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MULTIPLIERS OF A BANACH ALGEBRA IN THE SECOND CONJUGATE ALGEBRA AS AN IDEALIZER

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1. **Introduction.** In this paper we discuss the multipliers on a general Banach algebra A with an approximate identity. We are embedding a Banach algebra A into the second conjugate space A^{**} and introduce the Arens multiplication in A for which the second conjugate space A^{**} becomes a Banach algebra and then characterize the multiplier algebra of Banach algebra in its second conjugate algebra as an idealizer. If the algebra A is a C^*-algebra, it was known that A has an approximate identity and the multiplication in A is Arens regular, and the multipliers were investigated by Akemann, Pedersen and Tomiyama [1], Busby [2], Tomiuk and Wong [10], [11] and [12] etc.. Our purpose is to investigate the multipliers dealing with a general Banach algebra with bounded approximate identity and the multiplication of non Arens regular will be discussed.

For convenience we begin in section 2 by establishing the definitions and some notations, and the multipliers of a Banach algebra can be extended to be a multiplier of its second conjugate algebra which is also included. In section 3 we characterize the multiplier algebra as an idealizer in the second conjugate algebra. Section 4 is to extend the multipliers of a subalgebra of a Banach algebra to the multipliers of the Banach algebra by using the idealizer, and shows that the multipliers of subalgebra can be embedded in the Banach algebra with identity as an idealizer. The essential applications of idealizers are sections 5 and 6. In section 5 we study the homomorphism extension of Banach algebras to their multiplier algebras. Applying Theorem 5.3, we obtain some extension theorems of representations of algebra as linear operators in a Hilbert space in which the results of Johnson [5; Theorems 21 and 23] are included. Finally, by applying Theorem 5.2, we determine the multipliers of the vector-valued functions vanishing at infinity as a bounded continuous multiplier-valued functions which is a generalization of Theorem 3.3 and Corollary 3.4 of [1].

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2. Preliminaries and notations. Let A be a Banach algebra. Denote by A^* and A^{**} the first and second conjugate spaces of A. We denote the elements of A by $a, b, \cdots$; the elements of A^* by $\varphi, \psi, \cdots$; the elements of A^{**} by $F, G, \cdots$. Arens introduced a multiplication in the second conjugate space A^{**} as follows (cf. Civin and Yood [4]):

If $\varphi \in A^*$, $a \in A$, define $\varphi * a \in A^*$ by $(\varphi * a)b = \varphi(ab)$ for all $b \in A$.

If $F \in A^{**}$, $\varphi \in A^*$, define $F * \varphi \in A^*$ by $(F * \varphi)b = F(\varphi * b)$ for all $b \in A$.

If $F, G \in A^{**}$, define $F * G \in A^{**}$ by $(F * G)\varphi = F(G * \varphi)$ for all $\varphi \in A^*$.

This multiplication in A^{**} which makes A^{**} to be a Banach algebra. Throughout we denote by π the cannonical embedding of A into A^{**}. For some purpose, Arens considered also the following definition of multiplication in A^{**}:

If $\varphi \in A^*$, $a \in A$, define $\varphi \circ a \in A^*$ by $(\varphi \circ a)b = \varphi(ba)$ for all $b \in A$.

If $\varphi \in A^*$, $F \in A^{**}$, define $F \circ \varphi \in A^*$ by $(F \circ \varphi)b = F(\varphi \circ b)$ for all $b \in A$.

If $F, G \in A^{**}$, define $F \circ G \in A^{**}$ by $(F \circ G)\varphi = F(G \circ \varphi)$ for all $\varphi \in A^*$.

Again the multiplication $F \circ G$ in A^{**} as a product makes A^{**} to be a Banach algebra. If $F * G = G \circ F$ in A^{**}, the multiplication in A is called *regular*. It is known that the multiplication $F * G$ is continuous with respect to $\sigma(A^{**}, A^*)$ topology in F for fixed $G \in A^{**}$ and $\pi(a) * G$ is $\sigma(A^{**}, A^*)$ continuous in G for fixed $a \in A$. If the multiplication in A is regular, then $F * G$ is also $\sigma(A^{**}, A^*)$ continuous in G for fixed F (cf. Civin and Yood [4]). Note that the algebra A^{**} obtained by the Arens product needs not be commutative and not be semisimple even if the original algebra A is commutative and semisimple, and that $\pi(A)$ is only a subalgebra of A^{**} but is not an ideal in A^{**} in general (cf. Civin [3]). Keeping these phenomenon in mind, we start to investigate the multiplier algebra of A in A^{**}.

For convenience, throughout this paper we assume that the Banach algebra A has a (bounded) approximate identity $\{e_\alpha\}$ (without loss of generality, we may assume that $\|e_\alpha\| \leqq 1$). We say that a bounded linear operator T on a Banach algebra A is a left (right) multiplier of A if $T(ab) = Ta \cdot b$ $(T(ab) = a \cdot Tb)$ for all $a, b \in A$. The two-sided (= double) multiplier of A is an ordered pair $(T_l, T_r) = T$ of operators on A such that

$$a \cdot T_l b = T_r a \cdot b \quad \text{for all} \quad a, b \in A .$$

We denote by $M_l(A)$ $(M_r(A), M(A))$ the space of all left (right, double) multiplier operators on A. For any $a \in A$, we define

$$D_a = (L_a, R_a) \text{ where } L_a(x) = ax, \text{ and } R_a(x) = xa \text{ for all } x \in A\ .$$

Since A has an approximate identity, it is not hard to show that for $T = (T_l, T_r)$ in $M(A)$, $T_l \in M_l(A)$, $T_r \in M_r(A)$ and $\| T \| = \| T_r \| = \| T_l \|$ (cf. Johnson [5] and Busby [2]). Now we identify

$$a \leftrightarrow L_a \ (\text{resp. } a \leftrightarrow R_a, a \leftrightarrow D_a)\ ,$$

then A is a dense left (right, two-sided) ideal in $M_l(A)$ ($M_r(A)$, $M(A)$) with respect to the strict topology β. That is the topology generated by seminorms of the form

$$\| T \|_a = \| Ta \| \ (\| (T_l, T_r) \|_a = \| T_l a \| + \| T_r a \|$$

for double multipliers $T = (T_l, T_r) \in M(A)$). Some elementary properties and definitions for double multipliers, we refer to Johnson [5] and Busby [2].

Now we show that a multiplier $T \in M_l(A)$ (or $M_r(A)$) can be extended to a left (right) multiplier in A^{**} and hence for the two-sided multipliers in A^{**}.

PROPOSITION 2.1. *Let* $T \in M_l(A)$. *Then*
(i) $\varphi * Ta = T'\varphi * a$, (ii) $T'(G * \varphi) = G * T'\varphi$, (iii) $T''(F * G) = T''F * G$ *for any* $a \in A$, $\varphi \in A^*$, $F, G \in A^{**}$ *where* T' *and* T'' *are defined by*

$$\varphi(Ta) = (T'\varphi)a \quad \textit{and} \quad (T''F)\varphi = F(T'\varphi)\ .$$

PROOF. (i) For any $T \in M_l(A)$ and $\varphi \in A^*$, $a \in A$, we have

$$(\varphi * Ta)x = \varphi(Ta\, x) = \varphi(T(ax)) = (T'\varphi(ax)) = (T'\varphi * a)x$$

for all $x \in A$. This implies $T'\varphi * a = \varphi * Ta$.

(ii) For any $G \in A^{**}$, $\varphi \in A^*$, we have

$$\begin{aligned}(G * T'\varphi)x = G(T'\varphi * x) &= G(\varphi * Tx) \qquad \text{(by (i))}\\ &= T'(G * \varphi)x\end{aligned}$$

for all $x \in A$, then $T'(G * \varphi) = G * T'\varphi$.

(iii) For any $F, G \in A^{**}$, we have

$$\begin{aligned}T''(F * G)\varphi = (F * G)(T'\varphi) = F(G * T'\varphi) &= F(T'(G * \varphi)) \qquad \text{(by (ii))}\\ &= (T''F * G)\varphi\end{aligned}$$

for all $\varphi \in A^*$. Then $T''(F * G) = T''F * G$. q.e.d.

COROLLARY 2.2. *If* $T \in M_r(A)$, *then for* $a \in A$, $G \in A^{**}$ *we have*
(i) $T'(\varphi * a) = T'\varphi * a$, (ii) $T''F * \varphi = F * T'\varphi$, (iii) $T''(F * G) = F * T''G$.

