# Multiplying matrices in $O(n^{2.373})$ time

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**Abstract**

We develop new tools for analyzing matrix multiplication constructions similar to the Coppersmith-Winograd construction, and obtain a new improved bound on $\omega < 2.372873$.

## 1 Introduction

The product of two matrices is one of the most basic operations in mathematics and computer science. Many other essential matrix operations can be efficiently reduced to it, such as Gaussian elimination, LUP decomposition, the determinant or the inverse of a matrix [1]. Matrix multiplication is also used as a subroutine in many computational problems that, on the face of it, have nothing to do with matrices. As a small sample illustrating the variety of applications, there are faster algorithms relying on matrix multiplication for graph transitive closure (see e.g. [1]), context free grammar parsing [21], and even learning juntas [13].

Until the late 1960s it was believed that computing the product $C$ of two $n \times n$ matrices requires essentially a cubic number of operations, as the fastest algorithm known was the naive algorithm which indeed runs in $O(n^3)$ time. In 1969, Strassen [19] excited the research community by giving the first subcubic time algorithm for matrix multiplication, running in $O(n^{2.808})$ time. This amazing discovery spawned a long line of research which gradually reduced the matrix multiplication exponent $\omega$ over time. In 1978, Pan [14] showed $\omega < 2.796$. The following year, Bini et al. [4] introduced the notion of *border rank* and obtained $\omega < 2.78$. Schönhage [17] generalized this notion in 1981, proved his $\tau$-theorem (also called the asymptotic sum inequality), and showed that $\omega < 2.548$. In the same paper, combining his work with ideas by Pan, he also showed $\omega < 2.522$. The following year, Romani [15] found that $\omega < 2.517$. The first result to break $2.5$ was by Coppersmith and Winograd [9] who obtained $\omega < 2.496$. In 1986, Strassen [20] introduced his *laser* method which allowed for an entirely new attack on the matrix multiplication problem. He also decreased the bound to $\omega < 2.479$. Three years later, Coppersmith and Winograd [10] combined Strassen's technique with a novel form of analysis based on large sets avoiding arithmetic progressions and obtained the famous bound of $\omega < 2.376$ which has remained unchanged for more than twenty years.

In 2003, Cohn and Umans [8] introduced a new, group-theoretic framework for designing and analyzing matrix multiplication algorithms. In 2005, together with Kleinberg and Szegedy [7], they obtained several novel matrix multiplication algorithms using the new framework, however they were not able to beat 2.376.

Many researchers believe that the true value of $\omega$ is 2. In fact, both Coppersmith and Winograd [10] and Cohn et al. [7] presented conjectures which if true would imply $\omega = 2$. Recently, Alon, Shpilka and Umans [2] showed that both the Coppersmith-Winograd conjecture and one of the Cohn et al. [7] conjectures contradict a variant of the widely believed *sunflower* conjecture of Erdös and Rado [12]. Nevertheless, it could be that at least the remaining Cohn et al. conjecture could lead to a proof that $\omega = 2$.

**The Coppersmith-Winograd Algorithm.** In this paper we revisit the Coppersmith-Winograd (CW) approach [10]. We give a very brief summary of the approach here; we will give a more detailed account in later sections.

One first constructs an algorithm $A$ which given $Q$-length vectors $x$ and $y$ for constant $Q$, computes $Q$ values of the form $z_k = \sum_{i,j} t_{ijk} x_i y_j$, say with $t_{ijk} \in \{0, 1\}$, using a smaller number of products than would naively be necessary. The values $z_k$ do not necessarily have to correspond to entries from a matrix product. Then, one considers the algorithm $A^n$ obtained by applying $A$ to vectors $x, y$ of length $Q^n$, recursively $n$ times as follows. Split $x$ and $y$ into $Q$ subvectors of length $Q^{n-1}$. Then run $A$ on $x$ and $y$ treating them as vectors of length $Q$ with entries that are vectors of length $Q^{n-1}$. When the product of two entries is needed, use $A^{n-1}$ to compute it. This algorithm $A^n$ is called the *nth tensor power* of $A$. Its running time is essentially $O(r^n)$ if $r$ is the number of multiplications performed by $A$.

The goal of the approach is to show that for very large $n$ one can set enough variables $x_i$, $y_j$, $z_k$ to 0 so that running $A^n$ on the resulting vectors $x$ and $y$ actually computes a matrix product. That is, as $n$ grows, some subvectors $x'$ of $x$ and $y'$ of $y$ can be thought to represent square matrices and when $A^n$ is run on $x$ and $y$, a subvector of $z$ is actually the matrix product of $x'$ and $y'$.

If $A^n$ can be used to multiply $m \times m$ matrices in $O(r^n)$ time, then this implies that $\omega \leq \log_m r^n$, so that the larger $m$ is, the better the bound on $\omega$.

Coppersmith and Winograd [10] introduced techniques which, when combined with previous techniques by Schönhage [17], allowed them to effectively choose which variables to set to 0 so that one can compute very large matrix products using $A^n$. Part of their techniques rely on partitioning the index triples $i, j, k \in [Q]^n$ into groups and analyzing how "similar" each group $g$ computation $\{z_{kg} = \sum_{i,j:\ (i,j,k) \in g} t_{ijk} x_i y_j\}_k$ is to a matrix product. The similarity measure used is called the *value* of the group.

Depending on the underlying algorithm $A$, the partitioning into groups varies and can affect the final bound on $\omega$. Coppersmith and Winograd analyzed a particular algorithm $A$ which resulted in $\omega < 2.39$. Then they noticed that if one uses $A^2$ as the basic algorithm (the "base case") instead, one can obtain the better bound $\omega < 2.376$. They left as an open problem what happens if one uses $A^3$ as the basic algorithm instead.

**Our contribution.** We give a new way to more tightly analyze the techniques behind the Coppersmith-Winograd (CW) approach [10]. We demonstrate the effectiveness of our new analysis by showing that the 8th tensor power of the CW algorithm [10] in fact gives $\omega < 2.3729$. (The conference version of this paper claimed $\omega < 2.3727$, but due to an error, this turned out to be incorrect in the fourth decimal place.)

There are two main theorems behind our approach. The first theorem takes any tensor power $A^n$ of a basic algorithm $A$, picks a particular group partitioning for $A^n$ and derives a procedure computing formulas for (lower bounds on) the values of these groups.

The second theorem assumes that one knows the values for $A^n$ and derives an efficient procedure which outputs a (nonlinear) constraint program on $O(n^2)$ variables, the solution of which gives a bound on $\omega$.

We then apply the procedures given by the theorems to the second, fourth and eighth tensor powers of the Coppersmith-Winograd algorithm, obtaining improved bounds with each new tensor power.

Similar to [10], our proofs apply to any starting algorithm that satisfies a simple uniformity requirement which we formalize later. The upshot of our approach is that now any such algorithm and its higher tensor powers can be analyzed entirely by computer. (In fact, our analysis of the 8th tensor power of the CW algorithm is done this way.) The burden is now entirely offloaded to constructing base algorithms satisfying the requirement. We hope that some of the new group-theoretic techniques can help in this regard.

**Why wasn't an improvement on CW found in the 1990s?** After all, the CW paper explicitly posed the analysis of the third tensor power as an open problem.

The answer to this question is twofold. Firstly, several people have attempted to analyze the third tensor power (from personal communication with Umans, Kleinberg and Coppersmith). As the author found out from personal experience, analyzing the third tensor power reveals to be very disappointing. In fact no improvement whatsoever can be found. This finding led some to believe that $2.376$ may be the final answer, at least for the CW algorithm.

The second issue is that with each new tensor power, the number of new values that need to be analyzed grows quadratically. For the eighth tensor power for instance, $30$ separate analyses are required! Prior to our work, each of these analyses would require a separate application of the CW techniques. It would have required an enormous amount of patience to analyze larger tensor powers, and since the third tensor power does not give any improvement, the prospects looked bleak.

**Stothers' work.** We were recently made aware of the thesis work of A. Stothers [18] in which he claims an improvement to $\omega$. (More recently, a journal paper by Davie and Stothers provides a more detailed account of Stothers' work [11]). Stothers argues that $\omega < 2.3737$ by analyzing the $4$th tensor power of the Coppersmith-Winograd construction. Our approach can be seen as a vast generalization of the Coppersmith-Winograd analysis. In the special case of even tensor powers, part of our proof has benefited from an observation of Stothers' which we will point out in the main text.

There are several differences between our approach and Stothers'. The first is relatively minor: the CW approach requires the use of some hash functions; ours are different and simpler than Stothers'. The main difference is that because of the generality of our analysis, we do not need to fully analyze all groups of each tensor power construction. Instead we can just apply our formulas in a mechanical way. Stothers, on the other hand, did a completely separate analysis of each group.

Finally, Stothers' approach only works for tensor powers up to $4$. Starting with the $5$-th tensor power, the values of some of the groups begin to depend on more variables and a more careful analysis is needed.

(Incidentally, we also obtain a better bound from the $4$th tensor power, $\omega < 2.37293$, however we believe this is an artifact of our optimization software, as we end up solving an equivalent constraint program.)

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**Preliminaries** We use the following notation: $[n] := \{1, \ldots, n\}$, and $\binom{N}{[a_i]_{i\in[k]}} := \binom{N}{a_1,\ldots,a_k}$.

We define $\omega \geq 2$ to be the infimum over the set of all reals $r$ such that $n \times n$ matrix multiplication over $\mathbb{Q}$ can be computed in $n^r$ additions and multiplications for some natural number $n$. (However, the CW approach and our extensions work over any ring.)

A *three-term arithmetic progression* is a sequence of three integers $a \leq b \leq c$ so that $b - a = c - b$, or equivalently, $a + c = 2b$. An arithmetic progression is nontrivial if $a < b < c$.

The following is a theorem by Behrend [3] improving on Salem and Spencer [16]. The subset $A$ computed by the theorem is called a *Salem-Spencer* set.

**Theorem 1.** *There exists an absolute constant $c$ such that for every $N \geq \exp(c^2)$, one can construct in poly$(N)$ time a subset $A \subset [N]$ with no three-term arithmetic progressions and $|A| > N\exp(-c\sqrt{\log N})$.*

The following lemma is needed in our analysis.

**Lemma 1.** *Let $k$ be a constant. Let $B_i$ be fixed for $i \in [k]$. Let $a_i$ for $i \in [k]$ be variables such that $a_i \geq 0$ and $\sum_i a_i = 1$. Then, as $N$ goes to infinity, the quantity*

$$\binom{N}{[a_i N]_{i\in[k]}} \prod_{i=1}^{k} B_i^{a_i N}$$

*is maximized for the choices $a_i = B_i / \sum_{j=1}^{k} B_j$ for all $i \in [k]$ and for these choices it is at least*

$$\left(\sum_{j=1}^{k} B_j\right)^N \;/\; (N+1)^k.$$

*Proof.* We will prove the lemma by induction on $k$. Suppose that $k = 2$ and consider

$$\binom{N}{aN} x^{aN} y^{N(1-a)} = y^N \binom{N}{aN} (x/y)^{aN},$$

where $x \leq y$.

When $(x/y) \leq 1$, the function $f(a) = \binom{N}{aN}(x/y)^{aN}$ of $a$ is concave for $a \leq 1/2$. Hence its maximum is achieved when $\partial f(a)/\partial a = 0$. Consider $f(a)$: it is $N!/((aN)!(N(1-a))!)(x/y)^{aN}$. We can take the logarithm to obtain $\ln f(a) = \ln(N!) + Na\ln(x/y) - \ln(aN!) - \ln((N(1-a))!)$. $f(a)$ grows exactly when $a\ln(x/y) - \ln(aN!)/N - \ln(N(1-a))!/N$ does. Taking Stirling's approximation, we obtain

$$a\ln(x/y) - \ln(aN!)/N - \ln(N(1-a))!/N = a\ln(x/y) - a\ln(a) - (1-a)\ln(1-a) - \ln N - O((\log N)/N).$$

Since $N$ is large, the $O((\log N)/N)$ term is negligible. Thus we are interested in when $g(a) = a\ln(x/y) - a\ln(a) - (1-a)\ln(1-a)$ is maximized. Because of concavity, for $a \leq 1/2$ and $x \leq y$, the function is maximized when $\partial g(a)/\partial a = 0$, i.e. when

$$0 = \ln(x/y) - \ln(a) - 1 + \ln(1-a) + 1 = \ln(x/y) - \ln(a/(1-a)).$$

Hence $a/(1-a) = x/y$ and so $a = x/(x+y)$.

Furthermore, since the maximum is attained for this value of $a$, we get that for each $t \in \{0, \ldots, N\}$ we have that $\binom{N}{t} x^t y^{N-t} \leq \binom{N}{aN} x^{aN} y^{N(1-a)}$, and since $\sum_{t=0}^{N} \binom{N}{t} x^t y^{N-t} = (x+y)^N$, we obtain that for $a = x/(x+y)$,

$$\binom{N}{aN} x^{aN} y^{N(1-a)} \geq (x+y)^N/(N+1).$$

Now let's consider the case $k > 2$. First assume that the $B_i$ are sorted so that $B_i \leq B_{i+1}$. Since $\sum_i a_i = 1$, we obtain

$$\binom{N}{[a_i]_{i\in[k]}} \prod_{i=1}^{k} B_i^{a_i N} = \left(\sum_i B_i\right)^N \binom{N}{[a_i]_{i\in[k]}} \prod_{i=1}^{k} b_i^{a_i N},$$

where $b_i = B_i / \sum_j B_j$. We will prove the claim for $\binom{N}{[a_i]_{i\in[k]}} \prod_{i=1}^k b_i^{a_i N}$, and the lemma will follow for the $B_i$ as well. Hence we can assume that $\sum_i b_i = 1$.

Suppose that we have proven the claim for $k-1$. This means that in particular

$$\binom{N - a_1 N}{[a_j N]_{j\geq 2}} \prod_{j=2}^{k} b_j^{a_j N} \geq \left( \sum_{j=2}^{k} b_j \right)^{N - a_1 N} / (N+1)^{k-1},$$

and the quantity is maximized for $a_j N/(N - a_1 N) = b_j / \sum_{j\geq 2} b_j$ for all $j \geq 2$.

Now consider $\binom{N}{a_1 N} b_1^{a_1 N} \left(\sum_{j=2}^k b_j\right)^{N - a_1 N}$. By our base case we get that this is maximized and is at least $(\sum_{j=1}^k b_j)^N/N$ for the setting $a_1 = b_1$. Hence, we will get

$$\binom{N}{[a_j N]_{j\in[k]}} \prod_{j=1}^{k} b_j^{a_j N} \geq \left( \sum_{j=1}^{k} b_j \right)^{N} / (N+1)^{k},$$

for the setting $a_1 = b_1$ and for $j \geq 2$, $a_j N/(N - a_1 N) = b_j / \sum_{j\geq 2} b_j$ implies $a_j/(1-b_1) = b_j/(1-b_1)$ and hence $a_j = b_j$. We have proven the lemma. □

## 1.1 A brief summary of the techniques used in bilinear matrix multiplication algorithms

A full exposition of the techniques can be found in the book by Bürgisser, Clausen and Shokrollahi [6]. The lecture notes by Bläser [5] are also a nice read.

**Bilinear algorithms and trilinear forms.** Matrix multiplication is an example of a trilinear form. $n \times n$ matrix multiplication, for instance, can be written as

$$\sum_{i,j\in[n]} \sum_{k\in n} x_{ik} y_{kj} z_{ij},$$

which corresponds to the equalities $z_{ij} = \sum_{k\in n} x_{ik} y_{kj}$ for all $i, j \in [n]$. In general, a trilinear form has the form $\sum_{i,j,k} t_{ijk} x_i y_j z_k$ where $i, j, k$ are indices in some range and $t_{ijk}$ are the coefficients which define the trilinear form; $t_{ijk}$ is also called a tensor. The trilinear form for the product of a $k \times m$ by an $m \times n$ matrix is denoted by $\langle k, m, n \rangle$.

Strassen's algorithm for matrix multiplication is an example of a bilinear algorithm which computes a trilinear form. A bilinear algorithm is equivalent to a representation of a trilinear form of the following form:

$$\sum_{i,j,k} t_{ijk} x_i y_j z_k = \sum_{\lambda=1}^{r} (\sum_i \alpha_{\lambda,i} x_i)(\sum_j \beta_{\lambda,j} y_j)(\sum_k \gamma_{\lambda,k} z_k).$$

Given the above representation, the algorithm is then to first compute the $r$ products $P_\lambda = (\sum_i \alpha_{\lambda,i} x_i)(\sum_j \beta_{\lambda,j} y_j)$ and then for each $k$ to compute $z_k = \sum_\lambda \gamma_{\lambda,k} P_\lambda$.

For instance, Strassen's algorithm for $2 \times 2$ matrix multiplication can be represented as follows:

$$(x_{11}y_{11} + x_{12}y_{21})z_{11} + (x_{11}y_{12} + x_{12}y_{22})z_{12} + (x_{21}y_{11} + x_{22}y_{21})z_{21} + (x_{21}y_{12} + x_{22}y_{22})z_{22} =$$

$$(x_{11} + x_{22})(y_{11} + y_{22})(z_{11} + z_{22}) + (x_{21} + x_{22})y_{11}(z_{21} - z_{22}) + x_{11}(y_{12} - y_{22})(z_{12} + z_{22}) +$$

$$x_{22}(y_{21}-y_{11})(z_{11}+z_{21})+(x_{11}+x_{12})y_{22}(-z_{11}+z_{12})+(x_{21}-x_{11})(y_{11}+y_{12})z_{22}+$$
$$(x_{12}-x_{22})(y_{21}+y_{22})z_{11}.$$

The minimum number of products $r$ in a bilinear construction is called the rank of the trilinear form (or its tensor). It is known that the rank of $2 \times 2$ matrix multiplication is 7, and hence Strassen's bilinear algorithm is optimal for the product of $2\times 2$ matrices. A basic property of the rank $R$ of matrix multiplication is that $R(\langle k,m,n\rangle) = R(\langle k,n,m\rangle) = R(\langle m,k,n\rangle) = R(\langle m,n,k\rangle) = R(\langle n,m,k\rangle) = R(\langle n,k,m\rangle)$. This property holds in fact for any tensor and the tensors obtained by permuting the roles of the $x, y$ and $z$ variables.

Any algorithm for $n \times n$ matrix multiplication can be applied recursively $k$ times to obtain a bilinear algorithm for $n^k \times n^k$ matrices, for any integer $k$. Furthermore, one can obtain a bilinear algorithm for $\langle k_1k_2, m_1m_2, n_1n_2\rangle$ by splitting the $k_1k_2 \times m_1m_2$ matrix into blocks of size $k_1 \times m_1$ and the $m_1m_2 \times n_1n_2$ matrix into blocks of size $m_1\times n_1$. The one can apply a bilinear algorithm for $\langle k_2, m_2, n_2\rangle$ on the matrix with block entries, and an algorithm for $\langle k_1, m_1, n_1\rangle$ to multiply the blocks. This composition multiplies the ranks and hence $R(\langle k_1k_2, m_1m_2, n_1n_2\rangle) \leq R(\langle k_1, m_1, n_1\rangle)\cdot R(\langle k_2, m_2, n_2\rangle)$. Because of this, $R(\langle 2^k, 2^k, 2^k\rangle) \leq (R(\langle 2,2,2\rangle))^k = 7^k$ and if $N = 2^k$, $R(\langle N,N,N\rangle) \leq 7^{\log_2 N} = N^{\log_2 7}$. Hence, $\omega \leq \log_N R(\langle N,N,N\rangle)$.

In general, if one has a bound $R(\langle k,m,n\rangle) \leq r$, then one can symmetrize and obtain a bound on $R(\langle kmn, kmn, kmn\rangle) \leq r^3$, and hence $\omega \leq 3\log_{kmn} r$.

The above composition of two matrix product trilinear forms to form a new trilinear form is called the *tensor product* $t_1 \otimes t_2$ of the two forms $t_1, t_2$. For two generic trilinear forms $\sum_{i,j,k} t_{ijk}x_iy_jz_k$ and $\sum_{i',j',k'} t'_{ijk}x_{i'}y_{j'}z_{k'}$, their tensor product is the trilinear form

$$\sum_{(i,i'),(j,j'),(k,k')} (t_{ijk}t'_{i'j'k'})x_{(i,i')}y_{(j,j')}z_{(k,k')},$$

i.e. the new form has variables that are indexed by pairs if indices, and the coordinate tensors are multiplied.

The *direct sum* $t_1 \oplus t_2$ of two trilinear forms $t_1, t_2$ is just their sum, but where the variable sets that they use are disjoint. That is, the direct sum of $\sum_{i,j,k} t_{ijk}x_iy_jz_k$ and $\sum_{i,j,k} t'_{ijk}x_iy_jz_k$ is a new trilinear form with the set of variables $\{x_{i0}, x_{i1}, y_{j0}, y_{j1}, z_{k0}, z_{k1}\}_{i,j,k}$:

$$\sum_{i,j,k} t_{ijk}x_{i0}y_{j0}z_{k0} + t'_{ijk}x_{i1}y_{j1}z_{k1}.$$

A lot of interesting work ensued after Strassen's discovery. Bini et al. [4] showed that one can extend the form of a bilinear construction to allow the coefficients $\alpha_{\lambda,i}, \beta_{\lambda,j}$ and $\gamma_{\lambda,k}$ to be linear functions of the integer powers of an indeterminate, $\epsilon$. In particular, Bini et al. gave the following construction for three entries of the product of $2 \times 2$ matrices in terms of 5 bilinear products:

$$(x_{11}y_{11}+x_{12}y_{21})z_{11}+(x_{11}y_{12}+x_{12}y_{22})z_{12}+(x_{21}y_{11}+x_{22}y_{21})z_{21}+O(\epsilon) =$$
$$(x_{12}+\epsilon x_{22})y_{21}(z_{11}+\epsilon^{-1}z_{21})+x_{11}(y_{11}+\epsilon y_{12})(z_{11}+\epsilon^{-1}z_{12})+$$
$$x_{12}(y_{11}+y_{21}+\epsilon y_{22})(-\epsilon^{-1}z_{21})+(x_{11}+x_{12}+\epsilon x_{21})y_{11}(-\epsilon^{-1}z_{12})+$$
$$(x_{12}+\epsilon x_{21})(y_{11}+\epsilon y_{22})(\epsilon^{-1}z_{12}+\epsilon^{-1}z_{21}),$$

where the $O(\epsilon)$ term hides triples which have coefficients that depend on positive powers of $\epsilon$.

The minimum number of products of a construction of this type is called the *border rank* $\tilde{R}$ of a trilinear form (or its tensor). Border rank is a stronger notion of rank and it allows for better bounds on $\omega$. Most of the properties of rank also extend to border rank, so that if $\tilde{R}(\langle k, m, n\rangle) \leq r$, then $\omega \leq 3 * \log_{kmn} r$. For instance, Bini et al. used their construction above to obtain a border rank of 10 for the product of a $2 \times 2$ by a $2 \times 3$ matrix and, by symmetry, a border rank of $10^3$ for the product of two $12 \times 12$ matrices. This gave the new bound of $\omega \leq 3 \log_{12} 10 < 2.78$.

Schönhage [17] generalized Bini et al.'s approach and proved his $\tau$-theorem (also known as the asymptotic sum inequality). Up until his paper, all constructions used in designing matrix multiplication algorithms explicitly computed a single matrix product trilinear form. Schönhage's theorem allowed a whole new family of contructions. In particular, he showed that constructions that are direct sums of rectangular matrix products suffice to give a bound on $\omega$.

**Theorem 2** (Schönhage's $\tau$-theorem). *If $\tilde{R}(\bigoplus_{i=1}^{q} \langle k_i, m_i, n_i\rangle) \leq r$ for $r > q$, then let $\tau$ be defined as $\sum_{i=1}^{q}(k_i m_i n_i)^\tau = r$. Then $\omega \leq 3\tau$.*

## 2 Coppersmith and Winograd's algorithm

We recall Coppersmith and Winograd's [10] (CW) construction:

$$\lambda^{-2} \cdot \sum_{i=1}^{q}(x_0 + \lambda x_i)(y_0 + \lambda y_i)(z_0 + \lambda z_i) - \lambda^{-3} \cdot (x_0 + \lambda^2 \sum_{i=1}^{q} x_i)(y_0 + \lambda^2 \sum_{i=1}^{q} y_i)(z_0 + \lambda^2 \sum_{i=1}^{q} z_i)+$$

$$+(\lambda^{-3} - q\lambda^{-2}) \cdot (x_0 + \lambda^3 x_{q+1})(y_0 + \lambda^3 y_{q+1})(z_0 + \lambda^3 z_{q+1}) =$$

$$\sum_{i=1}^{q}(x_i y_i z_0 + x_i y_0 z_i + x_0 y_i z_i) + (x_0 y_0 z_{q+1} + x_0 y_{q+1} z_0 + x_{q+1} y_0 z_0) + O(\lambda).$$

The construction computes a particular symmetric trilinear form. The indices of the variables are either 0, $q+1$ or some integer in $[q]$. We define

$$p(i) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i \in [q] \\ 2 & \text{if } i = q+1 \end{cases}$$

The important property of the CW construction is that for any triple $x_i y_j z_k$ in the trilinear form, $p(i) + p(j) + p(k) = 2$.

In general, the CW approach applies to any construction for which we can define an integer function $p$ on the indices so that there exists a number $P$ so that for every $x_i y_j z_k$ in the trilinear form computed by the construction, $p(i) + p(j) + p(k) = P$. We call such constructions $(p, P)$-uniform.

**Definition 1.** *Let $p$ be a function from $[n]$ to $[N]$. Let $P \in [N]$ A trilinear form $\sum_{i,j,k\in[n]} t_{ijk} x_i y_j z_k$ is $(p, P)$-uniform if whenever $t_{ijk} \neq 0$, $p(i) + p(j) + p(k) = P$. A construction computing a $(p, P)$-uniform trilinear form is also called $(p, P)$-uniform.*

Any tensor power of a $(p, P)$-uniform construction is $(p', P')$ uniform for some $p'$ and $P'$. There are many ways to define $p'$ and $P'$ in terms of $p$ and $P$. For the $K$-th tensor power the variable indices are length $K$ sequences of the original indices: $\chi = \chi[1], \ldots, \chi[K]$, $\gamma = \gamma[1], \ldots, \gamma[K]$ and $\zeta = \zeta[1], \ldots, \zeta[K]$. Then, for instance, one can pick $p'$ to be an arbitrary linear combination, $p'[\chi] = \sum_i^K a_i \cdot \chi[i]$, and similarly $p'[\gamma] = \sum_i^K a_i \cdot \gamma[i]$ and $p'[\zeta] = \sum_i^K a_i \cdot \zeta[i]$. Clearly then $P' = P \sum_i a_i$, and the $K$-th tensor power construction is $(p', P')$-uniform.

In this paper we will focus on the case where $a_i = 1$ for all $i \in [K]$, so that for any index $\psi \in \{\chi, \gamma, \zeta\}$, $p'[\psi] = \sum_i^K \psi[i]$ and $P' = PK$. Similar results can be obtained for other choices of $p'$.

The CW approach proceeds roughly as follows. Suppose we are given a $(p, P)$-uniform construction and we wish to derive a bound on $\omega$ from it. (The approach only works when the range of $p$ is at least 2.) Let $C$ be the trilinear form computed by the construction and let $r$ be the number of bilinear products performed. If the trilinear form happens to be a direct sum of different matrix products, then one can just apply the Schönhage $\tau$-theorem [17] to obtain a bound on $\omega$ in terms of $r$ and the dimensions of the small matrix products. However, typically the triples in the trilinear form $C$ cannot be partitioned into matrix products on disjoint sets of variables.

The first CW idea is to partition the triples of $C$ into groups which look like matrix products but may share variables. Then the idea is to apply procedures to remove the shared variables by carefully setting variables to 0. In the end one obtains a smaller, but not much smaller, number of independent matrix products and can use Schönhage's $\tau$-theorem.

Two procedures are used to remove the shared variables. The first one defines a random hash function $h$ mapping variables to integers so that there is a large set $S$ such that for any triple $x_i y_j z_k$ with $h(x_i), h(y_j), h(z_k) \in S$ one actually has $h(x_i) = h(y_j) = h(z_k)$. Then one can set all variables mapped outside of $S$ to 0 and be guaranteed that the triples are *partitioned* into groups according to what element of $S$ they were mapped to, and moreover, the groups do not share any variables. Since $S$ is large and $h$ maps variables independently, there is a setting of the random bits of $h$ so that a lot of triples (at least the expectation) are mapped into $S$ and are hence preserved by this partitioning step. The construction of $S$ uses the Salem-Spencer theorem and $h$ is a cleverly constructed linear function.

After this first step, the remaining nonzero triples have been partitioned into groups according to what element of $S$ they were mapped to, and the groups do not share any variables. The second step removes shared variables within each group. This is accomplished by a greedy procedure that guarantees that a constant fraction of the triples remain. More details can be found in the next section.

When applied to the CW construction above, the above procedures gave the bound $\omega < 2.388$.

The next idea that Coppersmith and Winograd had was to extend the $\tau$-theorem to Theorem 2 below using the notion of *value* $V_\tau$. The intuition is that $V_\tau$ assigns a weight to a trilinear form $T$ according to how "close" an algorithm computing $T$ is to an $O(n^{3\tau})$ matrix product algorithm.

Suppose that for some $N$, the $N$th tensor power of $T$ [1] can be reduced to $\bigoplus_{i=1}^q \langle k_i, m_i, n_i \rangle$ by substitution of variables. Then, as in [10] we introduce the constraint

$$V_\tau(T) \geq \left( \sum_{i=1}^{q} (k_i m_i n_i)^\tau \right)^{1/N}.$$

Furthermore, if $\pi$ is the cyclic permutation of the $x$,$y$ and $z$ variables in $T$, then we also have $V_\tau(T) \geq (V_\tau(T \otimes \pi T \otimes \pi^2 T))^{1/3} \geq (V_\tau(T) V_\tau(\pi T) V_\tau(\pi^2 T))^{1/3}$.

We can give a formal definition of $V_\tau(T)$ as follows. Consider all positive integers $N$, and all possible ways $\sigma$ to zero-out variables in the $N$th tensor power of $T$ so that one obtains a direct sum of matrix products $\bigoplus_{i=1}^{q(\sigma)} \langle k_i^\sigma, m_i^\sigma, n_i^\sigma \rangle$. Then we can define

$$V_\tau(T) = \limsup_{N \to \infty, \sigma} \left( \sum_{i=1}^{q(\sigma)} (k_i^\sigma m_i^\sigma n_i^\sigma)^\tau \right)^{1/N}.$$

[1]Tensor powers of trilinear forms can be defined analogously to how we defined tensor powers of an algorithm computing them.

