PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 1, June 1973

MULTIPLIERS FOR THE SPACE OF ALMOST-CONVERGENT FUNCTIONS ON A SEMIGROUP

CHING CHOU AND J. PETER DURAN

ABSTRACT. Let S be a countably infinite left amenable cancellative semigroup, $FL(S)$ the space of left almost-convergent functions on S. The purpose of this paper is to show that the following two statements concerning a bounded real function f on S are equivalent: (i) $f \cdot FL(S) \subset FL(S)$; (ii) there is a constant α such that for each $\varepsilon>0$ there exists a set $A \subset S$ satisfying (a) $\varphi(X_A)=0$ for each left invariant mean φ on S and (b) $|f(x)-\alpha|<\varepsilon$ if $x \in S \backslash A$.

1. Let S be a semigroup, $m(S)$ the space of bounded real functions on S with the sup norm. $\varphi \in m(S)^*$ is called a left invariant mean on S if $\|\varphi\|=1$, $\varphi \geqq 0$ and $\varphi(l_s f)=\varphi(f)$ for $s \in S$ and $f \in m(S)$, where $l_s f \in m(S)$ is defined by $(l_s f)(t)=f(st)$, $t \in S$. The set of left invariant means on S is denoted by $ML(S)$. If $ML(S)$ is nonempty, then S is said to be left amenable [**2**]. A bounded real function f on a left amenable semigroup is called left almost-convergent if $\varphi(f)$ equals a fixed constant $d(f)$ as φ runs through $ML(S)$ [**2**]. The set of all left almost-convergent functions, denoted by $FL(S)$, is a vector subspace of $m(S)$ and it contains constant functions. But, in general, it is not closed under multiplication. The purpose of this paper is to study this aspect of $FL(S)$ and our main result is the following.

THEOREM. *Let S be a countable left-cancellative left amenable semigroup without finite left ideals. Then the following two statements concerning a function $f \in m(S)$ are equivalent*:

(i) *f is a multiplier of $FL(S)$, i.e., $f \cdot FL(S) \subset FL(S)$*;

(ii) *f is S-convergent to a constant α, i.e., for a given $\varepsilon>0$ there exists a set $A \subset S$ such that*

(a) *$\varphi(\chi_A)=0$ for each $\varphi \in ML(S)$, and*

(b) *$|f(x)-\alpha|<\varepsilon$ if $x \in S \backslash A$.*

2. From now on S will always denote a left-cancellative left amenable semigroup without finite left ideals.

Presented to the Society, January 17, 1972; received by the editors May 31, 1972.
AMS (MOS) subject classifications (1970). Primary 43A07; Secondary 46N05.
Key words and phrases. Amenable semigroups, almost-convergence, multipliers, invariant means, weak Cauchy sequences.

REMARKS. (1) A set $A\subset S$ is said to be left almost-convergent if its characteristic function χ_A is left almost-convergent. In this case we denote $d(\chi_A)$ by $d(A)$. Roughly speaking, a set $A\subset S$ is left almost-convergent if it is evenly distributed in S with respect to the semigroup structure and $d(A)$ indicates the density of $A\subset S$. In particular, $d(A)=0$ means that A is sparsely distributed in S. (The set A in the statement (ii) of the above Theorem is such a set.) For example, when $S=N$, the additive semigroup of positive integers, a set $A\subset N$ is left almost-convergent if and only if

$$\lim_n (1/n)\mathrm{Card}(\{k, k+1, \cdots, k+n-1\} \cap A) = d(A)$$

exists uniformly in k [7].

(2) Since S contains no finite left ideals, $d(B)=0$ for each finite subset $B\subset S$ (cf. [5]). Therefore if f converges to α at infinity, i.e., if given $\varepsilon>0$ there exists a finite set $B\subset S$ such that $|f(x)-\alpha|<\varepsilon$ whenever $x\notin B$, then f S-converges to α. On the other hand, the space of S-convergent functions is much smaller than $FL(S)$. Indeed, $FL(S)$ separates points of βS [6], the Stone-Čech compactification of the discrete set S, while it is easy to see that $f\in m(S)$ is S-convergent if and only if f is a constant on the set

$$K(S) = \mathrm{cl}_{\beta S} \cup \{\mathrm{supp}\ \varphi\colon \varphi\in ML(S)\},$$

(cf. [1]). Here we consider a bounded real function on S as a continuous function on βS and a mean on $m(S)$ as a probability measure on βS. In particular, if $\varphi\in ML(S)$, supp φ denotes the support of the measure φ.

(3) As in [2], let EG denote the smallest class of groups which contains all finite groups, all abelian groups and is closed under the following four ways of constructing new groups from given ones: (a) subgroup; (b) factor group; (c) group extension; and (d) direct limits.

Each group in EG is amenable and they constitute all the known amenable groups [2]. If we assume that S is an infinite group in EG, then a stronger result is known [1]: $f\in m(S)$ is S-convergent if and only if

(a) $f\cdot\chi_A\in FL(S)$ for each left almost-convergent set A,

(b) $f^n\in FL(S)$, $n=1, 2, \cdots$;

in particular, if $A\subset S$ and $A\cap B$ is left almost-convergent for each $\chi_B\in FL(S)$, then $d(A)=1$ or 0. It is not clear whether our Theorem yields the same conclusion. The proof of our Theorem is completely different from the proof in [1].

3. **Proof of the Theorem.** (ii)⇒(i) is easy, cf. [1].