By above proposition and corollary, we see that every multiplier of A can be extended as a multiplier of A^{**}. Thus the question arises that whether we can characterize the algebra $M_l(A)$ (resp. $M_r(A)$, $M(A)$) in A^{**}. To this end we define

$$M_l'(A) = \{F \in A^{**} \mid F*\pi(a) \in \pi(A) \text{ for all } a \in A\} ,$$

$$M_r'(A) = \{F \in A^{**} \mid \pi(a)*F \in \pi(A) \text{ for all } a \in A\} ,$$

$$M'(A) = \{F \in A^{**} \mid \pi(a)*F \text{ and } F*\pi(a) \in \pi(A) \text{ for all } a \in A\} .$$

Then $M_l'(A)$, $M_r'(A)$ are closed subalgebras in A^{**}, they are the (left, right) idealizers of A in A^{**}. For any $F \in M'(A)$, we identify

$$F \leftrightarrow D_F = (L_F, R_F)$$

where $L_F(\pi(a)) = F*\pi(a)$ and $R_F(\pi(a)) = \pi(a)*F$ for any $a \in A$, then $M'(A)$ is defined as the idealizer of A in A^{**}. Under this identification, the discussion of the double multiplier algebra $M(A)$ in A^{**} is almost the same as the discussion of $M_l(A)$ in A^{**}. We use the following notations for mappings

$$\theta_i\colon M_i'(A) \to M_i(A) \quad (i = l, r) \quad \text{and} \quad \theta\colon M'(A) \to M(A) .$$

The kernel of those mappings are denoted by $\theta_i^{-1}(0) = \mathscr{T}_i$ $(i = l, r)$, $\theta^{-1}(0) = \mathscr{T} = \{F \mid \pi(a)*F = F*\pi(a) = 0 \text{ for all } a \in A\}$. Denote by q the quotient mapping of

$$A^{**} \to A^{**}/\mathscr{T}_i \,(i = l, r) \quad \text{or} \quad A^{**} \to A^{**}/\mathscr{T} .$$

3. Characterization of multiplier algebras as the idealizers. Mátè [7] investigated the right multipliers of a Banach algebra A dealing with conjugate algebra A^{**}. In [7; Theorem 6], he showed that the $M_r(A)$ is isometric isomorphic to a certain closed subalgebra of $A^{**}/\mathscr{T}_r$ with Arens product. In this section we will do a systematic discussion concerning the multiplier algebras in A^{**}.

THEOREM 3.1. *Let A be a Banach algebra. Then there is an algebra homomorphism θ_l of $M_l'(A)$ onto $M_l(A)$. The kernel $\theta_l^{-1}(0) = \mathscr{T}_l$ of θ_l is the polar of the space spanned by $\{\pi(a)*\varphi; a \in A, \varphi \in A^*\}$ and $\mathscr{T}_l$ is a left ideal in A^{**}.*

PROOF. For any $F \in M_l'(A)$ and $a \in A$, $F*\pi(a) \in \pi(A)$. Hence there is an operator T on A determined by $F*\pi(a) = \pi(Ta)$. This T is a left multiplier in $M_l(A)$. Indeed, for $a, b \in A$,

$$F*\pi(a)*\pi(b) = F*\pi(ab) = \pi(T(ab))$$

and

$$F*\pi(a)*\pi(b) = \pi(Ta)*\pi(b) = \pi(Ta\, b)$$

implies $T(ab) = Ta \cdot b$ and $T \in M_l(A)$. Hence it is easy to show that

$$\theta_l\colon F \to T \text{ defined by } F * \pi(a) = \pi(Ta)$$

is an algebra homomorphism of $M_l'(A)$ into $M_l(A)$. We have only to show θ_l is an onto mapping. Let $T \in M_l(A)$ and let $\{a_\alpha\}$ be an approximate identity of A. Then $\{\pi(Ta_\alpha)\}$ is a bounded net in A^{**}. By Alaoglu's theorem there is a weak*-limit point F in A^{**} and then has a subnet $\{\pi(Ta_\beta)\}$ of $\{\pi(Ta_\alpha)\}$ such that $\lim_\beta \pi(Ta_\beta) = F$ in $\sigma(A^{**}, A^*)$ topology and

$$\begin{aligned}(F * \pi(x))\varphi &= F(\pi(x) * \varphi) = \lim_\beta \pi(Ta_\beta)(\pi(x) * \varphi) \\ &= \lim_\beta \pi(Ta_\beta x)\varphi = \lim_\beta T'\varphi(a_\beta x) = \pi(Tx)\varphi\end{aligned}$$

for any $x \in A$ and any $\varphi \in A^*$. Hence $F * \pi(x) = \pi(Tx)$ for all $x \in A$, this shows that θ_l is an onto homomorphism. The last part of theorem is trivial. q.e.d.

COROLLARY 3.2. *Let A be a Banach algebra. Then there is an algebra homomorphism θ_r of $M_r'(A)$ onto $M_r(A)$. The kernel $\theta_r^{-1}(0) = \mathscr{T}_r$ of θ_r is the polar of the space spanned by $\{\varphi * a \mid a \in A, \varphi \in A^*\}$ and $\mathscr{T}_r$ is a two-sided ideal of A^{**}.*

PROOF. The first part of this corollary follows from Theorem 3.1 by the same argument. For the second part, it follows from

$$F(\varphi * a) = (\pi(a) * F)\varphi \quad \text{for all } a \in A \text{ and } \varphi \in A^*$$

that $F \in \mathscr{T}_r$ if and only if F is in the polar of the space spanned by

$$\{\varphi * a \mid a \in A, \varphi \in A^*\} .$$

Evidently $\mathscr{T}_r$ is a right ideal in A^{**}. We need only to show that $\mathscr{T}_r$ is also a left ideal. Since for $F \in \mathscr{T}_r$,

$$F * \varphi(a) = F(\varphi * a) = 0 \text{ implies } F * \varphi = 0 \text{ for all } \varphi \in A^* ,$$

thus for $F \in \mathscr{T}_r$, $G \in A^{**}$ we have

$$\pi(a) * (G * F)\varphi = (\pi(a) * G)(F * \varphi) = 0 \text{ implies } G * F \in \mathscr{T}_r .$$

Hence $\mathscr{T}_r$ is a two-sided ideal in A^{**}. q.e.d.

COROLLARY 3.3. *There is an onto homomorphism θ of $M'(A)$ to $M(A)$. The kernel $\theta^{-1}(0) = \mathscr{T}$ is the polar of $\{\varphi * a \text{ or } \pi(a) * \varphi \mid a \in A, \varphi \in A^*\}$ and $\mathscr{T}$ is a two-sided ideal of A^{**}.*

PROOF. For the element $F \in M'(A)$, we identify that $F \leftrightarrow D_F$. Then

$$\pi(x) * L_F(\pi(y)) = R_F(\pi(x)) * \pi(y) \in \pi(A)$$

or

$$\pi(x)*(F*\pi(y)) = (\pi(x)*F)*\pi(y) \quad \text{for all} \quad x, y \in A .$$

Since A has approximate identity, there is $T_l \in M_l(A)$, $T_r \in M_r(A)$ such that

$$F*\pi(y) = \pi(T_l y) , \qquad \pi(x)*F = \pi(T_r x)$$

and hence $\pi(x\cdot T_l y) = \pi(T_r x \cdot y)$ or $x \cdot T_l y = T_r x \cdot y$. This shows that $T = (T_l, T_r) \in M(A)$ and the mapping defined by

$$\theta: F \to \theta(F) \equiv \theta(D_F) = T = (T_l, T_r)$$

is evidently a homomorphism under the multiplication given by

$$T^1 \circ T^2 = (T_l^1, T_r^1)\circ(T_l^2, T_r^2) = (T_l^1 \circ T_l^2, T_r^2 \circ T_r^1) .$$

Conversely, if $T = (T_l, T_r) \in M(A)$ and $\{a_\alpha\}$ is an approximate identity of A, then there is a subnet $\{a_\beta\}$ such that

$$\pi(T_l a_\beta) \to F \text{ and } \pi(T_r a_\beta) \to F' \text{ in } \sigma(A^{**}, A^*) \text{ topology} .$$

Thus $\pi(x\cdot T_l a_\beta) = \pi(T_r x \cdot a_\beta) \to \pi(x)*F = \pi(T_r x)$ in $\sigma(A^{**}, A^*)$ topology for any $x \in A$ implies

$$\pi(a_\beta)*F*\pi(y) = \pi(T_r a_\beta)*\pi(y) = \pi(a_\beta \cdot T_l y)$$

$\to F*\pi(y) = F'*\pi(y) = \pi(T_l y) \in \pi(A)$ in $\sigma(A^{**}, A^*)$-topology. Hence $F = F'$ and $F \in M'(A)$. This shows that θ is an onto homomorphism. The remains of theorem is immediately. q.e.d.