We can argue that for any permutation of the $x, y, z$ variables $\pi$, and any $N$ there is a corresponding permutation of the zeroed out variables $\sigma$ that gives the same (under the permutation $\pi$) direct sum of matrix products. Hence $V_\tau(T) \le V_\tau(\pi T)$ and since $T$ can be replaced with $\pi T$ and $\pi$ with $\pi^{-1}$, we must have $V_\tau(T) = V_\tau(\pi T)$, thus also satisfying the inequality $V_\tau(T) \ge (V_\tau(T)V_\tau(\pi T)V_\tau(\pi^2 T))^{1/3}$.

It is clear that values are superadditive and supermultiplicative, so that $V_\tau(T_1 \otimes T_2) \ge V_\tau(T_1)V_\tau(T_2)$ and $V_\tau(T_1 \oplus T_2) \ge V_\tau(T_1) + V_\tau(T_2)$.

With this notion of value as a function of $\tau$, we can state an extended $\tau$-theorem, implicit in [10].

**Theorem 3** ([10])**.** *Let $T$ be a trilinear form such that $T = \bigoplus_{i=1}^{q} T_i$ and the $T_i$ are independent copies of the same trilinear form $T'$. If there is an algorithm that computes $T$ by performing at most $r$ multiplications for $r > q$, then $\omega \le 3\tau$ for $\tau$ given by $qV_\tau(T') = r$.*

Theorem 2 has the following effect on the CW approach. Instead of partitioning the trilinear form into matrix product pieces, one could partition it into different types of pieces, provided that their value is easy to analyze. A natural way to partition the trilinear form $C$ is to group all triples $x_i y_j z_k$ for which $(i, j, k)$ are mapped by $p$ to the same integer 3-tuple $(p(i), p(j), p(k))$. This partitioning is particularly good for the CW construction and its tensor powers: in Claim 1 we show for instance that the trilinear form which consists of the triples mapped to $(0, J, K)$ for any $J, K$ is always a matrix product of the form $\langle 1, Q, 1\rangle$ for some $Q$.

Using this extra ingredient, Coppersmith and Winograd were able to analyze the second tensor power of their construction and to improve the estimate to the current best bound $\omega < 2.376$.

In the following section we show how with a few extra ingredients one can algorithmically analyze an arbitrary tensor power of any $(p, P)$-uniform construction. (Amusingly, the algorithms involve the solution of linear systems, indicating that faster matrix multiplication algorithms can help improve the search for faster matrix multiplication algorithms.)

# 3 Analyzing arbitrary tensor powers of uniform constructions

Let $\mathcal{K} \ge 2$ be an integer. Let $p$ be an integer function with range size at least 2. We will show how to analyze the $\mathcal{K}$-tensor power of any $(p, P)$-uniform construction by proving the following theorem:

**Theorem 4.** *Given a $(p, P)$-uniform construction and the values for its $\mathcal{K}$-tensor power, the procedure in Figure 1 outputs a constraint program the solution $\tau$ of which implies $\omega \le 3\tau$.*

Consider the the $\mathcal{K}$-tensor power of a particular $(p, P)$-uniform construction. Call the trilinear form computed by the construction $C$. Let $r$ be the bound on the (border) rank of the original construction. Then $r^{\mathcal{K}}$ is a bound on the (border) rank of $C$.

The variables in $C$ have indices which are $\mathcal{K}$-length sequences of the original indices. Moreover, for every triple $x_\chi y_\gamma z_\zeta$ in the trilinear form and any particular position $\ell$ in the index sequences, $p(\chi[\ell]) + p(\gamma[\ell]) + p(\zeta[\ell]) = P$. Recall that we defined the extension $\bar{p}$ of $p$ for the $\mathcal{K}$ tensor power as $\bar{p}(\psi) = \sum_{i=1}^{K} p(\psi[i])$ for index sequence $\psi$, and that the $\mathcal{K}$ tensor power is $(\bar{p}, P\mathcal{K})$-uniform.

Now, we can represent $C$ as a sum of trilinear forms $X^I Y^J Z^K$, where $X^I Y^J Z^K$ only contains the triples $x_\chi y_\gamma z_\zeta$ in $C$ for which $\bar{p}$ maps $\chi$ to $I$, $\gamma$ to $J$ and $\zeta$ to $K$. That is, if $C = \sum_{ijk} t_{ijk} x_i y_j z_k$, then $X^I Y^J Z^K = \sum_{i,j,k:\ \bar{p}(i)=I, \bar{p}(j)=J} t_{ijk} x_i y_j z_k$. We refer to $I$,$J$,$K$ as *blocks*.

Following the CW analysis, we will later compute the value $V_{IJK}$ (as a function of $\tau$) for each trilinear form $X^I Y^J Z^K$ separately. If the trilinear forms $X^I Y^J Z^K$ didn't share variables, we could just use Theorem 2 to estimate $\omega$ as $3\tau$ where $\tau$ is given by $r^{\mathcal{K}} = \sum_{IJ} V_{IJK}(\tau)$.

1. For each $I, J, K = P\mathcal{K} - I - J$, determine the value $V_{IJK}$ of the trilinear form $\sum_{i,j:\ p(i)=I, p(j)=J} t_{ijk} x_i y_j z_k$, as a nondecreasing function of $\tau$.

2. Define variables $a_{IJK}$ and $\bar{a}_{IJK}$ for $I \leq J \leq K = P\mathcal{K} - I - J$.

3. Form the linear system: for all $I$, $A_I = \sum_J \bar{a}_{IJK}$, where $\bar{a}_{IJK} = \bar{a}_{sort(IJK)}$.

4. Determine the rank of the linear system, and if necessary, pick enough variables $\bar{a}_{IJK}$ to place in $S$ and treat as constants, so the system has full rank.

5. Solve for the variables outside of $S$ in terms of the $A_I$ and the variables in $S$.

6. Compute the derivatives $p_{I'J'K'IJK}$.

7. Form the program:

   Minimize $\tau$ subject to

   $r^{\mathcal{K}} = \frac{1}{\prod_I A_I^{A_I}} \prod_{I \leq J \leq K} \left( \frac{\bar{a}_{IJK}^{\bar{a}_{IJK}}}{a_{IJK}^{a_{IJK}}} \right)^{perm(IJK)} \cdot V_{IJK}^{perm(IJK) \cdot a_{IJK}}$,

   $\bar{a}_{IJK} \geq 0, a_{IJK} \geq 0$ for all $I, J, K$

   $\bar{a}_{I'J'K'} > 0$ if $\bar{a}_{I'J'K'} \notin S$ and there is some $\bar{a}_{IJK} \in S, p_{I'J'K'IJK} > 0$,

   $\sum_{I \leq J \leq K} perm(IJK) \cdot \bar{a}_{IJK} = 1$,

   $\bar{a}_{IJK}^{perm(IJK)} \cdot \prod_{\bar{a}_{I'J'K'} \notin S, p_{I'J'K'IJK} > 0} (\bar{a}_{I'J'K'})^{perm(I'J'K') p_{I'J'K'IJK}}$
   $= \prod_{\bar{a}_{I'J'K'} \notin S, p_{I'J'K'IJK} < 0} (\bar{a}_{I'J'K'})^{-perm(I'J'K') p_{I'J'K'IJK}}$ for all $\bar{a}_{IJK} \in S$,

   $\sum_J a_{IJK} = \sum_J \bar{a}_{IJK}$ for all $I$( unless one is setting $a_{IJK} = \bar{a}_{IJK}$).

8. Solve the program to obtain $\omega \leq 3\tau$.
   (We note that if we only have access to $V_{IJK}$ when they are evaluated at a fixed $\tau$, we can perform the minimization via a binary search, solving each step for a fixed guess for $\tau$ and decreasing the guess while a feasible solution is found.)

Figure 1: The procedure to analyze the $\mathcal{K}$ tensor power.

However, the forms can share variables. For instance, $X^I Y^J Z^K$ and $X^I Y^{J'} Z^{K'}$ share the $x$ variables mapped to block $I$. We use the CW tools to zero-out some variables until the remaining trilinear forms no longer share variables, and moreover a nontrivial number of the forms remain so that one can obtain a good estimate on $\tau$ and hence $\omega$. We outline the approach in what follows.

Take the $N$-th tensor power $C^N$ of $C$ for large $N$; we will eventually let $N$ go to $\infty$. Now the indices of the variables of $C$ are $N$-length sequences of $\mathcal{K}$ length sequences. The blocks of $C^N$ are $N$-length sequences of blocks of $C$.

We will pick (rational) values $A_I \in [0,1]$ for every block $I$ of $C$, so that $\sum_I A^I = 1$. Then we will set to zero all $x, y, z$ variables of $C^N$ the indices of which map to blocks which **do not** have exactly $N \cdot A_I$ positions of block $I$ for every $I$. (For large enough $N$, $N \cdot A_I$ is an integer.)

For each triple of blocks of $C^N$ $(\bar{I}, \bar{J}, \bar{K})$ we will consider the trilinear subform of $C^N$, $X^{\bar{I}} Y^{\bar{J}} Z^{\bar{K}}$, where as before $C^N$ is the sum of these trilinear forms.

Consider values $a_{IJK}$ for all valid block triples $I, J, K$ of $C$ which satisfy

$$A_I = \sum_J a_{IJ(P\cdot\mathcal{K}-I-J)} = \sum_J a_{JI(P\cdot\mathcal{K}-I-J)} = \sum_J a_{(P\cdot\mathcal{K}-I-J)JI}.$$

The values $a_{IJK}$ will correspond to the number of index positions $\ell$ such that any trilinear form $X^{\bar{I}} Y^{\bar{J}} Z^{\bar{K}}$ of $C^N$ we have that $\bar{I}[\ell] = I, \bar{J}[\ell] = J, \bar{K}[\ell] = K$.

The $a_{IJK}$ need to satisfy the following additional two constraints:

$$1 = \sum_I A_I = \sum_{I,J,K} a_{IJK},$$

and

$$P\mathcal{K} = 3 \sum_I I \cdot A_I.$$

We note that although the second constraint is explicitly stated in [10], it actually automatically holds as a consequence of constraint 1 and the definition of $a_{IJK}$ since

$$3\sum_I IA_I = \sum_I IA_I + \sum_J JA_J + \sum_K KA_K =$$

$$\sum_I \sum_J Ia_{IJ(P\mathcal{K}-I-J)} + \sum_J \sum_I Ja_{IJ(P\mathcal{K}-I-J)} + \sum_K \sum_J Ka_{(P\mathcal{K}-J-K),J,K} =$$

$$\sum_I \sum_J (I + J + (P\mathcal{K} - I - J))a_{IJ(P\mathcal{K}-I-J)} = P\mathcal{K} \sum_{I,J} a_{IJ(P\mathcal{K}-I-J)} = P\mathcal{K}.$$

Thus the only constraint that needs to be satisfied by the $a_{IJK}$ is $\sum_{I,J,K} a_{IJK} = 1$.

Recall that $\binom{N}{[R_i]_{i\in S}}$ denotes $\binom{N}{R_{i_1}, R_{i_2}, \ldots, R_{i_{|S|}}}$ where $i_1, \ldots, i_{|S|}$ are the elements of $S$. When $S$ is implicit, we only write $\binom{N}{[R_i]}$.

By our choice of which variables to set to 0, we get that the number of $C^N$ block triples which still have nonzero trilinear forms is

$$\binom{N}{[N \cdot A_I]} \cdot \left( \sum_{[a_{IJK}]} \prod_I \binom{N \cdot A_I}{[N \cdot a_{IJK}]_J} \right),$$

where the sum ranges over the values $a_{IJK}$ which satisfy the above constraint. This is since the number of nonzero blocks is $\binom{N}{[N\cdot A_I]}$ and the number of block triples which contain a particular $X$ block is exactly $\prod_I \binom{N\cdot A_I}{[N\cdot a_{IJK}]_J}$ for every partition of $A_I$ into $[a_{IJK}]_J$ (for $K = P\mathcal{K} - I - J$).

Let $\aleph = \sum_{[a_{IJK}]} \prod_I \binom{N\cdot A_I}{[N\cdot a_{IJK}]_J}$. The current number of nonzero block triples is $\aleph \cdot \binom{N}{[N\cdot A_I]}$.

Our goal will be to process the remaining nonzero triples by zeroing out variables sharing the same block until the remaining trilinear forms corresponding to different block triples do not share variables. Furthermore, to simplify our analysis, we would like for the remaining nonzero trilinear forms to have the same value.

The triples would have the same value if we fix for each $I$ a partition $[a_{IJK}N]_J$ of $A_I N$: Suppose that each remaining triple $X^{\bar{I}}Y^{\bar{J}}Z^{\bar{K}}$ has exactly $a_{IJK}N$ positions $\ell$ such that $\bar{I}[\ell] = I, \bar{J}[\ell] = J, \bar{K}[\ell] = K$. Then each remaining triple would have value at least $\prod_{I,J} V_{IJK}^{a_{IJK}N}$ by supermultiplicativity.

Suppose that we have fixed a particular choice of the $a_{IJK}$. We will later show how to pick a choice which maximizes our bound on $\omega$.

The number of small trilinear forms (corresponding to different block triples of $C^N$) is $\aleph' \cdot \binom{N}{[N\cdot A_I]}$, where

$$\aleph' = \prod_I \binom{N\cdot A_I}{[N\cdot a_{IJK}]_J}.$$

Let us show how to process the triples so that they no longer share variables.

Pick $M$ to be a prime which is $\Theta(\aleph)$. Let $S$ be a Salem-Spencer set of size roughly $M^{1-o(1)}$ as in the Salem-Spencer theorem. The $o(1)$ term will go to 0 when we let $N$ go to infinity. In the following we'll let $|S| = M^{1-\varepsilon}$ and in the end we'll let $\varepsilon$ go to 0, similar to [10]; this is possible since our final inequality will depend on $1/M^{\varepsilon/N}$ which goes to 1 as $N$ goes to $\infty$ and $\varepsilon$ goes to 0.

Choose random numbers $w_0, w_1, \ldots, w_N$ in $\{0, \ldots, M-1\}$.

For an index sequence $\bar{I}$, define the hash functions which map the variable indices to integers, just as in [10]:

$$b_x(\bar{I}) = \sum_{\ell=1}^{N} w_\ell \cdot \bar{I}[\ell] \mod M,$$

$$b_y(\bar{I}) = w_0 + \sum_{\ell=1}^{N} w_\ell \cdot \bar{I}[\ell] \mod M,$$

$$b_z(\bar{I}) = 1/2(w_0 + \sum_{\ell=1}^{N} (P\mathcal{K} - w_\ell \cdot \bar{I}[\ell])) \mod M.$$

Set to 0 all variables with blocks mapping to outside $S$.

For any triple with blocks $\bar{I}, \bar{J}, \bar{K}$ in the remaining trilinear form we have that $b_x(\bar{I}) + b_y(\bar{J}) = 2b_z(\bar{K}) \mod M$. Hence, the hashes of the blocks form an arithmetic progression of length 3. Since $S$ contains no nontrivial arithmetic progressions, we get that for any nonzero triple

$$b_x(\bar{I}) = b_y(\bar{J}) = b_z(\bar{K}).$$

Thus, the Salem-Spencer set $S$ has allowed us to do some partitioning of the triples.

Let us analyze how many triples remain. Since $M$ is prime, and due to our choice of functions, the $x$,$y$ and $z$ blocks are independent and are hashed uniformly to $\{0, \ldots, M-1\}$. If the $I$ and $J$ blocks of a triple

$X^I Y^J Z^K$ are mapped to the same value, so is the $K$ block. The expected fraction of triples which remain is hence

$$(M^{1-\varepsilon}/M) \cdot (1/M), \text{ which is } 1/M^{1+\varepsilon}.$$

This holds for the triples that satisfy our choice of partition $[a_{IJK}]$.

The trilinear forms corresponding to block triples mapped to the same value in $S$ can still share variables. We do some pruning in order to remove shared blocks, similar to [10], with a minor change. For each $s \in S$, process the triples hashing to $s$ separately.

We first process the triples that obey our choice of $[a_{IJK}]$, until they do not share any variables. After that we also process the remaining triples, zeroing them out if necessary. (This is slightly different from [10].)

Greedily build a list $L$ of independent triples as follows. Suppose we process a triple with blocks $\bar{I}, \bar{J}, \bar{K}$. If $\bar{I}$ is among the $x$ blocks in another triple in $L$, then set to 0 all $y$ variables with block $\bar{J}$. Similarly, if $\bar{I}$ is not shared but $\bar{J}$ or $\bar{K}$ is, then set all $x$ variables with block $\bar{I}$ to 0. If no blocks are shared, add the triple to $L$.

Suppose that when we process a triple $\bar{I}, \bar{J}, \bar{K}$, we find that it shares a block, say $\bar{I}$, with a triple $\bar{I}, \bar{J}', \bar{K}'$ in $L$. Suppose that we then eliminate all variables sharing block $\bar{J}$, and thus eliminate $U$ new triples for some $U$. Then we eliminate at least $\binom{U}{2} + 1$ pairs of triples which share a block: the $\binom{U}{2}$ pairs of the eliminated triples that share block $\bar{J}$, and the pair $\bar{I}, \bar{J}, \bar{K}$ and $\bar{I}, \bar{J}', \bar{K}'$ which share $\bar{I}$.

Since $\binom{U}{2} + 1 \geq U$, we eliminate at least as many pairs as triples. The expected number of unordered pairs of triples sharing an $X$ (or $Y$ or $Z$) block and for which at least one triple obeys our choice of $[a_{IJK}]$ is

$$\left[(1/2)\left(\binom{N}{[N \cdot A_I]}\aleph'\right)(\aleph' - 1) + \left(\binom{N}{[N \cdot A_I]}\aleph'\right)(\aleph - \aleph')\right]/M^{2+\varepsilon} = \left(\binom{N}{[N \cdot A_I]}\aleph'\right)(\aleph - \aleph'/2 - 1/2)/M^{2+\varepsilon}.$$

Thus at most this many triples obeying our choice of $[a_{IJK}]$ have been eliminated. Hence the expected number of such triples remaining after the pruning is

$$\binom{N}{[N \cdot A_I]}\aleph'/M^{1+\varepsilon}[1 - \aleph/M + \aleph'/(2M)] \geq \binom{N}{[N \cdot A_I]}\aleph'/(CM^{1+\varepsilon}),$$

for some constant $C$ (depending on how large we pick $M$ to be in terms of $\aleph$). We can pick values for the variables $w_i$ in the hash functions which we defined so that at least this many triples remain. (Picking these values determines our algorithm.)

We have that

$$\max_{[a_{IJK}]} \prod_I \binom{N \cdot A_I}{[N \cdot a_{IJK}]_J} \leq \aleph \leq poly(N) \max_{[a_{IJK}]} \prod_I \binom{N \cdot A_I}{[N \cdot a_{IJK}]_J}.$$

Hence, we will approximate $\aleph$ by $\aleph_{\max} = \max_{[a_{IJK}]} \prod_I \binom{N \cdot A_I}{[N \cdot a_{IJK}]_J}$.

We have obtained

$$\Omega\left(\binom{N}{[N \cdot A_I]}\frac{\aleph'}{\aleph_{\max}} \cdot \frac{1}{\text{poly}(N)M^{\varepsilon}}\right)$$

trilinear forms that do not share any variables and each of which has value $\prod_{I,J} V_{IJK}^{a_{IJK}N}$.

If we were to set $\aleph' = \aleph_{\max}$ we would get $\Omega\left(\frac{\binom{N}{[N \cdot A_I]}}{\text{poly}(N)M^{\varepsilon}}\right)$ trilinear forms instead. We use this setting in our analyses, though a better analysis may be possible if you allow $\aleph'$ to vary.

We will see later that the best choice of $[a_{IJK}]$ sets $a_{IJK} = a_{sort(IJK)}$ for each $I, J, K$, where $sort(IJK)$ is the permutation of $IJK$ sorting them in lexicographic order (so that $I \leq J \leq K$). Since tensor rank is invariant under permutations of the roles of the $x, y$ and $z$ variables, we also have that $V_{IJK} = V_{sort(IJK)}$ for all $I, J, K$. Let $perm(I, J, K)$ be the number of unique permutations of $I, J, K$.

Recall that $r$ was the bound on the (border) rank of $C$ given by the construction. Then, by Theorem 2, we get the inequality

$$r^{\mathcal{K}N} \geq \binom{N}{[N \cdot A_I]} \frac{\aleph'}{\aleph_{\max}} \cdot \frac{1}{\text{poly}(N) M^{\varepsilon}} \prod_{I \leq J \leq K} (V_{IJK}(\tau))^{perm(IJK) \cdot N \cdot a_{IJK}}.$$

Let $\bar{a}_{IJK}$ be the choices which achieve $\aleph_{\max}$ so that $\aleph_{\max} = \prod_I \binom{N \cdot A_I}{[N \cdot \bar{a}_{IJK}]_J}$. Then, by taking Stirling's approximation we get that

$$(\aleph'/\aleph_{\max})^{1/N} = \prod_{IJK} \frac{\bar{a}_{IJK}^{\bar{a}_{IJK}}}{a_{IJK}^{a_{IJK}}}.$$

Taking the $N$-th root, taking $N$ to go to $\infty$ and $\varepsilon$ to go to 0, and using Stirling's approximation we obtain the following inequality:

$$r^{\mathcal{K}} \geq \frac{1}{\prod_I A_I^{A_I}} \prod_{I \leq J \leq K} \left( \frac{\bar{a}_{IJK}^{\bar{a}_{IJK}}}{a_{IJK}^{a_{IJK}}} \right)^{perm(IJK)} \cdot V_{IJK}^{perm(IJK) \cdot a_{IJK}}.$$

If we set $a_{IJK} = \bar{a}_{IJK}$, we get the simpler inequality

$$r^{\mathcal{K}} \geq \prod_{I \leq J \leq K} (V_{IJK})^{perm(IJK) \cdot a_{IJK}} / \prod_I A_I^{A_I},$$

which is what we use in our application of the theorem as it reduces the number of variables and does not seem to change the final bound on $\omega$ by much.

The values $V_{IJK}$ are nondecreasing functions of $\tau$, where $\tau = \omega/3$. The inequality above gives an upper bound on $\tau$ and hence on $\omega$.

**Computing $\bar{a}_{IJK}$ and $a_{IJK}$.** Here we show how to compute the values $\bar{a}_{IJK}$ forming $\aleph_{\max}$ and $a_{IJK}$ which maximize our bound on $\omega$.

The only restriction on $a_{IJK}$ is that $A_I = \sum_J a_{IJK} = \sum_J \bar{a}_{IJK}$, and so if we know how to pick $\bar{a}_{IJK}$, we can let $a_{IJK}$ vary subject to the constraints $\sum_J a_{IJK} = \sum_J \bar{a}_{IJK}$. Hence we will focus on computing $\bar{a}_{IJK}$.

Recall that $\bar{a}_{IJK}$ is the setting of the variables $a_{IJK}$ which maximizes $\prod_I \binom{N \cdot A_I}{[N \cdot a_{IJK}]_J}$ for fixed $A_I$.

Because of our symmetric choice of the $A_I$, the above is maximized for $\bar{a}_{IJK} = \bar{a}_{sort(IJK)}$, where $sort(IJK)$ is the permutation of $I, J, K$ which sorts them in lexicographic order.

Let $perm(I, J, K)$ be the number of unique permutations of $I, J, K$. The constraint satisfied by the $a_{IJK}$ becomes

$$1 = \sum_I A_I = \sum_{I \leq J \leq K} perm(I, J, K) \cdot a_{IJK}.$$

The constraint above together with $\bar{a}_{IJK} = \bar{a}_{sort(IJK)}$ are the only constraints in the original CW paper. However, it turns out that more constraints are necessary for $\mathcal{K} > 2$.

The equalities $A_I = \sum_J \bar{a}_{IJK}$ form a system of linear equations involving the variables $\bar{a}_{IJK}$ and the fixed values $A_I$. If this system had full rank, then the $\bar{a}_{IJK}$ values (for $\bar{a}_{IJK} = \bar{a}_{sort(IJK)}$) would be determined from the $A_I$ and hence $\aleph$ would be exactly $\prod_I \binom{N \cdot A_I}{[N \cdot \bar{a}_{IJK}]_J}$, and a further maximization step would not be necessary. This is exactly the case for $\mathcal{K} = 2$ in [10]. This is also why in [10], setting $a_{IJK} = \bar{a}_{IJK}$ was necessary.

However, the system of equations may not have full rank. Because of this, let us pick a minimum set $S$ of variables $\bar{a}_{\bar{I}\bar{J}\bar{K}}$ (with $I \leq J \leq K$) so that viewing these variables as constants would make the system (in terms of $\bar{a}_{sort(IJK)}$) have full rank.

Then, all variables $\bar{a}_{IJK} \notin S$ would be determined as linear functions depending on the $A_I$ and the variables in $S$.

Consider the function $G$ of $A_I$ and the variables in $S$, defined as

$$G = \prod_I \binom{N \cdot A_I}{[N \cdot \bar{a}_{IJK}]_{\bar{a}_{IJK} \notin S}, [N \cdot \bar{a}_{IJK}]_{\bar{a}_{IJK} \in S}}.$$

$G$ is only a function of $\{\bar{a}_{IJK} \in S\}$ for fixed $\{A_i\}_i$. We want to know for what values of the variables of $S$, $G$ is maximized.

$G$ is maximized when $\prod_{IJ} (\bar{a}_{IJK} N)!$ is minimized, which in turn is minimized exactly when $F = \sum_{IJ} \ln((N\bar{a}_{IJK})!)$ is minimized, where $K = P\mathcal{K} - I - J$.

Using Stirling's approximation $\ln(n!) = n \ln n - n + O(\ln n)$, we get that $F$ is roughly equal to

$$N[\sum_{IJ} \bar{a}_{IJK} \ln(\bar{a}_{IJK}) - \bar{a}_{IJK} + \bar{a}_{IJK} \ln N + O(\log(N\bar{a}_{IJK})/N)] =$$

$$N \ln N + N[\sum_{IJ} \bar{a}_{IJK} \ln(a_{IJK}) - \bar{a}_{IJK} + O(\log(N\bar{a}_{IJK})/N)],$$

since $\sum_{IJ} \bar{a}_{IJK} = \sum_I A_I = 1$. As $N$ goes to $\infty$, for any fixed setting of the $\bar{a}_{IJK}$ variables, the $O(\log N/N)$ term vanishes, and $F$ is roughly $N \ln N + N(\sum_{IJ} \bar{a}_{IJK} \ln(\bar{a}_{IJK}) - \bar{a}_{IJK})$. Hence to minimize $F$ we need to minimize $f = (\sum_{IJ} \bar{a}_{IJK} \ln(\bar{a}_{IJK}) - \bar{a}_{IJK})$.

We want to know for what values of $\bar{a}_{IJK}$, $f$ is minimized. Since $f$ is convex for positive $a_{IJK}$, it is actually minimized when $\frac{\partial f}{\partial \bar{a}_{IJK}} = 0$ for every $\bar{a}_{IJK} \in S$. Recall that the variables not in $S$ are linear combinations of those in $S$.[2]

Taking the derivatives, we obtain for each $\bar{a}_{IJK}$ in $S$:

$$0 = \frac{\partial f}{\partial \bar{a}_{IJK}} = \sum_{I'J'K'} \ln(\bar{a}_{I'J'K'}) \frac{\partial \bar{a}_{I'J'K'}}{\partial \bar{a}_{IJK}}.$$

We can write this out as

$$1 = \prod_{I'J'K'} (\bar{a}_{I'J'K'})^{\frac{\partial \bar{a}_{I'J'K'}}{\partial \bar{a}_{IJK}}}.$$

Since each variable $\bar{a}_{I'J'K'}$ in the above equality for $\bar{a}_{IJK}$ is a linear combination of variables in $S$, the exponent $p_{I'J'K'IJK} = \frac{\partial \bar{a}_{I'J'K'}}{\partial \bar{a}_{IJK}}$ is actually a constant, and so we get a system of polynomial equality constraints which define the variables in $S$ in terms of the variables outside of $S$: for any $\bar{a}_{IJK} \in S$, we get

[2] We could have instead written $f = \sum_{IJ} \bar{a}_{IJK} \ln(\bar{a}_{IJK})$ and minimized $f$, and the equalities we would have obtained would have been exactly the same since the system of equations includes the equation $\sum_{IJ} \bar{a}_{IJK} = 1$, and although $\partial f / \partial a$ is $\sum_{IJ} \frac{\partial \bar{a}_{IJK} \ln a_{IJK}}{\partial a} = \sum_{IJ} \frac{\partial \bar{a}_{IJK}}{\partial a} (\ln \bar{a} a_{IJK} - 1)$, the $-1$ in the brackets would be canceled out: if $\bar{a}_{0,0,P\mathcal{K}} = (1 - \sum_{IJ:\,(I,J) \neq (0,0)} \bar{a}_{IJK})$, then $\frac{\partial \bar{a}_{0,0,P\mathcal{K}} \ln \bar{a}_{0,0,P\mathcal{K}}}{\partial a} = \ln(\bar{a}_{0,0,P\mathcal{K}}) \frac{\partial \bar{a}_{0,0,P\mathcal{K}}}{\partial a} + \sum_{I'J':\,(I',J') \neq (0,0)} \frac{\partial \bar{a}_{I'J'K'}}{\partial a}$.

$$\bar{a}_{IJK} \cdot \prod_{\bar{a}_{I'J'K'} \notin S, p_{I'J'K'IJK} > 0} (\bar{a}_{I'J'K'})^{p_{I'J'K'IJK}} = \prod_{\bar{a}_{I'J'K'} \notin S, p_{I'J'K'IJK} < 0} (\bar{a}_{I'J'K'})^{-p_{I'J'K'IJK}}. \tag{1}$$

Now, recall that we also have $\bar{a}_{IJK} = \bar{a}_{sort(IJK)}$ so that we can rewrite Equation 1 only in terms of the variables with $I \leq J \leq K$ without changing the arguments above:

$$\bar{a}_{IJK}^{perm(IJK)} \cdot \prod_{\bar{a}_{I'J'K'} \notin S, p_{I'J'K'IJK} > 0} (\bar{a}_{I'J'K'})^{perm(I'J'K')p_{I'J'K'IJK}} = \prod_{\bar{a}_{I'J'K'} \notin S, p_{I'J'K'IJK} < 0} (\bar{a}_{I'J'K'})^{-p_{I'J'K'IJK}perm(I'J'K')}. \tag{2}$$

Given values for the variables not in $S$, we can use (2) to get valid values for the variables in $S$, provided that for every $\bar{a}_{IJK} \in S$ and any $\bar{a}_{I'J'K'} \notin S$ with $p_{I'J'K'IJK} > 0$ we have $\bar{a}_{I'J'K'} > 0$. Also, given such values for the variables in $S$ and the corresponding values for the variables not in $S$, we obtain values for the $A_I$. For that choice of the $A_I$, $G$ is maximized for exactly the variable settings we have picked. Now all we have to do is find the correct values for the variables outside of $S$ and for $\bar{a}_{IJK}$, given the constraints $A_I = \sum_J \bar{a}_{IJK}$.

