Multiplied-Poisson Noise in Pulse, Particle, and Photon Detection

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Abstract—Multiplication effects in point processes are important in a number of areas of electrical engineering and physics. We examine the properties and applications of a point process that arises when each event of a primary Poisson process generates a random number of subsidiary events, with a given time course. The multiplication factor is assumed to obey the Poisson probability law, and the dynamics of the time delay are associated with a linear filter of arbitrary impulse response function; special attention is devoted to the rectangular and exponential cases. The process turns out to be a doubly stochastic Poisson point process whose stochastic rate is shot noise; it has application in pulse, particle, and photon detection. Explicit results are obtained for the single and multifold counting statistics (distribution of the number of events registered in a fixed counting time), the time statistics (forward recurrence time and interevent probability densities), and the power spectrum (noise properties). These statistics can provide substantial insight into the underlying physical mechanisms generating the process. An example of the applicability of the model is provided by cathodoluminescence (the phenomenon responsible for the television image) where a beam of electrons (the primary process) impinges on a phosphor, generating a shower of visible photons (the secondary process). Each electron produces a random number of photons whose emission times are determined by the (possibly random) lifetime of the phosphor, so that multiplication effects and time delay both come into play. We use our formulation to obtain the forward-recurrence-time probability density for cathodoluminescence in $YVO_4:Eu^{3+}$, the excess cathodoluminescence noise in ZnS:Ag, and the counting distribution for radioluminescence photons produced in a glass photomultiplier tube. Agreement with experimental data is very good in all cases. A variety of other applications and extensions of the model are considered.

I. Introduction

MENTION THE TERM *point process* and the electrical engineer is likely to imagine highly localized events of a particular kind occurring along a time axis: the emission of electrons from a cathode or perhaps the registration of nuclear particles in a detector. But such processes form an intimate part of our daily lives as well. Few of us think of walking along the sidewalk or driving down the road in mathematical terms, yet to the operations researcher studying traffic flow, these acts are representable in terms of important point processes. Every field of scientific endeavor boasts its own examples.

Probably the simplest and most widely occurring point process is the one-dimensional homogeneous Poisson process [1]-[3]. It plays the role that the Gaussian process does in the theory of continuous stochastic processes. The homogeneous Poisson point process (HPP) is characterized by a single quantity, its rate, which is constant. One of its distinguishing features is that it evolves in time without aftereffects. This means that the occurrence times and number of events before an arbitrary time have no bearing on the subsequent occurrence times and number of events. It is said to have zero memory.

Because of its simple properties, the HPP turns out to be a suitable building block for more complex point processes that appear to resemble the Poisson process little, if at all. We all recall, no doubt, a very important property of this process: pass an HPP through a linear filter and out comes shot noise. This result is probably just a shade younger than electrical engineering itself.

There are many generalizations of the HPP. One case that has been studied extensively is the doubly stochastic Poisson point process (DSPP), which differs from the HPP in that the rate is no longer constant. Rather, it takes on a stochastic nature of its own. This process was first examined by Cox (and by Bartlett in the discussion to Cox's paper) [4]. Cox provided an example of its use in textile technology. The designation DSPP was introduced to emphasize that two kinds of randomness occur: randomness associated with the Poisson point process itself and an independent randomness associated with its rate. The earliest connection of the DSPP with electrical engineering appears to be in the work of Darling and Siegert [5] and Siegert [6]. The DSPP subsequently became the basis for understanding the photodetection of light, and many of the early papers in this area [7]-[15] drew heavily on results derived in connection with classical radio receiver communications and noise [16]-[18]. At the same time, the DSPP arose in the context of the detection of acoustic noise in the auditory system [19]. A number of excellent secondary sources, detailing the history and properties of the DSPP, are available [3], [20]-[27].

In certain areas of study, multiplication effects in point processes play an important role; in most such cases, a given (primary) point process is multiplied in some fashion to give rise to a second (subsidiary) point process. Cathodoluminescence [28], [29], which is responsible for the television, oscilloscope, and radar images, provides a simple but important example. Each electron incident on the phosphor in the screen releases a random number of photons, which form the image that we observe. Multiplication can also occur in successive stages, so that each generation of particles produces a new generation, and so on. Such branching (cascade) processes can be used to describe the noise properties of devices like avalanche detectors and electron multipliers [30], [31].

There have been many theoretical characterizations of the multiplication of events. In the mathematical-statistics com-

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munity, the formulations have been complete but often too general [32] to be directly useful in electrical engineering. In the electron-device noise community, on the other hand, effective formulations have been developed, but they have been framed almost exclusively in terms of the lowest-order properties of these processes: the mean, variance, and spectrum [31], [33]; the multiplication, furthermore, has been assumed to be instantaneous in most of these treatments. Additional information about the physical nature of a process can be gleaned from its higher-order properties. Measurements of the probability density of the time between consecutive events, for example, or the number of events registered in a fixed counting time, can provide valuable clues about underlying physical mechanisms.

Indeed, subsidiary events are often not generated instantaneously, and the multiplication properties alone are insufficient to characterize such a process. In many physical situations, there will be delay relative to the primary event in accordance with a particular probability law, so that the time behavior of the multiplication process comes into play.

In this paper, we examine the properties of a particular multiplication model that incorporates time effects, and is both mathematically tractable and useful in electrical engineering and physics. The subsidiary process is taken to be a Poisson point process whose rate is a linearly filtered version of the primary process, which is assumed to be an HPP. Since an HPP passed through a linear filter yields shot noise, we call our process a shot-noise-driven doubly stochastic Poisson point process (SNDP). It is a special case of the DSPP. Each primary event is multiplied into subsidiary events, but with a time course prescribed by the linear filter impulse response function. Thus event multiplication and dynamics are combined into one model. Bartlett [34] proposed a similar two-dimensional model for use in plant ecology some time ago, whereas Vere-Jones and Davies constructed this kind of formulation (they called it a "trigger" model) in connection with the study of earthquake occurrences [35]. Bartlett explicitly showed that the SNDP is a special but important example of a cluster point process developed by Neyman and Scott [36], [37] to describe the distribution of galaxies in space. This identity is explored by Lawrance in his excellent contribution to the point-process book edited by Lewis [21].

In Section II, we describe random events arising from multiplication processes. The properties of the SNDP, including the counting statistics and time statistics, are provided in Section III. In Section IV, we treat the multifold statistics and spectrum. A number of important applications of the SNDP are considered in Section V, including scintillation detection, photomultiplier noise induced by ionizing radiation, cathodoluminescent emission, electronography, and X-ray radiography. The conclusion is provided in Section VI.

II. Random Events Arising from Multiplication Processes

A. *Instantaneous Multiplication*

Consider the primary point process whose events are illustrated in Fig. 1(a). The number of events (counts) within a time interval $[0, T]$ is a discrete random variable m having a moment-generating function (mgf), $Q_m(s) = \langle \exp(-sm) \rangle$. The angular brackets represent an ensemble average. Let each primary event produce independently A subsidiary events (as illustrated in Fig. 1(b)), where A is a discrete random variable that has an mgf $Q_A(s) = \langle \exp(-sA) \rangle$. The total number of

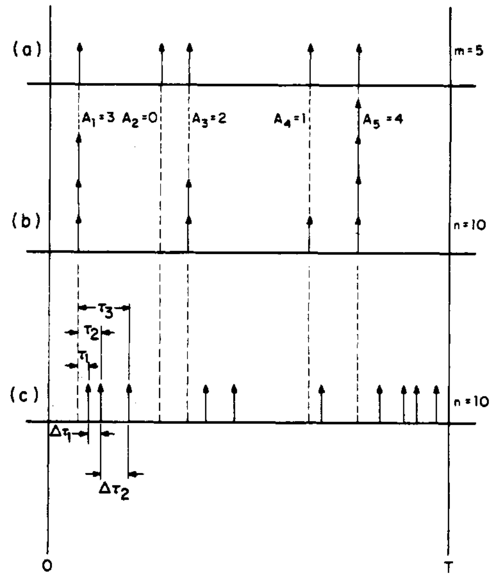

Fig. 1. Random multiplication of events. (a) Primary events. (b) Subsidiary events. (c) Randomly delayed subsidiary events.

secondary events n is given by

$$n = \sum_{k=1}^{m} A_k$$

where A_k are independent values of A. If A and m are statistically independent, then it can be shown [1], [31], [33] that the mgf of n, $Q_n(s) = \langle \exp(-sn) \rangle$, is given by

$$Q_n(s) = Q_m(-\ln Q_A(s)). \tag{1}$$

This equation can be used to relate the moments of n to those of m and A. The means are related by

$$\langle n \rangle = \langle A \rangle \langle m \rangle, \tag{2}$$

the second factorial moment obeys the equation

$$\langle n(n-1) \rangle = \langle A \rangle^2 \langle m(m-1) \rangle + \langle m \rangle \langle A(A-1) \rangle, \tag{3}$$

and the variances are expressed as

$$\operatorname{var}(n) = \langle A \rangle^2 \operatorname{var}(m) + \langle m \rangle \operatorname{var}(A). \tag{4}$$

Equation (4) is known as the Burgess variance theorem [38], [39].

When the primary events constitute a Poisson point process, the random variable m has a Poisson distribution with mgf

$$Q_m(s) = \exp[\langle m \rangle (e^{-s} - 1)] \tag{5}$$

which, when combined with (1), gives rise to

$$Q_n(s) = \exp[\langle m \rangle (Q_A(s) - 1)]. \tag{6}$$

The count variance associated with (6) is then

$$\operatorname{var}(n) = (1 + a_1) \langle n \rangle, \tag{7}$$

where

$$a_1 = \frac{\operatorname{var}(A)}{\langle A \rangle} + \langle A \rangle - 1. \tag{8}$$

Equation (7) demonstrates that for a Poisson primary process, whatever the distribution of the multiplication factor A, the variance of n is always proportional to the mean $\langle n \rangle$. The ratio $\operatorname{var}(n)/\langle n \rangle$ is known as the dispersion ratio or the Fano factor [31]. We denote parameter a_1 as the *excess* Fano factor.

When the multiplication factor A also has a Poisson distribution, (6) yields

$$Q_n(s) = \exp\left(\frac{\langle n\rangle}{\langle A\rangle}\left\{\exp\left[\langle A\rangle(e^{-s}-1)\right]-1\right\}\right), \qquad (9)$$

which is the mgf for the Neyman Type-A distribution [40], [41], [2], [19], [26], [37]. The variance is obtained from (7) and (8):

$$a_1 = \langle A\rangle. \qquad (10)$$

The Neyman Type-A distribution has a variance larger than that of a Poisson distribution of the same mean.

When the random variable A is describable instead by a Bernoulli distribution, which associates the values 1 and 0 with the probabilities $\langle A\rangle$ and $(1-\langle A\rangle)$, respectively, the factor a_1 equals 0 and n is Poisson. The Poisson distribution is therefore invariant against Bernoulli random multiplication or deletion [1]. It should also be noted [24] that all DSPP's are invariant under Bernoulli Selection (in the sense that the random integrated rate W is reduced to $\langle A\rangle W$, see Appendix A). When A has a geometric (Bose–Einstein) distribution, $a_1 = 2\langle A\rangle$; this is sometimes called the Polya–Aeppli distribution. Values of a_1 for other distributions can be easily obtained [2], [31], [33].

B. Multiplication with Random Delay

In the previous subsection (Fig. 1(a), (b)), a primary event was assumed to instantaneously excite a random number of subsidiary events. In many physical systems, a time delay will be inherent in the multiplication process, and that time delay will itself be random, as illustrated in Fig. 1(c). The primary events may or may not be included in the final point process.

