (\int_{B_{2\epsilon}(x_i)} |\nabla\chi_\epsilon|^3 dx)^{\frac{1}{3}} \\ &\le C(\int_{B_{2\epsilon}(x_i)} |u_n|^6 dx)^{\frac{1}{6}} \to 0 \end{aligned} \tag{5.3}$$

as $\epsilon \to 0$. Moreover, since $u_n \to u$ in $L^s(\Omega)$ for all $1 \le s < 6$ and χ has compact support, it follows from (5.2), (5.3) and (f) with $q \in (1, 2)$ that

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega u_n \nabla u_n \nabla \chi dx = &- \int_\Omega (|\nabla u|^2 + u^2 + \lambda\phi_u u^2 - \mu f(u)u - |u|^6)\chi dx \\ &+ \int_\Omega \chi d\nu - \int_\Omega \chi d\mu'. \end{aligned} \tag{5.4}$$

Letting $\epsilon \to 0$ in (5.4) we conclude that $\nu_i = \mu_i'$. It follows from (5.1) that $\nu_i \ge S^{3/2}$. Now we shall show that the earlier inequality is not possible, and therefore the set J_1 is empty. If not, let us suppose that $\nu_i \ge S^{3/2}$ for some $i \in J_1$. Thus, by (2.1), (2.2) and Young's inequality, for any $\epsilon > 0$, there exists $\beta_3(\epsilon) > 0$ such that

(5.5)

$$\begin{aligned} c &= I(u_n) - \frac{1}{2} I'(u_n)u_n + o_n(1) \\ &= \frac{-\lambda}{4} \int_\Omega \phi_{u_n} u_n^2 dx + \mu \int_\Omega (\frac{1}{2} f(u_n)u_n - F(u_n))dx + \frac{1}{3}\int_\Omega |u_n|^6 dx + o_n(1) \\ &\ge \frac{-\lambda}{4} C \int_\Omega |u_n|^4 dx - \mu(2c_1\|u_n\|_1 + Cc_2\|u_n\|_q^q) + \frac{1}{3}\int_\Omega |u_n|^6 dx + o_n(1) \\ &\ge \frac{1}{3}\int_\Omega |u_n|^6 dx - \mu(2\epsilon\|u_n\|_6^6 + \beta_3(\epsilon)C) + \frac{-\lambda}{4} C \int_\Omega |u_n|^4 dx + o_n(1). \end{aligned}$$

Choose $\epsilon = \frac{1}{12}$ and $0 < \mu < 1$, then by (5.5), we have

$$c \ge \frac{1}{6}\int_\Omega |u_n|^6 dx - \mu\beta_3(\frac{1}{12})C + \frac{-\lambda}{4} C \int_\Omega |u_n|^4 dx + o_n(1).$$

Letting $n \to \infty$, and using (5.1) we obtain

$$\begin{aligned} c &\ge \frac{1}{6}\int_\Omega |u|^6 dx - \mu\beta_3(\frac{1}{12})C + \frac{-\lambda}{4} C \int_\Omega |u|^4 dx + \frac{1}{6}\nu_i \\ &\ge \frac{1}{6} S^{3/2} + \frac{1}{6}\int_\Omega |u|^6 dx - \mu\beta_3(\frac{1}{12})C + \frac{-\lambda}{4} C \int_\Omega |u|^4 dx. \end{aligned}$$

In the above inequality, take $\mu \in (0, \mu_3^*)$ where $\mu_3^* = \min\{1, \mu_4^*, \frac{S^{3/2}}{12\beta_3(\frac{1}{12})C}\}$ (μ_4^* can be appeared in structure of the truncation functional I_1, μ^* replaced by μ_4^*, see Section 4), then, by the Hölder inequality, we deduce that

$$c \geq \frac{1}{12}S^{3/2} + \frac{1}{6}\int_\Omega |u|^6 dx - \frac{\lambda}{4}|\Omega|^{\frac{1}{3}} C(\int_\Omega |u|^6 dx)^{\frac{4}{6}}.$$

Let

$$g_2(t) = \frac{1}{6}t^6 - \frac{\lambda}{4}|\Omega|^{\frac{1}{3}} C t^4, \ t > 0.$$

This function attains its absolute minimum at the point $t_0 = (\lambda|\Omega|^{\frac{1}{3}} C)^{1/2}$. Thus, we conclude that

$$c \geq \frac{1}{12}S^{3/2} - \frac{1}{12}\lambda^3|\Omega|C^3.$$

Take $\lambda \in (0, \lambda^*)$, where $\lambda^* = (\frac{S^{3/2}}{2|\Omega|C^3})^{1/3}$, then $c \geq \frac{1}{24}S^{3/2}$. But this is a contradiction. Thus J_1 is empty and it follows that $u_n \to u$ in $L^6(\Omega)$. Therefore, arguing as in the proof of (PS)-condition in Theorem 1.1, we obtain $u_n \to u$ in $H_0^1(\Omega)$. □

Lemma 5.2. *If $I_1(u) < 0$, then $\|u\| < R_1$ and $I(v) = I_1(v)$ for all v in a sufficiently small neighborhood of u. Moreover, I_1 satisfies a local* (PS)-*condition for $c < 0$.*

Proof. Since $g_1(\|u\|) \leq I_1(u) < 0$, we know that $\|u\| < R_1$. Hence, there exists a sufficiently small neighborhood U of u such that $\|v\| < R_1$ for $v \in U$. Note that $\xi(\|v\|) \equiv 1$, then $I(v) \equiv I_1(v)$, $I'(v) \equiv I_1'(v)$ for $v \in U$. Moreover, if $\{u_n\}$ is a sequence such that $I_1(u_n) \to c < 0$ and $I_1'(u_n) \to 0$, then, for n sufficiently large, $I(u_n) = I_1(u_n) \to c < 0$ and $I'(u_n) = I_1'(u_n) \to 0$. Since I_1 is coercive, we have that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Lemma 5.1, we know that $u_n \to u$ in $H_0^1(\Omega)$. □

Lemma 5.3. *Assume that $\mu \in (0, \mu^*)$ and $\lambda \in (0, \lambda^*)$, then I_1 has infinitely many critical points.*

Proof. Arguing as in the proof in Lemma 4.1, we can construct an appropriate mini-max sequence $\{c_n'\}$ of negative critical values for the functional I_1, where c_n' has been defined in (4.9). In view of Lemma 5.2, if $-\infty < c_1' < c_2' < \cdots < c_n' < 0$, then each c_n' is a critical value of I_1 and I_1 has infinitely many critical points. On the other hand, if there are two constants $c_n' = c_{n+r}'$, then $c' = c_n' = \cdots = c_{n+r}'$. Note that, (see Lemma 5.6 in [30] for the proof)

$$\gamma(K_{c'}) \geq r + 1 \geq 2, \quad K_{c'} = \{u \in H_0^1(\Omega) : I_1'(u) = 0 \text{ and } J(u) = c'\}.$$

Moreover, $K_{c'}$ has infinitely many points, that is, I_1 has infinitely many critical points. □

Lemma 5.4. *Under the conditions of Theorem 1.3, the critical points of I_1 are critical points of I.*

The proof is the same as that of Lemma 4.2, and hence is omitted.

Lemmas 5.1-5.4 give the proof Theorem 1.3.

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