We cannot pick arbitrary values for the variables outside of $S$. They need to satisfy the following constraints:

- the obtained $A_I$ satisfy $\sum_I A_I = 1$,
- $\bar{a}_{I'J'K'} \notin S \implies \bar{a}_{I'J'K'} \geq 0$, and if $p_{I'J'K'IJK} > 0$ for some $\bar{a}_{IJK} \in S$, then $\bar{a}_{I'J'K'} > 0$,
- the variables in $S$ obtained from Equation 2 are nonnegative.

In summary, we obtain the procedure to analyze the $\mathcal{K}$ tensor power shown in Figure 1.

# 4 Analyzing the smaller tensors.

Consider the trilinear form consisting only of the variables from the $\mathcal{K}$ tensor power of $C$, with blocks $I, J, K$, where $I + J + K = P \cdot \mathcal{K}$. In this section we will prove the following theorem:

**Theorem 5.** *Given a $(p, P)$-uniform construction $C$, the procedure in Figure 2 computes the values $V_{IJK}$ for any tensor power of $C$.*

Recall that the indices of the variables of the $\mathcal{K}$ tensor power of $C$ are $\mathcal{K}$-length sequences of indices of the variables of $C$, and that the blocks of the $\mathcal{K}$ power are $\mathcal{K}$-length sequences of the blocks of $C$. Also recall that if $p$ was the function from $[n]$ to $[N]$ which maps the indices of $C$ to blocks, then we define $p^{\mathcal{K}}$ to be a function which maps the $\mathcal{K}$ power indices $\psi$ to blocks as $p^{\mathcal{K}}(\psi) = \sum_\ell p(\psi[\ell])$. We also define $p_{\mathcal{K}} : [n]^{\mathcal{K}} \to [N]^{\mathcal{K}}$ as $p_{\mathcal{K}}(\psi)[\ell] = p(\psi[\ell])$ for each $\ell \in [\mathcal{K}]$.

For any $I, J, K$ which form a valid block triple of the $\mathcal{K}$ tensor power, we consider the trilinear form $T_{I,J,K}$ consisting of all triples $x_i y_j z_k$ of the $\mathcal{K}$ tensor power of the construction for which $p^{\mathcal{K}}(i) = I, p^{\mathcal{K}}(j) = J, p^{\mathcal{K}}(k) = K$. We call an $X$ block $i$ of the $\mathcal{K}$ power *good* if $p^{\mathcal{K}}(i) = I$, and similarly, a $Y$ block $j$ and a $Z$ block $k$ are good if $p^{\mathcal{K}}(j) = J$ and $p^{\mathcal{K}}(k) = K$.

We will analyze the value $V_{IJK}$ of $T_{I,J,K}$. To do this, we first take the $\mathcal{K}N$-th tensor power of $T_{I,J,K}$, the $\mathcal{K}N$-th tensor power of $T_{K,I,J}$ and the $\mathcal{K}N$-th tensor power of $T_{J,K,I}$, and then tensor multiply these altogether. By the definition of value, $V_{I,J,K}$ is at least the $3\mathcal{K}N$-th root of the value of the new trilinear form.

Here is how we process the $\mathcal{K}N$-th tensor power of $T_{I,J,K}$. The powers of $T_{K,I,J}$ and $T_{J,K,I}$ are processed similarly.

We pick values $X_i \in [0,1]$ for each good block $i$ of the $\mathcal{K}$ tensor power of $C$ so that $\sum_i X_i = 1$. Set to 0 all $x$ variables except those that have exactly $X_i \cdot \mathcal{K}N$ positions of their ($\mathcal{K}N$-length) index mapped to $i$ by $p_{\mathcal{K}}$, for each good block $i$ of the $\mathcal{K}$ tensor power of $C$.

The number of nonzero $x$ blocks is $\binom{\mathcal{K}N}{[\mathcal{K}N\cdot X_i]_i}$.

Similarly pick values $Y_j$ for the $y$ variables, with $\sum_j Y_j = 1$, and retain only those with $Y_j\mathcal{K}N$ index positions mapped to $j$. Similarly pick values $Z_k$ for the $z$ variables, with $\sum_k Z_k = 1$, and retain only those with $Z_k\mathcal{K}N$ index positions mapped to $k$.

The number of nonzero $y$ blocks is $\binom{\mathcal{K}N}{[\mathcal{K}N\cdot Y_j]_j}$. The number of nonzero $z$ blocks is $\binom{\mathcal{K}N}{[\mathcal{K}N\cdot Z_k]_k}$.

For each $i$,$j$,$k$ that are valid block sequences of the $\mathcal{K}$ tensor power of $C$ such that $p^{\mathcal{K}}(i) = I, p^{\mathcal{K}}(j) = J, p^{\mathcal{K}}(k) = K = P\mathcal{K} - I - J$, let $\alpha_{ijk}$ be variables such that $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$ and $Z_k = \sum_i \alpha_{ijk}$.

After taking the tensor product of what is remaining of the $\mathcal{K}N$th tensor powers of $T_{I,J,K}$, $T_{K,I,J}$ and $T_{J,K,I}$, the number of $x, y$ or $z$ blocks is

$$\Gamma = \binom{\mathcal{K}N}{[\mathcal{K}N\cdot X_i]}\binom{\mathcal{K}N}{[\mathcal{K}N\cdot Y_j]}\binom{\mathcal{K}N}{[\mathcal{K}N\cdot Z_k]}.$$

The number of block triples which contain a particular $x, y$ or $z$ block is

$$\aleph = \sum_{[\alpha_{ijk}]_{ijk}} \prod_i \binom{\mathcal{K}NX_i}{[\mathcal{K}N\alpha_{ijk}]_j} \prod_j \binom{\mathcal{K}NY_j}{[\mathcal{K}N\alpha_{ijk}]_i} \prod_k \binom{\mathcal{K}NZ_k}{[\mathcal{K}N\alpha_{ijk}]_i},$$

where the sum is over the possible choices of $\alpha_{ijk}$ that respect $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$ and $Z_k = \sum_i \alpha_{ijk}$. We will approximate $\aleph$ as before by

$$\aleph_{\max} = \max_{[\alpha_{ijk}]_{ijk}} \prod_i \binom{\mathcal{K}NX_i}{[\mathcal{K}N\alpha_{ijk}]_j} \prod_j \binom{\mathcal{K}NY_j}{[\mathcal{K}N\alpha_{ijk}]_i} \prod_k \binom{\mathcal{K}NZ_k}{[\mathcal{K}N\alpha_{ijk}]_i}.$$

Let $\beta_{ijk}$ be the choice of $\alpha_{ijk}$ achieving the maximum above. With a slight abuse of notation let $\alpha_{ijk}$ be some choice of $\alpha_{ijk}$ that we will optimize over. Let for this choice of $\alpha_{ijk}$, $\aleph' = \prod_i \binom{\mathcal{K}NX_i}{[\mathcal{K}N\alpha_{ijk}]_j} \prod_j \binom{\mathcal{K}NY_j}{[\mathcal{K}N\alpha_{ijk}]_i} \prod_k \binom{\mathcal{K}NZ_k}{[\mathcal{K}N\alpha_{ijk}]_i}$

Later we will need $\Upsilon = \aleph'/\aleph_{\max}$, so let's see what it looks like as $N$ goes to $\infty$. Using Stirling's approximation, and the fact that for any fixed $j$ or $k$, $\sum_i \alpha_{ijk} = \sum_i \beta_{ijk}$, and for fixed $i$, $\sum_j \alpha_{ijk} = \sum_j \beta_{ijk}$, we get

$$\Upsilon^{1/(3\mathcal{K}N)} = \left(\frac{\aleph'}{\aleph_{\max}}\right)^{1/(3\mathcal{K}N)} \to \frac{\prod_{ij} \beta_{ijk}^{\beta_{ijk}}}{\prod_{ij} \alpha_{ijk}^{\alpha_{ijk}}}.$$

The number of triples is $\Gamma \cdot \aleph'$.

Set $M = \Theta(\aleph)$ to be a large enough prime greater than $\aleph$. Create a Salem-Spencer set $S$ of size roughly $M^{1-\varepsilon}$. Pick random values $w_0, w_1, w_2, \ldots, w_{\mathcal{K}N}$ in $\{0, \ldots, M-1\}$.

The blocks for $x, y$, or $z$ variables of the new big trilinear form are sequences of length $3\mathcal{K}N$; the first $\mathcal{K}N$ positions of a sequence contain $x$-blocks of the $\mathcal{K}$ tensor power, the second $\mathcal{K}N$ contain $y$-blocks and the last $\mathcal{K}N$ contain $z$-blocks of the $\mathcal{K}$ tensor power.

For an $x$-block sequence $i$, $y$-block sequence $j$ and $z$-block sequence $k$, we define

$$b_x(i) = \sum_{\ell=1}^{3\mathcal{K}N} w_\ell \cdot i[\ell] \mod M,$$

$$b_y(j) = w_0 + \sum_{\ell=1}^{3\mathcal{K}N} w_\ell \cdot j[\ell] \mod M,$$

$$b_z(k) = 1/2(w_0 + \sum_{\ell=1}^{3\mathcal{K}N} (P\mathcal{K} - (w_\ell \cdot k[\ell]))) \mod M.$$

We then set to 0 all variables that do not have blocks hashing to elements of $S$. Again, any surviving triple has all variables' blocks mapped to the same element of $S$. The expected fraction of triples remaining is $M^{1-\varepsilon}/M^2 = 1/M^{1+\varepsilon}$.

As before, we do the pruning of the triples mapped to each element $s$ of $S$ separately. Similarly to section 3, we greedily zero out variables first processing the triples that map to $s$ and obey the choice of $\alpha_{ijk}$, and then zeroing out any other remaining triples mapping to $s$. Just as the previous argument, after the pruning, over all $s$, we obtain in expectation at least $\Omega(\Upsilon\Gamma/(M^\varepsilon \text{ poly}(N)))$ independent triples all obeying the choice of $\alpha_{ijk}$.

Analogously to before, we will let $\varepsilon$ go to 0 and so the expected number of remaining triples is roughly $\Upsilon\Gamma/\text{ poly}(N)$. Hence we can pick a setting of the $w_i$ variables so that roughly $\Upsilon\Gamma/\text{ poly}(N)$ triples remain. We have obtained about $\Upsilon\Gamma/\text{ poly}(N)$ independent trilinear forms, each of which has value at least

$$\prod_{i,j,k,p} V_{i[p],j[p],k[p]}^{3\mathcal{K}N\alpha_{ijk}}.$$

This follows since values are supermultiplicative.

The final inequality becomes

$$V_{I,J,K}^{3\mathcal{K}N} \geq \Upsilon/\text{ poly}(N) \cdot \binom{\mathcal{K}N}{[\mathcal{K}N \cdot X_i]}\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Y_j]}\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Z_k]} \prod_{i,j,k,p} V_{i[p],j[p],k[p]}^{3\mathcal{K}N\alpha_{ijk}}.$$

$$V_{I,J,K} \geq \Upsilon^{1/(3\mathcal{K}N)}/\text{ poly}(N^{1/N}) \cdot \left(\binom{\mathcal{K}N}{[\mathcal{K}N \cdot X_i]}\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Y_j]}\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Z_k]} \prod_{i,j,k,p} V_{i[p],j[p],k[p]}^{3\mathcal{K}N\alpha_{ijk}}\right)^{1/(3\mathcal{K}N)}.$$

We want to maximize the right hand side, as $N$ goes to $\infty$, subject to the equalities $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$, $Z_k = \sum_i \alpha_{ijk}$, and $\sum_i X_i = 1$.

As $1/N^{1/N}$ goes to 1, it suffices to focus on

$$V_{I,J,K} \geq \Upsilon^{1/(3\mathcal{K}N)} \left(\binom{\mathcal{K}N}{[\mathcal{K}N \cdot X_i]}\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Y_j]}\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Z_k]} \prod_{i,j,k,p} V_{i[p],j[p],k[p]}^{3\mathcal{K}N\alpha_{ijk}}\right)^{1/(3\mathcal{K}N)}.$$

Now, since for any permutation $\pi$ on $[\mathcal{K}]$, $\prod_{i,j,k,p} V_{i[p],j[p],k[p]} = \prod_{i,j,k,p} V_{i[\pi(p)],j[\pi(p)],k[\pi(p)]}$, we can set $X_i = X_{\pi(i)}$ for any permutation $\pi$ on the block sequence. Similarly for $Y_j$ and $Z_k$. To do this, we set $\alpha_{ijk} = \alpha_{\pi(i),\pi(j),\pi(k)}$ for any permutation $\pi$.

Here we are slightly abusing the notation: whenever $\pi$ is called on a sequence of $\mathcal{K}$ numbers, it returns the permuted sequence, and whenever $\pi$ is called on a number $p$ from $[\mathcal{K}]$, $\pi(p)$ is a number from $[\mathcal{K}]$.

For a $\mathcal{K}$-length sequence $i$, let $perm(i)$ denote the number of distinct permutations over $i$. For instance, $perm(123) = 6$, whereas $perm(110) = 3$ since there are only three distinct permutations $110, 101, 011$. Similarly, for an ordered list of three $\mathcal{K}$ length sequences $i, j, k$, we let $perm(ijk)$ denote the number of distinct triples $(\pi(i), \pi(j), \pi(k))$ over all permutations $\pi$ on $\mathcal{K}$ elements. For instance, $perm(001, 220, 001) = 3$, whereas $perm(001, 210, 011) = 6$.

A set of $\mathcal{K}$-length block sequences is naturally partitioned into groups of sequences that are isomorphic in this permutation sense. Each group has a representative, namely the lexicographically smallest permutation, and all representatives are non-isomorphic. Let $S_I$ denote the set of group representatives for the set of $\mathcal{K}$-length sequences the values of which sum to $I$. ($S_J$ and $S_K$ are defined similarly.)

Similarly, a set of compatible triples of sequences can also be partitioned into groups of triples that are isomorphic under some permutation and for two triples $(i, j, k)$ and $(i', j', k')$ in different groups we have that for every permutation $\pi$, $(\pi(i), \pi(j), \pi(k)) \neq (i', j', k')$. The groups have representatives, namely the lexicographically smallest triple in the group, and all representatives are nonisomorphic. Let $S^3$ denote the set of group representatives of the triples the values of which sum to $(I, J, K)$.

Given these definitions, if $S = \{i_1, \ldots, i_s\}$, we can define the notation $\binom{N}{[n_i, perm(i)]_{i \in S}}$ as the number of ways to split up $N$ items into $perm(i_1)$ groups on $n_{i_1}$ elements, $perm(i_2)$ groups on $n_{i_2}$ elements, ..., and $perm(i_s)$ groups on $n_{i_s}$ elements. Whenever $S$ is implicit, we just write $\binom{N}{[n_i, perm(i)]}$.

Then we can rewrite the value inequality as

$V_{I,J,K} \geq \Upsilon^{1/(3\mathcal{K}N)}$.

$$\left[\binom{\mathcal{K}N}{[\mathcal{K}N \cdot X_i, perm(i)]}\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Y_j, perm(j)]} \cdot \binom{\mathcal{K}N}{[\mathcal{K}N \cdot Z_k, perm(k)]} \prod_{(i,j,k)\in S^3, p\in[\mathcal{K}]} V_{i[p],j[p],k[p]}^{3\mathcal{K}N\alpha_{ijk}perm(ijk)}\right]^{1/(3\mathcal{K}N)}.$$

We now rewrite using Stirling's approximation:

$$\binom{\mathcal{K}N}{[\mathcal{K}N \cdot X_i, perm(i)]} = (\mathcal{K}N)^{(\mathcal{K}N)} / \prod_i (\mathcal{K}N \cdot X_i)^{\mathcal{K}N \cdot X_i perm(i)} =$$

$$\mathcal{K}^{\mathcal{K}N} N^{\mathcal{K}N} / (\mathcal{K}^{\mathcal{K}N} \prod_i (NX_i)^{\mathcal{K}N \cdot X_i perm(i)}) =$$

$$N^{\mathcal{K}N} / \prod_i (NX_i)^{\mathcal{K}N \cdot X_i perm(i)} =$$

$$\prod_i perm(i)^{\mathcal{K}N \cdot X_i perm(i)} N^{\mathcal{K}N} / \prod_i (perm(i) NX_i)^{\mathcal{K}N \cdot X_i perm(i)} =$$

$$\prod_i perm(i)^{\mathcal{K}N \cdot X_i perm(i)} \binom{N}{[perm(i)NX_i]_i}^{\mathcal{K}}.$$

Similarly, we have

$$\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Y_j, perm(j)]} = \prod_j perm(j)^{\mathcal{K}N \cdot Y_j perm(j)} \binom{N}{[perm(j)NY_j]_j}^{\mathcal{K}}, \text{ and}$$

$$\binom{\mathcal{K}N}{[\mathcal{K}N \cdot Z_k, perm(k)]} = \prod_k perm(k)^{\mathcal{K}N \cdot Z_k perm(k)} \binom{N}{[perm(k)NZ_k]_k}^{\mathcal{K}}.$$

For each $ijk \in S^3$, set $V_{ijk} = \prod_{p \in [\mathcal{K}]} V_{i[p],j[p],k[p]}$. Then the inequality becomes

$$V_{I,J,K}^{3N} \geq \Upsilon^{1/\mathcal{K}} \prod_i perm(i)^{N \cdot X_i perm(i)} \binom{N}{[perm(i)NX_i]_i} \cdot \prod_j perm(j)^{N \cdot Y_j perm(j)} \binom{N}{[perm(j)NY_j]_j} \cdot$$

$$\prod_k perm(k)^{N \cdot Z_k perm(k)} \binom{N}{[perm(k)NZ_k]_k} \cdot \prod_{(i,j,k) \in S^3} V_{ijk}^{3N\alpha_{ijk} perm(ijk)}.$$

Now recall that we had a system of equalities $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$, and $Z_k = \sum_i \alpha_{ijk}$. We will show that if we restrict ourselves to the equations for $i \in S_I, j \in S_J, k \in S_K$ and if we replace each $\alpha_{ijk}$ with the $\alpha$ variable with an index which is the group representative of the group that $ijk$ is in, then if we omit any two equations, the remaining system has linearly independent equations. If the system has full rank, then each $\alpha_{ijk}$ can be represented uniquely as a linear combination of the $X_i, Y_j, Z_k$. Otherwise, we can pick a minimal number of $\alpha_{ijk}$ to view as constants (in a set $\Delta$) so that the system becomes full rank. Then we can represent all of the remaining $\alpha_{ijk}$ uniquely as linear combinations of the elements in $\Delta$ and the $X_i, Y_j, Z_k$.

The coefficient of each element $y \in \{X_i\}_i \cup \{Y_j\}_j \cup \{Z_k\}_k \cup \Delta$ in the linear combination that $\alpha_{ijk}$ is represented as is exactly $\partial\alpha_{ijk}/\partial y$. Hence, we can rewrite the inequality as

$$V_{I,J,K}^{3N} \geq \Upsilon^{1/\mathcal{K}} \binom{N}{[perm(\ell)NX_\ell]_\ell} \prod_\ell \left( perm(\ell)^{N \cdot X_\ell perm(\ell)} \prod_{i,j,k} V_{ijk}^{3NX_\ell perm(ijk)\partial\alpha_{ijk}/\partial X_\ell} \right) \cdot$$

$$\binom{N}{[perm(\ell)NY_\ell]_\ell} \prod_\ell \left( perm(\ell)^{N \cdot Y_\ell perm(\ell)} \prod_{i,j,k} V_{ijk}^{3NY_\ell perm(ijk)\partial\alpha_{ijk}/\partial Y_\ell} \right) \cdot$$

$$\binom{N}{[perm(\ell)NZ_\ell]_\ell} \prod_\ell \left( perm(\ell)^{N \cdot Z_\ell perm(\ell)} \prod_{i,j,k} V_{ijk}^{3NZ_\ell perm(ijk)\partial\alpha_{ijk}/\partial Z_\ell} \right) \prod_{y \in \Delta} \prod_{(i,j,k) \in S^3} V_{ijk}^{3Ny perm(ijk)\partial\alpha_{ijk}/\partial y}.$$

Now, we can maximize the right hand side using Lemma 1 by setting

$$nx_\ell = perm(\ell) \prod_{(i,j,k) \in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial X_\ell},$$

$$ny_\ell = perm(\ell) \prod_{(i,j,k) \in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial Y_\ell},$$

$$nz_\ell = perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial Z_\ell},$$

$$n\bar{x}_\ell = nx_\ell / \sum_j nx_j, n\bar{y}_\ell = ny_\ell / \sum_j ny_j, n\bar{z}_\ell = nz_\ell / \sum_j nz_j \text{ and}$$

$perm(i)X_i = n\bar{x}_i$, $perm(j)Y_j = n\bar{y}_j$ and $perm(k)Z_k = n\bar{z}_k$.

The inequality becomes

$V_{I,J,K} \geq \frac{\prod_{ij} \beta_{ijk}^{\beta_{ijk}}}{\prod_{ij} \alpha_{ijk}^{\alpha_{ijk}}}\cdot$

$$\left(\sum_\ell perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial X_\ell}\right)^{1/3} \left(\sum_\ell perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial Y_\ell}\right)^{1/3}\cdot$$

$$\left(\sum_\ell perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial Z_\ell}\right)^{1/3} \prod_{y\in\Delta} \prod_{(i,j,k)\in S^3} V_{ijk}^{yperm(ijk)\partial\alpha_{ijk}/\partial y}.$$

The variables in $\Delta$ are not free. They are constrained by the linear constraints that $y \geq 0$ for each $y \in \Delta$, and $\alpha_{ijk} \geq 0$ for all $\alpha_{ijk} \notin \Delta$ viewed as linear combinations of the elements of $\Delta$ (recall that all $X_i, Y_j, Z_k$ are already fixed).

The variables $\beta_{ijk}$ are also not fixed. Recall that they are the choices that achieve $\aleph_{\max}$ for the fixed choices for $X_i, Y_j, Z_k$ above. As $N$ grows, $\prod_i \binom{\mathcal{K}NX_i}{[\mathcal{K}N\alpha_{ijk}]_j} \prod_j \binom{\mathcal{K}NY_j}{[\mathcal{K}N\alpha_{ijk}]_i} \prod_k \binom{\mathcal{K}NZ_k}{[\mathcal{K}N\alpha_{ijk}]_i}$ is maximized when $\prod_{ij} \beta_{ijk}^{\beta_{ijk}}$ is minimized. Thus we can compute the values $\beta_{ijk}$ as follows.

For every $\alpha_{ijk}$ that we picked to be in $\Delta$, add the corresponding $\beta_{ijk}$ to a set $\bar{\Delta}$. Recall that the $\beta_{ijk}$ are solutions to the same linear system as the $\alpha_{ijk}$, i.e. $X_i = \sum_j \beta_{ijk}$, $Y_j = \sum_i \beta_{ijk}$, and $Z_k = \sum_i \beta_{ijk}$. Thus, we immediately get for any $\beta_{ijk} \notin \bar{\Delta}$ an expression as a linear function of the variables in $\bar{\Delta}$, the exact same linear function that corresponds to $\alpha_{ijk}$ in terms of the variables in $\Delta$. Now, to compute $\beta_{ijk}$ achieving $\aleph_{\max}$ we just form the convex system over the variables in $\bar{\Delta}$.

Minimize $\prod_{ij} \beta_{ijk}^{\beta_{ijk}}$, subject to $\beta_{ijk} \geq 0$ for all $i, j, k$.

(Above, for $\beta_{ijk} \notin \bar{\Delta}$, the LHS of the inequality is the linear function expressing $\beta_{ijk}$ in terms of the variables in $\bar{\Delta}$.)

Let $B_1 = \prod_{ij} \beta_{ijk}^{\beta_{ijk}}$ be the optimal value of the convex program above.

Now, we solve the following program over the variables in $\Delta$:

Maximize

$$\frac{1}{\prod_{ij} \alpha_{ijk}^{\alpha_{ijk}}}\cdot$$

$$\left(\sum_\ell perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial X_\ell}\right)^{1/3} \left(\sum_\ell perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial Y_\ell}\right)^{1/3}\cdot$$

$$\left(\sum_\ell perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3perm(ijk)/perm(\ell)\partial\alpha_{ijk}/\partial Z_\ell}\right)^{1/3} \prod_{y\in\Delta} \prod_{(i,j,k)\in S^3} V_{ijk}^{yperm(ijk)\partial\alpha_{ijk}/\partial y}, \text{ subject to}$$

$$\alpha_{ijk} \geq 0 \text{ for all } i, j, k.$$

Let $B_2$ be the solution of the above program. We can now return $V_{IJK} \geq B_1 \cdot B_2$.

To finish the proof we need to show that the linear system has linearly independent equations. Here we do it for the sequences over $\{0, 1, 2\}$ where in each index they sum to 2.

**Lemma 2.** *Suppose that $I > 0$. Consider the linear system $X_i = \sum_{j\,:\,(i,j,k)\in S^3} a^i_{ijk}\alpha_{ijk}$, $Y_j = \sum_{i\,:\,(i,j,k)\in S^3} b^j_{ijk}\alpha_{ijk}$ and $Z_k = \sum_{j\,:\,(i,j,k)\in S^3} c^k_{ijk}\alpha_{ijk}$ obtained by taking the system $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$ and $Z_k = \sum_j \alpha_{ijk}$ for $i \in S_I, j \in S_J, k \in S_K$ and setting $\alpha_{ijk} = \alpha_{\pi(i)\pi(j)\pi(k)}$ for the permutation $\pi$ that picks the representative of $(i, j, k)$ in $S^3$.*

*Then there is a way to omit two equations, one from the equations for $X_i$ and one from the equations for $Y_j$, so that the remaining equations over the $\alpha_{ijk}$ for $(i, j, k) \in S^3$ are linearly independent.*

In fact, *any* way to omit an equation for the $X_i$ variables and an equation for the $Y_j$ variables suffices: if for some choice $X_i, Y_j$ the remaining equations are linearly dependent, then since $X_i = \sum_k Z_k - \sum_{i'\neq i} X_{i'}$ and $Y_j = \sum_k Z_k - \sum_{j'\neq j} Y_{j'}$, then removing any other $X_{i'}, Y_{j'}$ would also result in linearly dependent equations.

*Proof.* Define $\bar{k}$ to be a sequence with exactly $\lfloor K/2 \rfloor$ twos and one 1 if $K$ is odd, otherwise no ones. Define $\bar{j}$ to be a sequence compatible to $\bar{k}$ that has one 1 and $(J-1)/2$ twos if $J$ is odd, and two ones and $(J-2)/2$ twos if $J$ is even. Define $\bar{i}$ to be the sequence compatible with $\bar{j}$ and $\bar{k}$. Pick the sorting of the sequences so that $(\bar{i}, \bar{j}, \bar{k}) \in S^3$. Omit the equations for $X_{\bar{i}}$ and $Y_{\bar{j}}$.

Now suppose for contradiction that there are coefficients $x_i, y_j, z_k$ so that for every $(i, j, k) \in S^3$, $x_i a^i_{ijk} + y_j b^j_{ijk} + z_k c^k_{ijk} = 0$. We want to show that $x_i = y_j = z_k = 0$ for all $i, j, k$.

We will describe a procedure that takes three compatible sequences $i, j, k$, assuming that $x_i = y_j = z_k = 0$ and transforming them into new compatible sequences $i', j', k'$ where $k'$ has one less 2 than $k$ and the number of 2s in $i$ and $j$ did not increase when going to $i'$ and $j'$. The procedure proceeds by taking one of the following steps:

1. Suppose there are positions $r, s$ so that $i[r] = j[r] = 1, i[s] = j[s] = 0, k[r] = 0, k[s] = 2$. Then let $i' = i$, $j'[r] = 0, j'[s] = 1, j'[t] = j[t]$ for $t \notin \{r, s\}$, and $k'[r] = k'[s] = 1, k'[t] = k[t]$ for $t \notin \{r, s\}$.

   Note that $i', j', k'$ are compatible, and that $j'$ and $j$ have the same number of 1s and 2s, so that after sorting, the equation $x_{i'} a^{i'}_{i'j'k'} + y_{j'} b^{j'}_{i'j'k'} + z_{k'} c^{k'}_{i'j'k'} = 0$ implies that $z_{k'} c^{k'}_{i'j'k'} = 0$ and hence $z_{k'} = 0$.

2. Suppose there are positions $r, s, t$ so that $i[r] = 0$,$i[s] = 1$,$i[t] = 0$, $j[r] = 2$,$j[s] = 0$,$j[t] = 0$, $k[r] = 0$,$k[s] = 1$,$k[t] = 2$. Then let $i' = i$, $j'[r] = 1$,$j'[s] = 0$,$j'[t] = 1$ and $k'[r] = 1$,$k'[s] = 1$,$k'[t] = 1$; on all other positions $j' = j$ and $k' = k$.

   Note that $i', j', k'$ are compatible. Consider the compatible $i'', j'', k''$ with $i'' = i$, $j''[r] = 1, j''[s] = 1, j''[t] = 0$, $k''[r] = 1, k''[s] = 0, k''[t] = 2$ and $k'' = k, j'' = j$ otherwise. Since $i'' = i$ and since $k''$ has the same number of 1s and 2s as $k$, the equation for $\alpha_{i'',j'',k''}$ gives $y_{j''} = 0$. Since $i' = i''$ and since $j'$ has the same number of 1s and 2s as $j''$, the equation for $\alpha_{i',j',k'}$ gives $z_{k''} = 0$.

3. Suppose that there are positions $r, s, t$ so that $i[r] = 2$,$i[s] = 0$,$i[t] = 0$, $j[r] = 0$,$j[s] = 1$,$j[t] = 0$, $k[r] = 0$,$k[s] = 1$,$k[t] = 2$. Then let $j' = j$, $i'[r] = 1$,$i'[s] = 0$,$i'[t] = 1$ and $k'[r] = 1$,$k'[s] = 1$,$k'[t] = 1$; on all other positions $i' = i$ and $k' = k$.

   Note that $i', j', k'$ are compatible. Consider the compatible $i'', j'', k''$ with $j'' = j$, $i''[r] = 1, i''[s] = 1, i''[t] = 0$, $k''[r] = 1, k''[s] = 0, k''[t] = 2$ and $k'' = k, i'' = i$ otherwise. Since $j'' = j$ and since $k''$

has the same number of 1s and 2s as $k$, the equation for $\alpha_{i'',j'',k''}$ gives $x_{i''} = 0$. Since $j' = j''$ and since $i'$ has the same number of 1s and 2s as $i''$, the equation for $\alpha_{i',j',k'}$ gives $z_{k''} = 0$.

