Kareem *Iraqi Journal of Science, 2021, Vol. 62, No. 8, pp: 2694-2701*
DOI: 10.24996/ijs.2021.62.8.22

ISSN: 0067-2904

Multipliers of an AT-algebra

Fatema F. Kareem
Department of Mathematics, Ibn-Al-Haitham college of Education, University ofBaghdad, Iraq

Received: 9/8/2020 Accepted: 15/2/2021

Abstract
In this work, we present the notion of a multiplier on AT-algebra and investigate several properties. Also, some theorems and examples are discussed. The notions of the kernel and the image of multipliers are defined. After that, some propositions related to isotone and regular multipliers are proved. Finally, the Left and the Right derivations of the multiplier are obtained.

Keywords: AT-algebra; multiplier;isotonemultiplier; regularmultiplier; Left and the Right derivations.

مضاعفات الجبر -AT

فاطمة فيصل كريم
قسم الرياضيات ، كلية التربية–ابن الهيثم، جامعة بغداد، العراق

الخلاصة
في هذا العمل ، قدمنا فكرة مضاعف الجبر AT ، وحققنا بعض الخصائص. ايضا, تمت مناقشة بعض النظريات والامثلة. تم تعريف كل من النواة وصورة المضاعفات. بعد ذلك, اثبتنا بعض الخصائص المتعلقة بالتساوي والمضاعفات المنتظمة. أخيرًا ، حصلنا على الاشتقاقات اليسرى واليمنى للمضاعف .

1. Introduction

Prabpayak and Leerawat introduced a new algebraic structure named KU-algebra. They studied a homomorphism of KU-algebra and discussed some ideals of this structure [1,2]. The notion of AT-algebra was introduced as a generalization of KU-algebra. Sevaral properties and many types of ideals on AT-algebras were discussed[3]. Investigations on multipliers were published by various researchers in the context of rings and semigroups [4]. Some authors studied the properties of multipliers and algebraic structures on rings and semirings see,[5-12].

The concept of derivation plays a significant role in analysis, algebraic geometry, and algebra. In 1957, Posnerintroduced the notion of derivation in the ring and the near-ring[13]. In addition, the notion of derivation was applied in BCI-algebras [14]. Furthermore, the notion of derivation in *B*-algebras was introducedandsome of their properties were investigated [15]. In 2014, Mostafa and Kareemstudiedthe notion of derivation in KU-algebras[16].

In this study, the multiplier of an AT-algebra and some of its properties are introduced. The kernel and the image of multipliers on AT-algebras are discussed. Also, some properties of theisotone and regular multipliers are given. In addition, the notion of the Left and the Right derivations of the multiplier in AT-algebra is studied.

2. Preliminaries

In this part, we review some basic definitions and theories of AT-algebra.

*Email: fa_sa20072000@yahoo.com

Definition 2.1[3]. An AT-algebra is a nonempty set$\aleph$ with a constant 0 and a binary operation$*$, satisfyingthe following condition : for all$\varepsilon, \sigma, \tau \in \aleph$,

(i) $(\varepsilon * \sigma) * ((\sigma * \tau) * (\varepsilon * \tau)) = 0$,

(ii)$0 * \varepsilon = \varepsilon$,

(iii) $\varepsilon * 0 = 0$.

Then the following properties are satisfied in AT-algebra($\aleph$,$*$,0):

(1) $((\sigma * \tau) * (\varepsilon * \tau)) \leq (\varepsilon * \sigma)$,

(2) $0 \leq \varepsilon$.

where$\varepsilon \leq \sigma$ if and only if $\sigma * \varepsilon = 0$ and $\varepsilon, \sigma, \tau \in \aleph$.

Proposition 2.2 [3]. Let ($\aleph$,$*$,0) be an AT-algebra. Then the following axioms hold:

i. $\tau * \tau = 0$,
ii. $\tau * (\varepsilon * \tau) = 0$,
iii. $\sigma * ((\sigma * \tau) * \tau) = 0$,
iv. $\varepsilon * \sigma = 0$ implies that $\varepsilon * 0 = \sigma * 0$,
v. $\varepsilon = 0 * (0 * \varepsilon)$,
vi. $0 * \varepsilon = 0 * \sigma$ implies that $\varepsilon = \sigma$.

where $\varepsilon, \sigma, \tau \in \aleph$.

Example 2.3[3]. Let$\aleph = \{0, a, b, c, d, e\}$ be a set with the operation $*$ defined by the following table

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	c	b	c
b	0	a	0	b	a	d
c	0	a	0	0	a	a
d	0	0	0	b	0	b
e	0	0	0	0	0	0

Then,($\aleph$,$*$,0) is anAT-algebra.