The nature of the randomness of the delay times that separate subsidiary events from their primary event can take different forms. In the Neyman–Scott model [36], [37], the primary process is taken to be an HPP, which is excluded from the final process. The multiplication factor can assume an arbitrary distribution, and the delay times, $\tau_1, \tau_2, \cdots, \tau_A$, measured from the primary event, are assumed to be statistically independent and identically distributed. This process has been used by Neyman and Scott to describe a broad variety of phenomena from the distribution of larvae on small plots of land to the distribution of galaxies in space. It has also been used by Vere-Jones [42], in connection with the occurrences of earthquakes in New Zealand. In the Bartlett–Lewis model [43], [44], primary events are again described by an HPP, but they are included in the final point process. Again, the multiplication factor has arbitrary statistics. In this case, time intervals between successive subsidiary pulses that belong to a common primary event, $\Delta\tau_1, \Delta\tau_2, \cdots, \Delta\tau_A$, are assumed statistically independent and identically distributed, i.e., they form a segment of a renewal process. This process has been used by Bartlett [43] for traffic studies and by Lewis [44] for the distribution of computer failure times.

C. The Shot-Noise-Driven Doubly Stochastic Poisson Point Process

We characterize the randomness of both the multiplication factor and the delay times by assuming that an inhomogeneous Poisson process produces the secondary events. The occurrence of a primary event (at $t = 0$, for example) is considered to generate a continuous (usually decaying) function $h(t)$ which acts as the rate of the inhomogeneous Poisson process.

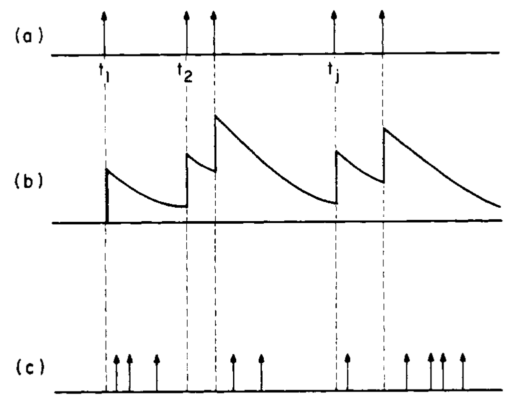

Fig. 2. Production of SNDP events. (a) Primary Poisson events. (b) Filtered Poisson events (shot noise). (c) Subsidiary doubly stochastic Poisson events whose rate is shot noise.

Fig. 3. Model for generation of the SNDP.

The multiplication factor has a Poisson distribution of mean

$$\langle A\rangle = \int_0^\infty h(t)\,dt. \qquad (11)$$

The delay times $\tau_1, \tau_2, \cdots, \tau_A$ are statistically independent and identically distributed with a probability density function [1]–[3], [27]

$$P(\tau) = h(\tau)\exp\left[-\int_0^\tau h(t)\,dt\right]\Big/\left[1 - e^{-\int_0^\infty h(t)\,dt}\right]. \qquad (12)$$

This process is therefore a special case of the Neyman–Scott model—a case in which A is Poisson with a mean given by (11), and with a delay-time distribution given by (12). The inter-event times are not identically distributed because of the nonstationarity of the rate; therefore, this process is not a special case of the Bartlett–Lewis model.

It will be useful to consider matters from an alternative point of view. The sum of the functions associated with primary events that occur at the times $\{t_j\}$

$$\lambda(t) = \sum_j h(t - t_j) \qquad (13)$$

constitutes a filtered Poisson process, or shot noise (as illustrated in Fig. 2). If the primary events are stationary, this will also be a stationary stochastic process The subsidiary events now form a DSPP whose stochastic rate is the shot noise $\lambda(t)$; this provides the rationale for the name SNDP. The structure of the model is schematically illustrated in the block diagram of Fig. 3, which shows its multiplicative Poisson nature. Physical systems that are describable by an SNDP model will exhibit a physical structure characterized by these three cascaded blocks. Several examples are presented in Section V.

III. Properties of the Shot-Noise-Driven Doubly Stochastic Poisson Point Process

A block diagram for the SNDP is presented in Fig. 3. A homogeneous Poisson point process of rate μ is filtered by a time-invariant linear filter, with a nonnegative impulse response function $h(t)$, resulting in a shot noise process $\lambda(t)$. The process $\lambda(t)$ forms the stochastic rate for a DSPP, $f(t)$. An important parameter is the average number of secondary events produced per primary event $\langle A \rangle$. For simplicity we shall use the symbol $\alpha = \langle A \rangle$, so that (see (11))

$$\alpha = \int_0^\infty h(t)\, dt. \tag{14}$$

The process is completely described by the parameter μ and the function $h(t)$. We assume, also for simplicity, that t is the only important variable in the stochastic process λ. In certain applications, e.g., photocounting detection, spatial variables may also play a role, and this can be incorporated into the model without difficulty. We are interested in determining the statistical properties of the SNDP, including 1) counting statistics, i.e., the statistics (probability distribution and moments) of the number n of events (counts) registered in the time interval $[0, T]$; 2) time statistics, i.e., the probability density functions for the forward recurrence time and the interevent time; and 3) multifold statistics, including the joint probability distribution of the numbers of counts $n_1, n_2, \cdots$ in multiple adjacent time intervals, and their moments. This task can be achieved by a straightforward application of the statistical properties of a general DSPP [3], [4], [24], [26], [27], and of shot noise [1], [16], [45]-[47]. These properties are reviewed in Appendexes A and B, respectively. In this section we discuss the singlefold statistics; in Section IV, the multifold statistics are presented.

As indicated in Appendix A, it is useful to define the random variable

$$W = \int_0^T \lambda(t)\, dt, \tag{15}$$

which represents the integrated rate of events of the process over the interval $[0, T]$. Once the moment-generating function of W, $Q_W(s) = \langle \exp(-sW) \rangle$, is determined, the singlefold statistics of a DSPP can be directly determined by use of the simple formulas presented in Appendix A. These include the mgf $Q_n(s)$ for the number of counts n, the probability distribution $p(n)$, and the factorial moments $F_n^{(m)}$. Also included are the probability density functions $P_1(\tau)$ and $P_2(\tau)$ for the forward recurrence time (time from an arbitrary point to the first event) and for the interevent time, respectively.

To determine the mgf of the integrated rate W, we require knowledge of the statistics of the rate $\lambda(t)$. In our case, fortunately, $\lambda(t)$ is a shot-noise process whose properties are well-known (see Appendix B). In particular the moment-generating function is

$$Q_\lambda(s) = \langle e^{-s\lambda} \rangle = \exp\left\{\mu \int_{-\infty}^{\infty} [e^{-sh(t)} - 1]\, dt\right\}. \tag{16}$$

The random variable W is the integral of a shot noise process $\lambda(t)$ over an interval $[0, T]$. We note that the integral of a shot noise is simply another shot noise. Thus we can regard W as a sample of a shot-noise process, generated by a homogeneous Poisson point process of rate μ, and filtered by a linear filter of impulse response $h_T(t)$, which is simply a convolution of $h(t)$ with an integrator over the time interval $[0, T]$. This

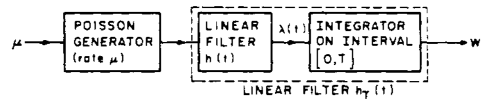

Fig. 4. Model for generation of the integrated rate W.

is shown schematically in Fig. 4, whence

$$h_T(t) = \int_0^T h(t + t')\, dt'. \tag{17}$$

Thus the mgf for W becomes

$$Q_W(s) = \exp\left\{\mu \int_{-\infty}^{\infty} [e^{-sh_T(t)} - 1]\, dt\right\}. \tag{18}$$

An alternative route to deriving (17) and (18) is to write $W = \sum_i \lambda(i\Delta t)\Delta t$ and use the multifold mgf for $\{\lambda(t_i)\}$, presented in (B9), in the limit $\{s_i\} = s$.

Given the mgf of W, represented in (18), we now proceed to determine the statistics of n, as a function of μ, T, and $h(t)$ by using (A4) and (A5), and the probability density functions for the forward recurrence time and the interevent time, by using (A7)-(A9).

A. Counting Statistics for the SNDP

The SNDP is a special DSPP whose integrated rate W is described by the moment-generating function given in (18). In this section we use (A4)-(A6) to determine the statistics (moments and probability distribution) for the number of counts n registered in the time interval $[0, T]$. We consider several limiting and special cases.

1) The Moments of n: Using (18) and (A4) we have derived the following recurrence relation for the mth-order factorial moment of n:

$$F_n^{(m+1)} = \langle n \rangle \sum_{l=0}^{m} a_l \binom{m}{l} F_n^{(m-l)}, \qquad F_n^{(0)} = 1, \tag{19}$$

where the coefficients a_l are

$$a_l = \frac{1}{\alpha T} \int_{-\infty}^{\infty} [h_T(t)]^{l+1}\, dt, \tag{20}$$

and the mean number of counts $\langle n \rangle$ is

$$\langle n \rangle = \mu \int_{-\infty}^{\infty} h_T(t)\, dt = \mu T \int_0^\infty h(t)\, dt = \mu \alpha T.$$

The first four factorial moments are explicitly written as

$$\begin{aligned}
F_n^{(1)} &= \langle n \rangle \\
F_n^{(2)} &= \langle n \rangle^2 + a_1 \langle n \rangle \\
F_n^{(3)} &= \langle n \rangle^3 + 3a_1 \langle n \rangle^2 + a_2 \langle n \rangle \\
F_n^{(4)} &= \langle n \rangle^4 + 6a_1 \langle n \rangle^3 + (4a_2 + 3a_1^2) \langle n \rangle^2 + a_3 \langle n \rangle,
\end{aligned} \tag{21}$$

which correspond to the central moments

$$\begin{aligned}
\langle \Delta n^2 \rangle &= \operatorname{var}(n) = (1 + a_1) \langle n \rangle \\
\langle \Delta n^3 \rangle &= (1 + 3a_1 + a_2) \langle n \rangle \\
\langle \Delta n^4 \rangle &= (1 + 7a_1 + 6a_2 + a_3) \langle n \rangle + 3(1 + a_1)^2 \langle n \rangle^2.
\end{aligned} \tag{22}$$

An important property of any DSPP, and therefore of the SNDP in particular, is that it is over-dispersed relative to the HPP. This means that the count variance, var (n), is greater than that for an HPP of the same mean, as is evident from (22).

An equally important property of the SNDP is the fact that the variance is proportional to the mean. This is the property discussed previously, in connection with a Poisson primary process and instantaneous multiplication with arbitrary statistics (see (7)). The parameter a_1 (the excess Fano factor) is, for the SNDP,

$$a_1 = \alpha/\mathcal{M}, \tag{23}$$

where

$$\mathcal{M}^{-1} = \frac{1}{T}\int_{-\infty}^{\infty} h_T^2(t)\,dt \bigg/ \left[\int_0^{\infty} h(t)\,dt\right]^2$$

$$= \frac{2}{T}\int_0^{T} (T-\tau)\xi(\tau)\,d\tau, \tag{24}$$

with

$$\xi(\tau) = \int_{-\infty}^{\infty} h(t)h(t+\tau)\,dt \bigg/ \left[\int_0^{\infty} h(t)\,dt\right]^2. \tag{25}$$

The excess Fano factor, which determines the excess multiplication noise, is therefore directly proportional to the multiplication parameter α (the average number of secondary events per primary event), and inversely proportional to the parameter $\mathcal{M}$, which we refer to as the number of degrees of freedom. It will be shown subsequently that in the limit of long counting times, i.e., $T \gg \tau_p$, where τ_p is the characteristic time scale of $h(t)$, $\mathcal{M}$ approaches 1. In the opposite limit, i.e., $T \ll \tau_p$, $\mathcal{M}$ approaches τ_p/T, a large number.

Another parameter of interest is the coefficient of skewness, S, which is given by

$$S = \langle \Delta n^3\rangle/[\langle \Delta n^2\rangle]^{3/2} = \langle n\rangle^{-1/2}\,(1+3a_1+a_2)/(1+a_1)^{3/2}. \tag{26}$$

For an HPP, $S = \langle n\rangle^{-1/2}$.