(i)⇒(ii). Let f be a multiplier of $FL(S)$. To show that f is S-convergent, it suffices to show that $f\equiv d(f)$ on supp φ. We claim that this follows from the following assertion:

(α) If $g\in FL(S)$ and $\varphi\in ML(S)$, then $\varphi(fg)=\varphi(f)\varphi(g)$.

Indeed, if (α) holds and if $\varphi \in ML(S)$ then

$$\varphi((f - d(f))^2) = (\varphi(f) - d(f))^2 = 0.$$

Therefore $f \equiv d(f)$ on supp φ as we wanted.

PROOF OF (α). We shall consider $l^1(S)^* = m(S)$ and $l^1(S)^{**} = m(S)^*$. If $\psi \in l^1(S)$ and $h \in m(S)$, then $h(\psi) = \psi(h) = \sum_{t \in S} h(t)\psi(t)$. Since S is left amenable and countable there exists a sequence φ_n in $l^1(S)$ such that $\|\varphi_n\|_1 = 1$, $\varphi_n \geqq 0$, and $\lim_n(\varphi_n(h) - \varphi_n(l_x h)) = 0$ for each $h \in m(S)$ and each $x \in S$ [**5**, Lemma 5.1]. We shall need the following two well-known facts (cf. [**3**, §9]):

(β) If $g \in FL(S)$, then $\lim_n \varphi_n(g) = d(g)$.

(γ) $FL(S)$ = the closed linear span of $\{l_x h - h : x \in S,\ h \in m(S)\} \cup \{\chi_S\}$.

Let $x \in S$ be fixed. Set $\psi_n = \varphi_n \cdot f - l_x(\varphi_n \cdot f)$, i.e., $\psi_n(t) = \varphi_n(t) f(t) - \varphi_n(xt) f(xt)$, $t \in S$. Then $\psi_n \in l^1(S)$. We claim that ψ_n is a weak Cauchy sequence in $l^1(S)$. Indeed, if $h \in m(S)$,

$$\begin{aligned} \psi_n(l_x h) &= \sum_{t \in S} (\varphi_n(t) f(t) h(xt) - \varphi_n(xt) f(xt) h(xt)) \\ &= \sum_{t \in S} \varphi_n(t) f(t)(h(xt) - h(t)) + \sum_{t \in S \setminus xS} \varphi_n(t) f(t) h(t) \\ &= \varphi_n(f \cdot (l_x h - h)) + \varphi_n(fh \cdot \chi_{S \setminus xS}) \equiv a_n + b_n. \end{aligned}$$

Note that $f \cdot (l_x h - h) \in FL(S)$, since f is a multiplier of $FL(S)$ and $l_x h - h \in FL(S)$. Therefore by (β) $\lim_n a_n = d(f \cdot (l_x h - h))$. Note also that $1 \geqq \varphi(\chi_{tS}) = \varphi(l_t \chi_{tS}) \geqq \varphi(\chi_S) = 1$, i.e., $\varphi(\chi_{S \setminus tS}) = 0$ for each $t \in S$. Therefore $\chi_{S \setminus tS}$ is left almost-convergent to zero. By (β) again, we get

$$|b_n| \leqq \|f\|_\infty \|h\|_\infty \varphi_n(\chi_{S \setminus tS}) \to 0 \quad \text{as } n \to \infty.$$

Therefore we have obtained:

(δ) $\lim_n \psi_n(l_x h) = d(f \cdot (l_x h - h))$, $h \in m(S)$.

Since S is left-cancellative, each $k \in m(S)$ is of the form $l_x h$ for some $h \in m(S)$. Therefore $\lim_n \psi_n(h)$ exists for each $h \in m(S)$, i.e., ψ_n is a weak Cauchy sequence as we claimed. Since $l^1(S)$ is weakly sequentially complete [**4**, p. 374], there exists $\psi \in l^1(S)$ such that $\psi = \lim_n \psi_n$ in the weak topology. Certainly, $\psi(t) = \lim_n \psi_n(t)$ for $t \in S$. On the other hand,

$$\lim_n \varphi_n(t) = \lim_n \varphi_n(\chi_{\{t\}}) = 0,$$

since $\chi_{\{t\}} \in FL(S)$ and $d(\chi_{\{t\}}) = 0$ (cf. Remark (1)). Hence

$$\psi(t) = \lim_n (\varphi_n(t) f(t) - \varphi_n(xt) f(xt)) = 0.$$

So, $\psi \equiv 0$. By (δ), $d(f \cdot (l_x h - h)) = 0 = d(f) \cdot d(l_x h - h)$. It is of course true that $d(f\cdot c\chi_S)=d(f)\cdot d(c\chi_S)$. Hence ($\alpha$) follows from ($\gamma$) and the above observation.

REFERENCES

1. C. Chou, *The multipliers for the space of almost convergent sequences*, Illinois J. Math. **16** (1972), 687–694.

2. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544. MR **19,** 1067.

3. ———, *Semigroups and amenability*, Proc. Sympos. Semigroups (Wayne State Univ., Detroit, Mich., 1968), Academic Press, New York, 1969, pp. 5–53. MR **42** #411.

3. N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

5. E. Granirer, *On amenable semigroups with a finite-dimensional set of invariant means.* I, Illinois J. Math. **7** (1963), 32–48. MR **26** #1744.

6. S. P. Lloyd, *Subalgebras in a subspace of* $C(X)$, Illinois J. Math **14** (1970), 259–267. MR **41** #8995.

7. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190. MR **10,** 367.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, AMHERST, NEW YORK 14226

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PUERTO RICO AT MAYAGUEZ, MAYAGUEZ, PUERTO RICO 00708