Proposition 2.4[3]. In any AT-algebra ($\aleph$,$*$,0), the following properties hold for all $\varepsilon, \sigma, \tau \in \aleph$:

a) $\varepsilon \leq \sigma$ implies that $\sigma * \tau \leq \varepsilon * \tau$,
b) $\varepsilon \leq \sigma$ implies that $\tau * \varepsilon \leq \tau * \sigma$,
c) $\tau * \varepsilon \leq \tau * \sigma$ implies that $\varepsilon \leq \sigma$,
d) $\varepsilon * \sigma \leq \tau$ imply $\tau * \sigma \leq \varepsilon$.

Definition 2.5[3]. A nonempty subset S of an AT-algebra $\aleph$ is called anAT-subalgebraif$\varepsilon * \sigma \in S$, whenever $\varepsilon, \sigma \in S$.

Definition 2.6[3]. A nonempty subset $I$of an AT-algebra $\aleph$ is called anAT-idealiffor all $\varepsilon, \sigma, \tau \in \aleph$:

(AT_1) $0 \in I$,

(AT_2)$(\varepsilon * (\sigma * \tau)) \in I$ and $\sigma \in I$imply that$\varepsilon * \tau \in I$.

Definition 2.7. AnAT-algebra ($\aleph$,$*$,0)is said to be AT-commutative if:

$(\varepsilon * \sigma) * \sigma = (\sigma * \varepsilon) * \varepsilon$, for all $\varepsilon, \sigma \in \aleph$.

Example 2.8. Let$\aleph = \{0, a, b, c\}$ be a set with the operation $*$, defined by the following table

$*$	0	a	b	c
0	0	a	b	c
a	0	0	a	c
b	0	0	0	c
c	0	a	b	0

By using the definition2.7, we can prove that($\aleph$,$*$,0) is anAT-commutative.

Theorem2.9. For anAT-algebra ($\aleph$,$*$,0), the followings are equivalentfor all $\varepsilon, \sigma \in \aleph$:

(a) $\aleph$is a commutative,

(b)$(\varepsilon * \sigma) * \sigma \leq (\sigma * \varepsilon) * \varepsilon$,

(c)$\big((\sigma * \varepsilon) * \varepsilon\big) * ((\varepsilon * \sigma) * \sigma) = 0$.

Proof. Clear by applying the definition 2.7.

3. A self-map ρ

Definition 3.1. Let ($\aleph$,$*$,0) be anAT-algebra. A self-map ρ of $\aleph$ is called a multiplier if$\rho(\varepsilon * \sigma) = \varepsilon * \rho(\sigma)$, for all $\varepsilon, \sigma \in \aleph$.

Example 3.2. Let$\aleph = \{0, a, b, c, d\}$ be a set with the operation $*$, defined by the following table:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	c
c	0	0	b	0	b
d	0	0	0	0	0

Based on definition 2.1,($\aleph$,$*$,0)is anAT-algebra and the self map ρ of $\aleph$is defined by:

$$\rho(\varepsilon) = \begin{cases} 0 & if \quad \varepsilon = 0, a, c \\ b & if \quad \varepsilon = b, d \end{cases}$$

Based on definition 2.1, ρ is a multiplier of $\aleph$.

Example 3.3. Based on Example 2.3,$\rho: \aleph \to \aleph$is defined as follows:

$\rho(0) = 0$, $\rho(a) = 0$, $\rho(b) = b$, $\rho(c) = c$, $\rho(d) = b$, $\rho(e) = d$ and $c = \rho(a * e) \neq a * \rho(e) = b$.

Then, ρ is not a multiplier of $\aleph$.

Lemma 3.4. If ρ_1 and ρ_2 are two multipliers of an AT-algebra $\aleph$,then $\rho_1 \circ \rho_2$ is a multiplier of $\aleph$.

Proof. Let ρ_1 and ρ_2 be two multipliers of$\aleph$.Then, $\rho_1 \circ \rho_2(\varepsilon * \sigma) = \rho_1(\rho_2(\varepsilon * \sigma)) = \rho_1((\varepsilon * \rho_2(\sigma)) = \varepsilon * \rho_1(\rho_2(\sigma)) = \varepsilon * (\rho_1 \circ \rho_2)(\sigma)$, for all $\varepsilon, \sigma \in \aleph$.Thus, $\rho_1 \circ \rho_2$ is a multiplier of $\aleph$.