2) The Probability Distribution of n: Using (A5), together with (18), we obtain the following recurrence relation for the counting distribution $p(n)$:

$$(n+1)p(n+1) = \langle n\rangle \sum_{l=0}^{n} C_l\, p(n-l)$$

$$p(0) = \exp\left\{\frac{\langle n\rangle}{\alpha T}\int_{-\infty}^{\infty} [e^{-h_T(t)} - 1]\,dt\right\}, \tag{27}$$

where the coefficients C_l are determined by the shape of the impulse response function $h_T(t)$ through

$$C_l = \frac{1}{l!\,\alpha T}\int_{-\infty}^{\infty} [h_T(t)]^{l+1} e^{-h_T(t)}\,dt. \tag{28}$$

Equations (27) and (28) are derived by forming the nth derivative of both sides of (18), applying the Leibnitz rule, and using (A5).

3) Counting Statistics in the Limit of Long Counting Time: When the counting time T is much longer than the characteristic time scale of the impulse response function $h(t)$, we can replace (17) by the approximation

$$h_T(t) = \begin{cases} \alpha, & -T < t < 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Consequently, (18) yields

$$Q_W(s) = \exp\left\{\frac{\langle n\rangle}{\alpha}(e^{-\alpha s} - 1)\right\}, \tag{29}$$

where $\langle n\rangle = \mu\alpha T$. Examination of (29) shows that W has a fixed-multiplicative Poisson distribution, i.e., it takes on the values $0, \alpha, 2\alpha, \cdots, k\alpha, \cdots$ where k has a Poisson distribution [48]. This is, of course, to be expected because of the assumption that $h(t)$ is very narrow. Using (A2) we obtain

$$Q_n(s) = \exp\left(\frac{\langle n\rangle}{\alpha}\left\{\exp[\alpha(e^{-s} - 1)] - 1\right\}\right), \tag{30}$$

which is precisely the mgf of a Neyman Type-A distribution of parameter α (see (9)). This is a very important result because it demonstrates that there is a unique SNDP counting distribution for arbitrary $h(t)$, in the limit of long counting time, and it is the Neyman Type-A [40], [41], [48], [49]. Various properties of this distribution are provided in Appendix C. We note that in this case the excess Fano factor a_1 is equal to the parameter α, i.e., the degrees-of-freedom parameter $\mathcal{M} = 1$.

4) Counting Statistics in the Limit of Short Counting Time: For a counting time T much shorter than the characteristic time scale of the impulse response function $h(t)$, (17) can be approximated by

$$h_T(t) \approx Th(t) \tag{31}$$

and (15) becomes

$$W \approx T\lambda(t). \tag{32}$$

Substitution in (18) results in

$$Q_W(s) = \exp\left\{\mu\int_{-\infty}^{\infty} [e^{-sTh(t)} - 1]\,dt\right\}. \tag{33}$$

The parameters $\{a_l\}$, which determine the moments of n (see (19)–(22)), are then approximated by

$$a_l = \frac{T^l}{\alpha}\int_0^{\infty} [h(t)]^{l+1}\,dt. \tag{34}$$

In particular, the excess Fano factor a_1, which determines the excess variance, is

$$a_1 = \alpha T/\tau_p \tag{35}$$

where

$$\tau_p = \left[\int_0^{\infty} h(t)\,dt\right]^2 \bigg/ \int_0^{\infty} h^2(t)\,dt \tag{36}$$

is the width of the function $h(t)$. Therefore, in this limit, the number of degrees of freedom becomes

$$\mathcal{M} = \tau_p/T, \tag{37}$$

which is large as mentioned earlier. The excess variance is reduced in the limit of short counting time [50]. Examples of counting statistics in this limit, for a number of filters of differ-

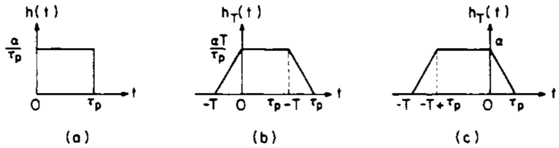

Fig. 5. (a) $h(t)$ for a rectangular filter. (b) $h_T(t)$ for a rectangular filter when $T \leqslant \tau_p$. (c) $h_T(t)$ for a rectangular filter when $T > \tau_p$.

ent shapes, are given in Appendix D. It is shown that when the filter is rectangular, the distribution becomes Neyman Type-A of parameter $a_1 = \alpha T/\tau_p$.

It should be noted when $a_1 \to 0$, corresponding to vanishing T or α, the counting distribution eventually approaches the Poisson limit.

5) Counting Statistics for Special Cases with Arbitrary Counting Time:

a) Rectangular filter: When $h(t)$ is rectangular, $h_T(t)$ becomes trapezoidal as shown in Fig. 5. Substitution in (18) results in

$$Q_W(s) = \begin{cases} \exp\left\{\dfrac{\langle n\rangle}{\alpha\beta}\,[(1-\beta)(e^{-s\alpha\beta}-1) - 2(e^{-s\alpha\beta}+s\alpha\beta-1)/s\alpha]\right\}, & \beta \leqslant 1 \\ \exp\left\{\dfrac{\langle n\rangle}{\alpha}\left[\left(1-\dfrac{1}{\beta}\right)(e^{-s\alpha}-1) - 2(e^{-s\alpha}+s\alpha-1)/s\alpha\beta\right]\right\}, & \beta \geqslant 1, \end{cases} \tag{38}$$

where $\beta = T/\tau_p$. In the limits of short counting time ($\beta \ll 1$) and long counting time ($\beta \gg 1$), (38) leads to the Neyman Type-A counting distribution with parameters $\alpha\beta$ and α, respectively.

The factorial moments of n are computed by using the recurrence relations provided in (19)–(21). The coefficients a_l are now

$$a_l = \begin{cases} \alpha^l\beta^l\left[(1-\beta) + \dfrac{2\beta}{(l+2)}\right], & \beta \leqslant 1 \\ \alpha^l\left[\left(1-\dfrac{1}{\beta}\right) + \dfrac{2}{\beta(l+2)}\right], & \beta \geqslant 1. \end{cases} \tag{39a}$$

The excess Fano factor $a_1 = \alpha/\mathcal{M}$, where the degrees-of-freedom parameter $\mathcal{M}$ is now given by [50]

$$\mathcal{M} = \begin{cases} 1/(\beta - \beta^2/3), & \beta \leqslant 1 \\ \beta/(\beta - 1/3), & \beta \geqslant 1. \end{cases} \tag{39b}$$

The dependence of $\mathcal{M}$ on β is presented in Fig. 6. Note that when $\beta \gg 1$, $\mathcal{M} \to 1$. When $\beta \ll 1$, $\mathcal{M} = 1/\beta$.

The three parameters (μ, α, and τ_p) that characterize the SNDP can be determined by measuring the mean $\langle n\rangle$ and the variance var(n) at two counting times T. From a measurement of a_1 for $T \gg \tau_p$, α can be determined, so that a subsequent measurement of a_1 for $T \ll \tau_p$ will yield τ_p. Using $\langle n\rangle = \mu\alpha T$ provides μ.

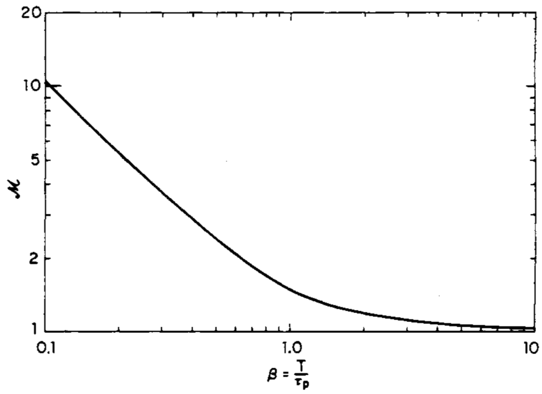

Fig. 6. Dependence of the degrees-of-freedom parameter $\mathcal{M}$ on the ratio $\beta = T/\tau_p$. The excess Fano factor $a_1 = \alpha/\mathcal{M}$. Filter is rectangular. Observe that $\mathcal{M} = \tau_p/T$ for $T/\tau_p \ll 1$ and $\mathcal{M} = 1$ for $T/\tau_p \gg 1$.

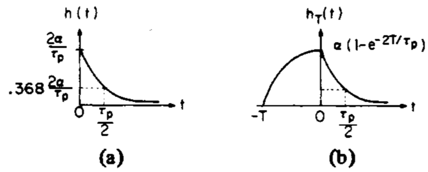

Fig. 7. (a) $h(t)$ for an exponential filter. (b) $h_T(t)$ for an exponential filter.

The probability distribution $p(n)$ can be obtained by using the recurrence relation provided in (27), where now

$$C_l = \begin{cases} (1-\beta)(\alpha\beta)^l e^{-\alpha\beta}/l! + 2\gamma(l+2,\alpha\beta)/l!\alpha^2\beta, & \beta \leqslant 1 \\ \left(1-\dfrac{1}{\beta}\right)\alpha^l e^{-\alpha}/l! + 2\gamma(l+2,\alpha)/l!\alpha^2\beta, & \beta \geqslant 1, \end{cases} \tag{40}$$

and where $\gamma(m+1, x)$ is the incomplete gamma function specified in (D8). The zero-count probability $p(0)$ is determined from (38) by setting $p(0) = Q_W(1)$.

b) Exponentially decaying filter: When the filter is exponential, $h(t) = (2\alpha/\tau_p)\exp(-2t/\tau_p)$, and as illustrated in Fig. 7,

$$h_T(t) = \begin{cases} \alpha(1 - e^{-2\beta})\,e^{-2t/\tau_p}, & 0 \leqslant t \\ \alpha[1 - e^{-2\beta}e^{-2t/\tau_p}], & -T \leqslant t \leqslant 0 \\ 0, & t \leqslant -T, \end{cases} \tag{41}$$

where again $\beta = T/\tau_p$. A rather complex expression for $Q_W(s)$ can be obtained by substitution in (18). The moments of n are obtained from (19)–(21), which lead to

$$a_l = \alpha^l\left[(1-e^{-2\beta})^{l+1}/(l+1) + 2\beta + \sum_{k=1}^{l+1}(-1)^k\binom{l+1}{k}(1-e^{-2\beta k})/k\right]\Big/2\beta. \tag{42}$$

The excess Fano factor $a_1 = \alpha/\mathcal{M}$, where the degrees-of-freedom parameter $\mathcal{M}$ is now given by [50]

$$\mathcal{M} = 2\beta/(e^{-2\beta} + 2\beta - 1). \tag{43}$$

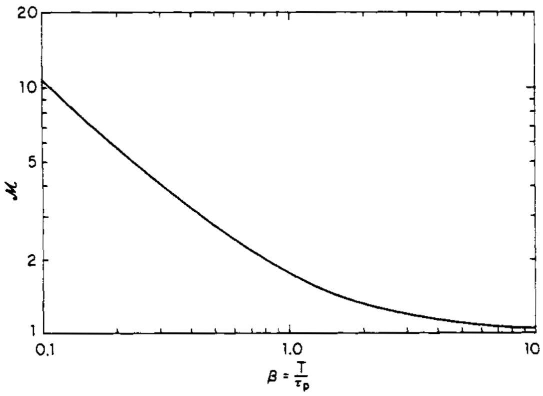

Fig. 8. Dependence of the degrees-of-freedom parameter $\mathcal{M}$ on the ratio $\beta = T/\tau_p$. The excess Fano factor $a_1 = \alpha/\mathcal{M}$. Filter is exponential. Observe that $\mathcal{M} = \tau_p/T$ for $T/\tau_p \ll 1$ and $\mathcal{M} = 1$ for $T/\tau_p \gg 1$. Note the similarity to Fig. 6.