Now we show that one of the above steps is always applicable if we keep running the procedure starting from $\bar{i}, \bar{j}, \bar{k}$, and as long as $k$ has more than the minimum number of 2s it can possibly have. First, by construction, $\bar{j}$ contains at least one 1. The procedure never decreases the number of 1s in $i$ and $j$ so that there is always at least one 1 in $j$. If there is a 1 in the same position in $i$ and $j$, then step 1 can be applied. Otherwise, wherever $j$ is a 1, $i$ is a 0. If $i$ contains a 2, then step 3 can be applied. If $i$ does not contain a 2, then $i$ contains a 1 since $I > 0$, and if $j$ contains a 2, then step 2 can be applied. Otherwise, $i$ and $j$ contain only 0s and 1s in distinct positions. However, then $k$ contains the maximum number of 1s that it can have, and hence the minimum number of 2s. □

### 4.1 Reducing the number of variables via recursion.

The number of variables in the above approach is roughly the number of triples in $S^3$. Consider the number of $\mathcal{K}$-length sequences in $S_I$. Each sequence is determined by the number of ways to represent $I$ as the sum of $\mathcal{K}$ integers from $\{0, \dots, n\}$ and this is no more than $I^n$. For every such choice only some choices of sequences in $S_J$ are compatible. However, even if we ignore compatibility, the number of triples in $S^3$ is never more than $(IJ)^n = O((\mathcal{K}n)^{2n})$. For the special case of $n = 2, P = 2$ as in the Coppersmith-Winograd construction, the only sequences of $S_J$ compatible with a sequence $s$ in $S_I$ are those that have 0 wherever $s$ is 2. Hence the only variability is wherever $s$ is 0 or 1, and so the number of triples in $S^3$ is $O(I^2 J) \leq O(\mathcal{K}^3)$.

Here we consider a recursive approach inspired by an observation by Stothers. We show that this approach is viable for even tensor powers and that it reduces the number of variables in the value computation substantially, from $O(\mathcal{K}^{2n})$ to $O(\mathcal{K}^2)$. This is significant even for the case of the Coppersmith-Winograd construction when the number of variables was $\Theta(\mathcal{K}^3)$.

Here we outline the approach. Suppose that we have analyzed the values for some powers $\mathcal{K}'$ and $\mathcal{K} - \mathcal{K}'$ of the trilinear form from the construction with $\mathcal{K}' < \mathcal{K}$. We will show how to inductively analyze the values for the $\mathcal{K}$ power, using the values for these smaller powers.

Consider the $\mathcal{K}$ tensor power of the trilinear form $C$. It can actually be viewed as the tensor product of the $\mathcal{K}'$ and $\mathcal{K} - \mathcal{K}'$ tensor powers of $C$.

Recall that the indices of the variables of the $\mathcal{K}$ tensor power of $C$ are $\mathcal{K}$-length sequences of indices of the variables of $C$. Also recall that if $p$ was the function which maps the indices of $C$ to blocks, then we define $p^{\mathcal{K}}$ to be a function which maps the $\mathcal{K}$ power indices $\psi$ to blocks as $p^{\mathcal{K}}(\psi) = \sum_{\ell} p(\psi[\ell])$.

An index of a variable in the $\mathcal{K}$ tensor power of $C$ can also be viewed as a pair $(l, m)$ such that $l$ is an index of a variable in the $\mathcal{K}'$ tensor power of $C$ and $m$ is an index of a variable in the $\mathcal{K} - \mathcal{K}'$ tensor power of $C$. Hence we get that $p^{\mathcal{K}}((l, m)) = p^{\mathcal{K}'}(l) + p^{\mathcal{K}-\mathcal{K}'}(m)$.

For any $I, J, K$ which form a valid block triple of the $\mathcal{K}$ tensor power, we consider the trilinear form $T_{I,J,K}$ consisting of all triples $x_i y_j z_k$ of the $\mathcal{K}$ tensor power of the construction for which $p^{\mathcal{K}}(i) = I, p^{\mathcal{K}}(j) = J, p^{\mathcal{K}}(k) = K$.

$T_{I,J,K}$ consists of the trilinear forms $T_{i,j,k} \otimes T_{I-i,J-j,K-k}$ for all $i, j, k$ that form a valid block triple for the $\mathcal{K}'$ power, and such that $I - i, J - j, K - k$ form a valid block triple for the $\mathcal{K} - \mathcal{K}'$ power. Call such blocks $i, j, k$ *good*. Then:

$$T_{IJK} = \sum_{\text{good } ijk} T_{i,j,k} \otimes T_{I-i,J-j,K-k}.$$

1. Define variables $\alpha_{ijk}$ for $(i,j,k) \in S^3$ and $X_i, Y_j, Z_k$ for all $i,j,k \in S$.
2. Form the linear system consisting of
$X_i = \sum_{j:(i,j,k)\in S^3} perm(ijk)/perm(i)\alpha_{ijk}$,
$Y_j = \sum_{i:(i,j,k)\in S^3} perm(ijk)/perm(j)\alpha_{ijk}$ and
$Z_k = \sum_{i:(i,j,k)\in S^3} perm(ijk)/perm(k)\alpha_{ijk}$, where $i,j$ range over elements of $S_I$ and $S_J$ respectively with the lexicographically smallest element of $S_I$ and $S_J$ omitted, and $k$ ranges over all elements of $S_K$.
3. Determine the rank of the system.
4. If the system does not have full rank, then pick enough variables $\alpha_{ijk}$ to treat as constants; place them in a set $\Delta$.
5. Solve the system for the variables outside of $\Delta$ in terms of the ones in $\Delta$ and $X_i, Y_j, Z_k$. Now we have $\alpha_{ijk} = \alpha_{ijk}([X_i],[Y_j],[X_k], y \in \Delta)$.
6. Let $V_{i,j,k} = \prod_{p\in[\mathcal{K}]} V_{i[p],j[p],k[p]}$. Compute for every $\ell$,
$$nx_\ell = perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3\frac{perm(ijk)}{perm(\ell)}\frac{\partial\alpha_{ijk}}{\partial X_\ell}},$$
$$ny_\ell = perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3\frac{perm(ijk)}{perm(\ell)}\frac{\partial\alpha_{ijk}}{\partial Y_\ell}},$$
$$nz_\ell = perm(\ell) \prod_{(i,j,k)\in S^3} V_{ijk}^{3\frac{perm(ijk)}{perm(\ell)}\frac{\partial\alpha_{ijk}}{\partial Z_\ell}}.$$
7. Compute for every variable $y \in \Delta$,
$$ny = \prod_{(i,j,k)\in S^3} V_{ijk}^{perm(ijk)\frac{\partial\alpha_{ijk}}{\partial y}}.$$
8. Compute for each $\alpha_{ijk}$ its setting $\alpha_{ijk}(\Delta)$ as a function of the $y \in \Delta$ when $perm(\ell)X_\ell = nx_\ell/\sum_i nx_i$, $perm(\ell)Y_\ell = ny_\ell/\sum_j ny_j$ and $perm(\ell)Z_\ell = nz_\ell/\sum_k nz_k$.
9. Then set
$$V'_{IJK} = (\sum_\ell nx_\ell)^{1/3}(\sum_\ell ny_\ell)^{1/3}(\sum_\ell nz_\ell)^{1/3}\prod_{y\in\Delta} ny^y / \prod_{ij} \alpha_{ijk}^{\alpha_{ijk}},$$
as a function of $y \in \Delta$.
10. Form the following linear constraints $L$ on $y \in \Delta$
$$y \geq 0 \text{ for all } y \in \Delta,$$
$$\alpha_{ijk}(\Delta) \geq 0 \text{ for every } \alpha_{ijk} \notin \Delta.$$
11. Solve the convex program, and let its solution be $B_1$:
Minimize $\prod_{ij} \alpha_{ijk}^{\alpha_{ijk}}$ subject to $L$.
12. Solve the program, and let its solution be $B_2$:
Maximize $V'_{IJK}$ subject to $L$.
13. Return that $V_{IJK} \geq B_1 \cdot B_2$.



Figure 2: The procedure for computing lower bounds on $V_{IJK}$ for arbitrary tensor powers.

(The sum above is a regular sum, not a disjoint sum, so the trilinear forms in it may share indices.) The above decomposition of $T_{IJK}$ was first observed by Stothers [18, 11].

Let $Q_{ijk} = T_{ijk} \otimes T_{I-i,J-j,K-k}$. By supermultiplicativity, the value $W_{ijk}$ of $Q_{ijk}$ satisfies $W_{ijk} \geq V_{ijk}V_{I-i,J-j,K-k}$. If the trilinear forms $Q_{ijk}$ didn't share variables, then we would immediately obtain a lower bound on the value $V_{IJK}$ as $\sum_{ijk} V_{ijk}V_{I-i,J-j,K-k}$. However, the trilinear forms $Q_{ijk}$ may share variables, and we'll apply the techniques from the previous section to remove the dependencies.

To analyze the value $V_{IJK}$ of $T_{I,J,K}$, we first take the $N$-th tensor power of $T_{I,J,K}$, the $N$-th tensor power of $T_{K,I,J}$ and the $N$-th tensor power of $T_{J,K,I}$, and then tensor multiply these altogether. By the definition of value, $V_{I,J,K}$ is at least the $3N$-th root of the value of the new trilinear form.

Here is how we process the $N$-th tensor power of $T_{I,J,K}$. The powers of $T_{K,I,J}$ and $T_{J,K,I}$ are processed similarly.

We pick values $X_i \in [0, 1]$ for each block $i$ of the $\mathcal{K}'$ tensor power of $C$ so that $\sum_i X_i = 1$. Set to 0 all $x$ variables except those that have exactly $X_i \cdot N$ positions of their index which are mapped to $(i, I-i)$ by $(p^{\mathcal{K}'}, p^{\mathcal{K}-\mathcal{K}'})$, for all $i$.

The number of nonzero $x$ blocks is $\binom{N}{[N \cdot X_i]_i}$.

Similarly pick values $Y_j$ for the $y$ variables, with $\sum_j Y_j = 1$, and retain only those with $Y_j$ index positions mapped to $(j, J-j)$. Similarly pick values $Z_k$ for the $z$ variables, with $\sum_k Z_k = 1$, and retain only those with $Z_k$ index positions mapped to $(k, K-k)$.

The number of nonzero $y$ blocks is $\binom{N}{[N \cdot Y_j]_j}$. The number of nonzero $z$ blocks is $\binom{N}{[N \cdot Z_k]_k}$.

For $i, j, k = P\mathcal{K}' - i - j$ which are valid blocks of the $\mathcal{K}'$ tensor power of $C$ with good $i, j, k$, let $\alpha_{ijk}$ be variables such that $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$ and $Z_k = \sum_i \alpha_{ijk}$.

After taking the tensor product of what is remaining of the $N$th tensor powers of $T_{I,J,K}$, $T_{K,I,J}$ and $T_{J,K,I}$, the number of $x, y$ or $z$ blocks is

$$\Gamma = \binom{N}{[N \cdot X_i]}\binom{N}{[N \cdot Y_j]}\binom{N}{[N \cdot Z_k]}.$$

The number of block triples which contain a particular $x, y$ or $z$ block is

$$\aleph = \sum_{\{\alpha_{ijk}\}} \prod_i \binom{NX_i}{[N\alpha_{ijk}]_j} \prod_j \binom{NY_j}{[N\alpha_{ijk}]_i} \prod_k \binom{NZ_k}{[N\alpha_{ijk}]_i},$$

where the sum is over all the possible choices of $\alpha_{ijk}$ satisfying $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$ and $Z_k = \sum_i \alpha_{ijk}$.

Hence the number of triples is $\Gamma \cdot \aleph$. As in Section 3, we will focus on a choice for $\alpha_{ijk}$, over which we will optimize, and let $\aleph'$ be the value of the summand in $\aleph$ corresponding to the choice $\alpha_{ijk}$. We also let $\beta_{ijk}$ be the choice that maximizes $\aleph$ for fixed $X_i, Y_j, Z_k$ as $N$ goes to infinity. We will approximate $\aleph$ by

$$\aleph_{\max} = \prod_i \binom{NX_i}{[N\alpha_{ijk}]_j} \prod_j \binom{NY_j}{[N\alpha_{ijk}]_i} \prod_k \binom{NZ_k}{[N\alpha_{ijk}]_i},$$

since $\aleph_{\max}$ is within a poly$(N)$ factor of $\aleph$.

Again, we let $\Upsilon = \aleph'/\aleph_{\max}$.

Set $M = \Theta(\aleph)$ to be a large enough prime greater than $\aleph$. Create a Salem-Spencer set $S$ of size roughly $M^{1-\varepsilon}$. Pick random values $w_0, w_1, w_2, \ldots, w_{3N}$ in $\{0, \ldots, M-1\}$.

The blocks for $x, y$, or $z$ variables of the new big trilinear form are sequences of length $3N$; the first $N$ positions of a sequence contain pairs $(i, I-i)$, the second $N$ contain pairs $(j, J-j)$ and the last $N$ contain pairs $(k, K-k)$. We can thus represent the block sequences $I$ of the $\mathcal{K}$ tensor power as $(I_1, I_2)$ where $I_1$ is a sequence of length $3N$ of blocks of the $\mathcal{K}'$ power of $C$ and $I_2$ is a sequence of length $3N$ of blocks of the $\mathcal{K}-\mathcal{K}'$ power of $C$ (the first $N$ are $x$ blocks, the second $N$ are $y$ blocks and the third $N$ are $z$ blocks).

For a particular block sequence $I = (I_1, I_2)$, we define the hash functions that depend only on $I_1$:

$$b_x(I) = \sum_{\ell=1}^{3N} w_\ell \cdot I_1[\ell] \mod M,$$

$$b_y(I) = w_0 + \sum_{\ell=1}^{3N} w_\ell \cdot I_1[\ell] \mod M,$$

$$b_z(I) = 1/2(w_0 + \sum_{\ell=1}^{3N}(P\mathcal{K}' - (w_\ell \cdot I_1[\ell]))) \mod M.$$

We then set to 0 all variables that do not have blocks hashing to elements of $S$. Again, any surviving triple has all variables' blocks mapped to the same element of $S$. The expected fraction of triples remaining is $M^{1-\varepsilon}/M^2 = 1/M^{1+\varepsilon}$. This also holds for the triples that have $\alpha_{ijk}$ positions in which they look like $x^i y^j z^k$.

As before, we do the pruning of the triples mapped to each element of $S$ separately, first zeroing out triples satisfying the choice for $\alpha_{ijk}$ and then any remaining ones. The number of remaining block triples over all elements of $S$ is $\Omega(\Upsilon\Gamma\aleph_{\max}/M^{1+\varepsilon}) = \Omega(\Upsilon\Gamma/(\operatorname{poly}(N)M^{\varepsilon}))$. Analogously to before, we will let $\varepsilon$ go to 0, and so the expected number of remaining triples is roughly $\Upsilon\Gamma/\operatorname{poly}(N)$. Hence we can pick a setting of the $w_i$ variables so that roughly $\Upsilon\Gamma/\operatorname{poly}(N)$ triples remain. We have obtained about $\Upsilon\Gamma/\operatorname{poly}(N)$ independent trilinear forms, each of which has value at least

$$\prod_{i,j,k} (V_{i,j,k} \cdot V_{I-i,J-j,K-k})^{3N\alpha_{ijk}}.$$

This follows since values are supermultiplicative.

The final inequality becomes

$$V_{I,J,K}^{3N} \geq (\Upsilon/\operatorname{poly}(N)) \cdot \binom{N}{[N\cdot X_i]}\binom{N}{[N\cdot Y_j]}\binom{N}{[N\cdot Z_k]} \prod_{i,j,k}(V_{i,j,k}\cdot V_{I-i,J-j,K-k})^{3N\alpha_{ijk}}.$$

$$V_{I,J,K} \geq (\Upsilon^{1/(3N)}/\operatorname{poly}(N^{1/N}))\cdot \left(\binom{N}{[N\cdot X_i]}\binom{N}{[N\cdot Y_j]}\binom{N}{[N\cdot Z_k]} \prod_{i,j,k}(V_{i,j,k}\cdot V_{I-i,J-j,K-k})^{3N\alpha_{ijk}}\right)^{1/(3N)}.$$

We want to maximize the right hand side, as $N$ goes to $\infty$, subject to the equalities $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$, $Z_k = \sum_i \alpha_{ijk}$, $\sum_i X_i = 1$, $X_i = \sum_j \beta_{ijk}$, $Y_j = \sum_i \beta_{ijk}$, $Z_k = \sum_i \beta_{ijk}$, and subject to $\beta_{ijk}$ maximizing $\aleph$.

As $N$ goes to infinity, $N^{1/N}$ goes to 1, and hence it suffices to analyze

$$V_{I,J,K} \geq \Upsilon^{1/(3N)} \left(\binom{N}{[N\cdot X_i]}\binom{N}{[N\cdot Y_j]}\binom{N}{[N\cdot Z_k]} \prod_{i,j,k}(V_{i,j,k}\cdot V_{I-i,J-j,K-k})^{3N\alpha_{ijk}}\right)^{1/(3N)}.$$

Consider the equalities $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$, and $Z_k = \sum_i \alpha_{ijk}$. If we fix $X_i, Y_j, Z_k$ over all $i, j, k$, this forms a linear system. As in our original analysis, we can remove an equation for some $X_i$ and an equation for some $Y_j$, and we will hope that the remaining equations are linearly independent. This will not necessarily be the case but we will prove that, with some modification, it is the case when $\mathcal{K}$ is even. In the following let's assume that the equations are linearly independent.

The linear system does not necessarily have full rank, and so we pick a minimum set $\Delta$ of variables $\alpha_{ijk}$ so that if they are treated as constants, the linear system has full rank, and the variables outside of $\Delta$ can be written as linear combinations of variables in $\Delta$ and of $X_i, Y_j, Z_k$.

Now we have that for every $\alpha_{ijk}$,

$$\alpha_{ijk} = \sum_{y \in \Delta \cup \{X_{i'}, Y_{j'}, Z_{k'}\}_{i',j',k'}} y \frac{\partial \alpha_{ijk}}{\partial y},$$

where for all $\alpha_{ijk} \notin \Delta$ we use the linear function obtained from the linear system.

Let $\delta_{ijk} = \sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y}$. Let $W_{i,j,k} = V_{i,j,k} \cdot V_{I-i,J-j,K-k}$. Then,

$$W_{i,j,k}^{\alpha_{ijk}} = W_{ijk}^{\sum_i X_i \frac{\partial \alpha_{ijk}}{\partial X_i}} W_{ijk}^{\sum_i Y_j \frac{\partial \alpha_{ijk}}{\partial Y_j}} W_{ijk}^{\sum_k Z_k \frac{\partial \alpha_{ijk}}{\partial Z_k}} W_{i,j,k}^{\delta_{ijk}}.$$

Define $nx_\ell = \prod_{i,j,k} W_{ijk}^{3\frac{\partial \alpha_{ijk}}{\partial X_\ell}}$ for any $\ell$. Set $\bar{nx}_\ell = \frac{nx_\ell}{\sum_{\ell'} nx_{\ell'}}$.

Define similarly $ny_\ell = \prod_{i,j,k} W_{ijk}^{3\frac{\partial \alpha_{ijk}}{\partial Y_\ell}}$ and $nz_\ell = \prod_{i,j,k} W_{ijk}^{3\frac{\partial \alpha_{ijk}}{\partial Z_\ell}}$, setting $\bar{ny}_\ell = \frac{ny_\ell}{\sum_{\ell'} ny_{\ell'}}$ and $\bar{nz}_\ell = \frac{nz_\ell}{\sum_{\ell'} nz_{\ell'}}$.

Consider the right hand side of our inequality for $V_{IJK}$:

$$\Upsilon^{1/(3N)} \binom{N}{[N \cdot X_i]} \binom{N}{[N \cdot Y_j]} \binom{N}{[N \cdot Z_k]} \prod_{i,j,k} W_{i,j,k}^{3N\alpha_{ijk}} =$$

$$\Upsilon^{1/(3N)} \binom{N}{[N \cdot X_i]} \prod_\ell nx_\ell^{NX_\ell} \binom{N}{[N \cdot Y_j]} \prod_\ell ny_\ell^{NY_\ell} \cdot \binom{N}{[N \cdot Z_k]} \prod_\ell nz_\ell^{NZ_\ell} \prod_{i,j,k} W_{i,j,k}^{(\sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y})}.$$

By Lemma 1, the above is maximized for $X_\ell = \bar{nx}_\ell$, $Y_\ell = \bar{ny}_\ell$, and $Z_\ell = \bar{nz}_\ell$ for all $\ell$, and for these settings $\binom{N}{[N \cdot X_i]} \prod_\ell nx_\ell^{NX_\ell}$, for instance, is essentially $(\sum_\ell nx_\ell)^N / \text{poly}(N)$, and hence after taking the $3N$th root and letting $N$ go to $\infty$, we obtain

$$V_{I,J,K} \geq \Upsilon^{1/(3N)} (\sum_\ell nx_\ell)^{1/3} (\sum_\ell ny_\ell)^{1/3} (\sum_\ell nz_\ell)^{1/3} \prod_{i,j,k} W_{i,j,k}^{(\sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y})}.$$

If $\Delta = \emptyset$ and if we pick $\alpha_{ijk} = \beta_{ijk}$, then the above gives a complete formula for $V_{I,J,K}$. Otherwise, to maximize the lower bound on $V_{I,J,K}$ we need to pick values for the variables in $\Delta$ so that the values for the variables outside of $\Delta$ (which are obtained from our settings of the $X_i, Y_j, Z_k$ and the values for the $\Delta$ variables) are nonnegative, and the variables $\beta_{ijk}$ maximize $\aleph$ for the fixed values of $X_i, Y_j, Z_k$.

To obtain the values $\beta_{ijk}$ we proceed as before. First we put a variable $\beta_{ijk}$ in $\bar{\Delta}$ iff the corresponding $\alpha_{ijk}$ is in $\Delta$. We also then represent the remaining $\beta_{ijk}$ as a linear combination of the variables in $\bar{\Delta}$ (using the same linear function as for the corresponding $\alpha_{ijk}$ in terms of the variables in $\Delta$). Then we solve the following systems.

Let $\bar{V}_{IJK} = (\sum_\ell nx_\ell)^{1/3} (\sum_\ell ny_\ell)^{1/3} (\sum_\ell nz_\ell)^{1/3} \prod_{i,j,k} W_{i,j,k}^{(\sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y})}$.

1. a convex program over the variables in $\bar{\Delta}$ that minimizes $\prod_{ij} \beta_{ijk}^{\beta_{ijk}}$ under the linear constraints that all $\beta_{ijk} \geq 0$; let the solution of this system be $B_1$;

2. a concave program that over the variables in $\Delta$ that maximizes $\bar{V}_{IJK} / \prod_{ij} \alpha_{ijk}^{\alpha_{ijk}}$ subject to the linear constraints that all $\alpha_{ijk} \geq 0$; let the solution be $B_2$;

Finally, return that $V_{IJK} \geq B_1 B_2$.

The approach allows us to obtain a procedure similar to the one we had before but with fewer variables. Next, we proceed to focus on the case when $\mathcal{K}$ is even. There we show how to further reduce the number of variables, now by a constant factor, and in the process to make sure that the linear system defined by the dependence of the $X_i, Y_j, Z_k$ variables on $\alpha_{ijk}$ is linearly independent.

### 4.2 Even tensor powers

Given the approach outlined in the previous subsection, we will outline the changes that occur for even powers so that we both reduce the number of variables and make sure that the linear system has linearly independent equations. We prove the following theorem.

**Theorem 6.** *Given a $(p, P)$-uniform construction $C$, the procedure in Figure 3 computes lower bounds on the values $V_{IJK}$ for any even tensor power $\mathcal{K}$ of $C$, given lower bounds on the values for the $\mathcal{K}/2$ power. Hence in $O(\log \mathcal{K})$ iterations of the procedure, one can compute lower bounds for the values of any tensor power which is a power of 2.*

To analyze the value $V_{IJK}$ of $T_{I,J,K}$, we first take the $2N$-th tensor power (instead of the $N$th) of $T_{I,J,K}$, the $2N$-th tensor power of $T_{K,I,J}$ and the $2N$-th tensor power of $T_{J,K,I}$, and then tensor multiply these altogether. By the definition of value, $V_{I,J,K}$ is at least the $6N$-th root of the value of the new trilinear form.

Here is how we process the $2N$-th tensor power of $T_{I,J,K}$, the powers of $T_{K,I,J}$ and $T_{J,K,I}$ are processed similarly.

We pick values $X_i \in [0, 1]$ for each block $i$ of the $\mathcal{K}' = \mathcal{K}/2$ tensor power of $C$ so that $\sum_i X_i = 2$ and $X_i = X_{I-i}$ for every $i \leq I/2$. Set to 0 all $x$ variables except those that have exactly $X_i \cdot N$ positions of their index which are mapped to $(i, I - i)$ by $(p^{\mathcal{K}'}, p^{\mathcal{K}'})$, for all $i$.

The number of nonzero $x$ blocks is $\binom{2N}{[N \cdot X_i]_{i<I/2}, [N \cdot X_i]_{i<I/2}, 2N \cdot X_{I/2}}$.

Similarly pick values $Y_j$ for the $y$ variables, with $Y_j = Y_{J-j}$, and retain only those with $Y_j$ index positions mapped to $(j, J - j)$. Similarly pick values $Z_k$ for the $z$ variables, with $Z_k = Z_{K-k}$, and retain only those with $Z_k$ index positions mapped to $(k, K - k)$.

The number of nonzero $y$ blocks is $\binom{2N}{[N \cdot Y_j]_{j<J/2}, [N \cdot Y_j]_{j<J/2}, 2N \cdot Y_{J/2}}$. The number of nonzero $z$ blocks is $\binom{2N}{[N \cdot Z_k]_{k<K/2}, [N \cdot Z_k]_{k<K/2}, 2N \cdot Z_{K/2}}$.

For $i, j, k = P\mathcal{K}' - i - j$ which are valid blocks of the $\mathcal{K}'$ tensor power of $C$ let $\alpha_{ijk}$ be variables such that $X_i = \sum_j \alpha_{ijk}$, $Y_j = \sum_i \alpha_{ijk}$ and $Z_k = \sum_i \alpha_{ijk}$.

After taking the tensor power of what is remaining of the $2N$th tensor powers of $T_{I,J,K}$, $T_{K,I,J}$ and $T_{J,K,I}$, the number of $x, y$ or $z$ blocks is

$$\Gamma = \binom{2N}{[N \cdot X_i]} \binom{2N}{[N \cdot Y_J]} \binom{2N}{[N \cdot Z_K]}.$$

The number of triples which contain a particular $x, y$ or $z$ block is now

1. Define variables $\alpha_{ijk}$ and $X_i, Y_j, Z_k$ for all valid triples $i, j, k$, i.e. the good triples with $i \leq I/2$ and if $i = I/2$, then $j \leq J/2$.

2. Form the linear system consisting of
$X_i = \sum_{j \in J(i)} \alpha_{ij\star}$ when $i < \lfloor I/2 \rfloor$,

   $Y_j = \sum_{i \in I(j)} \alpha_{ij\star} + \sum_{i \in I(J-j)} \alpha_{i,J-j,\star}$ when $j < \lfloor J/2 \rfloor$, and

   $Z_k = \sum_{i \in I(k)} \alpha_{i\star k} + \sum_{i \in I(K-k)} \alpha_{i,\star,K-k}$ for $k < K/2$ and $Z_{K/2} = 2 \sum_{i \in I(K/2)} \alpha_{i\star K/2}$.

3. If the system does not have full rank, then pick enough variables $\alpha_{ijk}$ to put in $\Delta$ and hence treat as constants.

4. Solve the system for the variables outside of $\Delta$ in terms of the ones in $\Delta$ and $X_i, Y_j, Z_k$. Now we have $\alpha_{ijk} = \alpha_{ijk}([X_i], [Y_j], [X_k], y \in \Delta)$ for all $\alpha_{ijk} \notin \Delta$.

5. Compute for every $\ell$,

$$nx_\ell = \prod_{i \leq I/2, j, k} W_{ijk}^{3 \frac{\partial \alpha_{ijk}}{\partial X_\ell}} \text{ for } \ell < \lfloor I/2 \rfloor \text{ and } nx_{\lfloor I/2 \rfloor} = 1 \text{ if } I \text{ is odd, } nx_{I/2} = 1/2 \text{ if } I \text{ is even,}$$

$$ny_\ell = \prod_{i \leq I/2, j, k} W_{ijk}^{3 \frac{\partial \alpha_{ijk}}{\partial Y_\ell}} \text{ for } \ell < \lfloor J/2 \rfloor, \text{ and, } ny_{\lfloor J/2 \rfloor} = 1 \text{ if } J \text{ is odd, } ny_{J/2} = 1/2 \text{ if } J \text{ is even,}$$

$$nz_\ell = \prod_{i \leq I/2, j, k} W_{ijk}^{3 \frac{\partial \alpha_{ijk}}{\partial Z_\ell}} \text{ for } \ell < K/2 \text{ and } nz_{K/2} = \prod_{i \leq I/2, j, k} W_{ijk}^{6 \frac{\partial \alpha_{ijk}}{\partial Z_{K/2}}}/2.$$

6. Compute for every variable $y \in \Delta$,

$$ny = \prod_{i \leq I/2, j, k} (V_{i,j,k} V_{I-i,J-j,K-k})^{\frac{\partial \alpha_{ijk}}{\partial y}}.$$

7. Compute for each $\alpha_{ijk}$ its setting $\alpha_{ijk}(\Delta)$ as a function of the $y \in \Delta$ when $X_\ell = nx_\ell / \sum_i nx_i$, $Y_\ell = ny_\ell / \sum_j ny_j$ and $Z_\ell = nz_\ell / \sum_k nz_k$.

8. $$\text{Then set } V'_{IJK}(\Delta) = 2 \left( \sum_{\ell \leq I/2} nx_\ell \right)^{1/3} \left( \sum_{\ell \leq J/2} ny_\ell \right)^{1/3} \left( \sum_{\ell \leq K/2} nz_\ell \right)^{1/3} \frac{\prod_{y \in \Delta} ny^y}{\prod_{ij} \alpha_{ijk}(\Delta)^{\alpha_{ijk}(\Delta)}},$$

   as a function of $y \in \Delta$.

9. Form the linear constraints $L$ on $y \in \Delta$ given by

   $y \geq 0$ for all $y \in \Delta$,
   $\alpha_{ijk}(\Delta) \geq 0$ for every $\alpha_{ijk} \notin \Delta$.