Proposition 3.5. Let $\aleph$ be an AT-algebra and ρ be a multiplier of $\aleph$. Then,for all $\varepsilon, \sigma \in \aleph$, we have

(i) $\rho(0) = 0$

(ii) $\rho(\varepsilon * 0) = 0$,

(iii) $\rho(\varepsilon) \leq \varepsilon$

(iv) $\varepsilon \leq \sigma \Rightarrow \rho(\varepsilon) \leq \sigma$

(v) $\rho(\rho(\varepsilon) * \varepsilon) = 0$

(vi) $\rho(\varepsilon * \sigma) \leq \rho(\varepsilon) * \rho(\sigma)$.

Proof

(i) Based on definition 3.1, $\rho(0) = \rho(\rho(0) * 0) = \rho(0) * \rho(0) = 0$.

(ii) Based on (i), $\rho(\varepsilon * 0) = \rho(0) = 0$, where $\varepsilon \in \aleph$.

(iii) Based on definition 3.1, $0 = \rho(0) = \rho(\varepsilon * \varepsilon) = \varepsilon * \rho(\varepsilon)$, hence $\rho(\varepsilon) \leq \varepsilon$, where $\varepsilon \in \aleph$.

(iv) Suppose that $\varepsilon \leq \sigma$ for every$\varepsilon, \sigma \in \aleph$, then$\sigma * \varepsilon = 0$. Thus, $0 = \rho(0) = \rho(\sigma * \varepsilon) = \sigma * \rho(\varepsilon)$, hence $\rho(\varepsilon) \leq \sigma$.

(v) Based on definition 3.1, $\rho(\rho(\varepsilon) * \varepsilon) = \rho(\varepsilon) * \rho(\varepsilon) = 0$.
where $\varepsilon \in \aleph$.

(vi) Based on (iii), $\rho(\varepsilon) \leq \varepsilon$ for all $\varepsilon \in \aleph$, andbased onproposition 2.4,we have $\varepsilon * \rho(\sigma) \leq \rho(\varepsilon) * \rho(\sigma)$, thus $\rho(\varepsilon * \sigma) \leq \rho(\varepsilon) * \rho(\sigma)$.

Definition 3.6. Let $\aleph$ be an AT-algebra and ρ be a multiplier of $\aleph$. Then, ρ is said to be an isotone if $\varepsilon \leq \sigma \Rightarrow \rho(\varepsilon) \leq \rho(\sigma)$ for all $\varepsilon, \sigma \in \aleph$.

Lemma 3.7. Let $\aleph$be an AT-algebra and ρ be a multiplier of $\aleph$. For all $\varepsilon, \sigma \in \aleph$, if $\rho(\varepsilon * \sigma) = \rho(\varepsilon) * \rho(\sigma)$, then ρ is anisotone .

Proof. Let $\rho(\varepsilon * \sigma) = \rho(\varepsilon) * \rho(\sigma)$. If $\varepsilon \leq \sigma \Rightarrow \sigma * \varepsilon = 0$, for all $\varepsilon, \sigma \in \aleph$. Then , we have
$\rho(\varepsilon) = \rho(0 * \varepsilon)$ by definition 2.1 (iii),
$= \rho((\sigma * \varepsilon) * \varepsilon)$ by hypothesis above,
$= \rho(\sigma * \varepsilon) * \rho(\varepsilon)$ by hypothesis above,
$= [\rho(\sigma) * \rho(\varepsilon)] * \rho(\varepsilon)$ by hypothesis above,
and$\leq \rho(\sigma)$ by definition 2.1 (3).
Thus, $\rho(\varepsilon) \leq \rho(\sigma)$ and then ρ is anisotone.

Proposition 3.8. Let ρ be a multiplierof $\aleph$.If ρ is an endomorphism on $\aleph$, then ρ is an isotone.

Proof. Let $\varepsilon \leq \sigma \Rightarrow \sigma * \varepsilon = 0$ and $0 = \rho(0) = \rho(\sigma * \varepsilon) = \rho(\sigma) * \rho(\varepsilon)$.Hence, $\rho(\varepsilon) \leq \rho(\sigma)$,then ρ is an isotone.

4. The kernel and the image of the multiplier ρ

Definition 4.1. Let $\aleph$be an AT-algebra and ρ be a multiplierof $\aleph$. Then,the kernelof ρ, denoted by$ker(\rho)$, is defined by$ker(\rho) = \{\varepsilon \in \aleph : \rho(\varepsilon) = 0\}$.

Lemma 4.2. Let $\aleph$be an AT-algebra and ρ be a multiplier of $\aleph$. Then,$ker(\rho)$ is an AT-subalgebra of$\aleph$.