The behavior of shot-noise light and thermal light has recently been contrasted in the case of an exponentially decaying correlation [50]. As with the rectangular case (or with arbitrary $h(t)$, for that matter), when $T \gg \tau_p$, $\mathcal{M} \to 1$. When $T \ll \tau_p$, $\mathcal{M} = \tau_p/T$. The dependence of $\mathcal{M}$ on T/τ_p is presented in Fig. 8. The value of $\mathcal{M}$ decreases montonically from τ_p/T to 1. Again, we can make effective use of a mean and variance measurement at two different counting times to extract all of the pertinent parameters.

The counting distribution $p(n)$ can be determined by use of the recurrence relation in (27) together with the coefficients C_l determined from (28), which are

$$C_l = \frac{1}{2\alpha\beta l!}\,\gamma(l+1, \alpha(1 - e^{-2\beta})) + \frac{\alpha^l e^{-\alpha}}{l!}$$

$$\cdot\left\{1 + \frac{1}{2\beta}\sum_{k=0}^{l+1}(-1)^k\binom{l+1}{k}\left[\eta_k(\alpha) - e^{-2k\beta}\,\eta_k(\alpha e^{-2\beta})\right]\right\}, \quad (44)$$

where $\gamma(l+1, x)$ is given by (D8) and

$$\eta_k(x) = \sum_{m=0}^{\infty}\frac{x^m}{m!(m+k)}, \qquad k = 1, 2, \cdots, \quad (45)$$

$$\eta_0(x) = \sum_{m=1}^{\infty}\frac{x^m}{m!m}. \quad (46)$$

The initial value $p(0)$ for the recurrence relation is computed from (27):

$$p(0) = \exp\left\{\frac{\langle n\rangle}{2\alpha\beta}\left[\eta_0[-\alpha(1 - e^{-2\beta})] + e^{-\alpha}[\eta_0(\alpha) - \eta_0(\alpha e^{-2\beta})] - 2\beta(1 - e^{-\alpha})\right]\right\}. \quad (47)$$

Fig. 9 illustrates the effect of the parameters α and $\beta = T/\tau_p$ on the counting distribution for a fixed overall mean count $\langle n\rangle = 5$. For vanishingly small β and $\alpha\beta$, the distribution approaches the Poisson [50]. For very large β (solid curves), the distribution is Neyman Type-A of parameter α. As α increases, the Neyman Type-A distribution becomes substantially different from the Poisson and begins to exhibit scallops as illustrated by the solid curves in Fig. 9 [41]. For completeness, we note that the counting distribution for an inhomogeneous single decaying exponential was obtained previously for arbitrary T [51].

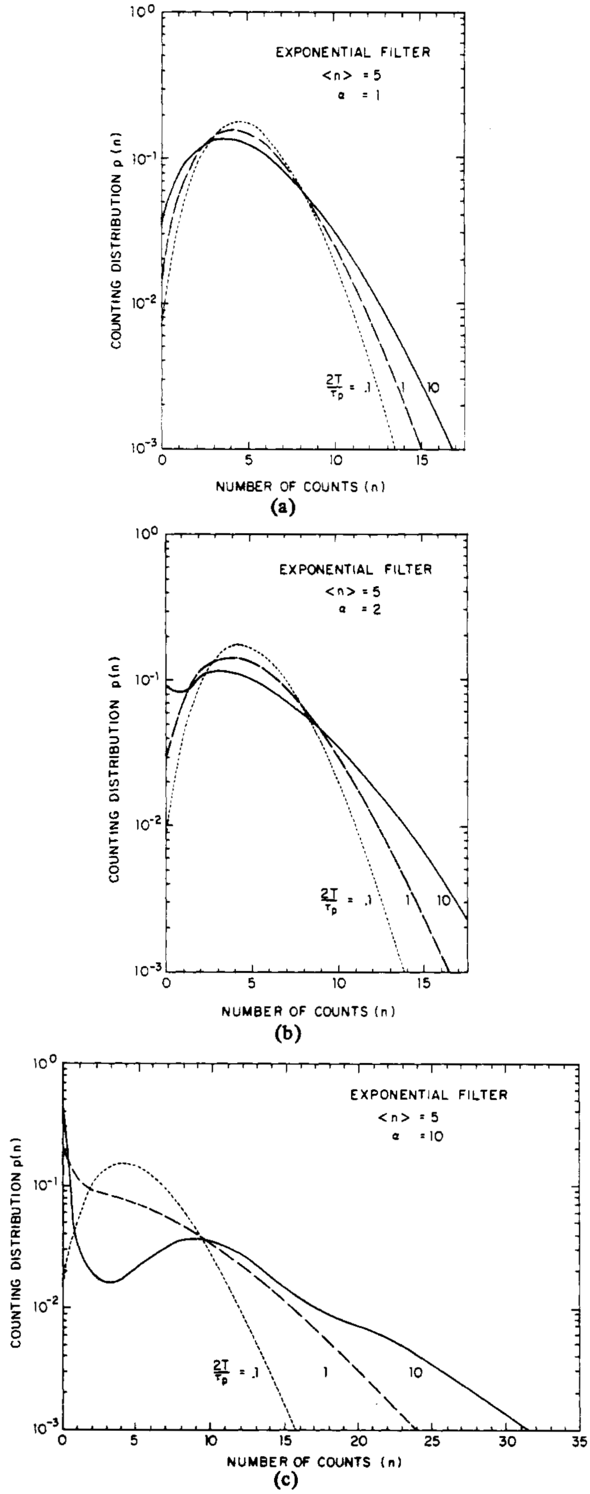

Fig. 9. Counting distribution $p(n)$ versus count number n for the SNDP. The impulse response function of the shot-noise filter is exponential with time constant $\tau_p/2$. In all cases the mean number of events in the counting time T is $\langle n\rangle = 5$. Distributions are shown when the ratio $2T/\tau_p$ takes on the values 0.1 (dotted curves), 1 (dashed curves), and 10 (solid curves). (a) Multiplication parameter $\alpha = 1$. (b) $\alpha = 2$. (c) $\alpha = 10$. For T/τ_p large, the counting distribution approaches the Neyman Type-A.

6) Counting Statistics in the Limit of Dense Shot Noise: It is well known that a shot-noise process approaches a Gaussian process [46] in the limit in which its pulses overlap considerably, i.e., when $\mu\tau_p \gg 1$. In this limit, the stochastic driving rate $\lambda(t)$ of our SNDP is Gaussian, and consequently its integral

W will also be Gaussian, with mean $\langle n \rangle$ and variance $a_1 \langle n \rangle$. The counting distribution $p(n)$ for this case can be determined by finding the Poisson transform (A6) of a (truncated) Gaussian distribution. Recurrence relations for this $p(n)$ have been obtained by Risken [52], in the course of evaluating the photon statistics of laser light. Risken showed that when the mean is much larger than the square root of the variance, $p(n)$ itself approaches a Gaussian distribution. For the SNDP, this means that when $\mu \tau_p \gg 1$ and $\langle n \rangle \gg a_1$, $p(n)$ becomes Gaussian, independent of β and the form of $h(t)$. For large β, we already know that the counting distribution is well described by the Neyman Type-A; this latter distribution does indeed converge in distribution to the Gaussian as $\langle n \rangle \to \infty$ [41].

B. Time Statistics for the SNDP

In this section, we determine the statistics of the forward recurrence time and the interevent time, for the SNDP. Because the SNDP is a special case of the general DSPP, the probability densities $P_1(\tau)$ and $P_2(\tau)$ can be determined from the mgf of the integrated rate W by using the formulas presented in (A7)–(A9). Since we have already obtained explicit expressions for $Q_W(s)$, the calculation of $P_1(\tau)$ and $P_2(\tau)$ is a straightforward exercise. Using (17) and (18), we readily obtain

$$P_1(\tau) = \mu \left[\int_{-\infty}^{\infty} h(t+\tau)\, e^{-h_\tau(t)}\, dt \right] \cdot \exp\left[\mu \int_{-\infty}^{\infty} (e^{-h_\tau(t)} - 1)\, dt \right] \tag{48}$$

$$P_2(\tau) = \frac{1}{\alpha} \left\{ \mu \left[\int_{-\infty}^{\infty} h(t+\tau)\, e^{-h_\tau(t)}\, dt \right]^2 - \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial \tau} h(t+\tau) - h^2(t+\tau) \right] e^{-h_\tau(t)}\, dt \right\} \cdot \exp\left[\mu \int_{-\infty}^{\infty} (e^{-h_\tau(t)} - 1)\, dt \right]. \tag{49}$$

It can be shown that for $\tau = 0$,

$$\langle \tau \rangle P_1(0) = 1 \tag{50}$$

$$\langle \tau \rangle P_2(0) = 1 + \frac{\mathrm{var}(\lambda)}{\langle \lambda \rangle^2}, \tag{51}$$

where $\langle \tau \rangle = 1/\langle \lambda \rangle = 1/\mu\alpha$ is the mean value of τ. The deviation of $\langle \tau \rangle P_2(0)$ from 1 determines the degree of bunching of the point process. By using (B1) and (36) we can write (51) in the form

$$\langle \tau \rangle P_2(0) = 1 + \frac{1}{\mu \tau_p}. \tag{52}$$

Therefore the parameter $\mu \tau_p$, the average number of primary events within the relaxation time of the filter, determines the degree of bunching of the SNDP.

1) Rectangular Filter: When $h(t)$ is a rectangular function of width τ_p and area α, we can directly insert (38) into (A7)–(A9) to obtain $P_1(\tau)$ and $P_2(\tau)$. The outcome is provided in Appendix E, (E1) and (E2). The most interesting general conclusion that we can draw from the results is that for $\tau > \tau_p$, both $P_1(\tau)$ and $P_2(\tau)$ are exponentially decaying. This is expected because for $\tau > \tau_p$, most events are associated with different primaries.

When $\tau_p \ll \langle \tau \rangle$, τ is more likely to be much greater than τ_p, so that the entire distributions $P_1(\tau)$ and $P_2(\tau)$ become exponential (with decay time $\langle \tau \rangle \alpha/(1 - e^{-\alpha})$), as is the case for a Poisson point process. Similarly, when $\alpha \ll 1$, the exponential term in (E1) can be linearized, and $P_1(\tau)$ and $P_2(\tau)$ become approximately exponential.

2) Exponential Filter: We can similarly obtain expressions for $P_1(\tau)$ and $P_2(\tau)$ for an exponential filter. These are also provided in Appendix E, (E3) and (E4).

In Figs. 10 and 11, we plot $P_1(\tau)$ and $P_2(\tau)$ versus τ, for different values of τ_p and α when $\langle \tau \rangle = 1$. Fig. 10 represents curves for $P_1(\tau)$, where τ is the forward recurrence time, whereas Fig. 11 represents curves for $P_2(\tau)$, where τ is the interevent time. When α is very small, both distributions, $P_1(\tau)$ and $P_2(\tau)$, are approximately exponential. As α increases, the probability densities become skewed toward the $\tau = 0$ axis, which is a manifestation of bunching. When the mean time interval and

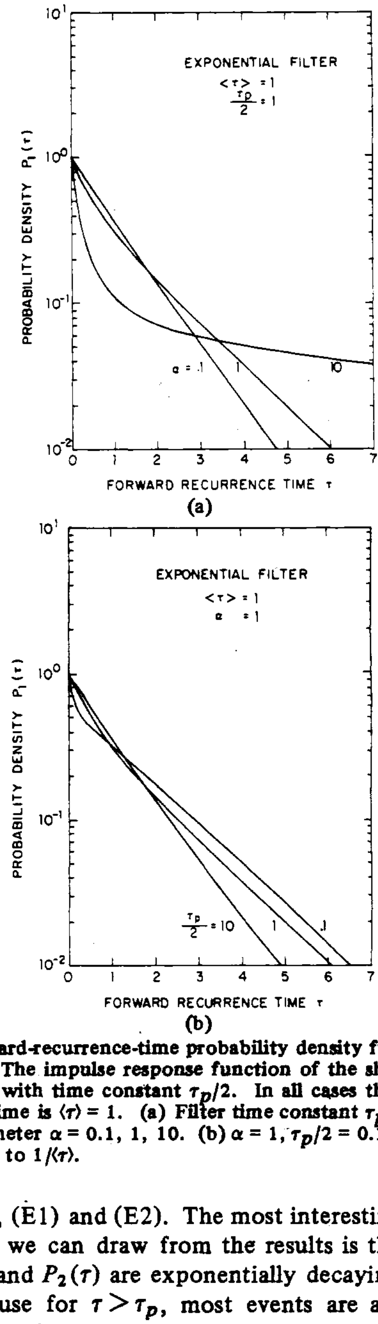

Fig. 10. Forward-recurrence-time probability density function $P_1(\tau)$ for the SNDP. The impulse response function of the shot-noise filter is exponential with time constant $\tau_p/2$. In all cases the mean forward recurrence time is $\langle \tau \rangle = 1$. (a) Filter time constant $\tau_p/2 = 1$, multiplication parameter $\alpha = 0.1, 1, 10$. (b) $\alpha = 1$, $\tau_p/2 = 0.1, 1, 10$. $P_1(0)$ is always equal to $1/\langle \tau \rangle$.