10. Solve the system
Minimize $\prod_{ij} \alpha_{ijk}(\Delta)^{\alpha_{ijk}(\Delta)}$ subject to $L$.
Let the solution be $B_1$.

11. Solve the system
Maximize $V'_{IJK}(\Delta)$ subject to $L$.
Let the solution be $B_2$.

12. Return that $V_{IJK} \geq B_1 B_2$.



Figure 3: The procedure to compute $V_{IJK}$ for even tensor powers.

$$\aleph = \sum_{\{\alpha_{ijk}\}} \prod_{i<I/2} \binom{NX_i}{[N\alpha_{ijk}]_j}^2 \prod_{j<J/2} \binom{NY_j}{[N\alpha_{ijk}]_i}^2 \prod_{k<K/2} \binom{NZ_k}{[N\alpha_{ijk}]_i}^2 \binom{NX_{I/2}}{[N\alpha_{(I/2)jk}]_j} \binom{NY_{J/2}}{[N\alpha_{i(J/2)k}]_i} \binom{NZ_{K/2}}{[N\alpha_{ij(K/2)}]_i}.$$

Hence the number of triples is $\Gamma \cdot \aleph$.

Set $M = \Theta(\aleph)$ to be a large enough prime greater than $\aleph$. Create a Salem-Spencer set $S$ of size roughly $M^{1-\varepsilon}$ and perform the hashing just as before. Then set to $0$ all variables that do not have blocks hashing to elements of $S$. Again, any surviving block triple has all variables' blocks mapped to the same element of $S$. The expected fraction of block triples remaining is $M^{1-\varepsilon}/M^2$ which will be $1/M$ when we let $\varepsilon$ go to $0$.

As before, let $\Upsilon = \aleph'/\aleph_{\max}$. We let $\beta_{ijk}$ be the values maximizing the inner summand in $\aleph$ and hence attaining $\aleph_{\max}$. We let $\alpha_{ijk}$ be values we optimize over.

After the usual pruning we have obtained $\Omega(\Upsilon\Gamma/\operatorname{poly}(N))$ independent trilinear forms, each of which has value at least

$$\prod_{i,j,k} (V_{i,j,k} \cdot V_{I-i,J-j,K-k})^{3N\alpha_{ijk}},$$

where $\alpha_{ijk}$ are the values maximizing $\aleph$.

Because of symmetry, $\alpha_{ijk} = \alpha_{I-i,J-j,K-k}$, so letting $W_{ijk} = V_{i,j,k} \cdot V_{I-i,J-j,K-k}$, we can write the above as

$$\prod_{i<I/2,j,k} (W_{i,j,k})^{6N\alpha_{ijk}} \prod_{j<J/2,k} (W_{I/2,j,k})^{6N\alpha_{I/2,jk}} (W_{I/2,J/2,k})^{3N\alpha_{I/2,J/2,k}}.$$

We can make a change of variables now, so that $\alpha_{I/2,J/2,k}$ is halved, and whereever we had $\alpha_{I/2,J/2,k}$ before, now we have $2\alpha_{I/2,J/2,k}$.

The value inequality becomes

$$V_{I,J,K}^{6N} \geq (\Upsilon/\operatorname{poly}(N)) \cdot \binom{2N}{[N \cdot X_i]} \binom{2N}{[N \cdot Y_j]} \binom{2N}{[N \cdot Z_k]} \prod_{i \leq I/2,j,k} (W_{i,j,k})^{6N\alpha_{ijk}}.$$

Using Stirling's approximation, we obtain that the right hand side is roughly

$$\Upsilon \cdot \frac{(2N)^{2N}}{(NX_{I/2})^{NX_{I/2}} \prod_{i<I/2} (NX_i)^{2NX_i}} \frac{(2N)^{2N}}{(NY_{J/2})^{NY_{J/2}} \prod_{j<J/2} (NY_j)^{2NY_j}} \times$$
$$\frac{(2N)^{2N}}{(NZ_{K/2})^{NZ_{K/2}} \prod_{k<K/2} (NZ_k)^{2NZ_k}} \prod_{i \leq I/2,j,k} W_{i,j,k}^{6N\alpha_{ijk}}.$$

Taking square roots and restructuring:

$$\sqrt{\Upsilon} \cdot 2^{3N - N(X_{I/2}+Y_{J/2}+Z_{K/2})/2} \binom{N}{[N \cdot X_i]_{i<I/2}, NX_{I/2}/2} \binom{N}{[N \cdot Y_j]_{j<J/2}, NY_{J/2}/2} \times$$
$$\binom{N}{[N \cdot Z_k]_{k<K/2}, NZ_{K/2}/2} \prod_{i \leq I/2,j,k} W_{i,j,k}^{3N\alpha_{ijk}}.$$

Because of the symmetry, we can focus only on the variables $\alpha_{ijk}$ for which

- $i \leq I/2$
- if $i = I/2$, then $j \leq J/2$.

A triple $(i,j,k)$ is *valid* if $i$ and $j$ satisfy the above two conditions and $(i,j,k)$ is good. When two of the indices in a triple are fixed (say $i,j$), we will replace the third index by $\star$. If $i \leq I/2$ is fixed, $J(i)$ will refer to the indices $j$ for which $(i,j,\star)$ is valid. Similarly one can define $K(i)$, $I(j)$,$K(j)$, $I(k)$ and $J(k)$.

We obtain the following linear equations.

$X_i = \sum_{j \in J(i)} \alpha_{ij\star}$ when $i < I/2$ and $X_{I/2} = 2\sum_{j \in J(I/2)} \alpha_{(I/2)j\star}$, $Y_j = \sum_{i \in I(j)} \alpha_{ij\star} + \sum_{i \in I(J-j)} \alpha_{i,J-j,\star}$ when $j < J/2$ and $Y_{J/2} = 2\sum_{i \in I(J/2)} \alpha_{i(J/2)\star}$, and $Z_k = \sum_{i \in I(k)} \alpha_{i\star k} + \sum_{i \in I(K-k)} \alpha_{i,\star,K-k}$ for $k < K/2$ and $Z_{K/2} = 2\sum_{i \in I(K/2)} \alpha_{i\star K/2}$.

If we fix $X_i, Y_j, Z_k$ over all $i \leq I/2, j \leq J/2, k \leq K/2$, this forms a linear system. This linear system has the property that $\sum_i X_i = \sum_j Y_j = \sum_k Z_k$, so we focus on the smaller system that excludes the equations for $X_{\lfloor I/2 \rfloor}$ and $Y_{\lfloor J/2 \rfloor}$. In the following lemma we show that the equations in this system are linearly independent.

**Lemma 3.** *The linear expressions* $X_i = \sum_{j \in J(i)} \alpha_{ij\star}$ *for* $i < \lfloor I/2 \rfloor$, $Y_j = \sum_{i \in I(j)} \alpha_{ij\star} + \sum_{i \in I(J-j)} \alpha_{i,J-j,\star}$ *for* $j < \lfloor J/2 \rfloor$, $Z_k = \sum_{i \in I(k)} \alpha_{i\star k} + \sum_{i \in I(K-k)} \alpha_{i,\star,K-k}$ *for* $k < K/2$ *and* $Z_{K/2} = 2\sum_{i \in I(K/2)} \alpha_{i\star K/2}$ *are linearly independent.*

*Proof.* The proof will proceed by contradiction. Let $P' = P\mathcal{K}/2$. Assume that there are coefficients $x_i$,$y_j$,$z_k$ for $0 \leq i \leq \lfloor I/2 \rfloor - 1$, $0 \leq j \leq \lfloor J/2 \rfloor - 1$, $\max\{0, P' - I - J\} \leq k \leq \lfloor k/2 \rfloor$, such that $\sum_i x_i X_i + \sum_j y_j Y_j + \sum_k z_k Z_k = 0$.

This means that for all valid triples $i,j,k$, the coefficient in front of $a_{ijk}$ must be 0. Consider the coefficient in front of $a_{\lfloor I/2 \rfloor \lfloor J/2 \rfloor (P' - \lfloor I/2 \rfloor - \lfloor J/2 \rfloor)}$. This coefficient is $z_{\lfloor K/2 \rfloor}$, unless both $I$ and $J$ are odd. If both $I$ and $J$ are odd, consider the coefficient in front of $a_{\lfloor I/2 \rfloor \lceil J/2 \rceil (P' - \lfloor I/2 \rfloor - \lfloor J/2 \rfloor)}$. That coefficient is $z_{\lfloor K/2 \rfloor}$. Hence, $z_{\lfloor K/2 \rfloor} = 0$.

Now, we will show by induction that for all $t$,

$$x_{\lfloor I/2 \rfloor - t} = y_{\lfloor J/2 \rfloor - t} = z_{\lfloor K/2 \rfloor - t} = 0.$$

The base case is for $t = 0$. This holds since the system does not contain equations for $X_{\lfloor I/2 \rfloor}$ and $Y_{\lfloor J/2 \rfloor}$, and since we showed that $z_{\lfloor K/2 \rfloor} = 0$.

Suppose that $x_{\lfloor I/2 \rfloor - t} = y_{\lfloor J/2 \rfloor - t} = z_{\lfloor K/2 \rfloor - t} = 0$. We will show that $x_{\lfloor I/2 \rfloor - t - 1} = y_{\lfloor J/2 \rfloor - t - 1} = z_{\lfloor K/2 \rfloor - t - 1} = 0$. Whenever an index for $x_i$,$y_j$,$z_k$ is not defined, we can assume that the corresponding variable is 0.

Suppose that $I$ and $J$ are not both odd.

Consider the coefficient in front of $a_{ijk}$ for $i = \lfloor I/2 \rfloor$, $j = \lfloor J/2 \rfloor - t - 1$, $k = K - (P' - \lfloor I/2 \rfloor - \lfloor J/2 \rfloor + t + 1) = \lfloor K/2 \rfloor - t - 1$. It is $y_{\lfloor J/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1}$.

Consider the coefficient in front of $a_{ijk}$ for $i = \lfloor I/2 \rfloor - t - 1$, $j = \lfloor J/2 \rfloor$, $k = K - (P' - \lfloor I/2 \rfloor - \lfloor J/2 \rfloor + t + 1) = \lfloor K/2 \rfloor - t - 1$. It is $x_{\lfloor I/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1}$.

Consider the coefficient in front of $a_{ijk}$ for $i = \lfloor I/2 \rfloor - t - 1$, $k = \lfloor K/2 \rfloor$, $j = J - (P' - \lfloor I/2 \rfloor - \lfloor K/2 \rfloor + t + 1) = \lfloor J/2 \rfloor - t - 1$. It is $x_{\lfloor I/2 \rfloor - t - 1} + y_{\lfloor J/2 \rfloor - t - 1}$.

Hence,

$$y_{\lfloor J/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1} = x_{\lfloor I/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1} = x_{\lfloor I/2 \rfloor - t - 1} + y_{\lfloor J/2 \rfloor - t - 1} = 0.$$

Therefore, $x_{\lfloor I/2 \rfloor - t - 1} = y_{\lfloor J/2 \rfloor - t - 1} = z_{\lfloor K/2 \rfloor - t - 1} = 0$.

Suppose now that both $I$ and $J$ are odd. Then $K$ is even.

Consider the coefficient in front of $a_{ijk}$ for $i = \lfloor I/2 \rfloor = (I-1)/2$, $j = \lceil J/2 \rceil - t - 1 = (J+1)/2 - t - 1$, $k = K - (P' - (I-1)/2 - (J+1)/2 + t + 1) = \lfloor K/2 \rfloor - t - 1$. It is $y_{\lfloor J/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1}$.

Consider the coefficient in front of $a_{ijk}$ for $i = \lfloor I/2 \rfloor - t - 1$, $j = \lceil J/2 \rceil$, $k = K - (P' - \lfloor I/2 \rfloor - \lceil J/2 \rceil + t + 1) = \lfloor K/2 \rfloor - t - 1$. It is $x_{\lfloor I/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1}$.

Consider the coefficient in front of $a_{ijk}$ for $i = \lfloor I/2 \rfloor - t - 1$, $k = \lceil K/2 \rceil$, $j = J - (P' - (I-1)/2 - K/2 + t + 1) = \lfloor J/2 \rfloor - t - 1$. It is $x_{\lfloor I/2 \rfloor - t - 1} + y_{\lfloor J/2 \rfloor - t - 1}$.

Hence, again

$$y_{\lfloor J/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1} = x_{\lfloor I/2 \rfloor - t - 1} + z_{\lfloor K/2 \rfloor - t - 1} = x_{\lfloor I/2 \rfloor - t - 1} + y_{\lfloor J/2 \rfloor - t - 1} = 0.$$

Therefore, $x_{\lfloor I/2 \rfloor - t - 1} = y_{\lfloor J/2 \rfloor - t - 1} = z_{\lfloor K/2 \rfloor - t - 1} = 0$. □

Because the equations are linearly independent, the rank of the system is exactly the number of equations. If the system has full rank, then we can determine each $\alpha_{ijk}$ as a linear combination of the $X_i, Y_j, Z_k$. Otherwise, we pick a minimum set $\Delta$ of variables $\alpha_{ijk}$ so that if they are treated as constants, the linear system has full rank and the variables outside of $\Delta$ can be written as linear combinations of variables in $\Delta$ and of $X_i, Y_j, Z_k$. (The choice of the variables to put in $\Delta$ can be arbitrary.)

Now, we have that for every valid $\alpha_{ijk}$,

$$\alpha_{ijk} = \sum_{y \in \Delta \cup \{X_i, Y_j, Z_k\}_{i,j,k}} y \frac{\partial \alpha_{ijk}}{\partial y},$$

where for all $\alpha_{ijk} \notin \Delta$ we use the linear function obtained from the linear system.

Let $\delta_{ijk} = \sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y}$. Then,

$$W_{i,j,k}^{3N\alpha_{ijk}} = W_{ijk}^{3N \sum_i X_i \frac{\partial \alpha_{ijk}}{\partial X_i}} W_{ijk}^{3N \sum_i Y_j \frac{\partial \alpha_{ijk}}{\partial Y_j}} W_{ijk}^{3N \sum_k Z_k \frac{\partial \alpha_{ijk}}{\partial Z_k}} W_{i,j,k}^{3N\delta_{ijk}}.$$

We now define $nx_\ell = \prod_{i \le I/2, j, k} W_{ijk}^{3 \frac{\partial \alpha_{ijk}}{\partial X_\ell}}$ for $\ell < \lfloor I/2 \rfloor$, $nx_{\lfloor I/2 \rfloor} = 1$ if $I$ is odd and $nx_{\lfloor I/2 \rfloor} = 1/2$ if $I$ is even.

Consider

$$F_X = \binom{N}{[N \cdot X_i]_{i<I/2}, NX_{I/2}/2} \left( \prod_{i \le I/2, j, k} (W_{i,j,k}^{(NX_{I/2}/2) 6 \frac{\partial \alpha_{i,j,k}}{\partial X_{I/2}}}) / 2^{NX_{I/2}/2} \right) \prod_{i \le I/2, j, k} W_{ijk}^{3N \sum_{\ell < I/2} X_\ell \frac{\partial \alpha_{ijk}}{\partial X_\ell}}.$$

By Lemma 1, $F_X$ is maximized for $X_\ell = nx_\ell / \sum_{\ell'} nx_{\ell'}$ for $\ell < I/2$ and $X_{I/2}/2 = nx_{I/2} / \sum_{\ell'} nx_{\ell'}$. Then $F_X$ is essentially $(\sum_{\ell \le I/2} nx_\ell)^N / \text{poly}(N)$.

Define similarly $ny_\ell = \prod_{i \le I/2, j, k} W_{ijk}^{3 \frac{\partial \alpha_{ijk}}{\partial Y_\ell}}$ for $\ell < \lfloor J/2 \rfloor$, $ny_{\lfloor J/2 \rfloor} = 1$ if $J$ is odd and $ny_{\lfloor J/2 \rfloor} = 1/2$ if $J$ is even, and $nz_\ell = \prod_{i \le I/2, j, k} W_{ijk}^{3 \frac{\partial \alpha_{ijk}}{\partial Z_\ell}}$ for $\ell < K/2$ and $nz_{K/2} = \prod_{i \le I/2, j, k} W_{ijk}^{6 \frac{\partial \alpha_{ijk}}{\partial Z_{K/2}}} / 2$.

We obtain that

$$V_{I,J,K}^{3N} \ge \sqrt{\Upsilon} 2^{3N} / \text{poly}(N) \left( \sum_{\ell \le I/2} nx_\ell \right)^N \left( \sum_{\ell \le J/2} ny_\ell \right)^N \left( \sum_{\ell \le K/2} nz_\ell \right)^N / \text{poly}(N) \prod_{i \le I/2, j, k} W_{i,j,k}^{3N (\sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y})}.$$

Taking the $3N$-th root and letting $N$ go to $\infty$, we finally obtain

$$V_{I,J,K} \geq 2\left(\sum_{\ell \leq I/2} nx_\ell\right)^{1/3}\left(\sum_{\ell \leq J/2} ny_\ell\right)^{1/3}\left(\sum_{\ell \leq K/2} nz_\ell\right)^{1/3}\prod_{i \leq I/2, j, k}(V_{i,j,k}V_{I-i,J-j,K-k})^{(\sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y})}\Upsilon^{1/(6N)}.$$

Now, as $N$ goes to $\infty$, $\Upsilon^{1/(6N)}$ goes to $\prod_{\text{valid } i,j,k} \beta_{ijk}^{\beta_{ijk}}/\alpha_{ijk}^{\alpha_{ijk}}$.

To obtain the lower bound on $V_{I,J,K}$ we need to pick values for the variables in $\Delta$, while still preserving the constraints that the values for the variables outside of $\Delta$ (which are obtained from our settings of the $X_I, Y_J, Z_K$ and the values for the $\Delta$ variables) are nonnegative. We also need that $\beta_{ijk}$ attain $\aleph_{\max}$ given the settings of the $X_I, Y_J, Z_K$. Since the $\beta_{ijk}$ are solutions of the same linear system as the $\alpha_{ijk}$, we proceed as before. We place $\beta_{ijk}$ in $\bar{\Delta}$ whenever $\alpha_{ijk} \in \Delta$, and express the $\beta_{ijk} \notin \bar{\Delta}$ in terms of the variables in $\bar{\Delta}$ using the same linear functions as the $\alpha_{ijk}$. We then minimize $\prod_{\text{valid } i,j,k} \beta_{ijk}^{\beta_{ijk}}$ subject to the constraints that all $\beta_{ijk} \geq 0$. This is a convex program over the variables in $\bar{\Delta}$. Let the solution of this program be $B_1$.

Then, we maximize

$$2\left(\sum_{\ell \leq I/2} nx_\ell\right)^{1/3}\left(\sum_{\ell \leq J/2} ny_\ell\right)^{1/3}\left(\sum_{\ell \leq K/2} nz_\ell\right)^{1/3}\prod_{i \leq I/2, j, k}(V_{i,j,k}V_{I-i,J-j,K-k})^{(\sum_{y \in \Delta} y \frac{\partial \alpha_{ijk}}{\partial y})}/\prod_{\text{valid } i,j,k}\alpha_{ijk}^{\alpha_{ijk}}$$

subject to $\alpha_{ijk} \geq 0$. This is only over the variables in $\Delta$. Let $B_2$ be the solution here.

Finally, we can output that $V_{I,J,K} \geq B_1 \cdot B_2$.

The procedure is shown in Figure 3.

## 5 Analyzing the CW construction

We can make the following observations about some of the values for any tensor power $\mathcal{K}$. First, $V_{IJK} = V_{IKJ} = V_{JKI} = V_{KJI} = V_{JIK} = V_{KIJ}$. For the special case $I = 0$ we get:

**Claim 1.** *Consider $V_{0JK}$ which is a value for the $\mathcal{K} = (J+K)/2$ tensor power for $J \leq K$. Then*

$$V_{0JK} \geq \left(\sum_{b \leq J, J=b \mod 2}\binom{(J+K)/2}{b, (J-b)/2, (K-b)/2} q^b\right)^\tau.$$

*Proof.* The trilinear form $T_{0JK}$ contains triples of the form $x_{0^\mathcal{K}} y_s z_t$ where $s$ and $t$ are $\mathcal{K}$ length sequences so that for fixed $s$, $t$ is predetermined. Thus, $T_{0JK}$ is in fact a matrix product of the form $\langle 1, Q, 1\rangle$ where $Q$ is the number of $y$ indices $s$. Let us count the $y$ indices containing $a$ positions mapped to a 0 block (hence 0s), $b$ positions mapped to a 1 block (integers in $[q]$) and $\mathcal{K} - a - b$ positions mapped to a 2 block (hence $q+1$s). The number of such $y$ indices is $\binom{\mathcal{K}}{a, b, \mathcal{K}-a-b} q^b$. However, since $a \cdot 0 + 1 \cdot b + 2 \cdot (\mathcal{K} - a - b) = J$, we must have $J + K - 2a - b = J$ and $a = (K-b)/2$. Thus, the number of $y$ indices containing $(K-b)/2$ 0s, $b$ positions in $[q]$ and $\mathcal{K} - a - b = (J-b)/2$ $(q+1)$s is $\binom{(J+K)/2}{b, (J-b)/2, (K-b)/2} q^b$. The claim follows since we can pick any $b$ as long as $(J-b)/2$ is a nonnegative integer. □

**Lemma 4.** *Consider $V_{1,x,2\mathcal{K}-x-1}$ for any $x$ and $\mathcal{K}$. For both approaches to lowerbounding $V_{1,x,2\mathcal{K}-x-1}$, the number of $\alpha_{ijk}$ variables and the number of equations is exactly $x+1$. Hence no variables need to be added to $\Delta$.*

*Proof.* Wlog, $x \leq 2\mathcal{K} - x - 1$ so that $x < \mathcal{K}$.

We first consider the general, nonrecursive approach to computing a lower bound on $V_{1,x,2\mathcal{K}-x-1}$.

Consider first the number of $X_*, Y_*, Z_*$ variables. There is only one $X_*$ variable- for the index sequence that contains a single 1 and all 0s otherwise. Consider now the $Y_*$ and $Z_*$ variables.

Since $x < \mathcal{K}$, there is an index sequence for the $Y_*$ variables for every $j$ ranging from 0 to $\lfloor x/2 \rfloor$ defined as the sequence with $j$ twos and $x - 2j < \mathcal{K} - j$ ones. For the $Z_*$ variables there is an index sequence for every $k$ defined as the sequence with $k$ twos and $2\mathcal{K} - x - 1 - 2k$ ones. Since the number of ones must be at least 0 and at most $\mathcal{K} - k$, we have that $0 \leq 2\mathcal{K} - x - 1 - 2k \leq \mathcal{K} - k$ and $k$ ranges from $\mathcal{K} - x - 1$ to $\mathcal{K} - \lceil (x+1)/2 \rceil$.

The number of equations is hence $\lfloor x/2 \rfloor + 1 + (\mathcal{K} - \lceil (x+1)/2 \rceil) - (\mathcal{K} - x - 1) = \lfloor x/2 \rfloor + 1 - \lceil (x+1)/2 \rceil + x + 1 = x + 1$.

Now let's consider the number of variables $\alpha_{ijk}$. Recall, there is only one sequence $i$, namely the one with one 1 and all zeros otherwise. Thus, for every sequence $j$ that contains at least one 1, there are exactly two variables $\alpha_{ijk}$: one for which the last positions of both $i$ and $j$ are 1, and one for which the last position of $j$ is 0 and the last position of $i$ is 1. If $x$ is even, there is a sequence $j$ with no ones and there is a unique variable $\alpha_{ijk}$ for it. Hence the number of $\alpha_{ijk}$ variables is twice the number of $Y_*$ variables if $x$ is odd, and that number minus 1 otherwise. That is, if $x$ is odd, the number of variables is $2(1 + \lfloor x/2 \rfloor) = 2(1 + (x-1)/2) = x + 1$, and if $x$ is even, it is $2(1 + \lfloor x/2 \rfloor) - 1 = x + 1$. In both cases, the number of $\alpha_{ijk}$ is exactly the number of equations in the linear system.

Now consider the recursive approach with even $\mathcal{K}$. Here the number of $X_*$ variables is 1 and the number of $Y_*$ variables is $\lfloor x/2 \rfloor + 1$. $Z_*$ ranges between $\mathcal{K} - x - 1$ and $\mathcal{K} - \lceil (x+1)/2 \rceil$ and so the number of these variables is $1 + \lfloor (x+1)/2 \rfloor$. The number of equations in the linear system is thus $\lfloor x/2 \rfloor + 1 + \lfloor (x+1)/2 \rfloor = x + 1$. The number of $\alpha_{ijk}$ variables is also $x+1$ since all variables have $i = 0$ and $j$ can vary from 0 to $x$. □

The calculations for the second tensor power were performed by hand. Those for the 4th and the 8th tensor power were done by computer (using Maple and C++ with NLOPT). We write out the derivations as lemmas for completeness.

**Second tensor power.** We will only give $V_{IJK}$ for $I \leq J \leq K$, and the values for other permutations of $I, J, K$ follow.

From Claim 1 know that $V_{004} = 1$ and $V_{013} = (2q)^\tau$, and $V_{022} = (q^2+2)^\tau$.

It remains to analyze $V_{112}$. As expected, we obtain the same value as in [10].

**Lemma 5.** $V_{112} \geq 2^{2/3} q^\tau (q^{3\tau} + 2)^{1/3}$.

*Proof.* We follow the proof in the previous section. Here $I = 1, J = 1, K = 2$. The only valid variables are $\alpha_{002}$ and $\alpha_{011}$, and we have that $Z_0 = \alpha_{002}$ and $Z_1 = 2\alpha_{011}$.

We obtain $nx_0 = ny_0 = 1$, $nz_1 = W_{011}^{2\cdot 3/2}/2 = V_{011}^6/2 = q^{6\tau}/2$ and $nz_0 = W_{002}^3 = V_{011}^3 = q^{3\tau}$.

The lower bound becomes

$$V_{112} \geq 2(q^{6\tau}/2 + q^{3\tau})^{1/3} = 2^{2/3} q^\tau (q^{3\tau} + 2)^{1/3}.$$

□

**The program for the second power:** The variables are $a = a_{004}, b = a_{013}, c = a_{022}, d = a_{112}$. $A_0 = 2(a+b)+c$, $A_1 = 2(b+d)$, $A_2 = 2c+d$, $A_3 = 2b$, $A_4 = a$.

We obtain the following program (where we take natural logs on the last constraint).

Minimize $\tau$ subject to

$q \geq 3, q \in \mathbb{Z}$,

$a, b, c, d \geq 0$,

$3a + 6b + 3c + 3d = 1$,

$$2\ln(q+2) + (2(a+b)+c)\ln(2(a+b)+c) + 2(b+d)\ln(2(b+d)) + (2c+d)\ln(2c+d) +$$
$$2b\ln 2b + a\ln a = 6b\tau\ln 2q + 3c\tau\ln(q^2+2) + d\ln(4q^{3\tau}(q^{3\tau}+2)).$$

Using Maple, we obtain the bound $\omega \leq 2.37547691273933114$ for the values $a = .000232744788234356428, b = .0125062362305418986, c = .102545675391892355, d = .205542440692123102, \tau = .791825637579776975$.

**The fourth tensor power.** From Claim 1 we have, $V_{008} = 1$, $V_{017} = \binom{4}{1,(1-1)/2,(7-1)/2} q^1)^\tau = (4q)^\tau$, $V_{026} = (\sum_{b\leq 2, b=2 \mod 2} \binom{4}{b,(2-b)/2,(6-b)/2} q^b)^\tau = (4+6q^2)^\tau$, $V_{035} = (\sum_{b\leq 3, b=3 \mod 2} \binom{4}{b,(3-b)/2,(5-b)/2} q^b)^\tau = (12q+4q^3)^\tau$, and $V_{044} = (\sum_{b\leq 4, b=4 \mod 2} \binom{4}{b,(4-b)/2,(4-b)/2} q^b)^\tau = (6+12q^2+q^4)^\tau$.

Let's consider the rest:

**Lemma 6.** $V_{116} \geq 2^{2/3}(8q^{3\tau}(q^{3\tau}+2) + (2q)^{6\tau})^{1/3}$.

*Proof.* Here $I = J = 1, K = 6$. The variables are $\alpha_{004}$ and $\alpha_{013}$.

The large variables are $Z_2$ and $Z_3$. The linear system is: $Z_2 = \alpha_{004}$, $Z_3 = 2\alpha_{013}$.

We can conclude that $\alpha_{013} = Z_3/2$ and $\alpha_{004} = Z_2$.

We obtain $nx_0 = ny_0 = 1$, and $nz_2 = W_{004}^3 = (V_{112})^3 = 4q^{3\tau}(q^{3\tau}+2)$, $nz_3 = W_{013}^{6/2}/2 = (V_{013}V_{103})^3/2 = (2q)^{6\tau}/2$. The lower bound becomes

$$V_{116} \geq 2(4q^{3\tau}(q^{3\tau}+2) + (2q)^{6\tau}/2)^{1/3} = 2^{2/3}(8q^{3\tau}(q^{3\tau}+2) + (2q)^{6\tau})^{1/3}.$$

□

**Lemma 7.** $V_{125} \geq 2^{2/3}(2(q^2+2)^{3\tau} + (4q^{3\tau}(q^{3\tau}+2)))^{1/3}((4q^{3\tau}(q^{3\tau}+2))/(q^2+2)^{3\tau} + (2q)^{3\tau})^{1/3}$.

*Proof.* Here $I = 1$, $J = 2$ and $K = 5$. The variables are $\alpha_{004}, \alpha_{013}, \alpha_{022}$ and $Y_0$ and $Z_1, Z_2$.

The linear system is as follows: $Y_0 = \alpha_{004} + \alpha_{022}$, $Z_1 = \alpha_{004}$, $Z_2 = \alpha_{013} + \alpha_{022}$.

We solve: $\alpha_{004} = Z_1$, $\alpha_{022} = Y_0 - Z_1$, $\alpha_{013} = Z_2 - Y_0 + Z_1$.

We obtain $nx_0 = 1$, $ny_1 = 1/2$, $ny_0 = W_{022}^3 W_{013}^{-3}$, $nz_1 = W_{004}^3 W_{022}^{-3} W_{013}^3$, $nz_2 = W_{013}^3$.

