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MULTIPLICITY RESULTS ON A FOURTH ORDER NONLINEAR ELLIPTIC EQUATION

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ABSTRACT. We are concerned with a fourth order semilinear elliptic boundary value problem under Dirichlet boundary condition $\Delta^2 u + c\Delta u = bu^+ + s$ in Ω, where Ω is a bounded open set in $\mathbf{R}^n$ with smooth boundary. We investigate the existence of solutions of the fourth order nonlinear equation when the nonlinearity bu^+ crosses eigenvalues of $\Delta^2 + c\Delta$ under the Dirichlet boundary condition and s is constant.

0. Introduction. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. In this paper, we are concerned with a fourth order semilinear elliptic boundary value problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= bu^+ + s \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{0.1}$$

where $u^+ = \max\{u, 0\}$, s is real, and c is not an eigenvalue of $-\Delta$ under Dirichlet boundary condition. The operator Δ^2 denotes the biharmonic operator. We assume that b is not an eigenvalue of $\Delta^2 + c\Delta$ under Dirichlet boundary condition.

The nonlinear equation with jumping nonlinearity has been extensively studied by many authors [**3, 4, 6, 7, 8**]. They studied the existence of solutions of the nonlinear equation with jumping nonlinearity for the second order elliptic operator [**6**], for one dimensional wave operators [**3, 4**], and for other operators [**7, 8**] when the source term is a multiple of the positive eigenfunction.

In [**13**], Tarantello considered the fourth order, nonlinear elliptic problem under the Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u &= b[(u+1)^+ - 1] \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{0.2}$$

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She showed by degree theory that if $b \geq \lambda_1(\lambda_1 - c)$ then (0.2) has a solution u such that $u(x) < 0$ in Ω. In [**9, 10**], Micheletti and Pistoia also investigated the existence of solutions of the above equation.

In this paper we investigate the existence of solutions of the fourth order nonlinear equation (0.1) when the nonlinearity bu^+ crosses eigenvalues of $\Delta^2 + c\Delta$ under the Dirichlet boundary condition.

In Section 1 we introduce the Banach space spanned by eigenfunctions of $\Delta^2 + c\Delta$ and investigate the existence of solutions of (0.1) when the nonlinearity bu^+ satisfies $\lambda_1 < c$, $b < \lambda_1(\lambda_1 - c)$ and when it satisfies $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$.

In Section 2 we investigate the multiplicity of solutions of (0.1) under the following two conditions.

Condition (1). $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$.

Condition (2). $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$ $(k = 1, 2, \dots)$ and $s < 0$.

We show by use of a variational reduction method that equation (0.1) under each condition of the above has at least two solutions.

1. The Banach space and the existence of solutions. In this section we introduce the Banach space spanned by eigenfunctions of the operator $\Delta^2 + c\Delta$, and we investigate the existence of solutions of the boundary value problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= bu^+ + s \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.1}$$

Here c is not an eigenvalue of $-\Delta$ under the Dirichlet boundary condition, and the nonlinearity bu^+ satisfies $\lambda_1 < c$, $b < \lambda_1(\lambda_1 - c)$ or $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$.

Let λ_k, $k = 1, 2, \dots$, denote the eigenvalues and ϕ_k, $k = 1, 2 \dots$, the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, under Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, $\lambda_i \to +\infty$ and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= \mu u \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

has infinitely many eigenvalues

$$\mu_k = \lambda_k(\lambda_k - c), \qquad k = 1, 2, \dots$$

and corresponding eigenfunctions $\phi_k(x)$.

The set of functions $\{\phi_k\}$ is an orthogonal base for $W_0^{1,2}(\Omega)$. Let us denote an element u of $W_0^{1,2}(\Omega)$ as

$$u = \sum h_k\phi_k, \qquad \sum h_k^2 < \infty.$$

Let c not be an eigenvalue of $-\Delta$, and define a subspace H of $W_0^{1,2}(\Omega)$ as follows

$$H = \Big\{u \in W_0^{1,2}(\Omega) : \sum |\lambda_k(\lambda_k - c)|h_k^2 < \infty\Big\}.$$

Then this is a complete normed space with a norm

$$|\,\|u\|\,| = \Big[\sum |\lambda_k(\lambda_k - c)|h_k^2\Big]^{1/2}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have the following simple properties.

Proposition 1.1. *Let c not be an eigenvalue of $-\Delta$ under the Dirichlet boundary condition. Then we have*

(i) $\Delta^2 u + c\Delta u \in H$ *implies* $u \in H$.

(ii) $|\|u\|| \geq C\|u\|_{L^2(\Omega)}$ *for some* $C > 0$.

(iii) $\|u\|_{L^2(\Omega)} = 0$ *if and only if* $|\|u\|| = 0$.

Proof. (i) Suppose c is not an eigenvalue of $-\Delta$, and let $u = \sum h_k\phi_k$. Then

$$\Delta^2 u + c\Delta u = \sum \lambda_k(\lambda_k - c)h_k\phi_k.$$

Hence

$$\begin{aligned}\infty > |\,\|\Delta^2 u + c\Delta u\|\,|^2 &= \sum |\lambda_k(\lambda_k - c)|(\lambda_k(\lambda_k - c))^2 h_k^2\\ &\geq C\sum |\lambda_k(\lambda_k - c)|h_k^2 = |\,\|u\|\,|^2,\end{aligned}$$

where $C = \inf_k\{\lambda_k(\lambda_k - c) : k = 1, 2, \cdots\}$. (ii) and (iii) are trivial. □

Lemma 1.1. *Let d not be an eigenvalue of $\Delta^2 + c\Delta$ and $u \in L^2(\Omega)$. Then $(\Delta^2 + c\Delta + d)^{-1}u \in H$.*

Proof. Suppose that d is not an eigenvalue of $\Delta^2 + c\Delta$ and finite. We know that the number of elements of $\{\lambda_k(\lambda_k - c) : |\lambda_k(\lambda_k - c)| < |d|\}$ is finite, where $\lambda_k(\lambda_k - c)$ is an eigenvalue of $\Delta^2 + c\Delta$. Let $u = \sum h_k\phi_k$. Then

$$(\Delta^2 + c\Delta + d)^{-1}u = \sum \frac{1}{\lambda_k(\lambda_k - c) + d} h_k\phi_k.$$

Hence we have the inequality

$$\begin{aligned} |\,\|(\Delta^2 + c\Delta + d)^{-1}u\|\,|^2 &= \sum |\lambda_k(\lambda_k - c)| \frac{1}{(\lambda_k(\lambda_k - c) + d)^2} h_k^2 \\ &\leq C \sum h_k^2 \end{aligned}$$

for some C, which means that

$$|\,\|(\Delta^2 + c\Delta + d)^{-1}u\|\,| \leq C_1\|u\|_{L^2(\Omega)}, \quad C_1 = \sqrt{C}. \qquad \square$$

With Lemma 1.1, we can obtain the following lemma.