Proof. Since$0 \in ker(\rho)$, thenker$(\rho) \neq \varphi$. Let $\varepsilon, \sigma \in$ ker(ρ), it follows that$\rho(\varepsilon) = 0$ and $\rho(\sigma) = 0$. Now,$\rho(\varepsilon * \sigma) = \varepsilon * \rho(\sigma) = \varepsilon * 0 = 0$, hence $\varepsilon * \sigma \in ker(\rho)$. Thus,$ker(\rho)$ is an AT-subalgebra of $\aleph$.

Remark 4.3. Let $\aleph$be an AT-algebra and ρ be a multiplier of $\aleph$.If ρ is one to one map,then$ker(\rho) = \{0\}$.

Proof. Suppose that ρ is one to one and $\varepsilon \in ker(\rho)$. Then, $\rho(\varepsilon) = 0 = \rho(0)$ and thus $\varepsilon = 0$, it follows that$ker(\rho) = \{0\}$.
The reverse of Remark~4.3is incorrect. Example 4.4~ shows ~the reverse**.**

Example 4.4. Let$\aleph = \{0, a, b\}$be a set with the operation$*$, defined by the following table:

$*$	0	a	b
0	0	a	b
a	0	0	0
b	0	0	0

It is easy to check that($\aleph$,$*$,0) is an AT-algebra and the multiplier ρ of $\aleph$is defined by

$$\rho(\varepsilon) = \begin{cases} 0 & if \quad \varepsilon = 0 \\ a & otherwise \end{cases}$$

Now,it is obvious that$ker(\rho) = \{0\}$and $\rho(a) = \rho(b) = a$ .Therefore, ρ is not one to one map.

Lemma 4.5. Let $\aleph$ be anAT-commutative and ρ be a multiplier of $\aleph$. If $\varepsilon \in ker(\rho)$ and $\sigma \leq \varepsilon$, then $\sigma \in ker(\rho)$.

Proof. Let $\varepsilon \in ker(\rho)$ and $\sigma \leq \varepsilon$. Then, $\rho(\varepsilon) = 0$ and $\varepsilon * \sigma = 0$.

$\rho(\sigma) = \rho(0 * \sigma) = \rho\big((\varepsilon * \sigma) * \sigma\big) = \rho\big((\sigma * \varepsilon) * \varepsilon\big) = (\sigma * \varepsilon) * \rho(\varepsilon) = (\sigma * \varepsilon) * 0 = 0$, thus $\sigma \in ker(\rho)$.

Remark 4.6. Let $\aleph$ be an AT-algebra and ρ be endomorphismmap of $\aleph$. Then $ker(\rho)$ is an AT-ideal of $\aleph$.

Proof. Clearly, $0 \in ker(\rho)$. Let $\sigma \in ker(\rho)$ and $(\varepsilon * (\sigma * \tau)) \in ker(\rho)$. Then, we have $\rho(\sigma) = 0$ and $\rho\big(\varepsilon * (\sigma * \tau)\big) = 0$, thus
$0 = \rho\big(\varepsilon * (\sigma * \tau)\big) = \Big(\rho(\varepsilon) * \big(\rho(\sigma) * \rho(\tau)\big)\Big) = \Big(\rho(\varepsilon) * \big(0 * \rho(\tau)\big)\Big) = \rho(\varepsilon) * \rho(\tau) = \rho(\varepsilon * \tau)$. This implies that $\varepsilon * \tau \in ker(\rho)$. Hence, $ker(\rho)$ is an AT-ideal of $\aleph$.

Definition 4.7. Let $\aleph$ be an AT-algebra and ρ be a multiplierof $\aleph$. ρ is named idempotent if $\rho\big(\rho(\varepsilon)\big) = \rho(\varepsilon)$, for all $\varepsilon \in \aleph$.

Example 4.8. Let $\aleph = \{0, a, b\}$ be a set with the operation $*$, defined by the following table:

*	0	a	b
0	0	a	b
a	0	0	0
b	0	0	0

It is easy to check that $(\aleph, *, 0)$ is an AT-algebra and the multiplier ρ of $\aleph$ isdefined by
$\rho(\varepsilon) = \varepsilon$
Then, ρ is idempotent.

Remark 4.9. Let $\aleph$ be an AT-algebraand ρ be a multiplier of $\aleph$. If ρ is idempotent, then
(i) $\mathrm{Im}(\rho) \cap \ker(\rho) = \{0\}$,
(ii) $\varepsilon \in \mathrm{Im}(\rho) \Leftrightarrow \rho(\varepsilon) = \varepsilon$, for all $\varepsilon \in \aleph$.