$ny_0 + ny_1 = (W_{022}/W_{013})^3 + 1/2 = ((2q(q^2+2))^\tau/((2q)^\tau)2^{2/3}q^\tau(q^{3\tau}+2)^{1/3})^3 + 1/2 = (q^2+2)^{3\tau}/(4q^{3\tau}(q^{3\tau}+2)) + 1/2$,

$nz_1 = (W_{004}W_{013}/W_{022})^3 = (V_{121}V_{013}V_{112}/(V_{022}V_{103}))^3 = (V_{112}^2/V_{022})^3 = (4q^{3\tau}(q^{3\tau}+2))^2/(q^2+2)^{3\tau}$. $nz_2 = (V_{013}V_{112})^3 = 4q^{3\tau}(q^{3\tau}+2)(2q)^{3\tau}$ and $nz_1 + nz_2 = (4q^{3\tau}(q^{3\tau}+2))[(4q^{3\tau}(q^{3\tau}+2))/(q^2+2)^{3\tau} + (2q)^{3\tau}]$.

We obtain

$$V_{125} \geq 2^{2/3}(2(q^2+2)^{3\tau} + (4q^{3\tau}(q^{3\tau}+2)))^{1/3}((4q^{3\tau}(q^{3\tau}+2))/(q^2+2)^{3\tau} + (2q)^{3\tau})^{1/3}.$$

□

**Lemma 8.** $V_{134} \geq 2^{2/3}((2q)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{1/3}(2+2(2q)^{3\tau}+(q^2+2)^{3\tau})^{1/3}$.

*Proof.* Here $I = 1, J = 3, K = 4$ and the relevant variables are $\alpha_{004}, \alpha_{013}, \alpha_{022}, \alpha_{031}$, and $Y_0, Z_0, Z_1, Z_2$.
The linear system is: $Y_0 = \alpha_{004} + \alpha_{031}$, $Z_0 = \alpha_{004}$, $Z_1 = \alpha_{013} + \alpha_{031}$, $Z_2 = 2\alpha_{022}$.
We solve: $\alpha_{004} = Z_0$, $\alpha_{031} = Y_0 - Z_0$, $\alpha_{013} = Z_1 - Y_0 + Z_0$, $\alpha_{022} = Z_2/2$.
$ny_0 = W_{031}^3 W_{013}^{-3} = (V_{103}/V_{121})^3$, $ny_1 = 1$.
$ny_0 + ny_1 = (V_{013}^3 + V_{112}^3)/V_{112}^3$.
$nz_0 = W_{004}^3 W_{031}^{-3} W_{013}^3 = (V_{130}V_{013}V_{121}/(V_{031}V_{103}))^3 = (V_{121})^3$,
$nz_1 = W_{013}^3 = (V_{013}V_{121})^3$, $nz_2 = W_{022}^{6/2}/2 = (V_{022}V_{112})^3/2$.
$nz_0 + nz_1 + nz_2 = V_{121}^3(1 + V_{013}^3 + V_{022}^3/2)$.
We obtain:

$$V_{134} \geq 2^{2/3}(V_{013}^3 + V_{112}^3)^{1/3}(2 + 2V_{013}^3 + V_{022}^3)^{1/3} \geq$$
$$2^{2/3}((2q)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{1/3}(2+2(2q)^{3\tau}+(q^2+2)^{3\tau})^{1/3}.$$

□

**Lemma 9.** $V_{224} \geq (2V_{022}^3 + V_{112}^3)^{2/3}(2 + 2V_{013}^3 + V_{022}^3)^{1/3}/V_{022} \geq (2(q^2+2)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{2/3}(2 + 2(2q)^{3\tau} + (q^2+2)^{3\tau})^{1/3}/(q^2+2)^{\tau}$.

*Proof.* $I = J = 2, K = 4$, so the variables are $\alpha_{004}, \alpha_{013}, \alpha_{022}, \alpha_{103}, \alpha_{112}$, and $X_0, Y_0, Z_0, Z_1, Z_2$.
The linear system is as follows:
$X_0 = \alpha_{004}+\alpha_{013}+\alpha_{022}$, $Y_0 = \alpha_{004}+\alpha_{103}+\alpha_{022}$, $Z_0 = \alpha_{004}$, $Z_1 = \alpha_{013}+\alpha_{103}$, $Z_2 = 2(\alpha_{022}+\alpha_{112})$.
We solve: $\alpha_{004} = Z_0$,
$\alpha_{022} = 1/2(X_0 + Y_0 - 2Z_0 - Z_1)$,
$\alpha_{112} = Z_2/2 - \alpha_{022} = 1/2(Z_2 - X_0 - Y_0 + 2Z_0 + Z_1)$,
$\alpha_{013} = X_0 - Z_0 - 1/2(X_0 + Y_0 - 2Z_0 - Z_1) = 1/2(X_0 - Y_0 + Z_1)$, and
$\alpha_{103} = 1/2(Y_0 - X_0 + Z_1)$.
$nx_0 = W_{022}^{3/2} W_{112}^{-3/2} W_{013}^{3/2} W_{103}^{-3/2} = V_{022}^3/V_{112}^3$,
$ny_0 = W_{022}^{3/2} W_{112}^{-3/2} W_{013}^{-3/2} W_{103}^{3/2} = V_{022}^3/V_{112}^3$,
$nz_0 = V_{004}^3 V_{022}^3 V_{112}^6/V_{022}^6 = V_{112}^6/V_{022}^3$,
$nz_1 = W_{022}^{-3/2} W_{112}^{3/2} W_{013}^{3/2} W_{103}^{3/2} = V_{013}^3 V_{112}^6/V_{022}^3$,
$nz_2 = V_{112}^6/2$.

$$V_{224} \geq 2(V_{022}^3/V_{112}^3 + 1/2)^{2/3}(V_{112}^6/V_{022}^3 + V_{013}^3V_{112}^6/V_{022}^3 + V_{112}^6/2)^{1/3} =$$
$$(2V_{022}^3/V_{112}^3 + 1)^{2/3}(2V_{112}^6/V_{022}^3 + 2V_{013}^3V_{112}^6/V_{022}^3 + V_{112}^6)^{1/3} =$$
$$(2V_{022}^3 + V_{112}^3)^{2/3}(2 + 2V_{013}^3 + V_{022}^3)^{1/3}/V_{022}.$$

□

**Lemma 10.** $V_{233} \geq (2(q^2+2)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{1/3}((2q)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{2/3}/(q^{\tau}(q^{3\tau}+2)^{1/3})$.

*Proof.* $I = 2$, $J = K = 3$, so the variables are $\alpha_{013}, \alpha_{022}, \alpha_{031}, \alpha_{103}, \alpha_{112}$, and $X_0, Y_0, Z_0, Z_1$.
The linear system is: $X_0 = \alpha_{013} + \alpha_{022} + \alpha_{031}$,
$Y_0 = \alpha_{031} + \alpha_{103}$,
$Z_0 = \alpha_{013} + \alpha_{103}$, $Z_1 = \alpha_{022} + \alpha_{031} + \alpha_{112}$.
We solve it as follows. Let $w = \alpha_{103}$ and $\Delta = \{w\}$. Then:

$\alpha_{031} = Y_0 - w$, $\alpha_{013} = Z_0 - w$,
$\alpha_{022} = X_0 - Y_0 - Z_0 + 2w$, $\alpha_{112} = Z_1 - X_0 + Z_0 - w$.
First, consider $nw$:
$nw = (V_{013}V_{220})^{-1} \cdot (V_{022}V_{211})^2 \cdot (V_{031}V_{202})^{-1} \cdot (V_{103}V_{130})^1 \cdot (V_{112}V_{121})^{-1} = 1$.
$nx_0 = (W_{022}/W_{112})^3 = (V_{022}/V_{112})^3, nx_1 = 1/2$,
$ny_0 = (W_{031}/W_{022})^3 = (V_{031}/V_{112})^3, ny_1 = 1$,
$nz_0 = (W_{013}W_{112}/W_{022})^3 = (V_{013}V_{112})^3$, $nz_1 = W_{112}^3 = V_{112}^6$.
$nx_0 + nx_1 = (V_{022}/V_{112})^3 + 1/2 = (2V_{022}^3 + V_{112}^3)/(2V_{112}^3)$,
$ny_0 + ny_1 = (V_{013}/V_{112})^3 + 1 = (V_{013}^3 + V_{112}^3)/V_{112}^3$.
$nz_0 + nz_1 = (V_{013}V_{112})^3 + V_{112}^6 = (V_{013}^3 + V_{112}^3) \cdot V_{112}^3$ Hence,

$$V_{233} \geq 2^{2/3}(2V_{022}^3 + V_{112}^3)^{1/3}(V_{013}^3 + V_{112}^3)^{2/3}/V_{112} \geq$$

$$2^{2/3}(2(q^2+2)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{1/3}((2q)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{2/3}/(2^{2/3}q^\tau(q^{3\tau}+2)^{1/3}) =$$
$$(2(q^2+2)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{1/3}((2q)^{3\tau} + 4q^{3\tau}(q^{3\tau}+2))^{2/3}/(q^\tau(q^{3\tau}+2)^{1/3}).$$

In order for the above formula to be valid, we need to be able to pick a value for $w = \alpha_{103}$ between 0 and 1 so that $Y_0 - w$, $Z_0 - w$, $X_0 - Y_0 - Z_0 + 2w$, $Z_1 - X_0 + Z_0 - w$ are all between 0 and 1, whenever $X_0, Y_0, Z_0, Z_1$ are set to $X_0 = nx_0/(nx_0+nx_1)$, $Z_0 = Y_0 = ny_0/(ny_0+ny_1)$, and $Z_1 = nz_1/(nz_0+nz_1)$.
The inequalities we need to satisfy are as follows:

- $w \geq 0, w \leq 1$,
- $w \leq Y_0$. Notice that $w \geq Y_0 - 1$ is always satisfied since $Y_0 \leq 1$.
- $w \geq 1/2(Y_0 + Z_0 - X_0)$ and $w \leq 1/2(1 + Y_0 + Z_0 - X_0)$,
- $w \leq Z_0 + Z_1 - X_0 = 1 - X_0$. Notice that $Z_1 - X_0 + Z_0 - w \leq 1$ always holds as $Z_1 - X_0 + Z_0 - w = 1 - X_0 - w \leq 1 - X_0 \leq 1$ as $X_0 \geq 0$.

When $q = 5$, for all $\tau \in [2/3, 1]$, $Y_0 > 0, 1/2(1 + Y_0 + Z_0 - X_0) > 0, 1 - X_0 > 0$, and $1/2(Y_0 + Z_0 - X_0) < 0$, so that $w = 0$ satisfies the system of inequalities. Thus the formula for our lower bound for $V_{233}$ holds. □

Now that we have the values, let's form the program. The variables are as follows:
$a$ for 008 (and its 3 permutations),
$b$ for 017 (and its 6 permutations),
$c$ for 026 (and its 6 permutations),
$d$ for 035 (and its 6 permutations),
$e$ for 044 (and its 3 permutations),
$f$ for 116 (and its 3 permutations),
$g$ for 125 (and its 6 permutations),
$h$ for 134 (and its 6 permutations),
$i$ for 224 (and its 3 permutations),
$j$ for 233 (and its 3 permutations).

We have

$$\begin{aligned}
A_0 &= 2a + 2b + 2c + 2d + e,\\
A_1 &= 2b + 2f + 2g + 2h,\\
A_2 &= 2c + 2g + 2i + j,\\
A_3 &= 2d + 2h + 2j,\\
A_4 &= 2e + 2h + i,\\
A_5 &= 2d + 2g,\\
A_6 &= 2c + f,\\
A_7 &= 2b,\\
A_8 &= a.
\end{aligned}$$

The rank is 8 since $\sum_I A_I = 1$. The number of variables is 10 so we pick two variables, $c, d$, to express the rest in terms of. We obtain:

$$\begin{aligned}
&a = A_8,\\
&b = A_7/2,\\
&f = A_6 - 2c,\\
&g = A_5/2 - d,\\
&e = A_0 - 2(a + b + c + d) = (A_0 - 2A_8 - A_7) - 2c - 2d,\\
&h = A_1/2 - b - f - g = (A_1/2 - A_7/2 - A_6 - A_5/2) + 2c + d,\\
&j = A_3/2 - d - h = (A_3/2 - A_1/2 + A_7/2 + A_6 + A_5/2) - 2c - 2d,\\
&i = A_4 - 2e - 2h = (A_4 - 2A_0 + 4A_8 + 3A_7 - A_1 + 2A_6 + A_5) + 2d.
\end{aligned}$$

We get the settings for $c$ and $d$:

$$c = (f^6 e^6 j^6 / h^{12})^{1/6} = fej/h^2,$$
$$d = (g^6 e^6 j^6 / (h^6 i^6))^{1/6} = egj/(hi).$$

Above we also get that $h, i > 0$.

We want to pick settings for integer $q \geq 3$ and rationals $a, b, e, f, g, h, i, j \in [0, 1]$ so that

- $3a + 6(b + c + d) + 3(e + f) + 6(g + h) + 3(i + j) = 1$,
- $(q + 2)^4 \prod_{I=0}^{8} A_I^{A_I} = V_{017}^{6b} V_{026}^{6c} V_{035}^{6d} V_{044}^{3e} V_{116}^{3f} V_{125}^{6g} V_{134}^{6h} V_{224}^{3i} V_{233}^{3j}$.

We obtain the following solution to the above program:

$q = 5, a = .13902732471126287828250 70 \cdot 10^{-6}, b = .170372737250679883223 8690 \cdot 10^{-4}, c = .4957293537057908947335441 \cdot 10^{-3}, d = .0046401687289426480759020 61, e = .0124900102014047580190115 4, f = .6775528221947777757442973 \cdot 10^{-3}, g = .0098617288151037893291667 89, h = .0462963391569208384326888 2, j = .1255544141080093435410128, i = .0719892105176034732930591 5$ which gives the bound $\tau = .79098$ and

$$\omega \leq 2.37294.$$

This bound is better than the one obtained by Stothers [18] (see also Davie and Stothers [11]).

**The eighth tensor power.** Let's first define the program to be solved. The variables are
$a$ for 0016 and its 3 permutations,
$b$ for 0115 and its 6 permutations,
$c$ for 0214 and its 6 permutations,
$d$ for 0313 and its 6 permutations,
$e$ for 0412 and its 6 permutations,
$f$ for 0511 and its 6 permutations,
$g$ for 0610 and its 6 permutations,
$h$ for 079 and its 6 permutations,
$i$ for 088 and its 3 permutations,
$j$ for 1114 and its 3 permutations,
$k$ for 1213 and its 6 permutations,
$l$ for 1312 and its 6 permutations,
$m$ for 1411 and its 6 permutations,
$n$ for 1510 and its 6 permutations,
$p$ for 169 and its 6 permutations,
$\bar{q}$ for 178 and its 6 permutations,
$r$ for 2212 and its 3 permutations,
$s$ for 2311 and its 6 permutations,
$t$ for 2410 and its 6 permutations,
$u$ for 259 and its 6 permutations,
$v$ for 268 and its 6 permutations,
$w$ for 277 and its 3 permutations,
$x$ for 3310 and its 3 permutations,
$y$ for 349 and its 6 permutations,
$z$ for 358 and its 6 permutations,
$\alpha$ for 367 and its 6 permutations,
$\beta$ for 448 and its 3 permutations,
$\gamma$ for 457 and its 6 permutations,
$\delta$ for 466 and its 3 permutations,
$\epsilon$ for 556 and its 3 permutations.

Here we will set $a_{IJK} = \bar{a}_{IJK}$ in Figure 1, so these will be the only variables aside from $q$ and $\tau$.

Let's figure out the constraints: First,
$a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, \bar{q}, r, s, t, u, v, w, x, y, z, \alpha, \beta, \gamma, \delta, \epsilon \geq 0$[3], and
$3a + 6(b + c + d + e + f + g + h) + 3(i + j) + 6(k + l + m + n + p + \bar{q}) + 3r + 6(s + t + u + v) + 3(w + x) + 6(y + z + \alpha) + 3\beta + 6\gamma + 3\delta + 3\epsilon = 1$.

Now,
$A_0 = 2(a + b + c + d + e + f + g + h) + i$,
$A_1 = 2(b + j + k + l + m + n + p + \bar{q})$,
$A_2 = 2(c + k + r + s + t + u + v) + w$,
$A_3 = 2(d + l + s + x + y + z + \alpha)$,

[3]Because the solver had issues with complex numbers when the variables were too close to zero, we actually made all the variables $a, \ldots, \epsilon \geq 10^{-12}$.

$$
\begin{aligned}
&A_4 = 2(e+m+t+y+\beta+\gamma)+\delta,\\
&A_5 = 2(f+n+u+z+\gamma+\epsilon),\\
&A_6 = 2(g+p+v+\alpha+\delta)+\epsilon,\\
&A_7 = 2(h+\bar{q}+w+\alpha+\gamma),\\
&A_8 = 2(i+\bar{q}+v+z)+\beta,\\
&A_9 = 2(h+p+u+y),\\
&A_{10} = 2(g+n+t)+x,\\
&A_{11} = 2(f+m+s),\\
&A_{12} = 2(e+l)+r,\\
&A_{13} = 2(d+k),\\
&A_{14} = 2c+j,\\
&A_{15} = 2b,\\
&A_{16} = a.
\end{aligned}
$$

We pick $\Delta = \{c, d, e, f, g, h, l, m, n, p, t, u, v, z\}$ to make the system have full rank.

After solving for the variables outside of $\Delta$ and taking derivatives we obtain the following constraints

$$
\begin{aligned}
&c\bar{q}^2 = iwj,\\
&d\bar{q}w\epsilon\beta = i\alpha\gamma^2 k,\\
&ew^2\epsilon^2\beta^2 = i\delta\gamma^4 r,\\
&fw\alpha\epsilon\beta^2 = i\delta\gamma^3 s,\\
&g\alpha^2\epsilon\beta^2 = i\delta^2\gamma^2 x,\\
&h\alpha\epsilon\beta^2 = i\delta\gamma^2 y,\\
&lw^2\epsilon\beta = \bar{q}\alpha\gamma^2 r,\\
&mw\alpha\epsilon\beta = \bar{q}\delta\gamma^2 s,\\
&n\alpha^2\beta = \bar{q}\delta\gamma x,\\
&p\alpha\beta = \bar{q}\delta y,\\
&t\alpha^2 = w\delta x,\\
&u\alpha\gamma = w\epsilon y,\\
&v\gamma^2 = w\epsilon\beta,\\
&z\delta\gamma = \alpha\epsilon\beta.
\end{aligned}
$$

These constraints say that $a^a b^b c^c \ldots \epsilon^\epsilon$ is minimized for fixed $\{A_I\}$. In order for these constraints to make sense, we need the following variables to be strictly positive:

$$\bar{q}, w, \alpha, \beta, \gamma, \delta, \epsilon > 0.$$

We enforce this by setting each of them to be $\geq 0.001$.

We want to minimize $\tau$ subject to the above constraints and

$$
\begin{aligned}
(q+2)^8 \leq \quad & V_{0115}^{6b} V_{0214}^{6c} V_{0313}^{6d} V_{0412}^{6e} V_{0511}^{6f} V_{0610}^{6g} V_{079}^{6h} V_{088}^{3i} V_{1114}^{3j} V_{1213}^{6k} V_{1312}^{6l} V_{1411}^{6m} V_{1510}^{6n} V_{169}^{6p} V_{178}^{6q} \cdot \\
& V_{2212}^{3r} V_{2311}^{6s} V_{2410}^{6t} V_{259}^{6u} V_{268}^{6v} V_{277}^{3w} V_{3310}^{3x} V_{349}^{6y} V_{358}^{6z} V_{367}^{6\alpha} V_{448}^{3\beta} V_{457}^{6\gamma} V_{466}^{3\delta} V_{556}^{3\epsilon} / \textstyle\prod_I A_I^{A_I}.
\end{aligned}
$$

We used Maple's NLPSolve function.

We get that if $q = 5, \tau = 2.372873/3$, the LHS above is $7^8 = 5,764,801$, and the following setting of the variables give that the RHS is $\geq 5,764,869 > 7^8$:

$a = 10^{-12}, \alpha = 0.02471103336215649762529364181336185726781094804932 3, b =$
$4.0933714418648223417623975259049800943950530358140 \cdot 10^{-12}, \beta =$
$0.0158803702039597470342593706930037996523555558450 66, c =$
$4.9700778192090709371705828474373904840448203393459 \cdot 10^{-10}, d =$
$2.4642347901810136898379263038819155118509349149757 \cdot 10^{-8}, \delta =$
$0.0540822926539293412186075231171986215567136797570 52, e =$
$5.9877570284688664381218758275595932304175525490625 \cdot 10^{-7}, \epsilon =$
$0.0697580087222669848490178437474089391802953310343 41, f =$
$0.7715644880853893315072258671175097158219294724049 4 \cdot 10^{-5}, g =$
$0.5298395012803432603749720904642878840501046588740 3 \cdot 10^{-4}, \gamma =$
$0.0400466415717113144334921151751759549207625076861 59, h =$
$0.1838700146234894330025205622073129252502178927652 0 \cdot 10^{-3}, i =$
$0.2900552412477732437340618534276949028781906844147 7 \cdot 10^{-3}, j =$
$5.3715725038689209206942456757488242329477901355954 \cdot 10^{-10}, k =$
$3.5695674552470591186014287061508526632570913566224 \cdot 10^{-8}, l =$
$0.1092300913409747883779706670887418535140005682509 9 \cdot 10^{-5}, m =$
$0.1768424695030493349745007805274683243560252368238 3 \cdot 10^{-4}, n =$
$0.1566381894537628744741543645623935779370603718456 5 \cdot 10^{-3}, p =$
$0.7340966210973296140822334734596449252707977859072 3 \cdot 10^{-3}, \bar{q} =$
$0.1676488335293794052624393619226719297329856033612 6 \cdot 10^{-2}, r =$
$0.1463988118978440503087736806854365589146473517448 3 \cdot 10^{-5}, s =$
$0.2984877039239095985994126746818869848714541299682 1 \cdot 10^{-4}, t =$
$0.3321801994724783454658100654952399989823644829239 3 \cdot 10^{-3}, u =$
$0.2008021197612406364455862321861981432820600462986 5 \cdot 10^{-2}, v =$
$0.6193050155187430563433908768649617659432276309259 2 \cdot 10^{-2}, w =$
$0.8965656612913809855146717674351341417157562867863 9 \cdot 10^{-2}, x =$
$0.4183293737968533358776695054220280856487827185468 0 \cdot 10^{-3}, y =$
$0.3177233439289517160066321287701068629949149069052 5 \cdot 10^{-2}, z =$
$0.0126393403854818286275189688565799872716906131480 88.$

**The values for the $8$th power.**

From Claim 1 we have:

$V_{0016} = 1$, $V_{0115} = (8q)^\tau$, $V_{0214} = (\sum_{b \le 2, b=0 \mod 2} \binom{8}{b,(2-b)/2,(14-b)/2} q^b)^\tau = (8 + 28q^2)^\tau$, $V_{0313} = (\binom{8}{1,1,6} q + \binom{8}{3,0,5} q^3)^\tau = (56q + 56q^3)^\tau$,

$V_{0412} = (70q^4 + 168q^2 + 28)^\tau$, $V_{0511} = (280q^3 + 168q + 56q^5)^\tau$, $V_{0610} = (56 + 420q^2 + 280q^4 + 28q^6)^\tau$, $V_{079} = (280q + 560q^3 + 168q^5 + 8q^7)^\tau$, $V_{088} = (70 + 560q^2 + 420q^4 + 56q^6 + q^8)^\tau$.

**Lemma 11.** $V_{1114} \ge 2^{2/3}(2V_{116}^3 + V_{017}^6)^{1/3}$.

*Proof.* $I = J = 1, K = 14$, and the variables are $\alpha_{008}, \alpha_{017}$. The system of equations is
$Z_6 = \alpha_{008}$,
$Z_7 = 2\alpha_{017}$.

Solving we obtain $\alpha_{008} = Z_6$ and $\alpha_{017} = Z_7/2$.

$nz_6 = W_{008}^3 = V_{116}^3$, and $nz_7 = W_{017}^3/2 = V_{017}^6/2$.

The inequality becomes:

$$V_{1114} \ge 2^{2/3}(2V_{116}^3 + V_{017}^6)^{1/3}.$$

□

**Lemma 12.** $V_{1213} \geq 2^{2/3}(V_{116}^3 + 2V_{026}^3)^{1/3}((V_{125}/V_{026})^3 + V_{017}^3)^{1/3}$.

*Proof.* $I = 1, J = 2, K = 13$, and the variables are $\alpha_{008}, \alpha_{017}, \alpha_{026}$. The system of equations becomes
$Y_0 = \alpha_{008} + \alpha_{026}$,
$Y_1 = 2\alpha_{017}$,
$Z_5 = \alpha_{008}$,
$Z_6 = \alpha_{017} + \alpha_{026}$.

We can solve the system:
$\alpha_{008} = Z_5, \alpha_{026} = Y_0 - Z_5, \alpha_{017} = Y_1/2$.
$ny_0 = W_{026}^3 = (V_{026}V_{017})^3$,
$ny_1 = W_{017}^3/2 = (V_{017}V_{116})^3/2$,
$nz_5 = W_{008}^3/W_{026}^3 = (V_{125}/(V_{026}V_{017}))^3, nz_6 = 1$.
$ny_0 + ny_1 = V_{017}^3(2V_{026}^3 + V_{116}^3)/2$.
$nz_5 + nz_6 = ((V_{026}V_{017})^3 + V_{125}^3)/(V_{026}V_{017})^3$.

$$V_{1213} \geq 2^{2/3}((2V_{026}^3 + V_{116}^3))^{1/3}(V_{017}^3 + V_{125}^3/V_{026}^3)^{1/3}.$$

□

**Lemma 13.** $V_{1312} \geq 2(V_{035}^3/V_{125}^3 + 1)^{1/3}(V_{134}^3V_{125}^3/V_{035}^3 + V_{017}^3V_{125}^3 + V_{026}^3V_{116}^3/2)^{1/3}$.

*Proof.* $I = 1, J = 3, K = 12$, and the variables are $\alpha_{008}, \alpha_{017}, \alpha_{026}, \alpha_{035}$. The system of equations is:
$Y_0 = \alpha_{008} + \alpha_{035}$,
$Y_1 = \alpha_{017} + \alpha_{026}$,
$Z_4 = \alpha_{008}$,
$Z_5 = \alpha_{017} + \alpha_{035}$,
$Z_6 = 2\alpha_{026}$.
We solve the system:
$\alpha_{008} = Z_4, \alpha_{026} = Z_6/2, \alpha_{017} = Y_1 - Z_6/2, \alpha_{035} = Y_0 - Z_4$.
$ny_0 = W_{035}^3 = (V_{035}V_{017})^3$,
$ny_1 = W_{017}^3 = (V_{017}V_{125})^3$,
$nz_4 = (W_{008}/W_{035})^3 = (V_{134}/(V_{035}V_{017}))^3$,
$nz_5 = 1$,
$nz_6 = (W_{026}/W_{017})^3/2 = (V_{026}V_{116}/(V_{017}V_{125}))^3/2$.
$ny_0 + ny_1 = V_{017}^3(V_{035}^3 + V_{125}^3)$,
$nz_4 + nz_5 + nz_6 = [2V_{017}^3 + 2(V_{134}/V_{035})^3 + (V_{026}V_{116}/V_{125})^3]/2V_{017}^3$.

$$V_{1312} \geq 2^{2/3}(V_{035}^3 + V_{125}^3)^{1/3}\left(2V_{017}^3 + \frac{2V_{134}^3}{V_{035}^3} + \frac{V_{026}^3V_{116}^3}{V_{125}^3}\right)^{1/3}.$$

□

**Lemma 14.**

$$V_{1411} \geq 2\left(\frac{V_{044}^3}{V_{134}^3} + 1 + \frac{V_{026}^3V_{125}^3}{(2V_{035}^3V_{116}^3)}\right)^{1/3}\left(\frac{V_{134}^6}{V_{044}^3} + V_{017}^3V_{134}^3 + V_{035}^3V_{116}^3\right)^{1/3}.$$

*Proof.* $I = 1$, $J = 4$, $K = 11$, the variables are $\alpha_{008}, \alpha_{017}, \alpha_{026}, \alpha_{035}, \alpha_{044}$. The linear system becomes
$Y_0 = \alpha_{008} + \alpha_{044}$,
$Y_1 = \alpha_{017} + \alpha_{035}$,
$Y_2 = 2\alpha_{026}$,
$Z_3 = \alpha_{008}$,
$Z_4 = \alpha_{017} + \alpha_{044}$,
$Z_5 = \alpha_{026} + \alpha_{035}$.

We solve it:
$\alpha_{008} = Z_3$,
$\alpha_{026} = Y_2/2$,
$\alpha_{044} = Y_0 - Z_3$,
$\alpha_{017} = Z_4 - Y_0 + Z_3$,
$\alpha_{035} = Z_5 - Y_2/2$.

$ny_0 = (W_{044}/W_{017})^3 = (V_{044}V_{017}/(V_{017}V_{134}))^3 = V_{044}^3/V_{134}^3$.
$ny_1 = 1$,
$ny_2 = W_{026}^3/(2W_{035}^3) = V_{026}^3V_{125}^3/(2V_{035}^3V_{116}^3)$,
$nz_3 = W_{008}^3W_{017}^3/W_{044}^3 = (V_{134}^2/V_{044})^3$,
$nz_4 = W_{017}^3 = V_{017}^3V_{134}^3$,
$nz_5 = W_{035}^3 = V_{035}^3V_{116}^3$.
$ny_0 + ny_1 + ny_2 = \frac{V_{044}^3}{V_{134}^3} + 1 + \frac{V_{026}^3V_{125}^3}{(2V_{035}^3V_{116}^3)}$,
$nz_3 + nz_4 + nz_5 = \frac{V_{134}^6}{V_{044}^3} + V_{017}^3V_{134}^3 + V_{035}^3V_{116}^3$.

The inequality becomes

$$V_{1411} \geq 2\left(\frac{V_{044}^3}{V_{134}^3} + 1 + \frac{V_{026}^3V_{125}^3}{(2V_{035}^3V_{116}^3)}\right)^{1/3}\left(\frac{V_{134}^6}{V_{044}^3} + V_{017}^3V_{134}^3 + V_{035}^3V_{116}^3\right)^{1/3}.$$

□

**Lemma 15.**

$$V_{1510} \geq 2\left(\frac{V_{035}^3}{V_{134}^3} + 1 + \frac{V_{026}^3V_{134}^3}{V_{044}^3V_{116}^3}\right)^{1/3}\left(\frac{V_{125}^3V_{134}^3}{V_{035}^3} + V_{017}^3V_{134}^3 + V_{044}^3V_{116}^3 + \frac{V_{035}^3V_{125}^3V_{044}^3V_{116}^3}{(2V_{026}^3V_{134}^3)}\right)^{1/3}.$$

*Proof.* $I = 1, J = 5, K = 10$. The variables are $\alpha_{008}, \alpha_{017}, \alpha_{026}, \alpha_{035}, \alpha_{044}, \alpha_{053}$. The linear system becomes

$Y_0 = \alpha_{008} + \alpha_{053}$,
$Y_1 = \alpha_{017} + \alpha_{044}$,
$Y_2 = \alpha_{026} + \alpha_{035}$,
$Z_2 = \alpha_{008}$,
$Z_3 = \alpha_{017} + \alpha_{053}$,
$Z_4 = \alpha_{026} + \alpha_{044}$,
$Z_5 = 2\alpha_{035}$.