Lemma 1.2. *Let $f \in L^2(\Omega)$. Let b be not an eigenvalue of $\Delta^2 + c\Delta$. Then all solutions in $W_0^{1,2}(\Omega)$ of*

$$\Delta^2 u + c\Delta u = bu^+ + f(x) \quad in \quad W_0^{1,2}(\Omega)$$

belong to H.

With the aid of Lemma 1.2, it is enough to investigate the existence of solutions of (1.1) in the subspace H of $W_0^{1,2}(\Omega)$, namely,

$$\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in} \quad H. \tag{1.2}$$

Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c)$, $\lambda_{k+1}(\lambda_{k+1} - c)$ be successive eigenvalues of $\Delta^2 + c\Delta$ such that there is no eigenvalue between

$\lambda_k(\lambda_k - c)$ and $\lambda_{k+1}(\lambda_{k+1} - c)$. Then $\lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c)$ and we have the uniqueness theorem.

Theorem 1.1. *Suppose $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$. Then Equation (1.2) has exactly one solution in $L^2(\Omega)$ for all real s. Furthermore Equation (1.2) has a unique solution in H.*

Proof. We consider the equation

$$-\Delta^2 u - c\Delta u + bu^+ = -s \quad \text{in} \quad L^2(\Omega). \tag{1.3}$$

Let $\delta = \{\lambda_k(\lambda_k - c) + \lambda_{k+1}(\lambda_{k+1} - c)\}/2$. Then Equation (1.3) is equivalent to

$$u = (-\Delta^2 - c\Delta + \delta)^{-1}[(\delta - b)u^+ - \delta u^- - s], \tag{1.4}$$

where $(-\Delta^2 - c\Delta + \delta)^{-1}$ is a compact, self-adjoint, linear map from $L^2(\Omega)$ into $L^2(\Omega)$ with norm $2/(\lambda_{k+1}(\lambda_{k+1} - c) - \lambda_k(\lambda_k - c))$. We note that

$$\begin{aligned}\|(\delta - b)(u_2^+ - u_1^+) &- \delta(u_2^- - u_1^-)\| \\ &\le \max\{|\delta - b|, |\delta|\}\|u_2 - u_1\| \\ &< \frac{1}{2}\{\lambda_{k+1}(\lambda_{k+1} - c) - \lambda_k(\lambda_k - c)\}\|u_2 - u_1\|.\end{aligned}$$

It follows that the right-hand side of (1.4) defines a Lipschitz mapping from $L^2(\Omega)$ into $L^2(\Omega)$ with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in L^2(\Omega)$ of (1.4).

On the other hand, by Lemma 1.2, the solution of (1.4) belongs to H. □

We now examine Equation (1.2) when $\lambda_1 < c$ and $b < \lambda_1(\lambda_1 - c) < 0$.

Theorem 1.2. *Assume that $\lambda_1 < c$ and $b < \lambda_1(\lambda_1 - c) < 0$. Then we have*

(i) *If $s < 0$, then equation (1.2) has no solution.*

(ii) *If* $s = 0$, *then equation* (1.2) *has only the trivial solution.*

Proof. Assume $s \le 0$. We rewrite (1.2) as

$$\{-\Delta^2 - c\Delta + \lambda_1(\lambda_1 - c)\}u + \{-\lambda_1(\lambda_1 - c) + b\}u^+ \\ - \{-\lambda_1(\lambda_1 - c)\}u^- = -s.$$

Multiply across by ϕ_1 and integrate over Ω. Since $(\{-\Delta^2 - c\Delta + \lambda_1(\lambda_1 - c)\}u, \phi_1) = 0$, we have

$$\int_\Omega [\{-\lambda_1(\lambda_1 - c) + b\}u^+ - \{-\lambda_1(\lambda_1 - c)\}u^-]\phi_1 = -s\int_\Omega \phi_1. \tag{1.5}$$

But $\{-\lambda_1(\lambda_1 - c) + b\}u^+ - \{-\lambda_1(\lambda_1 - c)\}u^- \le 0$ for all real valued function u and $\phi_1(x) > 0$ for $x \in \Omega$. Therefore the left-hand side of (1.5) is always less than or equal to zero. Hence, if $s < 0$, then there is no solution of (1.2) and if $s = 0$, then the only possibility is $u \equiv 0$. □

For the case $s > 0$ in Theorem 1.2, we shall investigate the existence of solutions of (1.2) in the next section.

If $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$ and $s > 0$, then the left-hand side of (1.5) is larger than or equal to zero and the right-hand side of it is negative.

Therefore we have the following theorem.

Theorem 1.3. *Assume that* $c < \lambda_1$ *and* $0 < \lambda_1(\lambda_1 - c) < b$, $b \ne \lambda_k(\lambda_k - c)$, $k = 2, 3, \dots$. *Then we have*

(i) *If* $s > 0$, *then equation* (1.2) *has no solution.*

(ii) *If* $s = 0$, *then equation* (1.2) *has only the trivial solution.*

Proof. Assume that $s \ge 0$. We rewrite (1.2) as

$$\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u + [\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- = s.$$

Multiply across by ϕ_1 and integrate over Ω. Since $(\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u, \phi_1) = 0$, we have

$$\int_\Omega \{[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^-\}\phi_1 = s\int \phi_1. \tag{1.6}$$

But $[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- \leq 0$ for any real valued function u. Also $\phi_1(x) > 0$ in Ω. Therefore, if $s > 0$, then equation (1.2) has no solution and if $s = 0$, then the only possibility is that $u = 0$. □

For the case $s < 0$ in Theorem 1.3, we shall investigate the existence of solutions of (1.1) in the next section.

2. Main results. In this section we investigate the multiplicity of solutions of the problem

$$\Delta^2 u + c\Delta = bu^+ + s \quad \text{in} \quad H \tag{2.1}$$

under the following two conditions.

Condition (1). $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$.

Condition (2). $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k = 1, 2, \ldots,$ and $s < 0$.

First we investigate the multiplicity of solutions of (2.1) under Condition (1).

Theorem 2.1. *Assume that* $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ *and* $s > 0$. *Then the problem* (2.1) *has at least two solutions.*

One solution is positive, and the existence of the other solution will be proved by critical point theory. For the proof of the theorem, we need several lemmas.