Proof. (i) If $\varepsilon \in \mathrm{Im}(\rho) \cap \ker(\rho)$, then $\rho(\sigma) = \varepsilon$ for some $\sigma \in \aleph$ and $\rho(\varepsilon) = 0$. It follows that $0 = \rho(\varepsilon) = \rho(\rho(\sigma)) = \rho(\sigma) = \varepsilon$, thus $\mathrm{Im}(\rho) \cap \ker(\rho) = \{0\}$.
(ii) Sufficiency is obvious. If $\varepsilon \in \mathrm{Im}(\rho)$, then $\rho(\sigma) = \varepsilon$ for some $\sigma \in \aleph$. Thus
$\rho(\varepsilon) = \rho(\rho(\sigma)) = \rho(\sigma) = \varepsilon$.

Remarke4.10. Let $M(\aleph)$ be the set of allmultiplieridempotent maps of AT-algebra $\aleph$ and $\blacksquare$ be a binary operation on $M(\aleph)$ defined by $(\rho \blacksquare \delta)(\varepsilon) = \rho(\varepsilon) * \delta(\varepsilon)$, for all $\rho, \delta \in M(\aleph)$ and $\varepsilon \in \aleph$. It is easy to show that $(M(\aleph), \blacksquare, 0)$ is anAT-algebra.

Theorem 4.11. Let $\rho, \delta \in M(\aleph)$, then
(i) If $\rho\big(\delta(\varepsilon)\big) = \delta(\rho(\varepsilon))$ for all $\varepsilon \in \aleph$, then $\rho \blacksquare \delta \in M(\aleph)$,
(ii) For all $\varepsilon \in \aleph$, if $Im(\delta) \subset Im(\rho)$ and $\rho\big(\delta(\varepsilon)\big) = \delta(\rho(\varepsilon))$, then $\rho \blacksquare \delta = 0$,
(iii) $Im(\delta) \cap \ker(\rho) \subset Im(\rho \blacksquare \delta)$.

Proof. (i) Assume that $\rho\big(\delta(\varepsilon)\big) = \delta(\rho(\varepsilon))$ for all $\varepsilon \in \aleph$. Then
$(\rho \blacksquare \delta)\big((\rho \blacksquare \delta)(\varepsilon)\big) = (\rho \blacksquare \delta)\big(\rho(\varepsilon) * \delta(\varepsilon)\big) = \rho\big(\rho(\varepsilon) * \delta(\varepsilon)\big) * \delta\big(\rho(\varepsilon) * \delta(\varepsilon)\big)$
$= \Big(\rho(\varepsilon) * \rho\big(\delta(\varepsilon)\big)\Big) * \Big(\rho(\varepsilon) * \delta\big(\delta(\varepsilon)\big)\Big) = \Big(\rho(\varepsilon) * \delta\big(\rho(\varepsilon)\big)\Big) * \big(\rho(\varepsilon) * \delta(\varepsilon)\big)$
$= \delta\big(\rho(\varepsilon) * \rho(\varepsilon)\big) * \big(\rho(\varepsilon) * \delta(\varepsilon)\big) = \delta(0) * (\rho \blacksquare \delta)(\varepsilon) = (\rho \blacksquare \delta)(\varepsilon)$.
Then $\rho \blacksquare \delta$ is idempotent, hence $\rho \blacksquare \delta \in M(\aleph)$.
(ii) Suppose that $Im(\delta) \subset Im(\rho)$ and $\rho\big(\delta(\varepsilon)\big) = \delta(\rho(\varepsilon))$, for all $\varepsilon \in \aleph$.
Since $\delta(\varepsilon) \in Im(\delta) \subset Im(\rho)$ for all $\varepsilon \in \aleph$, it follows from lemma 3.7 that
$(\rho \blacksquare \delta)(\varepsilon) = \rho(\varepsilon) * \delta(\varepsilon) = \rho(\varepsilon) * \rho\big(\delta(\varepsilon)\big) = \rho(\varepsilon) * \delta\big(\rho(\varepsilon)\big) = \delta\big(\rho(\varepsilon) * \rho(\varepsilon)\big) = \delta(0)$
,for all $\varepsilon \in \aleph$, hence $\rho \blacksquare \delta = 0$.

(iii) If $\sigma \in Im(\delta) \cap \ker(\rho)$, then $\delta(\varepsilon) = \sigma$and $\rho(\sigma) = 0$ for some $\varepsilon \in \aleph$. It follows that
$\sigma = \delta(\varepsilon) = 0 * \delta(\delta(\varepsilon)) = \rho(\sigma) * \delta(\sigma) = (\rho \blacksquare \delta)(\sigma) \in Im(\rho \blacksquare \delta)$.
Hence, the proof is completed.