Next we consider the coset space $A^{**}/\mathscr{T}_l$ with Arens product and the natural mapping $q: A^{**} \to A^{**}/\mathscr{T}_l$. Then it can be shown that q is an isometry of $\pi(A)$ into $A^{**}/\mathscr{T}_l$. Indeed, for any $b \in A$,

$$\|\pi(b)\| = \|b\| = \sup_{\|\varphi\|\leqq 1} |\pi(b)\varphi| .$$

We want to show that

$$\sup_{\|\varphi\|\leqq 1} |\pi(b)\varphi| = \sup_{\|\pi(a)*\varphi\|\leqq 1} |\pi(b)(\pi(a)*\varphi)| .$$

Let $\{a_\alpha\}$ be an approximate identity with $\|a_\alpha\| \leqq 1$. Then

$$\|b\| \geqq \|ba_\alpha\| = \sup_{\|\varphi\|\leqq 1} |(\pi(b)*\pi(a_\alpha))\varphi| \geqq \sup_{\|\pi(a_\alpha)*\varphi\|\leqq 1} |\pi(b)(\pi(a_\alpha)*\varphi)| .$$

The last supremum tends to $\sup_{\|\varphi\|\leqq 1} |\pi(b)\varphi| = \|b\|$. Consequently

$$\|b\| = \sup_{\|\varphi\|\leqq 1} |\pi(b)\varphi| = \sup_{\|\pi(a)*\varphi\|\leqq 1} |\pi(b)(\pi(a)*\varphi)| .$$

This shows that q maps $\pi(A)$ isometrically into $A^{**}/\mathscr{T}_l$. The same argument holds for the mapping of $\pi(A)$ to $A^{**}/\mathscr{T}_r$ (resp. $A^{**}/\mathscr{T}$). Henceforth we do not distinguish the elements between $\pi(A)$ in A^{**} and $q(\pi(A))$ in $A^{**}/\mathscr{T}_l$ (resp. in $A^{**}/\mathscr{T}_r$, $A^{**}/\mathscr{T}$) if there is no any confusion.

Since $\mathscr{T}_r$ and $\mathscr{T}$ are two-sided ideals of A^{**}, the space $B = A^{**}/\mathscr{T}_r$ (resp. $A^{**}/\mathscr{T}$) is a Banach algebra. Hence the right (resp. two-sided) multipliers of A can be embedded in B as an idealizer. Thus if set

$$M_r^B(A) = \{X \in B \mid \pi(a)*X \in \pi(A) \text{ for all } a \in A\}$$

$$M^B(A) = \{X \in B \mid L_X(\pi(a)),\ R_X(\pi(a)) \text{ in } \pi(A) \text{ for all } a \in A\}\ ,$$

then we have

COROLLARY 3.4. $M_r^B(A) = q(M_r'(A))$ (resp. $M^B(A) = q(M'(A))$).

PROOF. By definition, it is clear that $q(M_r'(A)) \subset M_r^B(A)$. We need only to show that $M_r^B(A) \subset q(M_r'(A))$. This fact is immediately, indeed for any $X \in M_r^B(A)$, $\pi(a)*X \in \pi(A)$ for any $a \in A$, thus there is an operator T on A such that $\pi(a)*X = \pi(Ta)$. Evidently, $T \in M_r(A)$ and hence there exists $F \in M_r'(A)$ such that

$$q(\pi(a))*X = q(\pi(Ta)) = q(\pi(a)*F) = q(\pi(a))*qF\ .$$

Hence $X = qF \in q(M_r'(A))$. The same argument holds for two-sided multipliers. q.e.d.

Note that if A is a subalgebra of a Banach algebra B, then in general,

$$M_l^B(A) = \{b \in B \mid ba \in A \text{ for all } a \in A\}$$

is a subset of $M_l(A)$, i.e., $M_l^B(A) \subset M_l(A)$. By this reason, the characterization of $M_l(A)$ as an idealizer of A in an algebra $B(\supset A)$ needs some additional conditions between A and B. Whence we established the Corollary 3.4.

The following theorem is essential in this section.

THEOREM 3.5. *Under the quotient mapping* $q\colon A^{**} \to A^{**}/\mathscr{T}_l$, *we have*

$$q(M_l'(A)) \cong M_l(A)\ .$$

PROOF. As $\mathscr{T}_l = \{F \in M_l'(A) \mid F*\pi(a) = 0 \text{ for all } a \in A\}$, it follows from Theorem 3.1 that the mapping $\theta_l\colon q(M_l'(A)) \to M_l(A)$ is an onto algebra isomorphism defined by the following identity

$$qF*\pi(a) = \pi(Ta)\ .$$

It remains to show that $\|qF\| = \|T\|$. Since A^{**} is a Banach algebra,

$$\| Tx \| = \| \pi(Tx) \| = \| qF*\pi(x) \| \leqq \| qF \| \| \pi(x) \| = \| qF \| \| x \| .$$

This implies $\| T \| \leqq \| qF \|$. On the other hand, for $F \in M_l'(A)$ and $\varphi \in A^*$,

$$\| F \| = \sup_{\|\varphi\| \leqq 1} | F(\varphi) | \geqq \sup_{\|\pi(a)*\varphi\| \leqq 1} | F(\pi(a)*\varphi) | = \| qF \| .$$

Let $\{a_\alpha\}$ be an approximate identity of A with $\| a_\alpha \| \leqq 1$. Then there is a subnet $\{a_\beta\}$ of $\{a_\alpha\}$ such that (cf. the proof of Theorem 3.1)

$$| F(\pi(a)*\varphi) | = \lim_\beta | \pi(Ta_\beta)(\pi(a)*\varphi) |$$

and

$$| \pi(Ta_\beta)(\pi(a)*\varphi) | \leqq \| Ta_\beta \| \| \pi(a)*\varphi \| \leqq \| T \| \| \pi(a)*\varphi \| .$$

Thus $\| F \| \leqq \| T \|$ and $\| qF \| \leqq \| T \|$, and hence $\| T \| = \| qF \|$. q.e.d.

For the right (or double) multipliers of A, we have the following

COROLLARY 3.6. *Let $B = A^{**}/\mathscr{T}_r$ (resp. $A^{**}/\mathscr{T}$) and $M_r^B = \{X \in B \mid \pi(a)*X$ in $\pi(A)$ for all $a \in A\}$ (resp. $M^B = \{X \in B \mid \pi(a)*X$ and $X*\pi(a)$ in $\pi(A)$ for $a \in A\}$). Then*

$$q(M_r'(A)) = M_r^B \cong M_r(A)$$

(resp. $q(M'(A)) = M^B \cong M(A)$, where $X \in M^B$ is identified by D_X).

PROOF. The proof follows from Corollary 3.4 and Theorem 3.5.

THEOREM 3.7. *Let A be a Banach algebra. If $\pi(A)$ is a left (resp. right, two-sided) ideal in A^{**}, then $\mathscr{T}_l$ is a two-sided ideal in A^{**} and*

$$M_l(A) \cong A^{**}/\mathscr{T}_l \text{ (resp. } M_r(A) \cong A^{**}/\mathscr{T}_r, M(A) \cong A^{**}/\mathscr{T}) .$$

PROOF. It is known that $\mathscr{T}_r$ and $\mathscr{T}$ are two-sided ideals. In addition, we note that if $\pi(A)$ is a left ideal in A^{**}, then $\mathscr{T}_l$ is also a two-sided ideal. In fact if $F \in A^{**}$, then for any $a \in A$, there is an element b in A such that $F*\pi(a) = \pi(b)$ and for $G \in \mathscr{T}_l$, we have $G*\pi(a) = 0$ for any $a \in A$. Therefore,

$$G*F*\pi(a) = G*\pi(b) = 0 \text{ implies } G*F \in \mathscr{T}_l .$$

This shows that $\mathscr{T}_l$ is a right ideal and hence it is a two-sided ideal. The conclusion follows from Theorem 3.5 (Corollary 3.6). q.e.d.