We solve it:

$\alpha_{008} = Z_2$,
$\alpha_{035} = Z_5/2$,
$\alpha_{053} = Y_0 - Z_2$,
$\alpha_{026} = Y_2 - Z_5/2$,
$\alpha_{017} = Z_3 - Y_0 + Z_2$,
$\alpha_{044} = Z_4 - Y_2 + Z_5/2$.
$ny_0 = (W_{053}/W_{017})^3 = V_{035}^3/V_{134}^3$,
$ny_1 = 1$,
$ny_2 = (W_{026}/W_{044})^3 = V_{026}^3 V_{134}^3/(V_{044}^3 V_{116}^3)$,
$nz_2 = (W_{008}W_{017}/W_{053})^3 = V_{125}^3 V_{134}^3/V_{035}^3$,
$nz_3 = W_{017}^3 = V_{017}^3 V_{134}^3$,
$nz_4 = W_{044}^3 = V_{044}^3 V_{116}^3$,
$nz_5 = W_{035}^3 W_{044}^3/(2W_{026}^3) = (V_{035}^3 V_{125}^3 V_{044}^3 V_{116}^3)/(2V_{026}^3 V_{134}^3)$.
$ny_0 + ny_1 + ny_2 = \frac{V_{035}^3}{V_{134}^3} + 1 + \frac{V_{026}^3 V_{134}^3}{V_{044}^3 V_{116}^3}$,
$nz_2 + nz_3 + nz_4 + nz_5 = \frac{V_{125}^3 V_{134}^3}{V_{035}^3} + V_{017}^3 V_{134}^3 + V_{044}^3 V_{116}^3 + \frac{V_{035}^3 V_{125}^3 V_{044}^3 V_{116}^3}{(2V_{026}^3 V_{134}^3)}$.

Hence we obtain

$$V_{1510} \geq 2\left(\frac{V_{035}^3}{V_{134}^3} + 1 + \frac{V_{026}^3 V_{134}^3}{V_{044}^3 V_{116}^3}\right)^{1/3}\left(\frac{V_{125}^3 V_{134}^3}{V_{035}^3} + V_{017}^3 V_{134}^3 + V_{044}^3 V_{116}^3 + \frac{V_{035}^3 V_{125}^3 V_{044}^3 V_{116}^3}{(2V_{026}^3 V_{134}^3)}\right)^{1/3}.$$

□

**Lemma 16.**

$$V_{169} \geq 2\left(\frac{V_{026}^3}{V_{125}^3} + 1 + \frac{V_{026}^3 V_{134}^3}{(V_{035}^3 V_{116}^3)} + \frac{V_{134}^6 V_{026}^3}{(2V_{044}^3 V_{125}^3 V_{116}^3)}\right)^{1/3}\left(\frac{V_{116}^3 V_{125}^3}{V_{026}^3} + V_{017}^3 V_{125}^3 + V_{035}^3 V_{116}^3 + \frac{V_{044}^3 V_{125}^3 V_{035}^3 V_{116}^3}{V_{026}^3 V_{134}^3}\right)^{1/3}.$$

*Proof.* $I = 1, J = 6, K = 9$ so the variables are $\alpha_{008}, \alpha_{017}, \alpha_{026}, \alpha_{035}, \alpha_{044}, \alpha_{053}, \alpha_{062}$. The linear system becomes
$Y_0 = \alpha_{008} + \alpha_{062}$,
$Y_1 = \alpha_{017} + \alpha_{053}$,
$Y_2 = \alpha_{026} + \alpha_{044}$,
$Y_3 = 2\alpha_{035}$,
$Z_1 = \alpha_{008}$,
$Z_2 = \alpha_{017} + \alpha_{062}$,
$Z_3 = \alpha_{026} + \alpha_{053}$,
$Z_4 = \alpha_{035} + \alpha_{044}$.

We solve it:
$\alpha_{008} = Z_1$,
$\alpha_{035} = Y_3/2$,
$\alpha_{062} = Y_0 - Z_1$,
$\alpha_{044} = Z_4 - Y_3/2$,
$\alpha_{026} = Y_2 - Z_4 + Y_3/2$,
$\alpha_{053} = Z_3 - Y_2 + Z_4 - Y_3/2$,
$\alpha_{017} = Z_2 - Y_0 + Z_1$,

$ny_0 = W_{062}^3/W_{017}^3 = V_{026}^3/V_{125}^3$,
$ny_1 = 1$,
$ny_2 = W_{026}^3/W_{053}^3 = V_{026}^3 V_{134}^3/(V_{035}^3 V_{116}^3)$,
$ny_3 = W_{035}^3 W_{026}^3/(2W_{044}^3 W_{053}^3) = V_{134}^6 V_{026}^3/(2V_{044}^3 V_{125}^3 V_{116}^3)$,
$nz_1 = W_{008}^3 W_{017}^3/W_{062}^3 = V_{116}^3 V_{125}^3/V_{026}^3$,
$nz_2 = W_{017}^3 = V_{017}^3 V_{125}^3$,
$nz_3 = W_{053}^3 = V_{035}^3 V_{116}^3$,
$nz_4 = W_{044}^3 W_{053}^3/W_{026}^3 = (V_{044}^3 V_{125}^3 V_{035}^3 V_{116}^3)/(V_{026}^3 V_{134}^3)$.
$ny_0 + ny_1 + ny_2 + ny_3 = \frac{V_{026}^3}{V_{125}^3} + 1 + \frac{V_{026}^3 V_{134}^3}{(V_{035}^3 V_{116}^3)} + \frac{V_{134}^6 V_{026}^3}{(2V_{044}^3 V_{125}^3 V_{116}^3)}$.
$nz_1 + nz_2 + nz_3 + nz_4 = \frac{V_{116}^3 V_{125}^3}{V_{026}^3} + V_{017}^3 V_{125}^3 + V_{035}^3 V_{116}^3 + \frac{V_{044}^3 V_{125}^3 V_{035}^3 V_{116}^3}{V_{026}^3 V_{134}^3}$.

$$V_{169} \geq 2\left(\frac{V_{026}^3}{V_{125}^3} + 1 + \frac{V_{026}^3 V_{134}^3}{(V_{035}^3 V_{116}^3)} + \frac{V_{134}^6 V_{026}^3}{(2V_{044}^3 V_{125}^3 V_{116}^3)}\right)^{1/3} \left(\frac{V_{116}^3 V_{125}^3}{V_{026}^3} + V_{017}^3 V_{125}^3 + V_{035}^3 V_{116}^3 + \frac{V_{044}^3 V_{125}^3 V_{035}^3 V_{116}^3}{V_{026}^3 V_{134}^3}\right)^{1/3}.$$

□

**Lemma 17.**

$$V_{178} \geq 2(V_{017}^3 + V_{116}^3 + V_{125}^3 + V_{134}^3)^{1/3} \left(1 + V_{017}^3 + V_{026}^3 + V_{035}^3 + \frac{V_{044}^3}{2}\right)^{1/3}.$$

*Proof.* $I = 1, J = 7, K = 8$, so the variables are $\alpha_{008}, \alpha_{017}, \alpha_{026}, \alpha_{035}, \alpha_{044}, \alpha_{053}, \alpha_{062}, \alpha_{071}$. The linear system is
$Y_0 = \alpha_{008} + \alpha_{071}$,
$Y_1 = \alpha_{017} + \alpha_{062}$,
$Y_2 = \alpha_{026} + \alpha_{053}$,
$Y_3 = \alpha_{035} + \alpha_{044}$,
$Z_0 = \alpha_{008}$,
$Z_1 = \alpha_{017} + \alpha_{071}$,
$Z_2 = \alpha_{026} + \alpha_{062}$,
$Z_3 = \alpha_{035} + \alpha_{053}$,
$Z_4 = 2\alpha_{044}$.

We solve the system:
$\alpha_{008} = Z_0$,
$\alpha_{044} = Z_4/2$,
$\alpha_{071} = Y_0 - Z_0$,
$\alpha_{035} = Y_3 - Z_4/2$,
$\alpha_{017} = Z_1 - Y_0 + Z_0$,
$\alpha_{053} = Z_3 - Y_3 + Z_4/2$,
$\alpha_{062} = Y_1 - Z_1 + Y_0 - Z_0$,
$\alpha_{026} = Z_2 - Y_1 + Z_1 - Y_0 + Z_0$.
$ny_0 = W_{071}^3 W_{062}^3/(W_{017} W_{026})^3 = V_{017}^3/V_{125}^3$,
$ny_1 = W_{062}^3/W_{026}^3 = V_{116}^3/V_{125}^3$,
$ny_2 = 1$,
$ny_3 = W_{035}^3/W_{053}^3 = V_{134}^3/V_{125}^3$,

$nz_0 = W_{008}^3 W_{017}^3 W_{026}^3/(W_{071}^3 W_{062}^3) = V_{017}^3 V_{017}^3 V_{116}^3 V_{026}^3 V_{125}^3/(V_{071}^3 V_{017}^3 V_{062}^3 V_{116}^3) = V_{125}^3,$
$nz_1 = W_{017}^3 W_{026}^3/W_{062}^3 = V_{017}^3 V_{125}^3,$
$nz_2 = W_{026}^3 = V_{026}^3 V_{125}^3,$
$nz_3 = W_{053}^3 = V_{035}^3 V_{125}^3,$
$nz_4 = W_{044}^3 W_{053}^3/(2W_{035}^3) = V_{044}^3 V_{125}^3/2.$

$ny_0 + ny_1 + ny_2 + ny_3 = (V_{017}^3 + V_{116}^3 + V_{125}^3 + V_{134}^3)/V_{125}^3,$
$nz_0 + nz_1 + nz_2 + nz_3 + nz_4 = V_{125}^3(1 + V_{017}^3 + V_{026}^3 + V_{035}^3 + V_{044}^3/2).$

$$V_{178} \geq 2(V_{017}^3 + V_{116}^3 + V_{125}^3 + V_{134}^3)^{1/3} \left(1 + V_{017}^3 + V_{026}^3 + V_{035}^3 + \frac{V_{044}^3}{2}\right)^{1/3}.$$

□

**Lemma 18.**

$$V_{2212} \geq 2\left(\frac{V_{026}^3}{V_{116}^3} + \frac{1}{2}\right)^{2/3} \left(\frac{V_{224}^3 V_{116}^3}{V_{026}^6} + \frac{V_{116}^3 V_{017}^3 V_{125}^3}{V_{026}^3} + \frac{V_{116}^3}{2}\right)^{1/3}.$$

*Proof.* $I = J = 2$, $K = 12$, and the variables are $a = \alpha_{008}$, $b = \alpha_{017}$, $c = \alpha_{026}$, $d = \alpha_{107}$, $e = \alpha_{116}$.

The linear system is:
$X_0 = a + b + c,$
$Y_0 = a + c + d,$
$Z_4 = a,$
$Z_5 = b + d,$
$Z_6 = 2(c + e).$

It has 5 variables and rank 5. We solve it:
$a = Z_4,$
$c = (X_0 + Y_0 - 2Z_4 - Z_5)/2,$
$e = Z_6/2 - c = (Z_6 - X_0 - Y_0 + 2Z_4 + Z_5)/2,$
$d = Y_0 - Z_4 - c = (-X_0 + Y_0 + Z_5)/2$
$b = Z_5 - d = (Z_5 + X_0 - Y_0)/2.$

$nx_0 = (W_{026}W_{017}/(W_{116}W_{107}))^{3/2} = (V_{026}^2 V_{017} V_{215}/(V_{116}^2 V_{107} V_{125}))^{3/2} = V_{026}^3/V_{116}^3.$
$nx_1 = 1/2,$
$ny_0 = (W_{026}W_{107}/(W_{017}W_{116}))^{3/2} = (V_{026}/V_{116})^3,$
$ny_1 = 1/2,$
$nz_4 = (W_{008}W_{116}/W_{026})^3 = (V_{224}V_{116}/V_{026}^2)^3,$
$nz_5 = (W_{017}W_{107}W_{116}/W_{026})^{3/2} = (V_{017}V_{125}V_{116}/V_{026})^3,$
$nz_6 = V_{116}^3/2.$

Hence:

$$V_{2212} \geq 2\left(\frac{V_{026}^3}{V_{116}^3} + \frac{1}{2}\right)^{2/3} \left(\frac{V_{224}^3 V_{116}^3}{V_{026}^6} + \frac{V_{116}^3 V_{017}^3 V_{125}^3}{V_{026}^3} + \frac{V_{116}^3}{2}\right)^{1/3}.$$

□

**Lemma 19.** *For $q = 5, \tau = 2.372873/3$, $V_{2311}(q, \tau) \geq 35517.87580$.*

*In general,*

$$V_{2311} \geq 2\left(\frac{V_{224}^3 V_{035}^3}{V_{125}^3 V_{134}^3} + 1/2\right)^{1/3} \left(\frac{V_{035}^3}{V_{125}^3} + 1\right)^{1/3} \left(\frac{V_{233}^3 V_{134}^3 V_{125}^6}{V_{035}^6 V_{224}^3} + \frac{V_{125}^6 V_{134}^3 V_{026}^3}{V_{224}^3 V_{035}^3}\right)^{1/3} \left(\frac{V_{116}V_{224}V_{035}}{(V_{134}V_{026}V_{125})}\right)^f.$$

*The variable $f$ is constrained as in our framework (see the proof).*

*Proof.* $I = 2, J = 3, K = 11$, so the variables are $a = \alpha_{008}, b = \alpha_{017}, c = \alpha_{026}, d = \alpha_{035}, e = \alpha_{107}, f = \alpha_{116}$.

The linear system is:
$X_0 = a + b + c + d$,
$Y_0 = a + d + e$,
$Z_3 = a$,
$Z_4 = b + e$,
$Z_5 = c + d + f$.

The system has 6 variables but only rank 5. We pick $f$ to be the variable in $\Delta$. We can now solve the system for the rest of the variables:
$a = Z_3$,
$d = X_0 + Y_0 - Z_4 - Z_5 - 2Z_3 + f$,
$e = Z_3 + Z_4 + Z_5 - X_0 - f$,
$b = X_0 - Z_3 - Z_5 + f$,
$c = 2Z_5 + 2Z_3 + Z_4 - X_0 - Y_0 - 2f$.
$nx_0 = (\frac{W_{017}W_{035}}{W_{026}W_{107}})^3 = (\frac{V_{224}V_{035}}{V_{125}V_{134}})^3$,
$nx_1 = 1/2$,
$ny_0 = (W_{035}/W_{026})^3 = (V_{035}/V_{125})^3$,
$ny_1 = 1$,
$nz_3 = \frac{(W_{008}W_{107}W_{026}^2)^3}{(W_{035}^2 W_{017})} = \frac{(V_{233}V_{134}V_{125}^2)^3}{(V_{035}^2 V_{224})^3}$,
$nz_4 = \frac{W_{026}^3 W_{107}^3}{W_{035}^3} = \frac{(V_{125}V_{017}V_{134})^3}{V_{035}^3}$,
$nz_5 = (\frac{W_{026}^2 W_{107}}{W_{017}W_{035}})^3 = (\frac{V_{125}^2 V_{134}V_{026}}{V_{224}V_{035}})^3$,
$nf = \frac{W_{116}W_{017}W_{035}}{W_{107}W_{026}^2} = \frac{V_{116}V_{224}V_{035}}{V_{134}V_{026}V_{125}}$.
The inequality is

$$V_{2311} \geq 2\left(\frac{V_{224}^3 V_{035}^3}{V_{125}^3 V_{134}^3} + 1/2\right)^{1/3}\left(\frac{V_{035}^3}{V_{125}^3} + 1\right)^{1/3}\left(\frac{V_{233}^3 V_{134}^3 V_{125}^6}{V_{035}^6 V_{224}^3} + \frac{V_{125}^6 V_{134}^3 V_{026}^3}{V_{224}^3 V_{035}^3}\right)^{1/3}\left(\frac{V_{116}V_{224}V_{035}}{(V_{134}V_{026}V_{125})}\right)^{f}.$$

The variable $f$ needs to minimize the quantity $b\ln b + c\ln c + d\ln d + e\ln e + f\ln f$, subject to the linear constraints that $a, b, c, d, e, f \in [0, 1]$, where $a, b, c, d, e$ are the linear expressions we obtained by solving the linear system above, and where $X_0 = nx_0/(nx_0 + nx_1)$, $Y_0 = ny_0/(ny_0 + ny_1)$, $nz_3 = nz_3/(nz_3 + nz_4 + nz_5)$, $nz_4 = nz_4/(nz_3 + nz_4 + nz_5)$, and $nz_5 = nz_5/(nz_3 + nz_4 + nz_5)$.

We maximize our value lower bound by first finding the value $f' = .433584923533081$ minimizing $F(f) = a^a b^b c^c d^d e^e f^f$, then finding the value $f'' = .433607696886902$ maximizing $V_{2311}(f)/F(f)$, and finally concluding that $V_{2311} \geq F(f') \geq V_{2311}(f'')/F(f'') \geq 35517.8758$.

□

**Lemma 20.** *For $q = 5, \tau = 2.372873/3$, $V_{2410} \geq 1.089681104 \cdot 10^5$.*

*In general,*

$$V_{2410} \geq 2\left(\frac{(V_{026}V_{224}V_{035})^{3/2}}{V_{125}^{3/2}} + \frac{V_{116}^{3/2}V_{134}^{3/2}}{2}\right)^{1/3}\left(\frac{V_{044}^3 V_{026}^{3/2}V_{125}^{3/2}}{(V_{224}^{3/2}V_{035}^{3/2})} + (V_{116}V_{134})^{3/2} + \frac{(V_{026}V_{224}V_{125})^{3/2}}{(2V_{035}^{3/2})}\right)^{1/3} \times$$

$$\left(\frac{V_{224}^3}{V_{044}^3V_{026}^3}+\frac{(V_{017}^2V_{233}^2V_{125})^{3/2}}{(V_{026}V_{224}V_{035}V_{116}V_{134})^{3/2}}+1+\frac{(V_{035}V_{125}^3)^{3/2}}{2(V_{026}V_{224}V_{116}V_{134})^{3/2}}\right)^{1/3}\left(\frac{V_{224}V_{035}V_{134}}{(V_{233}V_{044}V_{125})}\right)^f,$$

*where $f$ is constrained as in our framework (see the proof).*

*Proof.* $I=2, J=4, K=10$, and so the variables are $a=\alpha_{008}$, $b=\alpha_{017}$, $c=\alpha_{026}$, $d=\alpha_{035}$, $e=\alpha_{044}$, $f=\alpha_{107}$, $g=\alpha_{116}$, $h=\alpha_{125}$.

The linear system is: $X_0=a+b+c+d+e$,
$X_1=2(f+g+h)$,
$Y_0=a+e+f$,
$Y_1=b+d+g$,
$Y_2=2(c+h)$,
$Z_2=a$,
$Z_3=b+f$,
$Z_4=c+e+g$,
$Z_5=2(d+h)$,

We omit the equations for $X_1$ and $Y_2$. The system has rank 7 and has 8 unknowns. Hence we pick a variable, $f$, to place into $\Delta$.

We now solve the system:
$a=Z_2$,
$b=Z_3-f$,
$e=Y_0-Z_2-f$,
$d=1/2(X_0+Y_1-Z_4-Z_2-2Z_3+2f)$,
$h=Z_5/2-d=1/2(Z_5-X_0-Y_1+Z_4+Z_2+2Z_3-2f)$,
$g=Y_1-b-d=1/2(-X_0+Y_1+Z_4+Z_2)$,
$c=Z_4-e-g=-Y_0+f+1/2(X_0-Y_1+Z_4+Z_2)$.

Calculate:
$nx_0=(V_{026}V_{224}V_{035})^{3/2}/(V_{116}V_{134}V_{125})^{3/2}$,
$nx_1=1/2$,
$ny_0=V_{044}^3/V_{224}^3$,
$ny_1=(V_{035}V_{116}V_{134})^{3/2}/((V_{026}V_{224}V_{125})^(3/2))$,
$ny_2=1/2$,
$nz_2=V_{224}^(9/2)(V_{116}V_{134}V_{125})^{3/2}/((V_{035}V_{026})^{3/2}V_{044}^3)$,
$nz_3=V_{017}^3V_{233}^3V_{125}^3/V_{035}^3$,
$nz_4=(V_{026}V_{224}V_{116}V_{134}V_{125})^{3/2}/V_{035}^{3/2}$,
$nz_5=V_{125}^6/2$,
$nf=V_{224}^3V_{035}^3V_{134}^3/(V_{233}^3V_{125}^3V_{044}^3)$.

We obtain:

$$V_{2410}\geq 2\left(\frac{(V_{026}V_{224}V_{035})^{3/2}}{/(V_{116}V_{134}V_{125})^{3/2}}+\frac{1}{2}\right)^{1/3}\left(\frac{V_{044}^3}{V_{224}^3}+\frac{(V_{035}V_{116}V_{134})^{3/2}}{(V_{026}V_{224}V_{125})^(3/2)}+\frac{1}{2}\right)^{1/3}\times$$

$$\left(\frac{V_{224}^(9/2)(V_{116}V_{134}V_{125})^{3/2}}{/((V_{035}V_{026})^{3/2}V_{044}^3)}+\frac{V_{017}^3V_{233}^3V_{125}^3}{V_{035}^3}+\frac{(V_{026}V_{224}V_{116}V_{134}V_{125})^{3/2}}{V_{035}^{3/2}}+\frac{V_{125}^6}{2}\right)^{1/3}\left(\frac{V_{224}V_{035}V_{134}}{V_{233}V_{125}V_{044}}\right)^f.$$

We have some constraints on $f$ that we obtain from our settings in the linear system solution and from the settings $X_i := x_i/(nx_0 + nx_1)$ for $i \in \{0, 1\}$, $Y_j := ny_j/(ny_2 + ny_1 + ny_0)$ for $j \in \{0, 1, 2\}$ and $Z_k := nz_k/(nz_2 + nz_3 + nz_4 + nz_5)$ for $k \in \{2, 3, 4\}$.

Constraint 0 is from $a := Z_2 \geq 0$. It does not contain $f$. One can see that sinze $nz_i \geq 0$ for all $i \in 2, 3, 4, 5$, this constraint is always satisfied.

Constraint 1 is from $b := Z_3 - f \geq 0$ and is

$$f \leq Z_3.$$

For $f = 0.00327216658358239$, $\tau = 2.3729/3$ and $q = 5$, $Z_3 - f \geq 0.01$ and this constraint is satisfied.

Constraint 2 is from $e := Y_0 - Z_2 - f \geq 0$, and hence

$$f \leq Y_0 - Z_2.$$

For $f = 0.00327216658358239$, $\tau = 2.3729/3$ and $q = 5$, $Y_0 - Z_2 - f \geq 0.04$ and this constraint is satisfied.

Constraint 3 is from $d := 1/2(X_0 + Y_1 - Z_4 - Z_2 - 2Z_3 + 2f) \geq 0$, and it is

$$f \geq (Z_2 + 2Z_3 + Z_4 - X_0 - Y_1)/2.$$

For $f = 0.00327216658358239$, $\tau = 2.3729/3$ and $q = 5$, $1/2(X_0 + Y_1 - Z_4 - Z_2 - 2Z_3 + 2f) \geq 0.26$ and this constraint is satisfied.

Constraint 4 is from $h := Z_5/2 - d = 1/2(Z_5 - X_0 - Y_1 + Z_4 + Z_2 + 2Z_3 - 2f) \geq 0$, and is

$$f \leq (Z_5 - X_0 - Y_1 + Z_4 + Z_2 + 2Z_3)/2.$$

For $f = 0.00327216658358239$, $\tau = 2.3729/3$ and $q = 5$, $1/2(Z_5 - X_0 - Y_1 + Z_4 + Z_2 + 2Z_3 - 2f) \geq 0.29$ and this constraint is satisfied.

Constraint 5 is from $g := Y_1 - b - d = 1/2(-X_0 + Y_1 + Z_4 + Z_2) \geq 0$. It doesn't contain $f$ and is satisfied for all $\tau \in [2/3, 1]$ when $q = 5$.

Constraint 6 is from $c := Z_4 - e - g = -Y_0 + f + 1/2(X_0 - Y_1 + Z_4 + Z_2) \geq 0$. For $f = 0.00327216658358239$, $\tau = 2.3729/3$ and $q = 5$, $-Y_0 + f + 1/2(X_0 - Y_1 + Z_4 + Z_2) \geq 0.19$ and this constraint is satisfied.

The final constraint is that the following should be minimized under the above 7 constraints

$$a \ln a + b \ln b + c \ln c + d \ln d + e \ln e + f \ln f + g \ln g + h \ln h,$$

where $a, b, c, d, e, g, h$ are the linear functions of $f$ as above (for fixed $q$,$\tau$). For $q = 5$, $\tau = 2.372873/3$, we first compute the value $f'$ that maximized $F(f) = a^a b^b c^c d^d e^e f^f g^g h^h$ under our settings for $a, b, c, d, e, g, h$ and under the linear constraints on $f$ above. We obtain $f' = 0.00327237690111682$. Then, we minimize $V_{2410}(f)/F(f)$ over the linear constraints on $f$ obtaining $f'' = 0.00332911459461995$. Finally, we conclude that $V_{2410} \geq F(f') \cdot V_{2410}(f'')/F(f'') \geq 1.089681104 \cdot 10^5$.

□

**Lemma 21.** $V_{259} \geq 2.479007361 \cdot 10^5$ *for* $q = 5, \tau = 2.372873/3$.

*In general,*

$$V_{259} \geq 2 \left( \frac{V_{026}^3 V_{233}^3 V_{044}^3 V_{125}^3}{V_{035}^3 V_{224}^3 V_{116}^3 V_{134}^3} + \frac{1}{2} \right)^{1/3} \left( \frac{V_{035}^3}{V_{233}^3} + \frac{V_{044}^3 V_{125}^3}{V_{035}^3 V_{224}^3} + 1 \right)^{1/3} \times$$

$$\left(\frac{V_{224}^3V_{116}^3V_{134}^3}{V_{044}^3V_{026}^3}+\frac{V_{017}^3V_{224}^9V_{035}^6V_{116}^3V_{134}^3}{V_{026}^3V_{233}^3V_{044}^6V_{125}^6}+\frac{V_{035}^3V_{224}^3V_{116}^3V_{134}^3}{V_{044}^3V_{125}^3}+\frac{V_{035}^6V_{224}^6V_{116}^3V_{134}^3}{V_{026}^3V_{233}^3V_{044}^3V_{125}^3}\right)^{1/3}\times$$
$$\left(\frac{V_{026}V_{233}^2V_{044}^2V_{125}^3}{V_{224}^3V_{035}^3V_{116}V_{134}}\right)^g\left(\frac{V_{026}V_{233}V_{044}V_{125}^2}{V_{035}^2V_{224}^2V_{116}}\right)^j,$$

*where $g$ and $j$ are subject to the constraints of our framework (see the proof).*

*Proof.* $I=2,J=5,K=9$, and the variables are $a=\alpha_{008}$, $b=\alpha_{017}$, $c=\alpha_{026}$, $d=\alpha_{035}$, $e=\alpha_{044}$, $f=\alpha_{053}$, $g=\alpha_{107}$, $h=\alpha_{116}$, $j=\alpha_{125}$. The linear system is
$X_0=a+b+c+d+e+f$,
$Y_0=a+f+g$,
$Y_1=b+e+h$,
$Z_1=a$,
$Z_2=b+g$,
$Z_3=c+f+h$,
$Z_4=d+e+j$.

The system has rank 7 but it has 9 variables, so we pick two variables, $g$ and $j$, and we solve the system assuming them as constants.
$a=Z_1$,
$b=Z_2-g$
$h=Z_1+Z_2+Z_3+Z_4-X_0-g-j$
$e=Y_1-b-h=Y_1-Z_1-2Z_2-Z_3-Z_4+X_0+2g+j$,
$c=Z_3-f-h=-Z_4+2g+j+X_0-Y_0-Z_2$,
$d=Z_4-e-j=-2j-2g-X_0-Y_1+Z_1+Z_3+2Z_2+2Z_4$,
$f=Y_0-Z_1-g$.