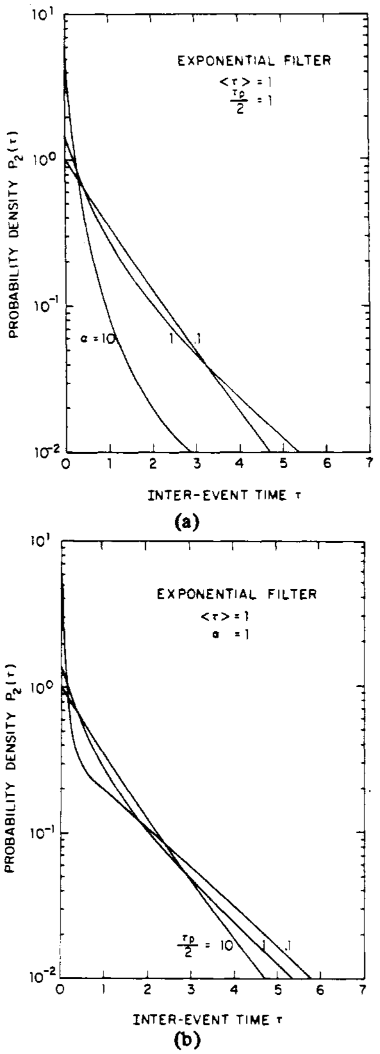

Fig. 11. Interevent (pulse-interval) probability density function $P_2(\tau)$ for the SNDP. The impulse response function of the shot-noise filter is exponential with time constant $\tau_p/2$. In all cases the mean interevent time is $\langle\tau\rangle = 1$. (a) Time constant $\tau_p/2 = 1$, multiplication parameter $\alpha = 0.1, 1, 10$. (b) $\alpha = 1$, $\tau_p/2 = 0.1, 1, 10$.

α are held fixed, decreasing τ_p similarly skews the densities toward the $\tau = 0$ axis, as is evident from the figures.

IV. Multifold Statistics for the SNDP

A. Counting Statistics

The joint statistical properties of the number of counts $\{n_j\}$ in N time intervals $[t_j, t_j + T_j]$, $j = 1, 2, \cdots, N$, for an SNDP can be determined by using the properties of doubly stochastic Poisson processes. Once the multifold mgf of the integrated rates

$$W_j = \int_{t_j}^{t_j+T_j} \lambda(t)\,dt, \qquad j = 1, 2, \cdots, N \tag{53}$$

is determined, the multifold factorial moments $\langle \Pi_{j=1}^{N} n_j!/(n_j - m_j)!\rangle$ and the joint counting distribution $p(n_1, n_2, \cdots, n_N)$ for $\{n_j\}$ can be determined by using (A12) and (A13). To determine the mgf of $\{W_j\}$, we need the mgf of $\{\lambda(t_j)\}$. For a shot-noise process $\lambda(t)$, the multifold mgf is known [(B9)]. By generalizing the analysis presented in Section III to the multifold case, we can easily show that the mgf of $\{W_j\}$ is given by

$$Q_{W_1,\cdots,W_N}(s_1,\cdots,s_N) = \exp\left\{\mu \int_{-\infty}^{\infty}\left[\exp\left(-\sum_{j=1}^{N} s_j h_{T_j}(t_j + x)\right) - 1\right] dx\right\} \tag{54}$$

where

$$h_{T_j}(t) = \int_0^{T_j} h(t + t')\,dt'. \tag{55}$$

We can therefore determine the joint statistics of $\{n_j\}$ explicitly.

B. Correlation Function and Power Spectrum

The simplest special case is $N = 2$. Setting $T_1 = T_2 = T$ in (55), and $m_1 = m_2 = 1$ in (A12), we obtain the correlation function between counts in time intervals T separated by a time delay $\tau = t_2 - t_1 \neq 0$,

$$G_n^{(2)}(\tau) = \langle n_1 n_2\rangle = \mu^2\left[\int_{-\infty}^{\infty} h_T(x)\,dx\right]^2 + \mu\int_{-\infty}^{\infty} h_T(x)\,h_T(\tau + x)\,dx, \tag{56}$$

where $h_T(t)$ is given by (17). By use of (17), we can rewrite the counting correlation function in the form

$$G_n^{(2)}(\tau) = \langle n\rangle^2 + \langle n\rangle\,\phi(\tau), \qquad \tau \neq 0. \tag{57}$$

Here

$$\phi(\tau) = \frac{\alpha}{T}\int_0^T\int_0^T \xi(\tau + t' - t)\,dt\,dt' = \frac{\alpha}{T}\int_0^T (T - x)\,[\xi(x + \tau) + \xi(x - \tau)]\,dx \tag{58}$$

where

$$\xi(\tau) = \int_{-\infty}^{\infty} h(t)\,h(t + \tau)\,dt \bigg/ \left[\int_0^{\infty} h(t)\,dt\right]^2 \tag{59}$$

is a normalized version of the autocorrelation function of $h(t)$.

For $\tau = 0$, $G_n^{(2)}(0) = \langle n^2\rangle$, which according to (21) and (25), is

$$G_n^{(2)}(0) = \langle n\rangle^2 + \langle n\rangle[1 + \phi(0)]. \tag{60}$$

Note that $\phi(0) = a_1$, the excess Fano factor. For the usual example of an exponential impulse response function, $h(t) = (2\alpha/\tau_p)\exp(-2t/\tau_p)$, (58)-(60) yield

$$\phi(\tau) = \begin{cases} \alpha b \exp(-2|\tau|/\tau_p), & |\tau| \geq T \\ \alpha\left\{1 - \dfrac{|\tau|}{T} + \dfrac{1}{2\beta}\left[e^{-2\beta}\cosh(2|\tau|/\tau_p) - e^{-2|\tau|/\tau_p}\right]\right\}, & |\tau| \leq T \\ \alpha/\aleph, & \tau = 0, \end{cases} \tag{61}$$

where

$$b = [\cosh(2\beta) - 1]/2\beta \qquad \aleph = 2\beta/(e^{-2\beta} + 2\beta - 1) \tag{62}$$

and $\beta = T/\tau_p$. The counting correlation function is therefore exponential. Similar results have been obtained by Vere-Jones and Davies [35].

When the sampling time T is much shorter than the time scale of the filter τ_p, (58) becomes approximately

$$\phi(\tau) \approx \alpha T \xi(\tau), \tag{63}$$

i.e., $\phi(\tau)$ is proportional to $\xi(\tau)$.

The power spectral density $S_{\Delta n}(\omega)$ of the process, representing fluctuations $n(t) - \langle n(t) \rangle$ in the number of counts in an interval $[t, t+T]$, can be determined by taking the Fourier transform of the autocovariance function $G_n^{(2)}(\tau) - \langle n \rangle^2$. This results in an expression of the form

$$S_{\Delta n}(\omega) = \langle n \rangle + \langle n \rangle \Phi(\omega) \tag{64}$$

where $\Phi(\omega)$ is the Fourier transform of $\phi(\tau)$. For a Poisson process, the second term of (64) disappears, and $S_{\Delta n}^P(\omega) = \langle n \rangle$. It is clear, therefore, that the second term represents excess noise. It is usual to define the relative excess noise as

$$[S_{\Delta n}(\omega) - S_{\Delta n}^P(\omega)]/S_{\Delta n}^P(\omega) = \Phi(\omega). \tag{65}$$

For a short counting time ($T \ll \tau_p$), equation (63) applies and (59) gives

$$\Phi(\omega) \approx \frac{T}{\alpha} |H(\omega)|^2, \tag{66}$$

where $H(\omega)$ is the transfer function of the filter (i.e., the Fourier transform of $h(t)$). For an exponential filter, the relative excess noise has a Lorentzian shape

$$\Phi(\omega) \approx \alpha T/[1 + \omega^2 \tau_p^2/4], \tag{67}$$

with an amplitude proportional to α and T.

In many situations a point process is observed not by counting, but rather by filtering the incoming pulses to generate a continuous signal, which we denote as $s(t)$. If such a filter has a rectangular impulse response of width T and unit area, then $s(t)$ would be identical to $n(t)$, the number of counts in an interval $[0, T]$. Therefore, the above results regarding counting statistics (including moments, correlation, and power spectral density) are also statistics of $s(t)$. In a typical situation, when the point process represents the passage of electrons in a circuit, $(e/T)n(t)$ represents the electric current. Here, e is the electronic charge, and T is the impulse response time of the circuit, when it is approximated by a rectangular function.

V. Applications of the SNDP

The mathematical description discussed to this point applies to a broad variety of phenomena. In the following, we deal with a number of applications of special interest in electrical engineering and physics. In particular, we use the SNDP model for describing the following processes: A) the detection of scintillation photons created by nuclear particles; B) photomultiplier noise produced by ionizing radiation; C) cathodoluminescence; and D) X-ray radiography and electronography. The SNDP model will also be useful in studies of related areas such as image intensification, the detection of particles by means of devices such as avalanche photodiodes and electron multipliers, and visual perception.

A. Scintillation Photon Counting

1) Description of the Process: The detection of ionizing radiation is often accomplished through a radiation–matter interaction in which a single high-energy particle produces a shower of particles of lower energy. A case in point is the scintillation detector, which is a combination of a scintillation crystal (e.g., NaI:Tl, plastic) with a photomultiplier tube

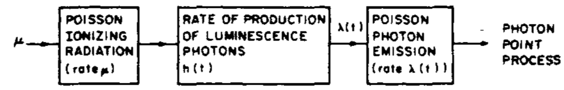

Fig. 12. Model for the photon point process generated by scintillation radiation, cathodoluminescence, and photoluminescence. Observe the relation to Fig. 3 which is the mathematical model studied here.

[53]. Conditions for the validity of the SNDP in describing scintillation detection are that the incident primary ionizing particles (e.g., electrons, gammas, protons) be representable as a homogeneous Poisson point process, and that each primary event have associated with it an impulse response function $h(t)$ that directly governs the rate of production of Poisson optical photons.

A model for the physical process is illustrated in Fig. 12. It is seen to be identical in form to the block diagram presented in Fig. 3, so that the resulting photon emissions form an SNDP. The function $h(t)$ will often be approximately a decaying exponential, so that the counting and time statistics are given by (27), (44)–(47) and (E1), (E3), (E4), respectively. The description is completely characterized by μ and $h(t)$ and therefore, in this case, by three parameters: μ, the rate of the primary process; α, the multiplication parameter; and τ_p, the lifetime of the process of secondary event generation. If we perform photon counting, we must adjoin T, the counting time. It is clear from Section III-A3, that in the limit of counting times much longer than the exponential decay time, the counting distribution will reduce to the Neyman Type-A for arbitrary $h(t)$. It is often assumed in the literature, generally tacitly, that scintillation photon counting statistics are describable by the fixed multiplicative Poisson distribution; this provides an appropriate approximation only for $T \gg \tau_p$ and $\alpha \gg 1$, however [41]. Of course, in those cases where the secondary events are not describable by a Poisson process, the SNDP is not the proper representation.