We ask that whether the kernel $\mathscr{T}_l = \theta_l^{-1}(0)$ ($\mathscr{T}_r = \theta_r^{-1}(0)$, $\mathscr{T} = \theta^{-1}(0)$) is a zero ideal of A^{**}. It is not hard to see that if $\{a_\alpha\}$ is an approximate identity of A and if $\pi(a_\alpha)*\varphi \to \varphi$ in A^* or $\varphi*a_\alpha \to \varphi$ in A^* with respect to the norm topology, then the space spanned by $\{\pi(a)*\varphi \mid \varphi \in A^*, a \in A\}$ or $\{\varphi*a \mid a \in A, \varphi \in A^*\}$ is dense in A^*. Thus $\mathscr{T}_l$ and $\mathscr{T}_r$ are zero ideals in A^{**}. Therefore we have

COROLLARY 3.8. *Let* $\{a_\alpha\}$ *be an approximate identity of* A. *If* $\pi(a_\alpha)*\varphi \to \varphi$ (*or* $\varphi * a_\alpha \to \varphi$) *in* A^* *with respect to the norm topology, then*

$$M_l'(A) \cong M_l(A) \text{ (or } M_r'(A) \cong M_r(A), M'(A) \cong M(A)) .$$

If the multiplication in A is Arens regular, then $F*G = G\circ F$ for any F, G in A^{**} and hence the product $F*G$ is continuous in G for fixed F and in F for fixed G. It can be shown then that $\mathscr{T}_l = \mathscr{T} = \mathscr{T}_r = (0)$, and we have

THEOREM 3.9. *Let* A *be a Banach algebra. If the multiplication in* A *is Arens regular, then*

$$M_l'(A) \cong M_l(A), M_r'(A) \cong M_r(A) \quad \text{and} \quad M'(A) \cong M(A) .$$

PROOF. Let $\{a_\alpha\}$ be an approximate identity of A with $\| a_\alpha \| \leqq 1$. Then $\{\pi(a_\alpha)\}$ is bounded in A^{**} and so there is a subnet $\{\pi(a_\beta)\}$ convergent to $I \in A^{**}$ in $\sigma(A^{**}, A^*)$ topology. Thus for $F \in A^{**}$, $\pi(a_\beta)*F = F\circ\pi(a_\beta) \to I*F = F\circ I = F$ in $\sigma(A^{**}, A^*)$ topology. The same result holds:

$$\pi(a_\beta)\circ F = F*\pi(a_\beta) \to I\circ F = F*I = F \text{ in } \sigma(A^{**}, A^*) \text{ topology} .$$

Hence

$$I*F = F*I = F = I\circ F = F\circ I .$$

Now for $F \in M_r'(A)$ (or $M_l'(A)$), we have

$$(\pi(a_\beta)*F)\varphi = \pi(a_\beta)(F*\varphi) \to I(F*\varphi) = F(\varphi)$$

and

$$(F*\pi(a_\beta))\varphi = (\pi(a_\beta)\circ F)\varphi = \pi(a_\beta)(F\circ\varphi) \to I(F\circ\varphi) = F(\varphi) .$$

Therefore

$$\pi(a_\beta)*F = 0 \text{ if and only if } I*F = F = 0$$

and

$$F*\pi(a_\beta) = 0 \text{ if and only if } I\circ F = F*I = F = 0 .$$

Then it follows from Theorem 3.5 and Corollary 3.6 that the theorem is proved. q.e.d.

By a consequence of Theorem 3.7 and the above theorem, we obtain the following

COROLLARY 3.10. *Suppose that* $\pi(A)$ *is a two-sided ideal in* A^{**} *and that the multiplication in* A *is Arens regular. Then*

$$M_l(A) \cong M_r(A) \cong M(A) \cong A^{**} .$$

REMARK 3.1. In the proofs of above theorems, sometime we need only the existence of the weak left (or right) approximate identity in A, that is there is a net $\{a_\alpha\}$ in A such that $\varphi(a_\alpha x) \to \varphi(x)$ (or $\varphi(x a_\alpha) \to \varphi(x)$) for $\varphi \in A^*$. But this is not essential for the arguments in our purpose. Hence throughout we assume that the Banach algebra A has an approximate identity for simply.

REMARK 3.2. From Tomiuk and Wong [10; Theorem 5.1] and Wong [11; Theorem 3.2] to Wong [12; Theorem 2.2] proved the fact that for a B^*-algebra A, $\pi(A)$ is a closed two-sided ideal of A^{**} if and only if A is a dual algebra and A is a dual algebra if and only if $M(A) \cong A^{**}$. It is known that every B^*-algebra A has an approximate identity (see Rickart [8; Theorem 4.8.14]), and the multiplication in A is Arens regular (see Civin and Yood [4; Theorem 7.1]). Hence the result $M(A) \cong A^{**}$ of Wong and Tomiuk is a special case of Corollary 3.10.

The following paragraph contains some applications of section 3.

4. Extension of multiplier algebras.

THEOREM 4.1. *Let B be a Banach algebra with approximate identity which is contained in its Banach subalgebra A. Then*

$$M_l(A) \subset M_l(B),\ M_r(A) \subset M_r(B) \quad \textit{and} \quad M(A) \subset M(B)\ .$$

PROOF. Since A is a Banach subalgebra of B, $A^{**} \subset B^{**}$ with Arens multiplication. By Theorem 3.5 (and Corollary 3.6), we have

$$q(M_l'(A)) \cong M_l(A)\ , \qquad q(M_l'(B)) \cong M_l(B)\ .$$

It is sufficient to show

$$q(M_l'(A)) \subset q(M_l'(B))\ .$$

Since $A \subset B$ and $\mathscr{T}_l(A) = \{F \in M_l'(A) \mid F*\pi(a) = 0, a \in A\}$, $\mathscr{T}_l(B) = \{F \in M_l'(B) \mid F*\pi(b) = 0, b \in B\}$, we have

$$\mathscr{T}_l(B) \subset \mathscr{T}_l(A) \quad \text{and} \quad A^{**}/\mathscr{T}_l(A) \subset B^{**}/\mathscr{T}_l(B)\ .$$

For $T \in M_l(A)$ there is a unique $F \in q(M_l'(A))$ such that

$$F*\pi(a) = \pi(Ta) \text{ for any } a \in A\ .$$

Let $\{a_\alpha\}$ be an approximate identity of B and it is contained in A. Then for any $b \in B$,

$$\pi(a_\alpha)*\pi(b) = \pi(a_\alpha b) \to \pi(b) \text{ uniformly (in } B\text{-norm)}\ .$$

Since B^{**} is a Banach algebra, it follows that

$$F*\pi(b) = F*\lim_\alpha (\pi(a_\alpha)*\pi(b)) = \lim_\alpha F*\pi(a_\alpha)*\pi(b)$$

with respect to the uniform topology in B^{**}. Since $F*\pi(a_\alpha)$ in $\pi(A)$, $F*\pi(a_\alpha)*\pi(b) \in \pi(B)$. As $\pi(B)$ is a closed subalgebra of B^{**}, $F*\pi(b) \in \pi(B)$. This shows that $F \in q(M_l'(B))$, and hence there exists T' in $M_l(B)$ such that

$$F*\pi(b) = \pi(T'b) \text{ for all } b \in B .$$

Evidently, the restriction $T'|_A = T$. Hence $M_l(A) \subset M_l(B)$. Similarly $M_r(A) \subset M_r(B)$ and $M(A) \subset M(B)$. q.e.d.

Let B be a Banach algebra with identity. Then the left regular representation $a \to L_a$ of B to $M(B)$ is an isometric isomorphism. That is $B \cong M_l(B)$. By the same reason after Corollary 3.4, we will establish the following

THEOREM 4.2. *Let B be a Banach algebra with identity e and A a Banach subalgebra of B with approximate identity $\{a_\alpha\}$. Suppose that for any $b \in B$, $bA = 0$ implies $b = 0$, and that $a_\alpha \to e$ in B with respect to the strict topology. Then*

$$M_l^B(A) \cong M_l(A) ,$$

where $M_l^B(A) = \{b \in B \mid ba \in A \text{ for all } a \in A\}$.