Lemma 2.1. *Let* $\lambda_k < c < \lambda_{k+1} (k \geq 1)$ *and* $b < \lambda_1(\lambda_1 - c)$. *Then the problem*

$$\Delta^2 u + c\Delta u = bu^+ \quad \text{in} \quad H \tag{2.2}$$

has only the trivial solution.

Proof. We rewrite (2.2) as

$$\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u + [\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- = 0 \quad \text{in} \quad H.$$

Multiply across by ϕ_1 and integrate over Ω. Since $(\{\Delta^2+c\Delta-\lambda_1(\lambda_1-c)\}u,\phi_1)=0$, we have

$$\int_\Omega\{[\lambda_1(\lambda_1-c)-b]u^+-\lambda_1(\lambda_1-c)u^-\}\phi_1=0. \tag{2.3}$$

But $[\lambda_1(\lambda_1-c)-b]u^+-\lambda_1(\lambda_1-c)u^-\geq 0$ for all real valued function u and $\phi_1(x)>0$ for $x\in\Omega$. Hence the left-hand side of (1.4) is always greater than or equal to zero.

Therefore the only possibility to hold (2.3) is that $u\equiv 0$. □

Now we study the existence of the positive solution of (2.1).

Lemma 2.2. *Let* $\lambda_1<c<\lambda_2$, $b<\lambda_1(\lambda_1-c)$ *and* $s>0$. *Then the unique solution* u_1 *of the problem*

$$\Delta^2u+c\Delta u=bu+s \quad in \quad L^2(\Omega) \tag{2.4}$$

is positive.

Proof. Let $\lambda_1<c<\lambda_2$ and $b<\lambda_1(\lambda_2-c)$. Then the problem

$$\Delta^2u+c\Delta u-bu=\mu u \quad \text{in} \quad L^2(\Omega)$$

has eigenvalues $\lambda_k(\lambda_k-c)-b$ and they are positive. Since the inverse $(\Delta^2+c\Delta-b)^{-1}$ of the operator $\Delta^2+c\Delta-b$ is positive, the solution $u=(\Delta^2+c\Delta-b)^{-1}(s)$ of (2.4) is positive. This proves the lemma. □

An easy consequence of Lemma 2.2 is

Lemma 2.3. *Let* $c<\lambda_1$, $b<\lambda_1(\lambda_1-c)$ *and* $s>0$. *Then the boundary value problem* (2.1) *has a positive solution* u_1.

Proof. The solution u_1 of the linear problem (2.4) is positive, hence it is also a solution of (2.1). □

Now, we investigate the existence of the other solution of problem (2.1) under the condition $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$ by the critical point theory.

Let us define the functional corresponding to (2.1) in $H \times R$

$$F_b(u,s) = \int_\Omega \left[\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2 - su\right] dx. \tag{2.6}$$

For simplicity, we shall write $F = F_b$ when b is fixed. Then F is well-defined. The solutions of (2.1) coincide with the critical points of $F(u,s)$.

Proposition 2.1. *Let b be fixed and $s \in R$. Then $F(u,s) = F_b(u,s)$ is continuous and Fréchet differentiable in H.*

Proof. Let $u \in H$. For $s \in R$, to prove the continuity of $F(u,s)$, we consider

$$\begin{aligned} F(u+v,s) - F(u,s) = \int_\Omega \Big[& u \cdot (\Delta^2 v + c\Delta v) + \frac{1}{2} v \cdot (\Delta^2 v + c\Delta v) \\ & - \frac{b}{2}(|(u+v)^+|^2 - |u^+|^2) - so \Big] dx. \end{aligned}$$

Let $u = \sum h_k \phi_k$, $v = \sum \tilde{h_k}\phi_k$. Then we have

$$\left| \int_\Omega u \cdot (\Delta^2 v + c\Delta v) dx \right| = \left| \sum \lambda_k(\lambda_k - c) h_k \tilde{h_k} | \leq | \, \|u\| \, | \cdot | \, \|v\| \, \right|.$$

$$\left| \int_\Omega \frac{1}{2} v \cdot (\Delta^2 v + c\Delta v)\, dx \right| = \left| \sum \lambda_k(\lambda_k - c) \tilde{h}_k^2 | \leq | \, \|u\| \, \right|^2.$$

On the other hand,

$$| \, |(u+v)^+|^2 - |u^+|^2 | \leq 2u^+|v| + |v|^2$$

and hence we have

$$\begin{aligned} \left| \int_\Omega [|(u+v)^+|^2 - |u^+|^2]\, dx \right| & \leq 2\|u^+\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \\ & \leq C(2| \, \|u\| \, | \cdot | \, \|v\| \, | + | \, \|v\| \, |^2) \end{aligned}$$

for some $C > 0$. With the above results, we see that $F(u,s)$ is continuous at u. Now we prove that $F(u,s)$ is Fréchet differentiable at $u \in H$ with

$$DF(u,s)v = \int_\Omega (\Delta^2 u + c\Delta u - bu^+ - s)v\,dx.$$

To prove the above equation, it is enough to compute the following:

$$\begin{aligned}
&|F(u+v,s) - F(u,s) - DF(u,s)v| \\
&\quad = \left| \int_\Omega \frac{1}{2} v \cdot (\Delta^2 v + c\Delta v) - \frac{b}{2}[|(u+v)^+|^2 - |u^+|^2 - 2u^+ v]\,dx \right| \\
&\quad \le \frac{1}{2} |\,\|v\|\,|^2 + \frac{|b|}{2} \int v^2\,dx \\
&\quad \le \frac{1}{2}(1 + |b|C)|\,\|v\|\,|^2
\end{aligned}$$

for some C, since $0 \le |(u+v)^+|^2 - |u^+|^2 - 2u^+ v \le |v|^2$. □

Let V be the one-dimensional subspace of $L^2(\Omega)$ spanned by ϕ_1 whose eigenvalue is $\lambda_1(\lambda_1 - c)$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ be the orthogonal projection of H onto V and $I - P : H \to W$ denote that of H onto W. Then every element $u \in H$ is expressed by $u = v + z$, where $v = Pu$, $z = (I - P)u$. Then problem (2.1) is equivalent to

$$\begin{aligned}
\Delta^2 v + c\Delta v &= P[b(v+z)^+ + s], \\
\Delta^2 z + c\Delta z &= (I-P)[b(v+z)^+ + s].
\end{aligned}$$

We look on the above equations as a system of two equations in two unknowns v and w.