5. The Left and The Right Derivations of multiplier mapsin AT-algebra

Theleft and the right derivationsof multiplier maps in AT-algebra are introduce in this section.

Definition 5.1. Let ρbe a multiplier map and $D_\rho$be a self map of $\aleph$. Then,$D_\rho$is calledL_ρ-derivation if :$D_\rho(\varepsilon * \sigma) = D_\rho(\varepsilon) * \rho(\sigma)$, for all $\varepsilon, \sigma \in \aleph$, and it is called$R_\rho$-derivation if:$D_\rho(\varepsilon * \sigma) = \rho(\varepsilon) * D_\rho(\sigma)$, for all $\varepsilon, \sigma \in \aleph$. If $D_\rho$is both L_ρ-derivation andR_ρ-derivationof $\aleph$, we say thatD_ρ is aρ-derivation of $\aleph$.

Definition 5.2.A self map $D_\rho$of $\aleph$ is called a regular if $D_\rho(0) = 0$.

Proposition5.3.A L_ρ-derivation of an AT-algebra$\aleph$ is a regular.

Proof. Let$D_\rho$be a L_ρ-derivation of $\aleph$, then for all $\varepsilon \in \aleph$
$D_\rho(0) = D_\rho(\varepsilon * 0) = D_\rho(\varepsilon) * \rho(0) = D_\rho(\varepsilon) * 0 = 0.$
Thus, a L_ρ-derivationof $\aleph$is a regular.

The reverse of Lemma~5.3 is incorrect. Example 5.4~ shows ~the reverse**.**

Example 5.4. Let$\aleph = \{0, a, b\}$be a set in Example4.4. We define a map$D_\rho: \aleph \to \aleph$by

$$D_\rho(\varepsilon) = \begin{cases} 0 & \varepsilon = 0 \\ b & otherwise \end{cases}$$

Then, it is easy to show that D_ρ is a regular map but not L_ρ-derivation of $\aleph$, since
$D_\rho(0 * a) = b$and $D_\rho(0) * \rho(a) = a$. Then,$D_\rho(0 * a) \neq D_\rho(0) * \rho(a)$.

Lemma 5.5. Let D_ρ be a regular map of an AT-algebra$\aleph$, then

(i) If $D_\rho$is a L_ρ-derivation of $\aleph$, then $D_\rho(\varepsilon) = \rho(\varepsilon)$for all $\varepsilon \in \aleph$.

(ii) If $D_\rho$is a L_ρ-derivation of $\aleph$, then $D_\rho((0 * \varepsilon) * \sigma) = \rho(\rho(\varepsilon) * \sigma)$for all $\varepsilon, \sigma \in \aleph$.

(iii) If $D_\rho$is a R_ρ-derivation of $\aleph$, then $D_\rho((0 * \varepsilon) * \sigma) = \rho(\varepsilon) * D_\rho(\sigma)$for all $\varepsilon, \sigma \in \aleph$.

Proof. (i)Let D_ρ be L_ρ-derivation of $\aleph$, then

$$D_\rho(\varepsilon) = D_\rho(0 * \varepsilon) = D_\rho(0) * \rho(\varepsilon) = 0 * \rho(\varepsilon) = \rho(\varepsilon).$$

(ii)Let D_ρ be L_ρ-derivation of $\aleph$, then

$$D_\rho((0 * \varepsilon) * \sigma) = D_\rho(0 * \varepsilon) * \rho(\sigma) = \left(D_\rho(0) * \rho(\varepsilon)\right) * \rho(\sigma) = (0 * \rho(\varepsilon)) * \rho(\sigma) = \rho(\varepsilon) * \rho(\sigma)$$
$$= \rho(\rho(\varepsilon) * \sigma)$$

(iii)Let D_ρ be R_ρ-derivation of $\aleph$, then
$D_\rho((0 * \varepsilon) * \sigma) = \rho(0 * \varepsilon) * D_\rho(\sigma) = \rho(\varepsilon) * D_\rho(\sigma)$

Lemma 5.6. Let D_ρ be a regular and L_ρ-derivation of $\aleph$, then

(i) $D_\rho(\varepsilon * \sigma) = \varepsilon * \rho(\sigma)$, for all $\varepsilon, \sigma \in \aleph$.

(ii) $D_\rho\left(\varepsilon * D_\rho(\varepsilon)\right) = 0$.