PROOF. Evidently $M_l^B(A) \subset M_l(A) \cong q(M_l'(A))$ (cf. Theorem 3.4), thus there is an isometric embedding $\tilde{\pi}$ of $M_l^B(A)$ into $A^{**}/\mathscr{T}_l$. Hence $\tilde{\pi}(M_l^B(A)) \subset q(M_l'(A))$. We want to show

$$\tilde{\pi}(M_l^B(A)) \supset q(M_l'(A)) .$$

For any $F \in q(M_l'(A))$, there is a unique $T \in M_l(A)$ such that $F*\pi(a) = \pi(Ta)$. By Theorem 4.1, we see that $T \in M_l(B)$. It follows that there is an element b in B such that $T = L_b$ and

$$F*\pi(a) = \pi(Ta) = \pi(L_b a) = \pi(ba) = \tilde{\pi}(b)*\pi(a)$$

for all $a \in A$. By assumption, the cancellation law holds, and so $F = \tilde{\pi}(b)$, shows $q(M_l'(A)) \subset \tilde{\pi}(M_l^B(A))$. Consequently $q(M_l'(A)) = \tilde{\pi}(M_l^B(A))$. The isometry between $M_l(A)$ and $M_l^B(A)$ follows from Theorem 3.5. Therefore

$$M_l(A) \cong M_l^B(A) . \qquad \text{q.e.d.}$$

In above theorem, if A is a left ideal of B, then $M_l^B(A) = B$ and hence the following corollary holds.

COROLLARY 4.3. *Under the assumption of Theorem 4.2, if we assume further that A is a left ideal of B, then $B \cong M_l(A)$. The same conclusion holds for right and two-sided multipliers.*

5. Homomorphism extension of Banach algebras. In this section the main task is to study the homomorphism extension of Banach algebras. Akemann, Pedersen and Tomiyama [1; Theorem 4.2] proved that there is a surjective homomorphism extension from a separable C^*-algebra to the multiplier algebra. For the isomorphism extension, one can refer to Johnson [5; Theorem 20] and Rudin [9; Theorem 4.6.4]. In [9] it was proved that the isomorphism extension is surjective for the case of group algebra. We will prove in Theorem 5.2 later by applying the idealizer to discuss the homomorphism extension in general Banach algebras.

The following lemma is immediately.

LEMMA 5.1. *Suppose that ρ is a continuous surjective homomorphism of a Banach algebra A to a Banach algebra B. If A has an approximate identity then B has an approximate identity.*

THEOREM 5.2. *Let A and B be two Banach algebras and A has an approximate identity. Suppose that there is a continuous surjective homomorphism ρ of A to B. Then ρ can be extended to a homomorphism $\bar{\rho}$ of $M_l(A)$ into $M_l(B)$. This $\bar{\rho}$ is continuous with respect to the strict topologies of $M_l(A)$ and $M_l(B)$. The same conclusion holds for right and two-sided multiplier algebras.*

PROOF. We denote by π_A, π_B the cannonical embeddings of A and B into A^{**} and B^{**} respectively. It follows from Civin and Yood [4; Theorem 6.1] that the continuous homomorphism ρ of A to B can be extended to be a homomorphism $\tilde{\rho}$ of A^{**} into B^{**}. By Lemma 5.1, as A has an approximate identity, so does B. Then the restriction $\tilde{\rho}|_{M_l'(A)}$ of $\tilde{\rho}$ is a homomorphism of $M_l'(A)$ into B^{**}. It is not hard to show that $\tilde{\rho}(M_l'(A)) \subset M_l'(B)$.

We have to show that $\tilde{\rho}$ deduces to a homomorphism ρ' of $A^{**}/\mathscr{T}_l(A)$ into $B^{**}/\mathscr{T}_l(B)$ and then the restriction $\rho'|_{q(M_l'(A))}$ is a homomorphism of $q(M_l'(A))$ into $q(M_l'(B))$. To this end it sufficies to show $\tilde{\rho}(\mathscr{T}_l(A)) \subset \mathscr{T}_l(B)$. For any $F \in \mathscr{T}_l(A)$, we have $F*\pi(a) = 0$ for any $a \in A$. Thus for any $b \in B$ there is $a \in A$, $b = \rho(a)$ such that

$$\tilde{\rho}F*\pi_B(b) = \tilde{\rho}F*\pi_B(\rho a) = \tilde{\rho}(F*\pi_A(a)) = 0 .$$

Hence $\tilde{\rho}F \in \mathscr{T}_l(B)$ shows $\tilde{\rho}(\mathscr{T}_l(A)) \subset \mathscr{T}_l(B)$. Therefore, $\tilde{\rho}$ induces a homomorphism ρ' of $A^{**}/\mathscr{T}_l(A)$ into $B^{**}/\mathscr{T}_l(B)$. It follows immediately that $\rho'(q(M_l'(A))) \subset q(M_l'(B))$. Since $q(M_l'(A)) \cong M_l(A)$ by Theorem 3.5, it follows that there is a homomorphism $\bar{\rho}$ which extends ρ to $M_l(A)$ into $M_l(B)$ such that

$$\rho(Ta) = \bar{\rho}(T)\rho(a) \text{ for } T \in M_l(A) \text{ and } a \in A .$$

Finally we show the continuity of $\bar{\rho}$ with respect to strict topology. If $\{T_\alpha\}$ is a net in $M_l(A)$ and $T_\alpha \to T$ strictly, then $T_\alpha a \to Ta$ uniformly in A for any $a \in A$, and hence $\rho(T_\alpha a) \to \rho(Ta)$ uniformly in B since ρ is continuous. Therefore

$$\| \rho(T_\alpha a) - \rho(Ta) \| = \| \bar{\rho}(T_\alpha)\rho a - \bar{\rho}(T)\rho a \| \to 0 .$$

This shows that $\bar{\rho}(T_\alpha) \to \bar{\rho}(T)$ strictly in $M_l(B)$ and hence $\bar{\rho}$ is continuous with respect to the strict topologies in $M_l(A)$ and $M_l(B)$. q.e.d.

We note that the continuous homomorphism of A onto B can not be extended in general to be a surjective homomorphism of $M_l(A)$ to $M_l(B)$. For a counter example one can refer to the fact below the proof of Theorem 4.2 in Akemann, Pedersen and Tomiyama [1]. In [1] they proved the surjective extension of homomorphism in the case of separable C^*-algebra (see [1; Theorem 4.2]). We remark here that if the homomorphism ρ is an isomorphism, then the above theorem is proved directly by Johnson [5; Theorem 20]. Furthermore, if the surjective isomorphism of A to B is bicontinuous then one can extend it to a surjective isomorphism of $M_l(A)$ to $M_l(B)$. In Rudin [9; Theorem 4.6.4] if $A = L^1(G_1)$, $B = L^1(G_2)$ are group algebras and $M_l(A) = M(G_1)$, $M_l(B) = M(G_2)$ the bounded regular Borel measure algebras on locally compact abelian groups $G_i (i = 1, 2)$, then the isomorphism of A onto B has a unique surjective isomorphism extension to $M_l(A)$ onto $M_l(B)$. Note that $L^1(G_i)$ are semisimple commutative Banach algebras and hence the onto isomorphism is bicontinuous.

The following theorems are important for the representations of Banach algebra as bounded linear operators in Hilbert space.

THEOREM 5.3. *Let A and B be Banach algebras with approximate identities. Let E be a Banach space such that the dual E^* of E is isometrically isomorphic to the Banach algebra B. If ρ is a continuous homomorphism of A into B, then ρ can be extended to a homomorphism $\bar{\rho}$ of $M_l(A)$ into B. This $\bar{\rho}$ is continuous strongly in B with respect to the strict topology in $M_l(A)$.*

For the proof of this theorem we need the following lemma.