Lemma 2.4. *Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then we have*

(i) *There exists a unique solution $z \in W$ of the equation*

$$\Delta^2 z + c\Delta z - (I-P)[b(v+z)^+ + s] = 0 \quad \textit{in} \quad W. \tag{2.7}$$

If, for fixed $s \in R$, we put $z = \theta(v,s)$, then θ is continuous on V. In particular, θ satisfies a uniform Lipschitz condition in v with respect to the L^2 norm (also the norm $|\|\cdot\||$).

(ii) *If $\tilde{F} : V \to R$ is defined by $\tilde{F}(v,s) = F(v + \theta(v,s), s)$, then $\tilde{F}$ has a continuous Fréchet derivative $D\tilde{F}$ with respect to v and*

$$D\tilde{F}(v,s)(h) = DF(v + \theta(v,s), s)(h) = 0 \quad \text{for all} \quad h \in V.$$

If v_0 is a critical point of $\tilde{F}$, then $v_0 + \theta(v_0, s)$ is a solution of the problem (2.1) and conversely every solution of (2.1) is of this form.

Proof. Let $\lambda_1 < c < \lambda_2$, $\alpha < b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Let $\delta = (b/2) < 0$ and $g(\xi) = b\xi^+$. If $g_1(\xi) = g(\xi) - \delta\xi$, then Equation (2.7) is equivalent to

$$z = (\Delta^2 + c\Delta - \delta)^{-1}(I - P)(g_1(v+z)^+ + s). \tag{2.8}$$

Since $(\Delta^2 + c\Delta - \delta)^{-1}(I - P)$ is a self-adjoint, compact, linear map from $(I-P)L^2(\Omega)$ onto itself, the eigenvalues of $(\Delta^2 + c\Delta - \delta)^{-1}(I - P)$ are $(\lambda_l(\lambda_l - c) - \delta)^{-1}$, where $\lambda_l(\lambda_l - c) \geq \lambda_2(\lambda_2 - c)$. Therefore its L^2 norm is $1/(\lambda_2(\lambda_2 - c) - \delta)$. Since

$$|g_1(\xi_2) - g_1(\xi_1)| \leq \max\{|b - \delta|, |\delta|\}|\xi_2 - \xi_1|,$$

it follows that the right-hand side of (2.8) defines, for fixed $v \in V$, a Lipschitz mapping of $(I - P)L^2(\Omega)$ into itself with Lipschitz constant $\gamma = |b|/2 \cdot (1/(\lambda_2(\lambda_2 - c) - (b/2))) < 1$.

Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I - P)L^2(\Omega)$ which satisfies (2.8).

Since the constant δ does not depend on v and s, it follows from standard arguments that, if $\theta(v,s)$ denotes the unique $z \in (I-P)L^2(\Omega)$ which solves (2.8), then θ is continuous with respect to v. In fact, if $z_1 = \theta(v_1, s)$ and $z_2 = \theta(v_2, s)$, then we have

$$\begin{aligned} |\|z_1 - z_2\|| &= \|(\Delta^2 + c\Delta - \delta)^{-1}(I - P)(g_1(v_1 + z_1) - g_1(v_2 + z_2)\| \\ &= \gamma\|(v_1 + z_1) - (v_2 + z_2)\| \\ &\leq \gamma(\|v_1 - v_2\| + \|z_1 - z_2\|). \end{aligned}$$

Hence we have

$$\|z_1 - z_2\| \le C\|v_1 - v_2\|, \quad C = \frac{\gamma}{1-\gamma},$$

which shows that $\theta(v,s)$ satisfies a uniform Lipschitz condition in v with respect to L^2-norm. With the above inequality we have

$$\begin{aligned} |\,\|z_1 - z_2\|\,| &= |\,\|(\Delta^2 + c\Delta - \delta)^{-1}(I-P)(g_1(v_1+z_1) - g_1(v_2+z_2))\|\,| \\ &\le C_1\|(I-P)(g_1(v_1+z_2) - g_2(v_2+z_2))\| \\ &\le C_1\frac{b}{2}\|(v_1+z_1) - (v_2+z_2)\| \\ &\le C_1\frac{b}{2}(\|v_1 - v_2\| + \|z_1 - z_2\|) \\ &\le C_1\frac{b}{2}(1+C)\|v_1 - v_2\| \end{aligned}$$

for some $C_1 > 0$. Hence we have

$$|\,\|z_1 - z_2\|\,| \le C_2|\,\|v_1 - v_2\|\,| \tag{2.9}$$

for some $C_2 > 0$. This shows that $\theta(v,s)$ satisfies a uniform Lipschitz condition in v with respect to the norm $|\,\|\;\|\,|$.

Let $v \in V$ and $z = \theta(v,s)$. If $w \in W$, then from (2.7) we see that

$$\int_\Omega [\Delta z \cdot \Delta w - c\nabla z \cdot \nabla w - (I-P)[b(v+z)^+ + s]] \cdot w\,dx = 0. \tag{2.10}$$

Since

$$\int_\Omega \Delta v \cdot \Delta w = 0 \quad \text{and} \quad \int_\Omega \nabla v \cdot \nabla w = 0,$$

we have

$$DF(v + \theta(v,s), s)(w) = 0 \quad \text{for} \quad w \in W. \tag{2.11}$$

From Proposition 2.1, $\tilde{F}(v,s)$ has a continuous Fréchet derivative $D\tilde{F}$, and

$$D\tilde{F}(v,s)(h) = DF(v + \theta(v,s), s)(h), \quad h \in V. \tag{2.12}$$

Suppose that, for some fixed $s > 0$, there exists $v_0 \in V$ such that $D\tilde{F}(v_0, s) = 0$. Then it follows (2.12) that

$$DF(v_0 + \theta(v_0, s), s)(v) = 0 \quad \text{for all} \quad v \in V.$$

Since (2.11) holds for all $w \in W$ and H is the direct sum of V and W, it follows that

$$DF(v_0 + \theta(v_0, s), s) = 0 \quad \text{in} \quad H.$$

Since (2.11) holds for all $w \in W$ and H is the direct sum of V and W, it follows that

$$DF(v_0 + \theta(v_0, s), s) = 0 \quad \text{in} \quad H.$$

Therefore $u = v_0 + \theta(v_0, s)$ is a solution of (2.1).

Conversely, our reasoning shows that if u is a solution of (2.1) and $v = Pu$, then $D\tilde{F}(v, s) = 0$ in V. □

Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c), \lambda_1(\lambda_1 - c) < 0 < \lambda_2(\lambda_2 - c)$ and $s > 0$. From Lemma 2.3, we see that (2.1) has a positive solution $u_1(x)$. From Lemma 2.4, $u_1(x)$ is of the form $u_1(x) = v_1 + \theta(v_1, s)$.