Proof. Let D_ρ be a regular and L_ρ-derivation of $\aleph$, thenwe have

(i)$D_\rho(\varepsilon * \sigma) = D_\rho(0 * (\varepsilon * \sigma)) = D_\rho(0) * \rho(\varepsilon * \sigma) = 0 * \rho(\varepsilon * \sigma) = \rho(\varepsilon * \sigma) = \varepsilon * \rho(\sigma)$

(ii)$D_\rho\left(\varepsilon * D_\rho(\varepsilon)\right) = D_\rho(\varepsilon) * \rho\left(D_\rho(\varepsilon)\right) = \rho\left(D_\rho(\varepsilon) * D_\rho(\varepsilon)\right) = \rho(0) = 0.$

Definition 5.7. Let D_ρ be a ρ-derivation of an AT-algebra$\aleph$. Then, D_ρ is said to be an isotone ρ-derivation if$\varepsilon \leq \sigma \Longrightarrow D_\rho(\varepsilon) \leq D_\rho(\sigma)$, for all $\varepsilon, \sigma \in \aleph$.

Lemma 5.8. Let $(\aleph, *, 0)$be an AT-algebra and D_ρ be ρ-derivation on $\aleph$. For all $\varepsilon, \sigma \in \aleph$, if $D_\rho(\varepsilon * \sigma) = D_\rho(\varepsilon) * D_\rho(\sigma)$, then D_ρ is anisotone ρ-derivation.

Proof. Let$D_\rho(\varepsilon * \sigma) = D_\rho(\varepsilon) * D_\rho(\sigma)$. If $\varepsilon \leq \sigma \Longrightarrow \sigma * \varepsilon = 0$ for all $\varepsilon, \sigma \in \aleph$. Then , we have
$D_\rho(\varepsilon) = D_\rho(0 * \varepsilon) = D_\rho((\sigma * \varepsilon) * \varepsilon) = D_\rho(\sigma * \varepsilon) * D_\rho(\varepsilon)$

$$= [D_\rho(\sigma) * D_\rho(\varepsilon)] * D_\rho(\varepsilon) \leq D_\rho(\sigma)$$

Thus,$D_\rho(\varepsilon) \leq D_\rho(\sigma)$, which implies that $D_\rho$is anisotone ρ-derivation.

Lemma 5.9. Let $(\aleph, *, 0)$ be anAT-algebra with a partial order $\leq$, and D_ρ be a self map of$\aleph$. Then for all $\varepsilon, \sigma \in \aleph$, we have

(i) If D_ρ is L_ρ-derivation of$\aleph$, then $D_\rho(\varepsilon * \sigma) \leq D_\rho(\varepsilon) * \rho(\sigma)$,

(ii) If D_ρ is R_ρ-derivation of$\aleph$, then $D_\rho(\varepsilon * \sigma) \leq \rho(\varepsilon) * D_\rho(\sigma)$.

(iii) If D_ρ is a regular and R_ρ-derivation of$\aleph$, then $ker(D_\rho) = \{\varepsilon \in \aleph: D_\rho(\varepsilon) = 0\}$is anAT-subalgebra of $\aleph$.

Proof. (i) If $D_\rho$is L_ρ-derivation of$\aleph$, we have

$$\left(D_\rho(\varepsilon) * \rho(\sigma)\right) * D_\rho(\varepsilon * \sigma) = \left(D_\rho(\varepsilon) * \rho(\sigma)\right) * \left(D_\rho(\varepsilon) * \rho(\sigma)\right) = 0$$

Then,$D_\rho(\varepsilon * \sigma) \leq D_\rho(\varepsilon) * \rho(\sigma)$.

(ii) If D_ρ is R_ρ-derivation of$\aleph$, we have

$$\left(\rho(\varepsilon) * D_\rho(\sigma)\right) * D_\rho(\varepsilon * \sigma) = \left(\rho(\varepsilon) * D_\rho(\sigma)\right) * \left(\rho(\varepsilon) * D_\rho(\sigma)\right) = 0$$

Then,$D_\rho(\varepsilon * \sigma) \leq \rho(\varepsilon) * D_\rho(\sigma)$.

(iv) Since D_ρ is a regular map, then $D_\rho(0) = 0$, it follows that$ker(D_\rho) \neq \varphi$.

Now, Let$\varepsilon, \sigma \in ker(D_\rho)$, then$D_\rho(\varepsilon) = 0, D_\rho(\sigma) = 0$. Since D_ρ is R_ρ-derivation of $\aleph$, then$D_\rho(\varepsilon * \sigma) = \rho(\varepsilon) * D_\rho(\sigma) = \rho(\varepsilon) * 0 = 0$. Hence,$\varepsilon * \sigma \in ker(D_\rho)$.

Therefore, $ker(D_\rho)$is anAT-subalgebra of $\aleph$.

Definition 5.10. Let $(\aleph, *, 0)$be an AT-algebra and ρ be a multiplierof $\aleph$. Then,an AT-ideal I is named an ρ-ideal if $\rho(I) \subseteq I$.