LEMMA 5.4. *Suppose that α is an embedding of E into $E^{**}(= B^*)$ and that E^0 is the polar of E* (i.e., *$\alpha(E)$ in B^*) in B^{**}. Then*

$$B^{**} = \pi(B) \oplus E^0 .$$

PROOF. The proof is immediately. For any $\Phi \in B^{**}(\supset E^0)$, set $b =$

$\Phi|_E$, the restriction of Φ on E, we see that $b \in B$. It follows that

$$\Psi = \Phi - \pi(b) \in E^0 \quad \text{or} \quad \Phi = \pi(b) + \Psi$$

for $\pi(b) \in \pi(B)$ and $\Psi \in E^0$. This expression is unique. In fact, if

$$\Phi = \pi(b_1) + \Psi_1 = \pi(b_2) + \Psi_2 ,$$

then $\pi(b_1 - b_2) = \Psi_2 - \Psi_1 \in E^0$. This implies that $b_1 - b_2 = 0$ or $\pi(b_1) = \pi(b_2)$ and $\Psi_2 = \Psi_1$, and $\pi(B) \cap E^0 = \{0\}$. Therefore

$$B^{**} = \pi(B) \oplus E^0 . \qquad \text{q.e.d.}$$

PROOF OF THEOREM 5.3. For $a, b \in B$ and $x \in E$, it is naturally to take

$$ab(x) = a(bx) .$$

Let α be the embedding of E into $E^{**} = B^*$. Then the transposed mapping α^t of α is a bounded linear transformation of B^{**} onto B. If E^0 is the polar of E in B^{**}, then for any $G \notin E^0$ in B^{**}, there exists an element b in B such that the restriction $G|_E = b$ and $G - \pi(b) \in E^0$ by Lemma 5.4. Evidently if $f \in E^0$, then $f|_E = 0$. That is $\operatorname{Ker}(\alpha^t) = E^0$. It can be shown that E^0 is a two-sided ideal in B^{**}. Indeed if $G \in B^{**}$, $f \in E^0$ then for any $x \in E$, we have

$$(G*f)(\alpha(x)) = \alpha^t(G*f)(x) = \alpha^t G(\alpha^t f x) = 0$$
$$\Longrightarrow G*f \in E^0$$

and

$$(f*G)(\alpha(x)) = \alpha^t f(\alpha^t G \cdot x) = 0$$
$$\Longrightarrow f*G \in E^0 .$$

Therefore the quotient algebra B^{**}/E^0 is isomorphic to B.

If $\tilde{\rho}$ is the extension homomorphism of ρ to A^{**} into B^{**}, then the restriction $\tilde{\rho}|_{M_l'(A)}$ is a homomorphism into $M_l'(B)$. Indeed for $F \in M_l'(A)$, there is a $T \in M_l(A)$ such that $F*\pi(a) = \pi(Ta)$ for any $a \in A$. This implies $\tilde{\rho}F*\tilde{\rho}\pi(a) = \tilde{\rho}F*\pi(\rho a) \in \pi(B)$. Hence $\tilde{\rho}F \in M_l'(B)$. We have to show $\tilde{\rho}(\mathscr{T}_l(A)) \subset E^0$.

For $F \in \mathscr{T}_l(A)$, we have $F*\pi(a) = 0$ for all $a \in A$, and $\tilde{\rho}F*\pi(\rho a) = 0$. But

$$(\tilde{\rho}F*\pi(\rho a))(\alpha(x)) = \tilde{\rho}F(\pi(\rho a)*\alpha(x)) = \alpha^t(\tilde{\rho}F)(\rho a \cdot x)$$
$$= \tilde{\rho}F(\alpha(\rho a \cdot x)) = 0$$

for any $x \in E$ and $a \in A$. This shows that $\tilde{\rho}F \in (\rho(A)E)^0$. If $\rho(A)E$ is dense in E, then $\tilde{\rho}F \in E^0$. Otherwise, we can put that the extension $\tilde{\rho}F$ vanishes outside $\rho(A)E$ in E $(\rho(A)E \subset E)$, and then $\tilde{\rho}F \in E^0$. Therefore, there is a homomorphism ρ' which maps

$$A^{**}/\mathscr{T}_l(A) \quad \text{into} \quad (B^{**}/(\rho(A)E)^0 \subset) B^{**}/E^0 \cong B \, .$$

The restriction $\rho'|_{q(M_l'(A))}$ which maps $q(M_l'(A)) \cong M_l(A)$ into B and hence there exists a homomorphism $\bar{\rho}$ which extends ρ to $M_l(A)$ into B. It can be identified by

$$\rho(Ta) = \bar{\rho}(T)\rho(a) \text{ for } T \in M_l(A) \text{ and } a \in A \, .$$

Finally, we show the continuity of $\bar{\rho}$.

If $\{T_\alpha\}$ is a net in $M(A)$ and $T_\alpha \to 0$ in strict topology, then $T_\alpha a \to 0$ uniformly in A for $a \in A$, and $\rho(T_\alpha a) \to 0$ in B-norm since ρ is continuous. Therefore, $\rho(T_\alpha a) = \bar{\rho}(T_\alpha)\rho(a)$ converges to 0 in B, hence $\bar{\rho}(T_\alpha) \to 0$ strongly in B. q.e.d.

In this theorem we have assumed that B is a dual of a Banach space E, if such B is also a B^*-algebra, then it is called a W^*-algebra. We now turn to study the homomorphism of involutive algebra. A homomorphism is said to be a **-homomorphism* if it commutes with involution *. If A is a B^*-algebra, the Arens multiplication in A is regular, it follows that the involution in A can be extended to an involution in A^{**} (see Civin and Yood [4; p. 868]), and hence the *-homomorphism of A to another B^*-algebra B can be extended to a *-homomorphism of A^{**} into B^{**}. In *-algebra A, we are dealing only with the multiplier algebra $M(A)$ in which the involution in $M(A)$ is defined to be $(T_l, T_r)^* = T^* = (T_r^*, T_l^*)$ with $T_i^*(x) = (T_i(x^*))^*$ for each $T \in M(A)$ and $x \in A$. Note that if $T \in M_l(A)$, then $T^* \in M_r(A)$ and vice versa.

As a corollary of Theorem 5.3, we have

COROLLARY 5.5. *Let A be a B^*-algebra and B a W^*-algebra. Then a *-homomorphism ρ of A into B has an extension to a *-homomorphism $\bar{\rho}$ of $M(A)$ into B, where $\bar{\rho}(D_a) = \rho(a)$ for any $a \in A$.*

Applying the consequence of above theorems, we will turn to discuss the representation of algebra as linear operators in a Hilbert space, and we will remark that the relationship between the representations of an algebra as linear operators in Hilbert space and corresponding representations of the multiplier algebra. We mention firstly some terminology which is used in describing representations of algebras as operators on Hilbert space (see [8]). Let H be a Hilbert space and ρ be a continuous algebraic homomorphism of an algebra A into $\mathscr{B}(H)$, the bounded linear operators in H, then ρ is called a *representation*. If A has an involution * and * commutes with ρ then the representation is called a **-representation*. If the linear subspace generated by $\{\rho(a)\xi \mid a \in A, \xi \in H\}$ is dense in H, then the representation is *essential*.

Since $\mathscr{B}(H)$ is a dual of Banach space, thus in Theorem 5.3 by setting $B = \mathscr{B}(H)$, we see that ρ is a representation, and Johnson [5; Theorem 21] can be proved by Theorem 5.3. We restate it as follows:

Let A be a Banach algebra with approximate identity. Then the representation ρ of A has an extension to a representation $\bar{\rho}$ of $M_l(A)$. If ρ is essential, then the extension $\bar{\rho}$ is unique and essential.

To prove this fact, we have only to show that ρ is continuous with respect to the operator norms of $M_l(A)$ and $\mathscr{B}(H)$. Let $\|T_\alpha\| \to 0$ in $M_l(A)$. Then $T_\alpha a \to 0$ in A uniformly with respect to a. For any $\xi \in H$,

$$\rho(T_\alpha a)\xi = \bar{\rho}(T_\alpha)\rho(a)\xi = \bar{\rho}(T_\alpha)\xi' \to 0$$

in H uniformly with respect to a. If $\xi' \in \rho(A)H$ with $\|\xi'\| \leqq 1$, then $\|\bar{\rho}(T_\alpha)\| \to 0$. If $\xi' \in (\rho(A)H)^\perp$, we then define $\bar{\rho}(T_\alpha) = 0$. Hence $\bar{\rho}$ is continuous with respect to the operator norms of $M_l(A)$ and $\mathscr{B}(H)$.

If ρ is essential, then $\rho(A)H$ is dense in H, and for $a \in A$, $T \in M_l(A)$,

$$\{\bar{\rho}(T)\xi \mid \xi \in H\} \supset \{\rho(a)\xi \mid \xi \in H\} ,$$

is an essential representation.

Since $\mathscr{B}(H)$ is a W^*-algebra, thus if $B = \mathscr{B}(H)$ in Corollary 5.5, we have:

Let A be a B^*-algebra and ρ be a *-representation of A. Then ρ has an extension to a *-representation $\bar{\rho}$ of $M(A)$, where $\bar{\rho}(D_a) = \rho(a)$ for any $a \in A$. If ρ is essential, then the extension ρ is unique and essential.

This statement is proved in Johnson [5; Theorem 23] for A as any Banach *-algebra with approximate identity. In our case, we need that an involution in A is extended to an involution in A^{**}, therefore in the context we assumed that A is a B^*-algebra. Actually it can be assumed that A is a Banach *-algebra with approximate identity and the multiplication in A is regular, then the above statement is also valid.