2) Experiment: A series of counting experiments of radioluminescence photons produced in glass has been recently carried out [50]. High energy β^- particles from a ^{90}Sr–^{90}Y equilibrium-mixture source irradiated the Corning 7056 glass faceplate of an EMR type 541N-01 photomultiplier tube from a distance of about 11.5 cm. The maximum β^- energies were 0.54 and 2.23 MeV for the ^{90}Sr and ^{90}Y, respectively, and the β^- flux was $\sim 8.2 \times 10^3$ cm$^{-2} \cdot$ s^{-1}. External light was excluded. The photomultiplier anode pulses were passed through a discriminator and standardized. Unavoidable system dead time was ~60 ns. The standardized pulses were counted during consecutive fixed counting intervals ($T = 400$ μs) and the counts were recorded. The experiment was performed repeatedly to obtain good statistical accuracy, and a histogram representing the relative frequency of the counts was constructed. The total duration of a run was about 4 min. In the particular experiment we illustrate, the observed mean count was 85.89 (this number was substantially higher than the mean dark count which could therefore be neglected) and the observed count variance was 429.58. The data are shown as the dots in Fig. 13. The solid curve represents the Neyman-Type A theoretical counting distribution with the count mean and variance fixed at the experimental values. It is clearly in accord with the data. When one assumes that $\tau_p \ll T$ ($\mathcal{M} = 1$), (C3) yields an experimental multiplication parameter $\alpha = 4.0$. A Poisson distribution with mean 85.89 (indicated by arrow) is plotted as the dashed curve in Fig. 13; clearly it bears no relation to the data.

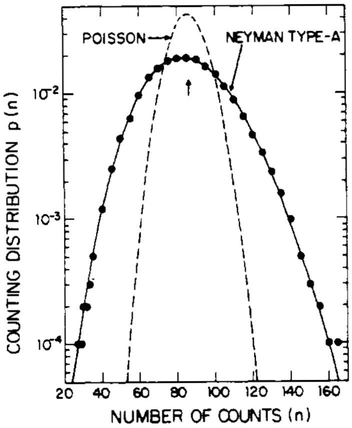

Fig. 13. Photon counting distribution $p(n)$ versus number of photon counts n. Data (dots) represent radioluminescence photon registrations from the glass faceplate of a photomultiplier tube, induced by $^{90}Sr - ^{90}Y\beta^-$ rays. The counting time $T = 400\ \mu s$. The experimental count mean and variance are 85.89 and 429.58, respectively. The solid curve represents the Neyman Type-A theoretical counting distribution with the same values of count mean and variance ($\alpha = 4.0$). The Poisson distribution with mean 85.89 (indicated by arrow) is shown as the dashed curve. After Fig. 2 of the paper by Teich and Saleh [50].

Because the primary process in this case consisted of high-energy charged particles (electrons), Čerenkov radiation could have been produced in addition to luminescence radiation. However, even if a large number of photoelectrons were generated by the Čerenkov photons arising from a single particle, they would nevertheless appear as a single (large) photoelectron pulse since the Čerenkov radiation emission time is much shorter than the transit time in the photomultiplier. In the presence of Čerenkov radiation, therefore, the (unmarked) total point process will include the primary as well as the subsidiary process. An analysis of this union process will be presented at a later time, but it will be well characterized by the SNDP for $\alpha \gg 1$.

B. Photomultiplier Noise Induced by Ionizing Radiation

In certain applications in which we wish to observe photon arrivals by using a photomultiplier tube, e.g., in astronomy conducted at high altitudes or in space, the description provided above may be characteristic of the noise rather than of the signal. Viehmann and Eubanks [54], [55] have discussed sources of noise in photomultiplier tubes in the ionizing radiation environment of space. Such noise may arise from several mechanisms such as luminescence and Čerenkov radiation in the photomultiplier window; secondary electron emission from the window, photocathode, and dynodes; Bremsstrahlung in turn causing such secondary electron emission; cosmic-ray bursts; and, of course, thermionic emission dark current. These effects clearly degrade both the dynamic range and the photometric accuracy of low-light-level measurements, and therefore must be properly modeled. It is evident from the experimental results reported in the previous subsection that the SNDP provides a sound point of departure in modeling a number of these sources of noise. Luminescence will be the dominant source of noise in many applications.

C. Cathodoluminescence

1) Description of the Process: Cathodoluminescence is an important process in which a beam of accelerated electrons incident on a luminescent material (e.g., a phosphor) induces the emission of light [28], [29]. It is the process responsible

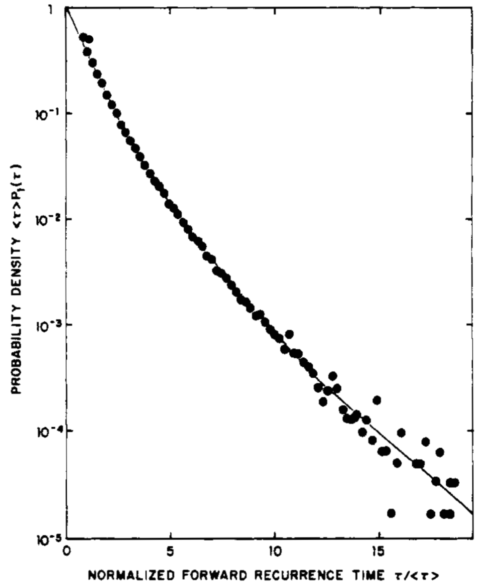

Fig. 14. Dots represent the relative frequency of th[illegible] to the first cathodoluminescent photoelectron event for a $YVO_4:Eu^{3+}$ phosphor with the exciting electron energy fixed at 10.2 keV. The mean photon-counting rate was $2.54 \times 10^4\ s^{-1}$ and the mean dark counting rate was $1.28 \times 10^3\ s^{-1}$. The solid curve is a plot of (E1), (E3), and (E4), which represent the SNDP theoretical probability density function $\langle\tau\rangle P_1(\tau)$ versus the normalized forward recurrence time $\tau/\langle\tau\rangle$ for an exponentially decaying shot-noise pulse, with $\bar{\tau} = 39.7\ \mu s$, $\alpha = 9.6$, and $\tau_p/2 = 0.5$ ms. Dark noise has been accounted for in the theoretical curve. Data adapted from Fig. 6 of the paper by van Rijswijk [57].

for the television and oscilloscope image. Conditions for the validity of the SNDP in describing cathodoluminescence are that the electron beam reaching the sample be represented as a homogeneous Poisson point process, and that each primary event have associated with it an impulse response function $h(t)$ that governs the rate of production of noninterfering Poisson optical photons. Cathodoluminescence may therefore be represented by the physical model in Fig. 12 if we simply replace the block labeled "Poisson ionizing radiation" by a block entitled "Poisson electron beam."

The counting statistics are then given by (27) and (28), whereas the time statistics are specified by (48) and (49). The description is completely characterized by μ and $h(t)$, and again, in the limit of counting times long in comparison with the time scale of $h(t)$, the counting distribution reduces to the Neyman Type-A for arbitrary $h(t)$.

In the special case where the function $h(t)$ is a decaying single-time-constant exponential, each electron entering the sample induces an exponentially decaying flux of luminescence photons, so that the counting and time statistics will be given by (27), (44)–(47) and (E1), (E3), (E4), respectively. These results are the same as those derived by van Rijswijk [56], [57] by means of an entirely different method. It appears that van Rijswijk's result can be used only for luminescence that decays precisely exponentially, whereas our general result will be valid for arbitrary $h(t)$. In particular, we can easily account for the finite "decay" (rise) time of the luminescence emission.

2) Forward Recurrence-Time Measurements: The statistics for the time to the first cathodoluminescent photon arrival were experimentally measured by van Rijswijk [57] for a $YVO_4:Eu^{3+}$ phosphor and, more recently, by Timmermans and Zijlstra [58] for ZnS:Ag. In Fig. 14 we present the normalized forward recurrence time data (dots) for $YVO_4:Eu^{3+}$ obtained by van Rijswijk [57] with an incident electron energy

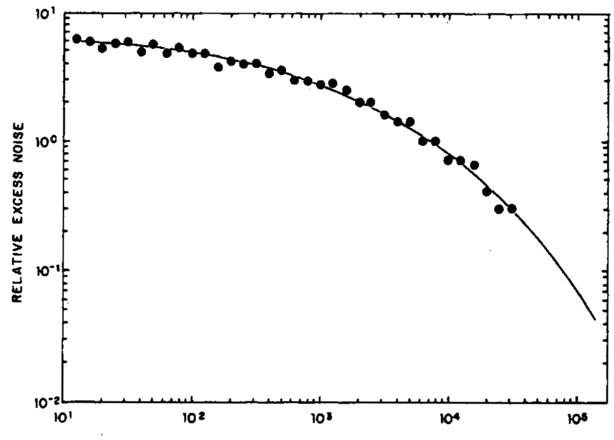

Fig. 15. Dots represent the experimental relative excess noise in the photomultiplier anode current for cathodoluminescence from ZnS:Ag. The solid curve is a Lorentzian. Figure adapted from Fig. 3 of the paper by Timmermans and Zijlstra [58].

of 10.2 keV. The mean photon counting rate was 2.54×10^4 s^{-1} and the mean dark counting rate was 1.28×10^3 s^{-1}. The solid curve is a plot of (E1), (E3), (E4), which represent the normalized probability density for the theoretical forward recurrence time $P_1(\tau)$ for an exponentially decaying shot-noise pulse, with $\bar{\tau} = 39.7$ μs, $\alpha = 9.6$, and $\tau_p/2 = 0.5$ ms. The effects of dark noise were not negligible in this case and were accounted for in the solid curve [57].

3) Excess-Noise Spectrum Measurements: Intensity fluctuations for cathodoluminescence were first investigated by means of the spectral properties of excess noise in the photomultiplier anode current [59], [60]. The theoretical expression for the spectrum in the general case (viz., for arbitrary $h(t)$ and arbitrary T) was derived in Section IV-B. The result for an exponentially decaying shot-noise pulse, in a form appropriate when measurements are made by a real-time spectrum analyzer, is presented in (67). Previous calculations were carried out only for purely exponentially decaying luminescence, which leads to the characteristic Lorentzian spectral shape.

Recent excess-noise measurements were performed by van Rijswijk [57] for $YVO_4:Eu^{3+}$ and by Timmermans and Zijlstra [58] for ZnS:Ag. In Fig. 15 we present ZnS:Ag experimental relative excess noise as a function of frequency (dots) obtained by Timmermans and Zijlstra. The solid curve is a Lorentzian which provides an excellent fit to the data, providing confirmation of the exponential decay shape and the luminescent-center lifetime.

D. X-Ray Radiography and Electronography

The recording of X-rays (or γ-rays) on a photographic emulsion involves a cascade of two processes [61]. The absorption of high-energy photons results in the emission of electrons (through the photoelectric effect, Compton scattering, or the formation of electron-positron pairs). The emitted electrons then traverse the emulsion, rendering a random number of silver halide grains developable. The higher the photon energy, the larger the average number of developable grains per photon [61]. It is a common observation in industrial radiography, that the higher the photon energy, the greater the graininess of the exposed film. This is a direct result of the exposure of a larger average number of grains per photon, and their clustering around that photon. With commercially available X-ray films this effect is easily visible to the naked eye [61].

It is apparent that the SNDP can serve as a simple model to account for this multiplicative cascade of events (assuming that other complex phenomena which may occur in the emulsion can be ignored). The primary process is then the random electrons which, because of the local Poisson nature of the incident photons in a region of fixed irradiance, can be reasonably assumed to form a Poisson point process of rate μ per unit area. These events are then spatially filtered, i.e., they are spatially spread, as they are transmitted through, and scattered by, the emulsion. The spread in energy of each primary electron then renders a random number of the grains in its path (on the average α) developable. The developable grains form the secondary process, and if this number is Poisson, the SNDP ensues. Note that here, time is replaced by space, and the SNDP is a spatial point process.