Lemma 2.5. *Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then there exists a small open neighborhood B of v_1 in V such that $v = v_1$ is a strict local point of minimum of $\tilde{F}$.*

Proof. Let $s > 0$. Then Equation (2.1) has a positive solution $u_1(x)$ which is of the form $u_1(x) = v_1 + \theta(v_1, s) > 0$, $\theta(v_1, s) \in W$. Since $I + \theta$, where I is an identity map on V, is continuous on V, there exists a small open neighborhood B of v_1 in V such that if $v \in B$, then $v + \theta(v, s) > 0$. Therefore, if $z = \theta(v, s)$, $z_1 = \theta(v_1, s)$ and $v + z = (v_1 + z_1) + (\tilde{v} + \tilde{z})$, then we have

$$\begin{aligned}
\tilde{F}(v, s) &= F(v + z, s) \\
&= \int_\Omega \left[\frac{1}{2}|\Delta(v + z)|^2 - \frac{c}{2}|\nabla(v + z)|^2 - \frac{b}{2}|v + z|^2 - s(v + z)\right] dx \\
&= \int_\Omega \left[\frac{1}{2}|\Delta(v_1 + z_1) + \Delta(\tilde{v} + \tilde{z})|^2 - \frac{c}{2}|\nabla(v_1 + z_1) + \nabla(\tilde{v} + \tilde{z})|^2\right. \\
&\qquad \left. - \frac{b}{2}|(v_1 + z_1) + (\tilde{v} + \tilde{z})|^2 - s\{(v_1 + z_1) + (\tilde{v} + \tilde{z})\}\right] dx
\end{aligned}$$

$$= \int_\Omega \left[\frac{1}{2}|\Delta(v_1+z_1)|^2 - \frac{c}{2}|\nabla(v_1+z_1)|^2 - \frac{b}{2}|v_1+z_1|^2 - s(v_1+z_1)\right] dx$$
$$+ \int_\Omega [\Delta(v_1+z_1)\cdot\Delta(\tilde{v}+\tilde{z}) - c\nabla(v_1+z_1)\cdot\nabla(\tilde{v}+\tilde{z}) - b(v_1+z_1)\cdot(\tilde{v}+\tilde{z}) - s(\tilde{v}+\tilde{z})]\, dx$$
$$+ \int_\Omega \left[\frac{1}{2}|\Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v}+\tilde{z})|^2 - \frac{b}{2}|\tilde{v}+\tilde{z}|^2\right] dx.$$

Here

$$\int_\Omega \left[\frac{1}{2}|\Delta(v_1+z_1)|^2 - \frac{c}{2}|\nabla(v_1+z_1)|^2 - \frac{b}{2}|v_1+z_1|^2 - s(v_1+z_1)\right] dx = F(v_1+z_1,s) = \tilde{F}(v_1,s)$$

and

$$\int_\Omega [\Delta(v_1+z_1)\cdot\Delta(\tilde{v}+\tilde{z}) - c\nabla(v_1+z_1)\cdot\nabla(\tilde{v}+\tilde{z}) - b(v_1+z_1)\cdot(\tilde{v}+\tilde{z}) - s(\tilde{v}+\tilde{z})]\, dx$$
$$= \int_\Omega [\Delta^2(v_1+z_1) + c\Delta(v_1+z_1) - b(v_1+z_1) - s]\cdot(\tilde{v}+\tilde{z})\, dx = 0,$$

since v_1+z_1 is a positive solution of (2.1). Since $\tilde{v}+\tilde{z}$ can be expressed by $\tilde{v}+\tilde{z} = e_1\phi_1 + e_2\phi_2 + \cdots$, we have

$$\tilde{F}(v,s) - \tilde{F}(v_1,s) = \int_\Omega \left[\frac{1}{2}|\Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v}+\tilde{z})|^2 - \frac{b}{2}|\tilde{v}+\tilde{z}|^2\right] dx$$
$$= \frac{1}{2}\{[\lambda_1(\lambda_1-c) - b]e_1^2 + [\lambda_2(\lambda_2-c) - b]e_2^2 + \cdots\} > 0,$$

since $b < \lambda_1(\lambda_1 - c)$ and $\lambda_1 < c < \lambda_2$. Therefore $v = v_1$ is a strict local point of minimum of $\tilde{F}$. This proves the lemma. □

We now define the functional on H as

$$F^*(u) = F(u,0) = \int_\Omega \left[\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2\right] dx.$$

Then the critical points of $F^*(u)$ coincide with solutions of the equation

$$\Delta^2 u + c\Delta u = bu^+ \quad \text{in} \quad H. \tag{2.13}$$

If $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$, then (2.13) has only the trivial solution, and hence $F^*(u)$ has only one critical point $u = 0$. Given $v \in V$, let $\theta^*(v) = \theta(v, 0) \in W$ be the unique solution of the equation

$$\Delta^2 z + c\Delta z - (I - P)[b(v + z)^+] = 0 \quad \text{in} \quad W.$$

Let us define the reduced functional $\tilde{F}^*(v)$ on V, by $F^*(v + \theta^*(v))$. We note that we can obtain the same result as Lemma 2.4 when we replace $\theta(v, s)$ and $\tilde{F}(v, \theta(v, s))$ by $\theta^*(v)$ and $\tilde{F}^*(v)$. We also note that $\tilde{F}^*(v)$ has only one critical point $v = 0$.

Lemma 2.6. *For* $d > 0$, $\quad \tilde{F}^*(dv) = d^2 \tilde{F}^*(v)$.

Proof. If $v \in V$ satisfy

$$\Delta^2 z + c\Delta z - (I - P)(b(v + \theta^*(v))^+) = 0 \quad \text{in} \quad W,$$

then for $d > 0$,

$$\Delta^2(dz) + c\Delta(dz) - (I - P)(b(dv + d\theta^*(v))^+) = 0 \quad \text{in} \quad W.$$

Therefore $\theta^*(dv) = d\theta^*(v)$ for $d > 0$. From the definition of $F^*(u)$ we see that

$$F^*(du) = d^2 F^*(u) \quad \text{for} \quad u \in H \quad \text{and} \quad d > 0.$$

Hence, for $v \in V$ and $d > 0$,

$$\tilde{F}^*(dv) = F^*(dv + \theta^*(dv)) = d^2 F^*(v + \theta^*(v)) = d^2 \tilde{F}^*(v). \qquad \square$$

Now we remember the notation F_b, which was defined in Equation (2.6). Until now, the notations F, F^* and $\tilde{F}^*$ denoted F_b, F_b^* and $\tilde{F}_b^*$, respectively. In the following lemma we use the latter notations.

