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MULTIPLIERS OF BERGMAN SPACES INTO LEBESGUE SPACES

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1. Introduction

Let U be the open unit disk in the complex plane $\mathbb{C}$ endowed with normalized Lebesgue measure m. L^p will denote the usual Lebesgue space with respect to m, with $0<p<+\infty$. The Bergman space consisting of the analytic functions in L^p will be denoted L^p_a. Let μ be some positive finite Borel measure on U. It has been known for some time (see [6] and [9]) what conditions on μ are equivalent to the estimate: There is a constant C such that

$$\left(\int |f|^q \, d\mu\right)^{1/q} \leqq C\left(\int |f|^p \, dm\right)^{1/p} \quad \text{for all} \quad f \in L^p_a; \tag{1.1}$$

provided $0<p\leqq q$. It has been of considerable interest (to the author at least) to obtain a similarly complete result for the remaining cases, namely $0<q<p$. One way the study of (1.1) arises is through consideration of the multiplier problem for Bergman spaces. That is, what conditions on a measurable function g are equivalent to $gL^p_a \subseteq L^q$? This reduces, via the closed graph theorem, to the estimate $(\int |gf|^q \, dm)^{1/q} \leqq C(\int |f|^p \, dm)^{1/p}$, which is (1.1) with $d\mu = |g|^q \, dm$. For g analytic, the problem was solved by K. R. M. Attele in [2] (see also [3]) where the obvious sufficient condition $g \in L^r_a$, $1/r = 1/q - 1/p$, was shown to be necessary. For a general measure μ, a sufficient condition is easy to come by. It can be shown that $\int |f|^q \, d\mu \leqq C \int |f|^q k \, dm$ where $k(z)$ is a function obtained from μ by averaging μ over a hyperbolic neighborhood of z (see the next section). The sufficient condition arises from Holder's inequality and is simply $k \in L^s$, $1/s + q/p = 1$. In this paper, I show that this condition is necessary.

2. Background

For $z, w \in U$ let $\rho(z,w) \equiv |(z-w)/(1-\bar{w}z)|$, the pseudohyperbolic distance between z and w. In this metric two points are far apart if the distance between them is nearly 1.

If $0<\varepsilon<1$ and $a \in U$, let $D_\varepsilon(a) = \{z : \rho(z,a) < \varepsilon\}$. Occasionally, when the exact value of ε is unimportant, I will write $D(a)$ for $D_\varepsilon(a)$. $D_\varepsilon(a)$ is an actual disk (i.e., in the Euclidean metric) with centre at

$$\frac{1-\varepsilon^2}{1-\varepsilon^2|a|^2} a \quad \text{and radius} \quad \varepsilon \frac{1-|a|^2}{1-\varepsilon^2|a|^2}.$$

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Thus, if a is fixed, $D_\varepsilon(a)$ behaves like a disk of radius $\sim\varepsilon$. And if ε is fixed the radius behaves like $1-|a|^2$. Its normalized area is

$$m(D_\varepsilon(a))=\varepsilon^2\left(\frac{1-|a|^2}{1-\varepsilon^2|a|^2}\right)^2.$$

Because $|f|^q$ is subharmonic for $f\in L_a^q$, it follows that $\int_{D(a)}|f|^q\,dm/m(D(a))$ exceeds the value of $|f|^q$ at the centre of $D(a)$. If ε is fixed, the distance from a to the centre of $D_\varepsilon(a)$ is at most $\varepsilon|a|$ times the radius of $D_\varepsilon(a)$. By subharmonicity again, there is a constant C depending only on ε such that

$$C\int_{D(a)}|f|^q\,dm/m(D(a))\geqq|f(a)|^q. \tag{2.1}$$

(This inequality is also valid for harmonic functions, except that C will also depend on q if $q<1$. Indeed, the proof of Lemma 2, page 152 of [5], shows that one only needs $|f|$ to be subharmonic.) Using (2.1) to estimate $|f|^q$ in $\int|f|^q\,d\mu$ and applying Fubini's theorem, one obtains

$$\int|f|^q\,d\mu\leqq C\int|f(z)|^q\int_{D(z)}\frac{1}{m(D(w))}\,d\mu(w)\,dm(z).$$

It is easy to verify that if $w\in D(z)$ then

$$\frac{m(D(z))}{m(D(w))}\leqq C$$

with C depending on ε. Thus, putting $k(z)=\mu(D(z))/m(D(z))$, one gets

$$\int|f|^q\,d\mu\leqq C\int|f|^q k\,dm. \tag{2.2}$$

An immediate result is the following:

If k belongs to L^s for $s=p/(p-q)$, then

$$\left(\int|f|^q\,d\mu\right)^{1/q}\leqq C\left(\int|f|^p\,dm\right)^{1/p}\quad\text{for all}\quad f\in L_a^p. \tag{2.3}$$

The constant C depends only on ε, q, and the value of $\int k^s\,dm$. The main theorem is the converse of (2.3):

Theorem. *Let μ be a positive measure on U and let $k(z)=\mu(D(z))/m(D(z))$ where $D(z)=D_\varepsilon(z)$ for some convenient $\varepsilon\in(0,1)$. Let $0<q<p$. Then a necessary and sufficient condition for there to exist a constant C satisfying*

$$\left(\int|f|^q\,d\mu\right)^{1/q}\leqq C\left(\int|f|^p\,dm\right)^{1/p} \tag{2.4}$$

for all $f\in L_a^p$ is that k belong to L^s, where $1/s+q/p=1$.

This will be proved in Section 3. The remainder of this section is devoted to showing that the condition $k \in L^s$ is independent of the choice of $\varepsilon \in (0, 1)$.

Lemma. *Let* $0 < \delta < \varepsilon < 1$ *and let* $k_\varepsilon(z) = \mu(D_\varepsilon(z))/m(D_\varepsilon(z))$ *with* k_δ *defined similarly. If* $s \geqq 1$, *then* $k_\varepsilon \in L^s$ *if and only if* $k_\delta \in L^s$.

Proof. Clearly $k_\delta(z) \leqq k_\varepsilon(z)[m(D_\varepsilon(z))/m(D_\delta(z))]$ and the formula for the area of pseudohyperbolic disks shows that $m(D_\varepsilon(z))/m(D_\delta(z))$ is a bounded function of z. Thus $k_\varepsilon \in L^s$ implies $k_\delta \in L^s$. Now suppose $k_\delta \in L^s$ and let $\phi(z) = \int_{D_\varepsilon(z)} k_\delta \, dm/m(D_\varepsilon(z))$. It is an easy exercise with Fubini's theorem to show that if k_δ is in L^1 then so is ϕ and it is even clearer that if k_δ is bounded so is ϕ. By any of a variety of interpolation theorems it follows that if $k_\delta \in L^s$, then also $\phi \in L^s$, $1 \leqq s < \infty$. Finally, the following estimates show that ϕ dominates k_ε:

$$\int_{D_\varepsilon(z)} k_\delta \, dm = \int_{D_\varepsilon(z)} \int_{D_\delta(w)} d\mu(t)/m(D_\delta(w)) \, dm(w)$$

$$= \iint \chi_{D_\varepsilon(z)}(w) \chi_{D_\delta(w)}(t) m(D_\delta(w))^{-1} \, dm(w) \, d\mu(t)$$

$$\geqq c \iint \frac{m(D_\varepsilon(z) \cap D_\delta(t))}{m(D_\delta(t))} \, d\mu(t).$$

It is clear that the integrand exceeds 1/3 when t lies in $D_\varepsilon(z)$, so

$$\int_{D_\varepsilon(z)} k_\delta \, dm \geqq c\mu(D_\varepsilon(z)). \quad \blacksquare$$

3. Interpolating sequences

In order to obtain an integrability condition on k from an inequality like (1.1), it has to be shown $|f|^q$ can be made "sufficiently arbitrary". Think of a discrete version of k obtained by decomposing the disk into hyperbolically "equal"-sized pieces $\{D_i\}$ as in [4] and putting k on each of these pieces equal to the average of μ on that piece. It is not hard to show that the condition on μ ($k \in L^s$) is equivalent to

$$\sum \left(\frac{\mu(D_i)}{m(D_i)} \right)^s m(D_i) < +\infty.$$

Then $\int |f|^q \, d\mu$ ought to be roughly $\sum \int_{D_i} |f|^q \, dm \mu(D_i)/m(D_i)$, so we would like to make $\int_{D_i} |f|^q \, dm/m(D_i)$ dominate an arbitrary sequence in the weighted $l^{s'}$ space with weights $m(D_i)$, $s' = p/q$. This can be done by making sure each D_i contains a point a_i so that $\{a_i\}$ is an interpolation sequence for L^p_a. The rest of the proof of the main theorem consists of making this intuition precise.

Definition. A sequence $\{a_i\}$ in U is said to be *separated* if there exists a $\delta > 0$ such that $\rho(a_i, a_j) > \delta$ when $i \neq j$. A separated sequence $\{a_i\}$ is called an *interpolation sequence* for L_a^p if whenever $\{c_i\}$ is a sequence of complex numbers such that $\sum |c_i|^p(1-|a_i|^2)^2 < +\infty$, then there exists $f \in L_a^p$ satisfying $f(a_i) = c_i$.