The detection of electron beams (in such fields as electron microscopy, electron diffraction, and β-ray dosimetry) may be similarly described [61], [62].

The photographic detection of both electrons and X-rays obey the single-hit theory [61], which means that a single detected particle is capable of rendering one or more silver halide grains developable. An important property of the SNDP is the fact that the variance-to-mean ratio (the Fano factor) is larger than one and is independent of the rate of the primary events (the intensity of the electron image). The multiplication parameter α, which determines the Fano factor, is larger than one for single-hit processes.

In radiography, X-ray images may also be detected by use of intensifying screens. X-ray radiation illuminates a fluorescent screen, which emits visible photons that are recorded on a photographic emulsion. An X-ray photon impinging on the fluorescent screen results in the emission of a random number of light photons. These spread as they propagate to the photographic emulsion, forming a patch of illumination with a spatial distribution $h(x, y)$, which is the point-spread function or the spatial impulse response function. If the incoming X-ray radiation is of uniform irradiance μ photons per unit area, its photons would be distributed according to a homogeneous Poisson process of rate μ. If we postulate that A, the number of light photons per X-ray photon, also has a Poisson distribution of mean value $\langle A \rangle = \alpha$, then the light photons arriving at the emulsion will obey an SNDP, with parameters μ, α, and $h(x, y)$.

It follows, then, that the light photons should exhibit spatial clustering. Such clustering has been observed experimentally, and is often referred to as quantum mottle. Indeed Kemperman and Trabka [63] have recently used the same model to describe the first- and second-order statistics of exposure fluctuations in the recording of X-ray images using intensifying screens.

VI. Conclusion

Thorough consideration has been given to the counting and time statistics for the SNDP. This process provides a natural description for phenomena involving the random multiplication (or reduction) of Poisson point events with a random time delay. The SNDP is a particular Neyman–Scott cluster process. The model has been applied to a number of important problems in electrical engineering, physics, and optics, including cathodoluminescence, photomultiplier-tube luminescence noise induced by ionizing radiation, the photon-counting scintillation detection of nuclear particles, X-ray radiography, and electronography. There are other single-stage random multiplication processes, involving random delay, for which this model provides a ready solution.

The model could be extended and strengthened in a number of ways. In this concluding section, we point to various restrictions that have been implicitly and explicitly imposed on the mathematical formulation, and discuss ways in which these restrictions can be relaxed.

The primary Poisson point process was taken to be homogeneous (HPP) and therefore stationary (see Fig. 3). Physical situations in which primary events are nonstationary also occur, and it is of interest to determine the statistical properties of the resultant process. This is necessary, for example, for studying cathodoluminescence when the exciting electron beam is pulsed. Another example that presents itself is the behavior of the neural discharge recorded from the mammalian retinal ganglion cell [64] excited by a brief pulse of light. We have recently solved for the counting distribution and the probability density function for the waiting time to the first event [65]. It is quite interesting that the counting distribution reduces to the Neyman Type-A when the rate is a pulse of short duration ($\tau_s \ll \tau_p + T$). Spatial inhomogeneities, correlations, and selection can also be important, so that it will be worthwhile to consider the addition of SNDP's and to extend the formulation to more than one dimension.

As is usual for the Neyman-Scott process, we have assumed that primary events are excluded from the final point process. A modified model may be constructed in which the resultant process is the union of the SNDP with the primary process. The statistics of the combined point process cannot be obtained by simple convolution because of the statistical dependence between primary and subsidiary events. In the long-counting-time limit, when the SNDP yields the Neyman Type-A counting distribution, it can be shown that the counting statistics associated with the combined process is the Thomas distribution [41], [66].

Moving to the linear filter in Fig. 3, we have tacitly assumed throughout that $h(t)$ is a deterministic impulse response function. Gilbert and Pollak [45] have shown that if the filter $h(t)$ contains a random parameter, then an equivalent *completely deterministic* impulse response function $h'(t)$ can always be found that generates $\lambda(t)$ with identical statistics. In particular, if $h(t) = h_0(t/\tau_p)$, where τ_p is random, $h'(t) = h_0(t/\langle|\tau_p|\rangle)$. This is an important result because it tells us that the analysis we have carried out applies also when the linear filter is not deterministic. In the context of cathodoluminescence, for example, not only can we ascribe an arbitrary time course to the phosphorescence by choosing $h(t)$ appropriately, but we can also permit $h(t)$ to contain a stochastic parameter. Indeed it is likely that local field inhomogeneities in a real phosphor will create some degree of randomness in τ_p.

Again examining Fig. 3, we see that the shot noise $\lambda(t)$ provides the rate for this second Poisson process. Our model can be modified to permit this second Poisson process to be self-excited, thereby allowing for aftereffects triggered by past events [3]. This modification will be immediately useful for calculating the effects of phenomena such as dead time (absolute refractoriness) and sick time (relative refractoriness). We have recently derived the interevent-time statistics for the SNDP with 1-memory recovery, and examined examples of their use in scintillation detection and vision [67]. A solution has also been obtained for the count mean and variance for an arbitrary dead-time-modified DSPP. These expressions have been experimentally verified for β^--induced radioluminescence photons produced in several transparent materials, in the presence of system dead time [68].[1] For large counting times, we also showed that the experimental photon-counting distributions were well described by the Neyman Type-A theoretical distribution, both in the absence and in the presence of dead time [68].

Given an SNDP ($f(t)$ in Fig. 3), we can easily account for the effects of a statistically independent Poisson point process representing, for example, broadband background light in a photomultiplier tube. The counting statistics for the superposition process can be simply determined by numerical convolution. An interesting case arises in the context of photomultiplier dark noise. Such dark counts can be considered to arise from a combination of photocathode and dynode thermionic emission, describable by the HPP, and multiplication noise. We have recently analyzed the experimental counting statistics for photomultiplier dark noise, and found it to be consistent with this model over a rather broad range of counts [69]. A related approach has been found to be useful for describing the detection of light by the human visual system at threshold [70].

We should, perhaps, say a few words about branching (cascade) processes, in which each subsidiary event generates its own cluster, and so on. This kind of nesting may be referred to as higher-order clustering; in this framework, the SNDP is first-order clustered. A generalization of the SNDP to allow for nesting will be valuable for deriving the precise counting and time statistics for electron devices that involve more than one stage of random multiplication and random time delay. Examples are the avalanche detector and the electron multiplier. We have recently analyzed some problems of this type [71].

Finally, we point out that the statistical behavior of the clustered processes considered here is sufficiently unique that one may conceive of innovative signal processors and receivers specifically designed to enhance such signals (or discriminate against such noise). Calculations of the performance characteristics of such devices (detection, discrimination, estimation) will have to be carried out, of course, to ascertain the value of any such proposed scheme. In the meantime, we have experimentally shown that dead time does selectively discriminate against such clusters [68].

Appendix A
General Properties of the DSPP

For a DSPP with stochastic rate $\lambda(t)$, the counting statistics (statistics of n) can best be described in terms of the statistics of the random variable

$$W = \int_0^T \lambda(t)\, dt \tag{A1}$$

which represents the integrated rate of events [3], [24], [26], [27]. The moment-generating function of n is related to that of W by [27]

$$Q_n(s) = Q_W(1 - e^{-s}) \tag{A2}$$

where $Q_W(s) = \langle e^{-Ws}\rangle$ is the mgf of the random variable W. The factorial moments

$$F_n^{(m)} = \left\langle \frac{n!}{(n-m)!} \right\rangle \tag{A3}$$

and the probability distribution $p(n)$, can be computed by use

[1] We note that the dashed curve shown in Fig. 3 of [68] is incorrect. The proper result for the variance-to-mean ratio for an SNDP with a rectangular impulse response function, in the absence of dead time, is given by (22), (23), and (39b) above. This curve does indeed lie everywhere above the solid curves shown in [68, figure 3], thereby confirming that the introduction of dead time always produces a decrease in the variance-to-mean ratio, as intuitively expected.

of the formulas [27]

$$F_n^{(m)} = \langle W^m \rangle = (-1)^m \frac{\partial^m}{\partial s^m} Q_W(s)\bigg|_{s=0} \tag{A4}$$

$$p(n) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial s^n} Q_W(s)\bigg|_{s=1}. \tag{A5}$$

Alternatively, $p(n)$ can be determined from the Poisson transform of $P(W)$

$$p(n) = \int_0^\infty \frac{W^n e^{-W}}{n!} P(W)\, dW. \tag{A6}$$

Statistics of the times of occurrence of the events of a DSPP can also be determined from the mgf of W. If $P_1(\tau)$ and $P_2(\tau)$ represent the probability density functions for the forward recurrence time (time from an arbitrary point to the first event) and the interevent time, respectively, then [3], [27]

$$P_1(\tau) = -\frac{\partial}{\partial \tau} p_0(\tau) \tag{A7}$$

$$P_2(\tau) = -\frac{1}{\langle\lambda\rangle}\frac{\partial}{\partial \tau} P_1(\tau) = -\frac{1}{\langle\lambda\rangle}\frac{\partial^2}{\partial \tau^2} p_0(\tau), \tag{A8}$$

where

$$p_0(T) = Q_W(1) = p(0), \tag{A9}$$

$\langle\lambda\rangle$ is the mean rate of events, and $p(0)$ is the probability of zero counts in the interval $[0, T]$. The key to determining the singlefold statistics of the DSPP therefore lies in the determination of the moment-generating function $Q_W(s)$ of its integrated rate W.

Joint statistics of the number of events $\{n_j\}$ in N time intervals $[t_j, t_j + T_j]$, $j = 1, 2, \cdots, N$, can also be obtained from the multifold mgf

$$Q_{W_1, \cdots, W_N}(s_1, \cdots, s_N) = \left\langle \exp\left(-\sum_{i=1}^N s_i W_i\right)\right\rangle \tag{A10}$$

of the integrated rates

$$W_j = \int_{t_j}^{t_j+T_j} \lambda(t)\, dt. \tag{A11}$$

The factorial moments are given by

$$\left\langle \prod_{j=1}^N \frac{n_j!}{(n_j - m_j)!}\right\rangle = \left\langle \prod_{j=1}^N W_j^{m_j}\right\rangle = \prod_{j=1}^N (-1)^{m_j} \cdot \frac{\partial^{m_j}}{\partial s_j^{m_j}} Q_{W_1, \cdots, W_N}(s_1, \cdots, s_N)\bigg|_{\{s_j=0\}} \tag{A12}$$

and the joint counting distribution is

$$p(n_1, \cdots, n_N) = \prod_{j=1}^N \frac{(-1)^{n_j}}{n_j!} \frac{\partial^{n_j}}{\partial s_j^{n_j}} Q_{W_1, \cdots, W_N}(s_1, \cdots, s_N)\bigg|_{\{s_j=1\}}. \tag{A13}$$

Equations (A12) and (A13) are generalizations of (A4) and (A5).

Appendix B
General Properties of Shot Noise

The statistics of shot noise $\lambda(t)$ are well known [16], [45]-[47]; they are summarized here for subsequent use.

A. Singlefold Statistics

The first-, second-, and third-order moments are, respectively,

$$\langle\lambda(t)\rangle = \mu \int_0^\infty h(y)\, dy = \mu\alpha$$

$$\langle\lambda^2(t)\rangle = \mu^2\alpha^2 + \mu \int_0^\infty h^2(y)\, dy$$

$$\langle\lambda^3(t)\rangle = \mu^3\alpha^3 + 3\mu^2 \int_0^\infty h^2(y)\, dy + \mu \int_0^\infty h^3(y)\, dy. \tag{B1}$$

The moment-generating function is

$$Q_\lambda(s) = \langle e^{-s\lambda(t)}\rangle = \exp\left\{\mu \int_{-\infty}^\infty [e^{-sh(t)} - 1]\, dt\right\}. \tag{B2}$$

The probability density function $P(\lambda)$ can be determined by solving the integral equation [45]

$$\lambda P(\lambda) = \mu \int_{-\infty}^\infty P[\lambda - h(t)]\, h(t)\, dt. \tag{B3}$$

Explicit solutions for this equation are known in only few cases, as indicated below.