Lemma 2.7. *Let $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$. Then there exist v_1 and v_2 in V such that $\tilde{F}_b^*(v_1) > 0$ and $\tilde{F}_b^*(v_2) < 0$.*

Proof. First, we choose $v_1 \in V$ such that $v_1 + \theta(v_1, 0) > 0$. In this case $z = \theta(v_1, 0) = 0$. Hence $v_1 + z = d_1\phi_1$, and we have

$$\begin{aligned}\tilde{F}_b^*(v_1) &= \int_\Omega \left[\frac{1}{2}|\Delta(v_1+z)|^2 - \frac{c}{2}|\nabla(v_1+z)|^2 - \frac{b}{2}|(v_1+z)^+|^2\right] dx \\ &= \int_\Omega \left[\frac{1}{2}|\Delta(v_1+z)|^2 - \frac{c}{2}|\nabla(v_1+z)|^2 - \frac{b}{2}|v_1+z|^2\right] dx \\ &= \int_\Omega \left[\frac{1}{2}(\Delta^2 + c\Delta)(v_1+z)\cdot(v_1+z) - \frac{b}{2}(v_1+z)\cdot(v_1+z)\right] dx \\ &= \frac{1}{2}\left[\{\lambda_1(\lambda_1-c) - b\}d_1^2\right] > 0.\end{aligned}$$

Next, we choose $v_2 \in V$ such that $v_2 + \theta(v_2, 0) < 0$. In this case $z = \theta(v_2, 0) = 0$. Hence if we write $v_2 + z = e_1\phi_1$, then we have

$$\begin{aligned}\tilde{F}_b^*(v_2) &= \int_\Omega \left[\frac{1}{2}|\Delta(v_2+z)|^2 - \frac{c}{2}|\nabla(v_2+z)|^2\right] dx \\ &= \int_\Omega \left[\frac{1}{2}(\Delta^2 + c\Delta)(v_2+z)\cdot(v_2+z)\right] dx \\ &= \frac{1}{2}[\lambda_1(\lambda_1-c)e_1^2] < 0,\end{aligned}$$

since $b < \lambda_1(\lambda_1 - c) < 0 < \lambda_2(\lambda_2 - c)$. □

Lemma 2.8. *Let $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then $\tilde{F}_b(v, s)$ is neither bounded above nor below on V.*

Proof. From Lemma 2.7, $\tilde{F}_b^*(v)$ has negative (positive) value. Suppose that $\tilde{F}_b^*(v)$ assumes negative values and that $\tilde{F}_b(v, s)$ is bounded below. Let v_0 denote a fixed point in V with $\|v_0\| = 1$. Let $z_n = nv_0 + \theta(nv_0, s)$, and let $z_n^* = v_0 + (\theta(nv_0, s)/n) = v_0 + w_n^*$. Since θ is Lipschitzian, the sequence $\{z_n^*\}_1^\infty$ is bounded in $L^2(\Omega)$. We have $DF(z_n, s)(y) = 0$ for all n and arbitrary $y \in W$. Dividing this equation by n gives

$$\int_\Omega \left[\Delta z_n^* \cdot \Delta y - c\nabla z_n^* \cdot \nabla y - bz_n^{*+}y - \frac{s}{n}y\right] dx = 0. \tag{2.14}$$

Setting $y = z_n$ we know that $\{z_n^*\}_{n=1}^{\infty}$ is bounded in $L^2(\Omega)$. Hence $\{w_n^*\}_1^{\infty}$ is bounded in $L^2(\Omega)$ so we may assume that it converges weakly to an element $w^* \in W$. If $z^* = w^* + v_0$ and we let $n \to \infty$ in (2.14), we obtain

$$(2.15) \qquad \int_\Omega [\Delta z^* \cdot \Delta y - c\nabla z^* \cdot \nabla y - b{z^*}^+ y]\, dx = 0$$

for arbitrary $y \in W$. Hence $w^* = \theta(v_0, 0)$. If we set $y = w_n$ in 2.14 and dividing by n, then we have

$$(2.16) \qquad \int_\Omega \left[|\Delta w_n^*|^2 - c|\nabla w_n^*|^2 - \left(b|z_n^*|^+ + \frac{s}{n} \right) w_n^* \right] dx = 0.$$

Letting $n \to \infty$ in (2.16), we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega [|\Delta w_n^*|^2 - c|\nabla w_n^*|^2]\, dx &= \lim_{n\to\infty} \int_\Omega \left[b\left(|z_n^*|^+ + \frac{s}{n} \right) w_n^* \right] dx \\ &= \int_\Omega b|z^*|^+ w^*\, dx \\ &= \int_\Omega [\Delta z^* \cdot \Delta w^* - c\nabla z^* \cdot \nabla w^*]\, dx \\ &= \int_\Omega [|\Delta w^*|^2 - c|\nabla w^*|^2]\, dx, \end{aligned}$$

where we have used (2.15). Hence

$$\lim_{n\to\infty} \int_\Omega [|\Delta z_n^*|^2 - |\nabla z_n^*|^2]\, dx = \int_\Omega [|\Delta z^*|^2 - c|\nabla z^*|^2]\, dx.$$

The assumption that $\tilde{F}(v, s)$ is bounded below implies the existence of a constant M such that

$$\tilde{F}_b(nv_0, s)/n^2 \geq M/n^2.$$

Letting $n \to \infty$, our previous reasoning shows that

$$\tilde{F}_b^*(v_0) = \tilde{F}_b(v_0, 0) = \lim_{n\to\infty} \tilde{F}_b(nv_0, s)/n^2 \geq 0.$$

Since v_0 was an arbitrary member of V with $\|v_0\| = 1$ and $\tilde{F}_b(kv, 0) = k^2\tilde{F}_b(v, 0)$, this contradicts the assumption that $\tilde{F}_b^*(v)$ is negative for some value of $v \in V$. Hence $\tilde{F}_b(v, s)$ cannot be bounded below. The proof that $\tilde{F}_b(v, s)$ cannot be bounded above if $\tilde{F}_b^*(v)$ assumes positive values is essentially the same. □

Proof of Theorem 2.1. Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. By Lemma 2.3, (2.1) has a positive solution $u_1(x) = v_1 + \theta(v_1, s)$. By Lemma 2.5, there exists a small open neighborhood B of v_1 in V such that $v = v_1$ is a strict local point of minimum of $\tilde{F}_b$. Since $\tilde{F}_b(v, s)$ is not bounded below, there exists a point $v_2 \in V$ with $v_1 \neq v_2$ and $\tilde{F}_b(v_1, s) = \tilde{F}_b(v_2, s)$. The Rolle's theorem and the fact that $\tilde{F}_b(v, s)$ has a continuous Fréchet derivative imply that there exists a strict local point of maximum $\tilde{F}_b$. Thus $\tilde{F}_b$ has at least two critical points. Therefore (2.1) has at least two solutions. □

Next, we investigate the multiplicity of solutions of (2.1) under Condition (2).