Because $|f(a_i)|^p m(D_\delta(a_i)) \leqq C \int_{D_\delta(a_i)} |f|^p\, dm$, it follows that if $\{a_i\}$ is 2δ-separated, then the operator $Rf = \{f(a_i)\}$ is a bounded map of L_a^p into the weighted sequence space $l^p\{(1-|a_i|^2)^2\}$. A sequence $\{a_i\}$ is an interpolation sequence if R is onto. It follows from the open mapping theorem that a constant M may be associated with any given interpolation sequence $\{a_i\}$ such that any $\{c_i\} \in l^p\{(1-|a_i|^2)^2\}$ with $\sum |c_i|^p(1-|a_i|^2)^2 \leqq 1$ is the image under R of a function $f \in L_a^p$ with $(\int |f|^p\, dm)^{1/p} \leqq M$. This M will be referred to as the interpolation constant of $\{a_i\}$.

It is a result of Eric Amar [1] (but see also [10]) that if $\{a_i\}$ is a separated sequence, then it is the union of finitely many interpolation sequences. Specifically, the following was shown.

Theorem. (E. Amar) *If $\{b_i\}$ is a δ-separated sequence, then $\{b_i\}$ is the union of $N = N(\delta, \eta)$ η-separated sequence, and if η is near enough to 1 then each η-separated sequence is an interpolation sequence for L_a^p. The size of η will depend on p and the interpolation constant M will depend only on η and p.*

Now fix $\eta > \frac{1}{2}$ once and for all, so near to 1 that any η-separated sequence is an interpolation sequence. This fixes an interpolation constant M. Let $\delta \in (0,1)$ be a small number; its actual size will be specified later and will depend only on η, M and the constant C in the estimate (2.4) of the main theorem. Construct a $\delta/2$-lattice, that is, a $\delta/2$-separated sequence $\{b_i\}$ such that the disks $\{D_{\delta/2}(b_i)\}$ cover U. Here is a simple construction: let $b_1 = 0$, and once b_1 through b_{n-1} are obtained, pick $b_n \notin \bigcup_1^{n-1} D_{\delta/2}(b_i)$ which minimizes $|b_n|$. Clearly $\{b_i\}$ will be $\delta/2$-separated. If $z_0 \notin \cup D_{\delta/2}(b_i)$ then all b_i lie in $\{z : |z| < |z_0|\}$ or else z_0 was needlessly overlooked in the selection. A contradiction has been reached in that infinitely many disjoint $D_{\delta/4}(b_i)$ have their centres in $|z| < |z_0|$. The proof of the following lemma is quite similar to arguments used in [7] and [8].

Lemma. *There is a constant A depending only on q and η such that if $\{a_i\}$ is an η-separated sequence and δ is sufficiently small, then for every $f \in L_a^p$*

$$\sum \int_{D_\delta(a_i)} |f(z) - f(a_i)|^q\, d\mu(z) \leqq A\delta^q \|f\|_{L^p}^q \left(\sum \mu(D_\delta(a_i))^s m(D_i)^{1-s}\right)^{1/s} \tag{3.1}$$

where $D_i = D_{\eta/2}(a_i)$.

Proof. It is clear by normal families and scaling that if $|z| < \delta < \eta/4$ and $D = \{z : |z| < \eta/2\}$, then

$$\left|\frac{f(z) - f(0)}{z}\right|^q \leqq C \int_D |f|^q\, dm$$

where C depends only on q, if that. Thus $|f(z) - f(0)|^q \leqq C\delta^q \int_D |f|^q\, dm$. The change of variables $z \to (z-a_i)/(1-\bar{a}_i z)$ gives

$$|f(z)-f(a_i)|^q \leqq C\delta^q \int_{D_i} |f|^q \frac{(1-|a_i|^2)^2}{|1-\bar{a}_i z|^4}\, dm$$

$$\leqq A\delta^q \int_{D_i} |f|^q\, dm/m(D_i) \tag{3.2}$$

where the estimate $(1-|a_i|^2)^2/|1-\bar{a}_i z|^4 \leqq \text{constant}\, m(D_i)^{-1}$ has been used for $z \in D_i$. The constant depends only on η. Integrating (3.2) with respect to μ over $D_\delta(a_i)$, and summing, one sees that the left-hand side of (3.1) is at most

$$A\delta^q \sum \int_{D_i} |f|^q\, dm\mu(D_\delta(a_i))m(D_i)^{-1} \leqq A\delta^q \sum \left(\int_{D_i} |f|^p\, dm\right)^{q/p} \mu(D_\delta(a_i))m(D_i)^{1/s-1}$$

$$\leqq A\delta^q \left(\sum \int_{D_i} |f|^p\, dm\right)^{q/p} \left(\sum \mu(D_\delta(a_i))^s m(D)^{1-s}\right)^{1/s},$$

(recall s is just the conjugate exponent of p/q).