1) Rectangular Filter: The filter

$$h(t) = \begin{cases} \lambda_0 = \dfrac{\alpha}{\tau_p}, & 0 \leqslant t \leqslant \tau_p \\ 0, & \text{elsewhere} \end{cases} \tag{B4}$$

corresponds to a fixed-multiplicative probability density

$$P(\lambda) = \frac{(\mu\tau_p)^k e^{-\mu\tau_p}}{k!} \delta(\lambda - k\lambda_0), \quad k = 0, 1, 2, \cdots, \tag{B5}$$

i.e., λ takes the discrete values $0, \lambda_0, 2\lambda_0, \cdots, k\lambda_0, \cdots$ where k has a Poisson distribution of mean $\mu\tau_p$.

2) Special Filter with Logarithmic Singularity and Exponential Tail: Another filter with rather simple properties is that described by the impulse response function [45]

$$h(t) = \begin{cases} -(\alpha/\tau_p) E_i^{-1}(-t/\tau_p), & t > 0 \\ 0, & t \leqslant 0 \end{cases} \tag{B6}$$

where $E_i^{-1}(x)$ is the inverse of the exponential integral

$$E_i(x) = \int_{-\infty}^x \frac{e^t}{t}\, dt. \tag{B7}$$

This leads to a gamma-distributed λ

$$P(\lambda) = \frac{M}{\langle\lambda\rangle\Gamma(M)} \left(\frac{M\lambda}{\langle\lambda\rangle}\right)^{M-1} \exp\left(-M\frac{\lambda}{\langle\lambda\rangle}\right), \quad M = \mu\tau_p. \tag{B8}$$

B. Multifold Statistics

The multifold moment-generating function is [46]

$$Q_{\lambda_1,\cdots,\lambda_N}(s_1,\cdots,s_N)=\left\langle \exp\left(-\sum_{j=1}^{N} s_j\lambda(t_j)\right)\right\rangle = \exp\left\{\mu\int_{-\infty}^{\infty}\left[\exp\left(-\sum_{j=1}^{N} s_j h(t_j+x)\right)-1\right]dx\right\}. \quad (B9)$$

In particular, the twofold mgf is given by

$$Q_{\lambda_1,\lambda_2}(s_1,s_2)=\exp\left\{\mu\int_{-\infty}^{\infty}[\exp(-s_1h(x) - s_2h(t_2-t_1+x))-1]\,dx\right\} \quad (B10)$$

and the correlation function is written as

$$\langle\lambda(t_1)\lambda(t_2)\rangle=\mu^2\alpha^2+\mu\int_{-\infty}^{\infty}h(x)h(t_2-t_1+x)\,dx. \quad (B11)$$

Appendix C
Properties of the Neyman Type-A Distribution

The Neyman Type-A distributed random variable n of parameter a has an mgf [40], [41], [49]

$$Q_n(s)=\exp\left(\frac{\langle n\rangle}{a}\{\exp[a(e^{-s}-1)]-1\}\right). \quad (C1)$$

Its factorial moments are given by the recurrence relation provided in (19), with

$$a_l=a^l. \quad (C2)$$

In particular, the variance is

$$\mathrm{var}(n)=(1+a)\langle n\rangle \quad (C3)$$

so that the excess Fano factor a_1 is equal to a. The Neyman Type-A probability distribution $p(n)$ is determined from the recurrence relation in (27). The coefficients C_l are

$$C_l=\frac{a^le^{-a}}{l!} \quad (C4)$$

so that (27) becomes [40]

$$(n+1)p(n+1)=\langle n\rangle\sum_{l=0}^{n}e^{-a}\frac{a^l}{l!}p(n-l)$$
$$p(0)=\exp\left[\frac{\langle n\rangle}{a}(e^{-a}-1)\right]. \quad (C5)$$

Note that when $a\to 0$, (C5) reduces to

$$(n+1)p(n+1)=\langle n\rangle p(n), \qquad p(0)=e^{-\langle n\rangle} \quad (C6)$$

which is the recurrence relation for the ordinary Poisson distribution. This same result may be obtained by letting $a\to 0$ in (C1). This leads to $Q_n(s)=\exp[\langle n\rangle(e^{-s}-1)]$, the mgf of the Poisson distribution.

Appendix D
Counting Statistics for Some Special Filters in the Limit of Short Counting Time

A. Rectangular Filter

When the impulse response function is rectangular of area α and width τ_p, each event of the primary homogeneous Poisson process initiates a rectangular pulse. Within the time interval of that pulse, secondary Poisson events occur at the rate $\mu\alpha/\tau_p$ per second. With $h(t)$ rectangular, (33) yields

$$Q_W(s)=\exp\left\{\frac{\langle n\rangle}{\alpha\beta}(e^{-\alpha\beta s}-1)\right\} \quad (D1)$$

where the mean number of counts $\langle n\rangle$ registered in the time interval T is

$$\langle n\rangle=\mu\alpha T, \quad (D2)$$

and where

$$\beta=T/\tau_p. \quad (D3)$$

Using (A2) we then arrive at $Q_n(s)$, which turns out to be

$$Q_n(s)=\exp\left(\frac{\langle n\rangle}{\alpha\beta}\{\exp[\alpha\beta(e^{-s}-1)]-1\}\right). \quad (D4)$$

This is precisely the mgf of the Neyman Type-A distribution of parameter $\alpha\beta$ (see (C1)).

We conclude that the shot-noise-driven Poisson point process results in the Neyman Type-A counting distribution when the impulse response function of the filter is rectangular in shape, and the counting time is very short.

B. Exponentially Decaying Filter

When $h(t)$ is an exponentially decaying function, $h(t)=(2\alpha/\tau_p)\exp(-2t/\tau_p)$, (33) gives

$$Q_W(s)=\exp\left\{-\frac{\langle n\rangle}{2\alpha\beta}\int_0^{2\alpha\beta s}\frac{1-e^{-x}}{x}dx\right\}, \quad (D5)$$

where again the mean count $\langle n\rangle=\mu\alpha T$ and $\beta=T/\tau_p$. The factorial and central moments of n are given by (19)–(22), with

$$a_l=(2\alpha\beta)^l/(l+1). \quad (D6)$$

Comparing these moments with those derived for the rectangular filter (viz., those of the Neyman Type-A distribution) we see that, for the same variance, the third central moment (and the skewness defined by (26)) is larger in the exponential case than in the rectangular case.

The probability distribution $p(n)$ is given by (27) in conjunction with the coefficients C_l from (28), which are now

$$C_l=\frac{1}{2\alpha\beta l!}\int_0^{2\alpha\beta}y^le^{-y}\,dy=\frac{1}{2\alpha\beta l!}\gamma(l+1,2\alpha\beta) \quad (D7)$$

Here $\gamma(l+1,x)$ is the incomplete gamma function, which may be computed from the series

$$\gamma(l+1,x)=l!\left[1-e^{-x}\sum_{k=0}^{l}x^k/k!\right]. \quad (D8)$$

Also

$$p(0) = \exp\left\{\frac{\langle n\rangle}{2\alpha\beta}\int_0^{2\alpha\beta}\frac{(e^{-z}-1)}{z}\,dz\right\}$$

$$= \exp\left\{\frac{\langle n\rangle}{2\alpha\beta}\sum_{k=1}^{\infty}\frac{(-2\alpha\beta)^k}{k(k!)}\right\}. \quad \text{(D9)}$$

Again, for vanishing T or α, the counting distribution becomes Poisson.

C. Special Filter with Logarithmic Singularity and Exponential Tail

This filter, described by (B6), corresponds to shot noise with a rate $\lambda(t)$ characterized by the gamma probability density function (B8). The integrated intensity $W \approx \lambda T$ will also have a gamma density with mean $\langle W\rangle$ and parameter $M = \mu\tau_p$. In this case, we obtain the counting distribution $p(n)$ by using the Poisson transform specified in (A6), since it is well known that the Poisson transform of the gamma density is the negative binomial distribution, as was first shown by Greenwood and Yule [72]:

$$p(n) = \binom{n+M-1}{n}\Bigg/\left(1+\frac{M}{\langle n\rangle}\right)^{-n}\left(1+\frac{\langle n\rangle}{M}\right)^{-M}. \quad \text{(D10)}$$

This distribution also turns out to provide a good approximation for the photocounting detection of multimode chaotic light of arbitrary spectrum; in that case M represents the number of degrees of freedom of the *chaotic* light [8], [12], [24], [27], [50], [73], [74]. This is to be distinguished from the degrees-of-freedom parameter $\mathfrak{M}$ for the shot-noise light considered here [50]. The factorial moments are

$$F_n^{(m)} = \frac{\Gamma(m+M)}{\Gamma(M)M^m}\langle n\rangle^m. \quad \text{(D11)}$$

The variance is given by [50]

$$\operatorname{var}(n) = \langle n\rangle + \langle n\rangle^2/M = (1+\alpha\beta)\langle n\rangle \quad \text{(D12)}$$

so that the excess Fano factor for this distribution is again $a_1 = \alpha\beta$. The SNDP is, in this example, equivalent to an inhomogeneous Polya process [2], [75].

Appendix E
Time Statistics for an SNDP with Rectangular and Exponential Filters

The probability density functions for the forward recurrence time and the interevent time are, respectively,

$$P_1(\tau) = \frac{-1}{\tau_p}R'd\exp(Rd)$$

$$P_2(\tau) = \frac{1}{\tau_p\alpha^2}(R'' + R'^2 d)\exp(Rd). \quad \text{(E1)}$$

For a rectangular filter of width τ_p and area α

$$R = \begin{cases} e^{-\alpha x}(\alpha - \alpha x - 2) - (\alpha + \alpha x - 2), & x \leqslant 1 \\ e^{-\alpha}(\alpha x - \alpha - 2) - (\alpha + \alpha x - 2), & x \geqslant 1 \end{cases}$$

$$R' = \begin{cases} e^{-\alpha x}\alpha(1 - \alpha + \alpha x) - \alpha, & x \leqslant 1 \\ \alpha(e^{-\alpha} - 1), & x \geqslant 1 \end{cases}$$

$$R'' = \begin{cases} \alpha^3(1-x)e^{-\alpha x}, & x \leqslant 1 \\ 0, & x \geqslant 1 \end{cases}$$

$$x = \tau/\tau_p,$$

$$d = \mu\tau_p/\alpha = \frac{1}{\alpha^2}(\tau_p/\langle\tau\rangle). \quad \text{(E2)}$$

For an exponential filter of time constant $\tau_p/2$ and area α,

$$R = \alpha\{\eta_0(\alpha e^{-x} - \alpha) + e^{-\alpha}[\eta_0(\alpha) - \eta_0(\alpha e^{-x})] - x(1 - e^{-\alpha})\}$$

$$R' = -\alpha\{\alpha e^{-x}\eta_0'(\alpha e^{-x} - \alpha) - \alpha e^{-\alpha}e^{-x}\eta_0'(\alpha e^{-x}) + 1 - e^{-\alpha}\}$$

$$R'' = \alpha^3 e^{-x}\eta_0''(\alpha e^{-x} - \alpha) \quad \text{(E3)}$$

with

$$\eta_0(y) = \sum_{m=1}^{\infty}\frac{y^m}{m!\,m}, \quad \eta_0'(y) = \frac{e^y - 1}{y}, \quad \eta_0''(y) = \frac{1 + (y-1)e^y}{y^2}, \quad \text{(E4)}$$

and $x = 2\tau/\tau_p$.

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