Condition (2). $c < \lambda_1$ (in this case $0 < \lambda_1(\lambda_1 - c)$), $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k = 1, 2, \cdots$, and $s < 0$.

Theorem 2.2. *Assume that $c < \lambda_1$, $0 < \lambda_1(\lambda_1 - c)$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k \geq 0$ and $s < 0$. Then the problem* (2.1) *has at least two solutions.*

One solution is a negative solution, and the existence of another solution will be shown by critical point theory.

To prove Theorem 2.2, we need several lemmas.

Lemma 2.9. *Let $c < \lambda_1$, $b \geq 0$ and $b \neq \lambda_1(\lambda_1 - c)$. Then the problem*

$$\Delta^2 u + c\Delta u = bu^+ \quad in \quad H \tag{2.18}$$

has only the trivial solution.

Proof. For $c < \lambda_1$, $0 < \lambda_1(\lambda_1 - c) < b$, the result follows from Theorem 1.3 (ii). We prove the lemma for the case $0 \leq b < \lambda_1(\lambda_1 - c)$. From

(2.1) we have

$$\lambda_1(\lambda_1 - c)\|u\|^2 \le \int_\Omega |\Delta u|^2 - c|\nabla u|^2 = b\int_\Omega u^+ \cdot u \le b\|u\|^2, \tag{2.19}$$

where $\| \ \|$ is the L^2 norm is Ω. It follows from (2.19) that $b\|u\|^2 \ge \lambda_1(\lambda_1 - c)\|u\|^2$, which yields $u = 0$. □

Now we investigate the existence of the negative solution of (2.1) under Condition (2).

Lemma 2.10. *Assume that* $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k \ge 1$, *and* $s < 0$. *Then the problem* (2.1) *has a negative solution* $u_2(x)$.

Proof. If u is a smooth function satisfying

$$\begin{aligned} \Delta^2 u + c\Delta u \ge 0 \quad &\text{in} \quad \Omega, \\ u = 0, \quad \Delta u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}$$

and $c < \lambda_1$, then $u > 0$ in Ω or $u = 0$. This immediately follows by first applying standard (strong) maximum principle to $w = \Delta u$ and consequently to u. Subsequently, for $c < \lambda_1$ and $s < 0$, it follows that if u_2 is the unique solution for

$$\begin{aligned} \Delta^2 u_2 + c\Delta u_2 = s \quad &\text{in} \quad \Omega, \\ u_2 = 0, \quad \Delta u_2 = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{2.20}$$

then $u_2 < 0$ in Ω. The unique negative solution u_2 solution of (2.20) is also a negative solution of (2.1). □

Now we investigate the existence of the other solution of the problem (2.1) under the condition $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k \ge 1$, and $s < 0$ will be shown by critical point theory. Now we consider the functional

$$F_b(u, s) = \int_\Omega \left[\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2 - su\right] dx,$$

which is well defined in $H \times R$, continuous and Fréchet differentiable in H (by Proposition 2.1).

Let V be the k-dimensional subspace of H spanned by eigenfunctions $\phi_1, \phi_2, \cdots, \phi_k$. Let W be the orthogonal complement of V in H. We note that Lemma 2.4 holds under Condition (2). From Lemma 2.10, we see that (2.1) has a negative solution $u_2(x)$. By Lemma 2.4, u_2 is of the form $u_2 = v_2 + \theta(v_2, s)$.

Lemma 2.11. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k \geq 1$, and $s < 0$. Then there exists a small open neighborhood D of v_2 in V such that $v = v_2$ is a strict local point of minimum of $\tilde{F}_b$.*

Proof. Let $s < 0$. Then the problem (2.1) has a negative solution $u_2(x)$ which is of the form $u_2(x) = v_2 + \theta(v_2, s) < 0$. Since $I + \theta$, where I is an identity map on V, is continuous, there exists a small open neighborhood D of v_2 in V such that if $v \in D$, $v + \theta(v, s) < 0$. Therefore if $z = \theta(v, s)$, $z_2 = \theta(v_2, s)$ and $v + z = (v_2 + z_2) + (\tilde{v} + \tilde{z})$, then we have

$$\begin{aligned}
\tilde{F}_b(v, s) &= F_b(v + z, s) \\
&= \int_\Omega \left[\frac{1}{2}|\Delta(v+z)|^2 - \frac{c}{2}|\nabla(v+z)|^2 - s(v+z)\right] dx \\
&= \int_\Omega \left[\frac{1}{2}|\Delta(v_2+z_2) + \Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(v_2+z_2) + \nabla(\tilde{v}+\tilde{z})|^2 \right. \\
&\qquad \left. - s\{(v_2+z_2) + (\tilde{v}+\tilde{z})\}\right] dx \\
&= \int_\Omega \left[\frac{1}{2}|\Delta(v_2+z_2)|^2 - \frac{c}{2}|\nabla(v_2+z_2)|^2 - s(v_2+z_2)\right] dx \\
&\quad + \int_\Omega [\Delta(v_2+z_2)\cdot\Delta(\tilde{v}+\tilde{z}) \\
&\qquad - c\nabla(v_2+z_2)\cdot\nabla(\tilde{v}+\tilde{z}) - s(\tilde{v}+\tilde{z})]\, dx \\
&\quad + \int_\Omega \left[\frac{1}{2}|\Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v}+\tilde{z})|^2\right] dx.
\end{aligned}$$