Definition 5.11. Let D_ρ be a self map of an AT-algebra$\aleph$. An ρ-ideal I of $\aleph$ is said to be D_ρ-invariant if $D_\rho(I) \subseteq I$.

Propotion 5.12. Let $D_\rho$be a regular L_ρ-derivation of anAT-algebra $\aleph$, then every ρ-ideal I of $\aleph$ is D_ρ-invariant.

Proof. By Lemma 5.5(i), we have $D_\rho(\varepsilon) = \rho(\varepsilon)$for all $\varepsilon \in \aleph$. Let $\sigma * \tau \in D_\rho(I)$. Then,$\sigma * \tau = D_\rho(\varepsilon)$, for some $\varepsilon \in I$. It follows that

$\left(\sigma * (\rho(\varepsilon) * \tau)\right) = \left(\rho(\varepsilon) * (\sigma * \tau)\right) = \left(\rho(\varepsilon) * D_\rho(\varepsilon)\right) = \left(\rho(\varepsilon) * \rho(\varepsilon)\right) = 0 \in I$. Since $\varepsilon \in I$, then $\rho(\varepsilon) \in \rho(I) \subseteq I$,as I is an ρ-ideal. It follows that $\sigma * \tau \in I$and since I is an AT-ideal of $\aleph$. Hence $D_\rho(I) \subseteq I$, thus I is D_ρ-invariant.

Corollary5.13. Let $D_\rho$be ρ-derivation of anAT-algebra$\aleph$, then$D_\rho$is a regular if and only if every ρ-ideal of $\aleph$ is D_ρ-invariant.

Proof. Assume that every ρ-ideal of $\aleph$ is D_ρ-invariant. Then, since the zero ideal $\{0\}$ isρ-ideal and D_ρ-invariant, we have $D_\rho(\{0\}) \subseteq \{0\}$, which implies that $D_\rho(0) = 0$. Thus, D_ρ is aregular. Combining this and Theorem 5.12, theproof is complete.

References

1. Prabpayak, C. and Leerawat, U. **2009**. On ideals and congruence in KU-algebras, *scientia Magna,* **5**(1): 54-57.
2. Prabpayak, C. and Leerawat, U. **2009**.On isomorphisms of KU-algebras, *scientia magna*, **5**(3): 25-31.
3. Hameed, A.T. **2016**.*AT-ideals and Fuzzy AT-ideals of AT-algebra*, Germany. LAP LEMBRT Academic Publishing.
4. Larsen, B. **1971**.*An introduction to the theory of multipliers*. Berlin, Spring-Verlag.
5. Blecher, D.P.**2001**. Multipliers and Dual Operator Algebras,*Journal of Functional Analysis*, **183**: 498-525.
6. Chaudhry, M.A. and Ali, F. **2012**. Multipliers in d-algebras, *World Applied Sciences Journal*, **18**(11): 1649-1653.
7. Cornish,W.H. **1980**.A multiplier approach to implicative BCK-algebras, *Mathematics Seminar Notes (Kobe)*, **8**(2): 157-169.
8. Janssen,K. and Vercuysse, J. **2009.***Multiplier Hopf and BI-Algebras*.Faculty of Engineering, Vrije Universiteit Brussel (VUB).
9. Kim, K. H. and Lim, H. J. **2013** . On Multipliers of BCC-algebras, *Honam Mathematical Journal*, **35**(2): 201-210.

10. Ahmed, H. A., & Majeed, A. H. **2020**. Γ-(λ,δ) – Derivation on Semi-Group ideals in Prime Γ-Near-Ring, *Iraqi Journal of Science*, **61**(3): 600-607.

11. Rasheed, M. K., & Majeed, A. H**. 2019.** Some results of (α, β) derivations on prime semirings. *Iraqi Journal of Science*, **60**(5): 1154-1160.

12. Enaam F. A. **2020.** A Study on n-Derivation in Prime Near – Rings, *Iraqi Journal of Science,* **60**(5): 1154-1160.

13. Posner, E.C. **1957** . Derivations in prime rings, *Proceedings of the American Mathematical Society*, **8**: 1093-1100.

14. Jun,Y. B. and Xin, X.L. **2004**. On derivations of BCI-algebras, *Information Sciences,* 159 (3): 167-176.

15. Al-Shehri, N.O. **2010**. Derivations of B-algebras, *JKAU: Sci.*, 22: 71-83.

16. Mostafa, S. M. and Kareem, F. F. **2014**. Left fixed maps and α -derivations of a KU-algebra,*Journal of Advances in mathematics*, 9(7):2817-2827.