Since the D_i are disjoint, the expression in the first parentheses is at most $\|f\|_{L^p}^p$. ∎

The proof of the main theorem may now be completed. To this end let μ be a measure satisfying the integral inequality (2.4) of the theorem. If we replace μ with $\chi_{\{|z|<r\}}\mu$, then (2.4) is still valid with the same constant. If we show that the estimate on $\|k\|_{L^s}$ is independent of r, we may let $r \to 1$ to obtain the theorem. Thus, without any loss of generality, μ is compactly supported in U and all of the sums below involving μ are finite. Let $\{b_i\}$ be the $\delta/2$-lattice constructed earlier and let $\{a_k\}$ be one of the $N((\delta/2), \eta)$ η-separated sequences whose union is $\{b_i\}$. Let M be its interpolation constant. From the lemma, if $f \in L_a^p$, $\|f\|_{L^p} \leqq M$, and $q \leqq 1$ then

$$\sum \int_{D_\delta(a_k)} |f|^q\, d\mu \geqq \sum \int_{D_\delta(a_k)} |f(a_k)|^q\, d\mu - \sum \int_{D_\delta(a_k)} |f-f(a_k)|^q\, d\mu$$

$$\geqq \sum |f(a_k)|^q \mu(D_\delta(a_k)) - A\delta^q M^q \left(\sum \mu(D_\delta(a_k))m(D_k)^{1-s}\right)^{1/s} \tag{3.3}$$

where $D_k = D_{\eta/2}(a_k)$ as in the lemma. Since $f(a_k)$ may assume the values of any sequence $\{c_k\}$ with $\sum |c_k|^p (1-|a_k|)^2)^2 = 1$, the sum $\sum |f(a_k)|^q \mu(D_\delta(a_k))$ may assume the value $(\sum \mu(D_\delta(a_k))^s (1-|a_k|^2)^{2(1-s)})^{1/s} \geqq \beta(\sum \mu(D_\delta(a_k))m(D_k)^{1-s})^{1/s}$. Here β depends only on η. Thus we have

$$C^q M^q \geqq \sum \int_{D_\delta(a_k)} |f|^q\, d\mu$$

$$\geqq (\beta - A\delta^q M^q)\left(\sum \mu(D_\delta(a_k))^s m(D_k)^{1-s}\right)^{1/s}.$$

We now choose $\delta^q = \beta(2AM^q)$, and sum over the N sequences $\{a_k\}$ to get

$$(\sum \mu(D_\delta(b_i))^s m(D_i)^{1-s})^{1/s} < 2NC^q M^q/\beta, \tag{3.4}$$

where $D_i = D_{\eta/2}(b_i)$. It remains to be shown that (3.3) implies $k \in L^s$. Set $\varepsilon = \delta/2$ and define $k(z) = \mu(D_\varepsilon(z))/m(D_\varepsilon(z))$. If $z \in D_\varepsilon(b_i)$, then $D_\varepsilon(z) \subseteq D_\delta(b_i)$ and so $k(z) \leqq \mu(D_\delta(b_i))/m(D_\varepsilon(z)) \leqq$ constant $\mu(D_\delta(b_i))/m(D_i)$. Thus $\sum \int_{D_\varepsilon(b_i)} k^s\, dm \leqq$ constant $\sum \mu(D_\delta(b_i))^s m(D_i)^{-s} m(D_\varepsilon(b_i)) \leqq$ constant $\sum \mu(D_\delta(b_i))^s m(D_i)^{1-s} <$ constant by (3.3). Since the disks $D_\varepsilon(b_i)$ cover U we get $\int_U k^s\, dm \leqq$ constant, where the constant depends only on N, C, M, q, β, η, and δ. That is, ultimately only on C, q, and p. If $q > 1$ only minor changes are needed in (3.3). The proof is completed.

4. Remarks

It should come as no surprise that the theorem remains valid, *mutatis mutandis*, when the disk is replaced by the unit ball in $\mathbb{C}^n$, Lebesgue measure m is replaced by a weighted measure $m_\alpha(1-|z|^2)^\alpha m$, and analytic functions are replaced by pluriharmonic functions. In fact, thanks to Richard Rochberg's extension **[10]** of Eric Amar's result on interpolation sequences, there is a formulation, left to the reader, of the theorem that is valid in weighted Bergman spaces on bounded symmetric domains in $\mathbb{C}^n$.

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