Here

$$\begin{aligned}
\int_\Omega \left[\frac{1}{2}|\Delta(v_2+z_2)|^2 - \frac{c}{2}|\nabla(v_2+z_2)|^2 - s(v_2+z_2)\right] dx& \\
= F_b(v_2+z_2, s) = \tilde{F}_b(v_2, s)&
\end{aligned}$$

and

$$\int_\Omega [\Delta(v_2+z_2)\cdot\Delta(\tilde{v}+\tilde{z}) - c\nabla(v_2+z_2)\cdot\nabla(\tilde{v}+\tilde{z}) - s(\tilde{v}+\tilde{z})]\,dx$$
$$= \int_\Omega [\Delta^2(v_2+z_2) + c\Delta(v_2+z_2) - s]\cdot(\tilde{v}+\tilde{z})\,dx = 0,$$

since v_2+z_2 is a negative solution of (2.1). Since $\tilde{v}+\tilde{z}$ can be expressed by $\tilde{v}+\tilde{z} = \sum_{i=1}^\infty e_i\phi_i$, we have

$$\tilde{F}_b(v,s) - \tilde{F}_b(v_2,s) = \int_\Omega \left[\frac{1}{2}|\Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v}+\tilde{z})|^2\right]dx$$
$$= \frac{1}{2}\{\lambda_1(\lambda_1-c)e_1^2 + \lambda_2(\lambda_2-c)e_2^2 + \cdots\} > 0,$$

since $0 < \lambda_1(\lambda_1-c)$. Therefore $\tilde{F}_b(v,s)$ has a strict local minimum at $v = v_2$. This proves the lemma. □

Lemma 2.12. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1}-c)$, $k \geq 1$. Then there exist v_p and v_q in V such that $\tilde{F_b}^*(v_p) < 0$ and $\tilde{F_b}^*(v_q) > 0$.*

Proof. First we choose $v_p \in V$ such that $v_p + \theta(v_p, 0) > 0$ and $\theta(v_p,0) = 0$. If $v_p + z = \sum_{i=1}^k f_i\phi_i$, where $\theta(v_p,0)=0$, then we have

$$\tilde{F_b}^*(v_p) = \int_\Omega \left[\frac{1}{2}|\Delta(v_p+z)|^2 - \frac{c}{2}|\nabla(v_p+z)|^2 - \frac{b}{2}|(v_p+z)^+|^2\right]dx$$
$$= \int_\Omega \left[\frac{1}{2}|\Delta v_p|^2 - \frac{c}{2}|\nabla v_p|^2 - \frac{b}{2}|v_p|^2\right]dx$$
$$= \int_\Omega \left[\frac{1}{2}(\Delta^2 + c\Delta)v_p\cdot v_p - \frac{b}{2}|v_p|^2\right]dx$$
$$= \frac{1}{2}\{[\lambda_1(\lambda_1-c)-b]f_1^2 + \cdots + [\lambda_k(\lambda_k-c)-b]f_k^2\} < 0.$$

Next, we choose $v_q \in V$ such that $v_q + \theta(v_q,0) < 0$. Let $z = \theta(v_q,0)$. If $v_q + z = \sum_{i=1}^\infty g_i\phi_i$, then we have

$$\tilde{F_b}^*(v_q) = \int_\Omega \left[\frac{1}{2}|\Delta(v_q+z)|^2 - \frac{c}{2}|\nabla(v_q+z)|^2\right]dx$$
$$= \int_\Omega \left[\frac{1}{2}(\Delta^2+c\Delta)(v_q+z)\cdot(v_q+z)\right]dx$$
$$= \frac{1}{2}[\lambda_1(\lambda_1-c)g_1^2 + \cdots + \lambda_k(\lambda_k-c)g_k^2] > 0,$$

since $0 < \lambda_l(\lambda_l - c)$. □

Lemma 2.13. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k \geq 1$, and $s < 0$. Then $\tilde{F}_b(v, s)$ is neither bounded above nor below on V.*

The proof of the lemma is the same as the proof of Lemma 2.8.

Lemma 2.14. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k = 1, 2, \ldots$ and $s < 0$. Then the functional $\tilde{F}_b(v, s)$, defined on V, satisfies the Palais-Smale condition: Any sequence $\{v_n\} \subset V$ for which $\tilde{F}_b(v_n, s)$ is bounded and $D\tilde{F}_b(v_n, s) \to 0$ possesses a convergent subsequence.*

Proof. Suppose that $\tilde{F}_b(v_n, s)$ is bounded and $D\tilde{F}_b(v_n, s) \to 0$ in V, where $\{v_n\}$ is a sequence in V. Since V is a k-dimensional subspace spanned by $\phi_1, \ldots, \phi_k$, we have, with $u_n = v_n + \theta(v_n, s)$,

$$\Delta^2 u_n + c\Delta u_n - bu_n^+ = s + DF_b(u_n, s).$$

Assuming [P.S.] condition does not hold, that is, $\|v_n\| \to \infty$, $|||v_n||| \to \infty$, we see that $\|u_n\| \to \infty$. Dividing by $\|u_n\|$ and taking $w_n = \|u_n\|^{-1} u_n$, we have

$$\Delta_2 w_n + c\Delta w_n - bw_n^+ = \|u_n\|^{-1}(s + DF_b(u_n, s)), \tag{2.17}$$

since $DF_b(u_n, s) \to 0$ as $n \to \infty$ and $\|u_n\| \to \infty$. Moreover, (2.17) shows that $\|\Delta^2 w_n + c\Delta w_n\|$ is bounded. Since $(\Delta^2 + c\Delta)^{-1}$ is a compact operator, passing to a subsequence we get that $w_n \to w_0$. Since $\|w_n\| = 1$ for all $n = 1, 2, \ldots$ it follows that $\|w_0\| = 1$. Taking the limit of both sides of (2.17), we find

$$\Delta^2 w_0 + c\Delta w_0 - bw_0^+ = 0$$

with $\|w_0\| \neq 0$. This contradicts the fact that the equation

$$\Delta^2 u + c\Delta u = bu^+$$

has only the trivial solution. □

Proof of Theorem 2.2. By Lemma 2.10, (2.1) has a negative solution $u_2(x) = v_2 + \theta(v_2, s)$. By Lemma 2.11, there exists a small open neighborhood D of v_2 in V such that $v = v_2$ is a strict local point of minimum of $\tilde{F}_b$. Also $\tilde{F}_b \in C^1(V, R)$ satisfies the Palais-Smale condition. Since $\tilde{F}_b(v, s)$ is neither bounded above nor below on V, Lemma 2.13, we can choose $v_3 \in V \backslash D$ such that

$$\tilde{F}_b(v_3, s) < \tilde{F}_b(v_2, s).$$

Let Γ be the set of all paths in V joining v_3 and v_2. The mountain pass theorem, cf. [**3**], implies that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_b(v, s)$$

is a critical value of $\tilde{F}_b$. Thus, $\tilde{F}_b$ has at least two critical values, and (2.1) has at least two solutions. □

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