6. Determination of multiplier algebras. This section contains an application of Theorem 5.2. Let X be a locally compact Hausdorff space. For each $t \in X$, $A(t)$ denotes a Banach algebra. Let $\{X, A(t)\}$ be a fibred space X. Assume that there is a family $\mathscr{F}$ of cross sections of $\{X, A(t)\}$ such that it forms an algebra under pointwise operations and satisfies the following conditions:

(0) There exists a net $\{a_\alpha\}_{\alpha \in I}$ in $\mathscr{F}$ such that for any $b \in \mathscr{F}$ and any $\varepsilon > 0$, there is $\beta_0 \in I$ such that

$$\sup_{t \in X} \| a_\beta(t)b(t) - b(t) \| < \varepsilon \quad \text{whenever} \quad \beta > \beta_0 .$$

(i) The set $\mathscr{F}(t) = \{a(t) \mid a \in \mathscr{F}\}$ is dense in $A(t)$ for each t.

(ii) The function $t \to \| a(t) \|$ belongs to $C_0(X)$ for all a in $\mathscr{F}$.

A cross section x of $\{X, A(t)\}$ is said to be *continuous* at $t_0 \in X$ with respect to $\mathscr{F}$ if for any $\varepsilon > 0$ there exists a neighborhood $N(t_0)$ and a in $\mathscr{F}$ such that

$$\| x(t) - a(t) \| < \varepsilon \text{ whenever } t \in N(t_0) .$$

Denote by $C_{\mathscr{F}}(X, A(t))$ the set of all continuous cross sections of $\{X, A(t)\}$ vanishing at infinity. Then it will form a Banach algebra with approximate identity under the supremum norm. An interesting special case arises when all fibres $A(t)$ are isomorphic to the same Banach algebra A, in which the family $\mathscr{F}$ is taken to be all norm continuous A-valued functions on X vanishing at infinity and we have a trivial fibred space $C_0(X, A)$.

Now consider the fibred space $\{X, \mathscr{B}(A(t))\}$, where $\mathscr{B}(A(t))$ denotes the bounded operators of $A(t)$. A cross section f in the fibred space $\{X, \mathscr{B}(A(t))\}$ is said to be *strictly continuous* at t_0 with respect to $\mathscr{F}$ if for every $\varepsilon > 0$ and each a in $\mathscr{F}$ there is an element b in $\mathscr{F}$ and a neighborhood $N(t_0)$ such that

$$\| (f(t) - b(t))a(t) \| < \varepsilon \text{ whenever } t \in N(t_0) .$$

Denote by $B_{\mathscr{F}}(X, \mathscr{B}(A(t)))$ the set of all bounded strictly continuous cross sections in $\{X, \mathscr{B}(A(t))\}$ with respect to $\mathscr{F}$.

By these preparation, we will characterize the multiplier algebra $M_l(C_{\mathscr{F}}(X, A(t)))$ as a space of bounded strictly continuous cross sections in $\{X, M_l(A(t))\}$. The case in C^*-algebra is discussed by Akemann, Pedersen and Tomiyama [1; Theorem 3.3 and Corollary 3.4]. Our result is a generalization of [1] from C^*-algebra to any general Banach algebra with approximate identity for which Theorem 5.2 is available. We need three lemmas.

LEMMA 6.1. *$C_{\mathscr{F}}(X, A(t))$ is a Banach algebra with approximate identity under supremum norm and pointwise operations.*

PROOF. For any x, y in $C_{\mathscr{F}}(X, A(t))$, it is not hard to show that xy belongs to $C_{\mathscr{F}}(X, A(t))$ and $\| xy \|_\infty \leqq \| x \|_\infty \| y \|_\infty$. The completeness of $C_{\mathscr{F}}(X, A(t))$ is immediate. Indeed if $\{x_n\}$ is a Cauchy sequence in $C_{\mathscr{F}}(X, A(t))$, then $\{x_n(t)\}$ converges to $x(t)$ in $A(t)$ for any t. Thus for any $\varepsilon > 0$ there exists n_0 such that

$$\| x_n(t) - x(t) \| < \varepsilon/2 \text{ whenever } n \geqq n_0 .$$

On the other hand, for any $t_0 \in X$ and for the given $\varepsilon > 0$, there exists a neighborhood $N(t_0)$ and corresponds a cross section a_{n_0} in $\mathscr{F}$ such that

$$\| x_{n_0}(t) - a_{n_0}(t) \| < \varepsilon/2 \quad \text{for} \quad t \in N(t_0) .$$

Then $\| x(t) - a_{n_0}(t) \| < \varepsilon$ for $t \in N(t_0)$. This shows that the cross section x is continuous. Since $\| x(t) \| \in C_0(X)$, $\{x_\alpha\}$ converges to x in $C_{\mathscr{F}}(X, A(t))$.

Finally we show $C_{\mathscr{F}}(X, A(t))$ has an approximate identity. Let a be any element in $C_{\mathscr{F}}(X, A(t))$ and ε any positive number. Then for any $t_0 \in X$ there is a neighborhood $N(t_0)$ and exists b in $\mathscr{F}$ such that

$$\| a(t) - b(t) \| < \varepsilon/8 \text{ whenever } t \in N(t_0) .$$

Since $\{N(t_0)\}_{t_0 \in X}$ is an open covering of X, thus if K is a compact subset of X such that

$$\| a(t) \| < \varepsilon/4 \text{ when } t \in X - K ,$$

then there is a finite subcovering $\{N(t_1), \cdots, N(t_n)\}$ such that $\bigcup_{i=1}^n N(t_i) \supset K$. Now for any $t \in K$, there is $N(t_i) \ni t$ for some i and $b_1 \in \mathscr{F}$ such that

$$\| a(t) - b_i(t) \| < \varepsilon/8 .$$

By condition (0), for any b_i in $\mathscr{F}$, there is β_i such that

$$\| a_\beta(t) b_i(t) - b_i(t) \| < \varepsilon/4 \text{ whenever } \beta > \beta_i ,$$

and assuming that $\| a_\alpha(t) \| \leqq 1$ for all $\alpha \in I$, then for $\beta > \beta_i$,

$$\begin{aligned} \| a_\beta(t) a(t) - a(t) \| &\leqq \| a_\beta(t) b_i(t) - b_i(t) \| + \| a_\beta(t)(b_i(t) - a(t)) \| \\ &\quad + \| b_i(t) - a(t) \| \\ &< \varepsilon/4 + \varepsilon/8 + \varepsilon/8 \\ &= \varepsilon/2 . \end{aligned}$$

Consequently, for any $t \in X$ and $\beta > \beta_1, \beta_2, \cdots, \beta_n$,

$$\begin{aligned} \sup_{t \in X} \| a_\beta(t) a(t) - a(t) \| &\leqq \sup_{t \in K} \| a_\beta(t) a(t) - a(t) \| + \sup_{t \in X-K} \| a_\beta(t) a(t) - a(t) \| \\ &< \varepsilon/2 + 2 \sup_{t \in X-K} \| a(t) \| \\ &< \varepsilon/2 + \varepsilon 2 = \varepsilon . \end{aligned}$$

q.e.d.

LEMMA 6.2. *$B_{\mathscr{F}}(X, M_l(A(t)))$ is a Banach algebra under pointwise operations and supremum norm.*

PROOF. For any F, G in $B_{\mathscr{F}}(X, M_l(A(t)))$, we want to show that FG belongs to $B_{\mathscr{F}}(X, M_l(A(t)))$. To this end, we show first that $B_{\mathscr{F}}$ is a space of linear mappings of $C_{\mathscr{F}}$ into $C_{\mathscr{F}}$, and hence it follows that $B_{\mathscr{F}}$ will be characterized as a left multiplier algebra of $C_{\mathscr{F}}$ in next lemma.

For any $F \in B_{\mathscr{F}}(X, M_l(A(t)))$ and $a \in C_{\mathscr{F}}(X, A(t))$ such that for any $\varepsilon > 0$, $t_0 \in X$, there is a neighborhood $N(t_0)$ and $b \in \mathscr{F}$ such that

$$\| Fa(t) - ba(t) \| = \| (F(t) - b(t))a(t) \| < \varepsilon/2$$

for $t \in N(t_0)$. Since $ba \in C_{\mathscr{F}}(X, A(t))$ and a vanishing at infinity, we obtain $Fa \in C_{\mathscr{F}}(X, A(t))$.

Next as $G \in B_{\mathscr{F}}(X, M_l(A(t)))$, thus for $t_0 \in X$, $\varepsilon > 0$ and $a \in \mathscr{F}$, there is $c \in C_{\mathscr{F}}(X, A(t))$ and a neighborhood $N(t_0)$ such that

$$\| (G(t) - c(t))a(t) \| < \varepsilon/\| F \|_\infty \text{ whenever } t \in N(t_0) .$$

Then

$$\| (FG(t) - F(t)c(t))a(t) \| < \varepsilon \text{ whenever } t \in N(t_0) .$$

Since we have shown that $Fc \in C_{\mathscr{F}}$, it follows that $FG \in B_{\mathscr{F}}(X, M_l(A(t)))$.

The condition $\| FG \|_\infty \leqq \| F \|_\infty \| G \|_\infty$ is immediately, and the completeness of $B_{\mathscr{F}}(X, M_l(A(t)))$ follows from the completeness of $M_l(A(t))$ for any t. Hence $B_{\mathscr{F}}(X, M_l(A(t)))$ is a Banach algebra with respect to the supremun norm topology. q.e.d.

LEMMA 6.3. *The algebra $B_{\mathscr{F}}(X, M_l(A(t)))$ can be isometrically embedded in $M_l(C_{\mathscr{F}}(X, A(t)))$.*

PROOF. For F in $B_{\mathscr{F}}(X, M_l(A(t)))$ and $a \in C_{\mathscr{F}}(X, A(t))$, it has been shown in Lemma 6.2 that $Fa \in C_{\mathscr{F}}(X, A(t))$. We identify F as an element T_F in $M_l(C_{\mathscr{F}}(X, A(t)))$ by

$$T_F y = Fy \quad \text{for all} \quad y \in C_{\mathscr{F}}(X, A(t)) .$$

Then $(T_F y)(t) = F(t)y(t)$ for all $t \in X$. It is immediate that $\| F \|_\infty = \| T_F \|$. Indeed,

$$\begin{aligned} \| T_F \| &= \sup_{\|y\|_\infty \leqq 1} \| T_F y \| = \sup_{\|y\|_\infty \leqq 1} \sup_{t \in X} \| T_F y(t) \| \\ &= \sup_{t \in X} \| F(t) \| = \| F \|_\infty . \end{aligned}$$

q.e.d.

THEOREM 6.4. $M_l(C_{\mathscr{F}}(X, A(t))) \cong B_{\mathscr{F}}(X, M_l(A(t)))$.

PROOF. Observe that Lemma 6.3, we have only to prove

$$M_l(C_{\mathscr{F}}(X, A(t))) \subset B_{\mathscr{F}}(X, M_l(A(t))) .$$

For any T in $M_l(C_{\mathscr{F}}(X, A(t)))$ and y in $C_{\mathscr{F}}(X, A(t))$, we have Ty in $C_{\mathscr{F}}(X, A(t))$. We will show that the identity

$$(Ty)(t) = F_T(t)y(t) \text{ for any } y \text{ in } C_{\mathscr{F}}(X, A(t)) \text{ and } t \in X ,$$

is well defined and then F_T defines a bounded strict continuous cross section in $\{X, M_l(A(t))\}$. To this end, for each t in X, consider a homomorphism ρ_t of $C_{\mathscr{F}}(X, A(t))$ onto $A(t)$ given by

$$\rho_t y = y(t) \text{ for } t \in X \text{ and } y \in C_{\mathscr{F}}(X, A(t)) .$$

By Theorem 5.2, we know that ρ_t has a homomorphism extension to their multiplier algebra such that

$$\rho_t(Ty) = \bar{\rho}_t(T)\rho_t(y) .$$

Define $\bar{\rho}_t(T) = F_T(t)$. Then $F_T(t) \in M_l(A(t))$ and F_T is a bounded function on X. Furthermore $\| T \| = \| F_T \|_\infty$. It remains to show that F_T is strictly continuous on X. Since F_T is bounded, we may assume that $\| F_T(t) \| \leqq 1$ for all $t \in X$. For x in $C_{\mathscr{F}}(X, A(t))$, $t_0 \in X$, $\varepsilon > 0$ we have $F_T x \in C_{\mathscr{F}}(X, A(t))$ and there is y in $\mathscr{F}$ and exists a neighborhood $N(t_0)$ such that

$$\| F_T x(t) - y(t) \| < \varepsilon \quad \text{for} \quad t \in N(t_0) .$$

On the other hand, the factorization property holds in a Banach algebra with a (bounded) approximate identity, thus for x, y in $\mathscr{F}(C_{\mathscr{F}}(X, A(t)))$ there is a in $\mathscr{F}(C_{\mathscr{F}}(X, A(t)))$ such that $y = ax$. Therefore for $t \in N(t_0)$,

$$\| F_T x(t) - y(t) \| = \| (F_T(t) - a(t))x(t) \| < \varepsilon .$$

Hence F_T is a bounded strictly continuous with respect to $\mathscr{F}$ in X.

q.e.d.

Now we turn to the case that as all $A(t)$ are isomorphic to the same Banach algebra A with approximate identity, we can specify $\mathscr{F}$ as the family of cross sections $\{af \mid a \in A, f \in C_0(X)\}$. Then $C_{\mathscr{F}}(X, A(t))$ coincides with $C_0(X, A)$.

Indeed for any $t_0 \in X$, there is a function $f \in C_0(X)$ such that $f(t) = 1$ in some neighborhood $N(t_0)$ of t_0 and $0 \leqq f \leqq 1$ on X since X is a locally compact Hausdorff space. Then for any $a \in A$, the mapping $f_a : t \longrightarrow af(t)$ of X into A defines a continuous A-valued function on X vanishing at infinity, i.e., $f_a \in C_{\mathscr{F}}(X, A(t)) \subset C_0(X, A)$. Conversely, for $a \in C_0(X, A)$, we see that a is continuous at any point $t_0 \in X$, thus for $\varepsilon > 0$, there is a neighborhood $N(t_0)$ such that

$$\| a(t) - a(t_0) \| < \varepsilon \quad \text{for} \quad t \in N(t_0) .$$

Choose $g \in C_0(X)$ such that $g = 1$ on $N(t_0)$ and $0 \leqq g \leqq 1$ on X, then we have

$$a(t_0)g \in \mathscr{F}, \text{ and } \| (a(t_0)g)(t) - a(t) \| < \varepsilon \text{ whenever } t \in N(t_0) .$$

Hence $a \in C_{\mathscr{F}}(X, A(t))$. Therefore under our specification of $\mathscr{F}$,

$$C_{\mathscr{F}}(X, A(t)) = C_0(X, A) .$$

This algebra $C_0(X, A)$ is isometric isomorphic to the tensor product

$C_0(X) \otimes_\lambda A$ with respect to the smallest cross norm λ.

It is immediate that the approximate identity of $C_0(X, A)$ satisfies the condition (0) provided A has a bounded approximate identity. Hence if we given a Banach algebra A with bounded approximate identity and let $B(X, M_l(A))$ denote the set of all bounded strictly continuous $M_l(A)$-valued functions on X, then we have the following

COROLLARY 6.5. $M_l(C_0(X, A)) \cong B(X, M_l(A))$.

PROOF. Let $F \in B(X, M_l(A))$ and $a \in C_0(X, A)$. It is not hard to show that Fa is continuous on X and vanishing at infinity, and so $Fa \in C_0(X, A)$. By definition, F defines a multiplier T in $M_l(C_0(X, A))$ and $\|T\| = \|F\|_\infty$.

Conversely, for $T \in M_l(C_0(X, A))$ and for any fixed a in A, one can choose $g \in C_0(X)$ such that $ag \in C_0(X, A)$, and hence $T(ag) \in C_0(X, A)$. By using Theorem 5.2, the following identity

$$T(ag)(t) = F_T(t)ag(t)$$

is well defined where F_T is a bounded $M_l(A)$-valued function on X. By the same argument of the proof in theorem, it is immediate that F_T is strictly continuous and $\|F_T\|_\infty = \|T\|$, i.e., $F_T \in B(X, M_l(A))$. q.e.d.

Note that the same conclusion holds for the right and two-sided multiplier algebras. The special case of Corollary 6.5 is that $M(C_0(X)) \cong C^b(X)$ for the commutative Banach algebra $C_0(X)$ of scalar-valued continuous function vanishing at infinity, where $C^b(X)$ denotes the bounded continuous functions on X.

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