$nx_0=\frac{V_{026}^3V_{233}^3V_{044}^3V_{125}^3}{V_{035}^3V_{224}^3V_{116}^3V_{134}^3}$,
$nx_1=1/2$,
$ny_0=V_{035}^3/V_{233}^3$,
$ny_1=\frac{V_{044}^3V_{125}^3}{V_{035}^3V_{224}^3}$,
$ny_2=1$,
$nz_1=\frac{V_{224}^3V_{116}^3V_{134}^3}{V_{044}^3V_{026}^3}$,
$nz_2=\frac{V_{017}^3V_{224}^9V_{035}^6V_{116}^3V_{134}^3}{V_{026}^3V_{233}^3V_{044}^6V_{125}^6}$,
$nz_3=\frac{V_{035}^3V_{224}^3V_{116}^3V_{134}^3}{V_{044}^3V_{125}^3}$,
$nz_4=\frac{V_{035}^6V_{224}^6V_{116}^3V_{134}^3}{V_{026}^3V_{233}^3V_{044}^3V_{125}^3}$,
$ng=\frac{V_{026}V_{233}^2V_{044}^2V_{125}^3}{V_{224}^3V_{035}^3V_{116}V_{134}}$,
$nj=\frac{V_{026}V_{233}V_{044}V_{125}^2}{V_{035}^2V_{224}^2V_{116}}$.

$$V_{259}\geq 2\left(\frac{V_{026}^3V_{233}^3V_{044}^3V_{125}^3}{V_{035}^3V_{224}^3V_{116}^3V_{134}^3}+\frac{1}{2}\right)^{1/3}\left(\frac{V_{035}^3}{V_{233}^3}+\frac{V_{044}^3V_{125}^3}{V_{035}^3V_{224}^3}+1\right)^{1/3}\times$$
$$\left(\frac{V_{224}^3V_{116}^3V_{134}^3}{V_{044}^3V_{026}^3}+\frac{V_{017}^3V_{224}^9V_{035}^6V_{116}^3V_{134}^3}{V_{026}^3V_{233}^3V_{044}^6V_{125}^6}+\frac{V_{035}^3V_{224}^3V_{116}^3V_{134}^3}{V_{044}^3V_{125}^3}+\frac{V_{035}^6V_{224}^6V_{116}^3V_{134}^3}{V_{026}^3V_{233}^3V_{044}^3V_{125}^3}\right)^{1/3}\times$$

$$\left(\frac{V_{026}V_{233}^2V_{044}^2V_{125}^3}{V_{224}^3V_{035}^3V_{116}V_{134}}\right)^g\left(\frac{V_{026}V_{233}V_{044}V_{125}^2}{V_{035}^2V_{224}^2V_{116}}\right)^j.$$

Now let's look at the constraints on $g$ and $j$:

Constraint 1: since $b = Z_2 - g \geq 0$ and $Z_2$ was set to $nz_2/(nz_1 + nz_2 + nz_3 + nz_4)$, we have that $g \leq nz_2/(nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 2: since $e = Y_1 - Z_1 - 2Z_2 - Z_3 - Z_4 + X_0 + 2g + j \geq 0$ and we set $X_0 = nx_0/(nx_0 + nx_1)$, $Y_1 = ny_1/(ny_0 + ny_1 + ny_2)$, $Z_2 = nz_2/(nz_1 + nz_2 + nz_3 + nz_4)$, and $Z_1 + Z_2 + Z_3 + Z_4 = 1$, we get
$2g + j \geq -ny_1/(ny_0 + ny_1 + ny_2) + nz_2/(nz_1 + nz_2 + nz_3 + nz_4) + 1 - nx_0/(nx_0 + nx_1)$,

Constraint 3: since $c = -Z_4 + 2g + j + X_0 - Y_0 - Z_2 \geq 0$ and $Z_4 = nz_4/(nz_1 + nz_2 + nz_3 + nz_4)$, we get:
$2g + j \geq (nz_2 + nz_4)/(nz_1 + nz_2 + nz_3 + nz_4) - nx_0/(nx_0 + nx_1) + ny_0/(ny_0 + ny_1 + ny_2)$,

Constraint 4: since $d = -2j - 2g - X_0 - Y_1 + Z_1 + Z_3 + 2Z_2 + 2Z_4 \geq 0$, we get:
$g + j \leq 0.5(1 + (nz_2 + nz_4)/(nz_1 + nz_2 + nz_3 + nz_4) - ny_1/(ny_0 + ny_1 + ny_2) - nx_0/(nx_0 + nx_1))$,

Constraint 5: since $f = Y_0 - Z_1 - g \geq 0$, we get:
$g \leq ny_0/(ny_0 + ny_1 + ny_2) - nz_1/(nz_1 + nz_2 + nz_3 + nz_4)$,

Constraint 6: since $h = Z_1 + Z_2 + Z_3 + Z_4 - X_0 - g - j \geq 0$, we get
$g + j \leq 1 - nx_0/(nx_0 + nx_1)$.

We first find the values of $g$ and $j$ that maximize $F(g, j) = a^a b^b c^c d^d e^e f^f g^g h^h$ under the above 6 constraints. These settings are $g' = 0.426490605423260 \cdot 10^{-3}, j' = .467539258441983$ and give $F = 0.218734398504605437$. Then, we find the settings for $g$ and $j$ that minimize $v(g, j) = V_{259}(g, j)/(a^a b^b c^c d^d e^e f^f g^g h^h)$ subject to the above 6 constraints. These are $g'' = 0.000421947422353580, j'' = 0.467186674565410$ and give $v(g'', j'') = 1.13334133895458700 \cdot 10^6$. Finally, we obtain that $V_{259} \geq v(g'', j'')F(g'j') \geq 2.479007361 \cdot 10^5$ for $q = 5, \tau = 2.372873/3$.

□

**Lemma 22.** *For $\tau = 2.372873/3$ and $q = 5$, $V_{268} \geq 4.108912286 \cdot 10^5$.*

*In general,*

$$V_{268} \geq 2\left(\frac{V_{035}^3}{V_{125}^3V_{116}^3V_{134}^3} + \frac{V_{035}^{3/2}}{2V_{026}^{3/2}V_{224}^{3/2}V_{116}^{3/2}V_{134}^{3/2}V_{125}^{3/2}}\right)^{1/3}\times$$

$$\left(\frac{V_{026}^{3/2}V_{224}^{3/2}V_{035}^{3/2}V_{116}^3V_{116}^{3/2}}{V_{134}^{3/2}V_{125}^{3/2}} + V_{116}^3V_{125}^3 + \frac{V_{026}^{3/2}V_{224}^{3/2}V_{116}^{3/2}V_{134}^{3/2}V_{125}^{3/2}}{V_{035}^{3/2}} + \frac{V_{233}^3V_{116}^3}{2}\right)^{1/3}\times$$

$$\left(\frac{V_{026}^{3/2}V_{125}^{9/2}V_{134}^{9/2}}{V_{224}^{3/2}V_{035}^{9/2}V_{116}^{3/2}} + \frac{V_{017}^3V_{125}^3V_{134}^3}{V_{035}^3} + \frac{V_{026}^{3/2}V_{224}^{3/2}V_{116}^{3/2}V_{125}^{3/2}V_{134}^{3/2}}{V_{035}^{3/2}} + V_{125}^3V_{134}^3 + \frac{V_{044}^3V_{224}^{3/2}V_{116}^{3/2}V_{125}^{3/2}V_{134}^{3/2}}{2V_{026}^{3/2}V_{035}^{3/2}}\right)^{1/3}\times$$

$$\left(\frac{V_{125}V_{026}V_{134}}{V_{224}V_{035}V_{116}}\right)^g\left(\frac{V_{134}^2V_{026}}{V_{233}V_{044}V_{116}}\right)^k,$$

*where $g$ and $k$ are constrained by the constraints of our framework (see the proof).*

*Proof.* $I = 2, J = 6, K = 8$, and the variables are $a = \alpha_{008}$, $b = \alpha_{017}$, $c = \alpha_{026}$, $d = \alpha_{035}$, $e = \alpha_{044}$, $f = \alpha_{053}$, $g = \alpha_{062}$, $h = \alpha_{107}$, $i = \alpha_{116}$, $j = \alpha_{125}$, $k = \alpha_{134}$. The linear system becomes
$X_0 = a + b + c + d + e + f + g$,
$Y_0 = a + g + h$,
$Y_1 = b + f + i$,
$Y_2 = c + e + j$,
$Z_0 = a$,
$Z_1 = b + h$,
$Z_2 = c + g + i$,
$Z_3 = d + f + j$,
$Z_4 = 2(e + k)$.

The system has rank 9 and 11 variables, and so we pick two of the variables, $g$ and $k$ to put in $\Delta$. We solve the system:
$a = Z_0$,
$e = (e + k) - k = Z_4/2 - k$,
$h = (a + g + h) - a - g = Y_0 - Z_0 - g$,
$b = (b + h) - h = Z_1 - Y_0 + Z_0 + g$,
$d = (c+g+i)+(d+f+j)-g-(b+f+i)-(c+e+j)+b+e = (Z_0+Z_1+Z_2+Z_3+Z_4/2)-Y_0-Y_1-Y_2-k$,
$c = ((c + e + j) - e + (a + b + c + d + e + f + g) - a - b - d - e - g - (d + f + j) + d)/2 = (Y_2 - Z_4 + 2k + X_0 - 2Z_0 - Z_1 + Y_0 - 2g - Z_3)/2$,
$f = (a+b+c+d+e+f+g)-a-b-c-d-e-g = X_0 - Z_0 - (Z_1 - Y_0 + Z_0 + g) - (Y_2 - Z_4 + 2k + X_0 - 2Z_0 - Z_1 + Y_0 - 2g - Z_3)/2 - ((Z_0 + Z_1 + Z_2 + Z_3 + Z_4/2) - Y_0 - Y_1 - Y_2 - k) - (Z_4/2 - k) - g = -2Z_0 + k + 3Y_0/2 - g - 3Z_1/2 - Z_4/2 + X_0/2 + Y_1 + Y_2/2 - Z_3/2 - Z_2$,
$i = (c + g + i) - c - g = Z_2 - (Y_2 - Z_4 + 2k + X_0 - 2Z_0 - Z_1 + Y_0 - 2g - Z_3)/2 - g = (-Y_2 + Z_4 - 2k - X_0 + 2Z_0 + Z_1 - Y_0 + Z_3 + 2Z_2)/2$,
$j = (d + f + j) - f - d = Z_3 - ((Z_0 + Z_1 + Z_2 + Z_3 + Z_4/2) - Y_0 - Y_1 - Y_2 - k) - (-2Z_0 + k + 3Y_0/2 - g - 3Z_1/2 - Z_4/2 + X_0/2 + Y_1 + Y_2/2 - Z_3/2 - Z_2) = Z_0 + Z_1/2 - Y_0/2 + Y_2/2 + g - X_0/2 + Z_3/2$,

$nx_0 = (V_{026}V_{224})^{3/2}(V_{035}V_{125})^{3/2}/((V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2})$,
$nx_1 = 1/2$,
$ny_0 = (V_{026}V_{224})^{3/2}(V_{035}V_{125})^{9/2}V_{116}^3/(V_{125}^3V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2})$,
$ny_1 = V_{125}^3/V_{233}^3$,
$ny_2 = (V_{026}V_{224})^{3/2}(V_{035}V_{125})^{3/2}(V_{134}V_{125})^{3/2}/(V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2})$,
$ny_3 = 1/2$,
$nz_0 = V_{125}^3V_{233}^3V_{134}^3/(V_{224}^3V_{035}^3)$,
$nz_1 = V_{017}^3V_{125}^3V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2}/((V_{026}V_{224})^{3/2}(V_{035}V_{125})^{9/2})$,
$nz_2 = V_{233}^3V_{116}^3$,
$nz_3 = V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2}/((V_{026}V_{224})^{3/2}(V_{035}V_{125})^{3/2})$,
$nz_4 = V_{233}^3V_{044}^3V_{116}^3/(2V_{026}^3)$,

$ng = V_{125}V_{026}V_{134}/(V_{224}V_{035}V_{116})$,
$nk = V_{026}V_{134}^2/(V_{233}V_{044}V_{116})$.

We obtain

$$V_{268} \geq 2\left(\frac{(V_{026}V_{224})^{3/2}(V_{035}V_{125})^{3/2}}{(V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2}} + 1/2\right)^{1/3} \times$$

$$\left(\frac{(V_{026}V_{224})^{3/2}(V_{035}V_{125})^{9/2}V_{116}^3}{V_{125}^3V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2}} + \frac{V_{125}^3}{V_{233}^3} + \frac{(V_{026}V_{224})^{3/2}(V_{035}V_{125})^{3/2}(V_{134}V_{125})^{3/2}}{V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2}} + 1/2\right)^{1/3} \times$$

$$\left(\frac{V_{125}^3V_{233}^3V_{134}^3}{V_{224}^3V_{035}^3} + \frac{V_{017}^3V_{125}^3V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2}}{(V_{026}V_{224})^{3/2}(V_{035}V_{125})^{9/2}} + V_{233}^3V_{116}^3 + \frac{V_{035}^3V_{233}^3(V_{116}V_{125})^{3/2}(V_{134}V_{125})^{3/2}}{(V_{026}V_{224})^{3/2}(V_{035}V_{125})^{3/2}}\right.$$

$$\left.+\frac{V_{233}^3V_{044}^3V_{116}^3}{2V_{026}^3}\right)^{1/3} \times \left(\frac{V_{125}V_{026}V_{134}}{V_{224}V_{035}V_{116}}\right)^g \left(\frac{V_{134}^2V_{026}}{V_{233}V_{044}V_{116}}\right)^k.$$

The linear constraints on $g$ and $k$ are as follows:

Constraint 1: since $e = Z_4/2 - k \geq 0$, we get that $k \leq Z_4/2$, but since we set $Z_4/2 = nz_4/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$, we get

$k \leq nz_4/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 2: since $d = (Z_0 + Z_1 + Z_2 + Z_3 + Z_4/2) - Y_0 - Y_1 - Y_2 - k \geq 0$, we get that $k \leq (Z_0 + Z_1 + Z_2 + Z_3 + Z_4/2) - Y_0 - Y_1 - Y_2$ and by our choices for the $Y_i, Z_i$, we get

$k \leq ny_3/(ny_0 + ny_1 + ny_2 + ny_3)$.

Constraint 3: since $h = Y_0 - Z_0 - g \geq 0$ and we set $Y_0 = ny_0/(ny_0 + ny_1 + ny_2 + ny_3)$ and $Z_0 = nz_0/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$, we get

$g \leq ny_0/(ny_0 + ny_1 + ny_2 + ny_3) - nz_0/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 4: since $b = Z_1 - Y_0 + Z_0 + g \geq 0$ and we set $Z_1 = nz_1/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$, we get

$g \geq ny_0/(ny_0 + ny_1 + ny_2 + ny_3) - (nz_0 + nz_1)/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 5: since $c = (X_0 - 2Z_0 - Z_1 - Z_3 - Z_4 + Y_0 + Y_2 + 2k - 2g)/2 \geq 0$, we get

$g - k \leq (ny_0 + ny_2)/(2(ny_0 + ny_1 + ny_2 + ny_3)) + (-2nz_0 - nz_1 - nz_3 - 2nz_4)/(2(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)) + nx_0/(2(nx_0 + nx_1))$.

Constraint 6: since $f = X_0/2 + 3Y_0/2 + Y_1 + Y_2/2 - 2Z_0 - 3Z_1/2 - Z_3/2 - Z_2 - Z_4/2 + k - g \geq 0$, we get that

$g - k \leq (nx_0/2)/(nx_0 + nx_1) + (3ny_0 + 2ny_1 + ny_2)/(2(ny_0 + ny_1 + ny_2 + ny_3)) - (4nz_0 + 3nz_1 + nz_3 + 2nz_2 + 2nz_4)/(2(nz_0 + nz_1 + nz_2 + nz_3 + nz_4))$.

Constraint 7: since $i = (-X_0 - Y_0 - Y_2 + 2Z_0 + Z_1 + 2Z_2 + Z_3 + Z_4)/2 - k \geq 0$, we get

$k \leq -(ny_0 + ny_2)/(2(ny_0 + ny_1 + ny_2 + ny_3)) - nx_0/(2(nx_0 + nx_1)) + (2nz_0 + nz_1 + 2nz_2 + nz_3 + 2nz_4)/(2(nz_0 + nz_1 + nz_2 + nz_3 + nz_4))$.

Constraint 8: since $j = -Y_0/2 + Y_2/2 - X_0/2 + Z_0 + Z_1/2 + Z_3/2 + g \geq 0$, we get

$g \geq -nx_0/(2(nx_0 + nx_1)) + (-ny_0 + ny_2)/(2(ny_0 + ny_1 + ny_2 + ny_3)) - (2nz_0 + nz_1 + nz_3)/(2(nz_0 + nz_1 + nz_2 + nz_3 + nz_4))$.

Setting $q = 5$ and $\tau = 2.372873/3$ we first find the values $g = g'$ and $k = k'$ for which $F(g,k) = a^ab^bc^cd^de^ef^fg^gh^hi^ij^jk^k$ is maximized for our settings for $a,b,c,d,e,f,h,i,j$ above, given the 8 constraints on $g$ and $k$ above. We obtain $g' = 0.305326266603096 \cdot 10^{-3}, k' = 0.327470701886469$, and $F(g',k') = 0.223663404773788599$. We then minimize $G(g,k) = V_{268}/(a^ab^bc^cd^de^ef^fg^gh^hi^ij^jk^k)$ for the settings and over the 8 constraints on $g,k$. We obtain that $G$ is minimized for $gg = 0.000305373765871005, k = 0.327509292231427$ and for these settings it is $1.83709636797664943 \cdot 10^6$. We then obtain that

$$V_{268} \geq G(g'k')F(g,k) = 4.108912286 \cdot 10^5.$$

□

**Lemma 23.** *When $q = 5$ and $\tau = 2.372873/3$, $V_{277} \geq 4.850714396 \cdot 10^5$.*

*In general,*

$$V_{277} \geq 2\left(1+\frac{V_{116}^3}{2V_{026}^3}\right)^{1/3}\left(\frac{V_{017}^3V_{026}^3V_{125}^3}{V_{116}^3}+V_{026}^3V_{125}^3+\frac{V_{026}^3V_{125}^6}{V_{116}^3}+\frac{V_{134}^6V_{026}^6V_{125}^6}{V_{035}^3V_{224}^3V_{116}^6}\right)^{1/3}\times$$

$$\left(\frac{V_{017}^3}{V_{125}^3}+\frac{V_{116}^3}{V_{125}^3}+1+\frac{V_{035}^3V_{224}^3V_{116}^3}{V_{026}^3V_{125}^6}\right)^{1/3}\left(\frac{V_{035}^2V_{224}^2V_{116}^2}{V_{134}^2V_{026}^2V_{125}^2}\right)^c\left(\frac{V_{044}V_{233}V_{116}}{V_{134}^2V_{026}}\right)^d.$$

*where $a, c, d$ are constrained by constraints from our framework (see the proof).*

*Proof.* $I = 2$, $J = K = 7$, and the variables are $a = \alpha_{017}, b = \alpha_{026}, c = \alpha_{035}, d = \alpha_{044}, e = \alpha_{053}, f = \alpha_{062}, g = \alpha_{071}, h = \alpha_{107}, i = \alpha_{116}, j = \alpha_{125}, k = \alpha_{134}$. The linear system is
$X_0 = a + b + c + d + e + f + g$,
$Y_0 = g + h$,
$Y_1 = a + f + i$,
$Y_2 = b + e + j$,
$Z_0 = a + h$,
$Z_1 = b + g + i$,
$Z_2 = c + f + j$,
$Z_3 = d + e + k$.

The system has rank 8 and 11 variables, so we pick 3 variables, $a, c, d$ , and put them in $\Delta$. We solve the system:

$h = (a + h) - a = Z_0 - a$,
$g = (g + h) - h = Y_0 - Z_0 + a$,
$k = (a + h) + (b + g + i) + (c + f + j) + (d + e + k) - (g + h) - (a + f + i) - (b + e + j) - c - d = Z_0 + Z_1 + Z_2 + Z_3 - Y_0 - Y_1 - Y_2 - c - d$,
$e = (d+e+k)-d-k = Z_3-d-(Z_0+Z_1+Z_2+Z_3-Y_0-Y_2-c-d) = -Z_0-Z_1-Z_2+Y_0+Y_1+Y_2+c$,
$b = (a + b + c + d + e + f + g) - (a + f + i) + (b + g + i) - 2g - c - d - (d + e + k) + d + k)/2 = X_0/2 - Y_1 + Z_1 - (3/2)Y_0 + (3/2)Z_0 - a - c - d/2 + Z_2/2 - Y_2/2$,
$i = (b + g + i) - b - g = -X_0/2 + Y_1 + Y_0/2 - Z_0/2 + c + d/2 - Z_2/2 + Y_2/2$,
$f = (a + f + i) - a - i = -a + X_0/2 - Y_0/2 + Z_0/2 - c - d/2 + Z_2/2 - Y_2/2$,
$j = (c + f + j) - c - f = Z_2/2 + a - X_0/2 + Y_0/2 - Z_0/2 + d/2 + Y_2/2$.

$nx_0 = V_{026}^3V_{125}^3/(V_{116}^3V_{125}^3)$,
$nx_1 = 1/2$,
$ny_0 = V_{035}^3V_{224}^3V_{017}^3V_{116}^3/(V_{026}^3V_{125}^3V_{134}^6)$,
$ny_1 = V_{035}^3V_{224}^3V_{116}^6/(V_{026}^3V_{125}^3V_{134}^6)$,
$ny_2 = V_{035}^3V_{224}^3V_{116}^3/(V_{026}^3V_{134}^6)$,
$ny_3 = 1$,
$nz_0 = V_{026}^3V_{125}^3V_{017}^3V_{134}^6/(V_{035}^3V_{224}^3V_{116}^3)$,
$nz_1 = V_{026}^3V_{125}^3V_{134}^6/(V_{035}^3V_{224}^3)$,
$nz_2 = V_{026}^3V_{125}^6V_{134}^6/(V_{035}^3V_{224}^3V_{116}^3)$,
$nz_3 = V_{134}^6$,

$na = 1$,
$nc = V_{035}^2 V_{224}^2 V_{116}^2/(V_{134}^2 V_{026}^2 V_{125}^2)$,
$nd = V_{044} V_{233} V_{116}/(V_{134}^2 V_{026})$.

$$V_{277} \geq 2\left(\frac{V_{026}^3 V_{125}^3}{V_{116}^3 V_{125}^3} + 1/2\right)^{1/3} \left(\frac{V_{035}^3 V_{224}^3 V_{017}^3 V_{116}^3}{V_{026}^3 V_{125}^3 V_{134}^6} + \frac{V_{035}^3 V_{224}^3 V_{116}^6}{V_{026}^3 V_{125}^3 V_{134}^6} + \frac{V_{035}^3 V_{224}^3 V_{116}^3}{V_{026}^3 V_{134}^6} + 1\right)^{1/3} \times$$

$$\left(\frac{V_{026}^3 V_{125}^3 V_{017}^3 V_{134}^6}{V_{035}^3 V_{224}^3 V_{116}^3} + \frac{V_{026}^3 V_{125}^3 V_{134}^6}{V_{035}^3 V_{224}^3} + \frac{V_{026}^3 V_{125}^6 V_{134}^6}{V_{035}^3 V_{224}^3 V_{116}^3} + V_{134}^6\right)^{1/3} \left(\frac{V_{035}^2 V_{224}^2 V_{116}^2}{V_{134}^2 V_{026}^2 V_{125}^2}\right)^c \left(\frac{V_{044} V_{233} V_{116}}{V_{134}^2 V_{026}}\right)^d.$$

We now consider the constraints on $a, c, d$.

Constraint 1: since $k = Z_0 + Z_1 + Z_2 + Z_3 - Y_0 - Y_1 - Y_2 - c - d \geq 0$, by our settings of the $Y_i, Z_i$ we get that

$c + d \leq ny_3/(ny_0 + ny_1 + ny_2 + ny_3)$.

Constraint 2: since $e = -Z_0 - Z_1 - Z_2 + Y_0 + Y_1 + Y_2 + c \geq 0$, by our settings of the $Y_i, Z_i$ we get that

$c \geq ny_3/(ny_0 + ny_1 + ny_2 + ny_3) - nz_3/(nz_0 + nz_1 + nz_2 + nz_3)$.

Constraint 3: since $h = Z_0 - a \geq 0$ and we set $Z_0 = nz_0/(nz_0 + nz_1 + nz_2 + nz_3)$, we get that

$a \leq nz_0/(nz_0 + nz_1 + nz_2 + nz_3)$.

Constraint 4: since $g = Y_0 - Z_0 + a \geq 0$ and we set $Y_0 = ny_0/(ny_0 + ny_1 + ny_2 + ny_3)$, we get that

$a \geq nz_0/(nz_0 + nz_1 + nz_2 + nz_3) - ny_0/(ny_0 + ny_1 + ny_2 + ny_3)$.

Constraint 5: since $b = X_0/2 - (3/2)Y_0 - Y_1 - Y_2/2 + (3/2)Z_0 + Z_1 + Z_2/2 - a - c - d/2 \geq 0$, we get that

$a + c + d/2 \leq (3nz_0 + 2nz_1 + nz_2)/(2(nz_0 + nz_1 + nz_2 + nz_3)) + (-ny_1 - 3ny_0/2 - ny_2/2)/(ny_0 + ny_1 + ny_2 + ny_3) + nx_0/(2(nx_0 + nx_1))$.

Constraint 6: since $i = -X_0/2 + Y_0/2 + Y_1 + Y_2/2 - Z_0/2 - Z_2/2 + c + d/2 \geq 0$, we get that

$c + d/2 \geq -(ny_0 + 2ny_1 + ny_2)/(2(ny_0 + ny_1 + ny_2 + ny_3)) + nx_0/(2(nx_0 + nx_1)) + (nz_0 + nz_2)/(2(nz_0 + nz_1 + nz_2 + nz_3))$.

Constraint 7: since $j = -X_0/2 + Y_0/2 + Y_2/2 - Z_0/2 + Z_2/2 + a + d/2 \geq 0$, we get that

$a + d/2 \geq (nz_0 - nz_2)/(2(nz_0 + nz_1 + nz_2 + nz_3)) - (ny_0 + ny_2)/(2(ny_0 + ny_1 + ny_2 + ny_3)) + nx_0/(2(nx_0 + nx_1))$.

Constraint 8: since $f = -a - c - d/2 + X_0/2 - Y_0/2 - Y_2/2 + Z_0/2 + Z_2/2 \geq 0$, we get that

$a + c + d/2 \leq (-ny_0 - ny_2)/(2(ny_0 + ny_1 + ny_2 + ny_3)) + (nz_0 + nz_2)/(2(nz_0 + nz_1 + nz_2 + nz_3)) + nx_0/(2(nx_0 + nx_1))$.

Now, let $F(a, c, d) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j k^k$ for the settings of $b, e, f, g, h, i, j, k$ in terms of $a, c, d$ as above. We first find the settings $a', c', d'$ that maximize $F$ under the constraints from 1 to 8 above. Then, we find the settings $a'', c'', d''$ that minimize $V_{277}(a, c, d)/F(a, c, d)$ under the same 8 constraints. Finally, we conclude that $V_{277} \geq F(a'c'd')/F(a'', c'', d'')V_{277}(a'', c'', d'')$.

From Maple we get that $a' = 0.970387372073004 \cdot 10^{-5}$, $c' = 0.0816678275096020$, $d' = .346697875753760$, $a'' = 0.971619435812525 \cdot 10^{-5}$, $c'' = 0.0822791888760284$, $d'' = .345469920458090$, and thus $V_{277} \geq 4.850714396 \cdot 10^5$.

□

**Lemma 24.** *When $q = 5$ and $\tau = 2.372873/3$, we have $V_{3310} \geq 1.242573275 \cdot 10^5$.*

*Furthermore,*

$$V_{3310} \geq 2\left(1 + \frac{V_{116}^3 V_{224}^3}{V_{026}^3 V_{134}^3}\right)^{1/3} \left(\frac{V_{134}^3 V_{026}^3}{V_{116}^3 V_{224}^3} + 1\right)^{1/3} \times$$

$$\left(\frac{V_{233}^3 V_{116}^3 V_{224}^3}{V_{134}^3 V_{026}^3} + V_{017}^3 V_{233}^3 + V_{026}^3 V_{134}^3 + \frac{V_{125}^6 V_{026}^3 V_{134}^3}{2V_{116}^3 V_{224}^3}\right)^{1/3} \left(\frac{V_{035}^2 V_{116}^2 V_{224}^2}{V_{125}^2 V_{026}^2 V_{134}^2}\right)^d,$$

*where $b$ and $d$ are constrained as in our framework (see the proof).*

*Proof.* $I = J = 3, K = 10$, and the variables are $a = \alpha_{008}, b = \alpha_{017}, c = \alpha_{026}, d = \alpha_{035}, e = \alpha_{116}, f = \alpha_{125}, g = \alpha_{134}, h = \alpha_{107}$. The linear system is as follows:
$X_0 = a + b + c + d$,
$Y_0 = a + d + g + h$,
$Z_2 = a$,
$Z_3 = b + h$,
$Z_4 = c + e + g$,
$Z_5 = 2(d + f)$.

The system has rank 6 and 8 variables, so we pick two variables, $b$ and $d$, and add them to $\Delta$. We solve the system:
$a = Z_2$,
$h = (b + h) - b = Z_3 - b$,
$f = (d + f) - d = Z_5/2 - d$,
$c = (a + b + c + d) - a - b - d = X_0 - Z_2 - b - d$,
$g = (a + d + g + h) - a - d - h = Y_0 - Z_2 - Z_3 + b - d$,
$e = (c+e+g)-c-g = Z_4-(X_0-Z_2-b-d)-(Y_0-Z_2-Z_3+b-d) = 2Z_2+Z_3+Z_4-X_0-Y_0+2d$.

$nx_0 = V_{026}^3 V_{134}^3/(V_{116}^3 V_{224}^3)$,
$nx_1 = 1$,
$ny_0 = V_{026}^3 V_{134}^3/(V_{116}^3 V_{224}^3)$,
$ny_1 = 1$,
$nz_2 = V_{233}^3 V_{116}^6 V_{224}^6/(V_{026}^6 V_{134}^6)$,
$nz_3 = V_{116}^3 V_{224}^3 V_{017}^3 V_{233}^3/(V_{026}^3 V_{134}^3)$,
$nz_4 = V_{116}^3 V_{224}^3$,
$nz_5 = V_{125}^6/2$,

$nb = 1$,
$nd = V_{035}^2 V_{116}^2 V_{224}^2/(V_{125}^2 V_{026}^2 V_{134}^2)$.

$$V_{3310} \geq 2\left(\frac{V_{026}^3 V_{134}^3}{V_{116}^3 V_{224}^3} + 1\right)^{1/3} \left(\frac{V_{134}^3 V_{026}^3}{V_{116}^3 V_{224}^3} + 1\right)^{1/3} \times$$
$$\left(\frac{V_{233}^3 V_{116}^6 V_{224}^6}{V_{134}^6 V_{026}^6} + \frac{V_{116}^3 V_{224}^3 V_{017}^3 V_{233}^3}{V_{026}^3 V_{134}^3} + V_{116}^3 V_{224}^3 + V_{125}^6/2\right)^{1/3} \times$$
$$\left(\frac{V_{035}^2 V_{116}^2 V_{224}^2}{V_{125}^2 V_{026}^2 V_{134}^2}\right)^d.$$

We now give the constraints on $b, d$:
Constraint 1: since $h = Z_3 - b \geq 0$ and $Z_3 = nz_3/(nz_2 + nz_3 + nz_4 + nz_5)$, we get $b \leq nz_3/(nz_2 + nz_3 + nz_4 + nz_5)$.

Constraint 2: since $f = Z_5/2 - d \geq 0$ and $Z_5/2 = nz_5/(nz_2 + nz_3 + nz_4 + nz_5)$, we get

$d \le nz_5/(nz_2+nz_3+nz_4+nz_5)$.

Constraint 3: since $c = X_0 - Z_2 - b - d \ge 0$, we get that
$b + d \le nx_0/(nx_0+nx_1) - nz_2/(nz_2+nz_3+nz_4+nz_5)$.
Constraint 4: since $g = Y_0 - Z_2 - Z_3 + b - d \ge 0$ and since $Y_0 = ny_0/(ny_0+ny_1)$ and $Z_2 = nz_2/(nz_2+nz_3+nz_4+nz_5)$, we get
$d - b \le ny_0/(ny_0+ny_1) - (nz_2+nz_3)/(nz_2+nz_3+nz_4+nz_5) = C_4$.

Constraint 5: since $e = Z_4 - (X_0 - Z_2 - b - d) - (Y_0 - Z_2 - Z_3 + b - d) = 2d - X_0 - Y_0 + 2Z_2 + Z_3 + Z_4 \ge 0$, we get that
$d \ge (nx_0/(nx_0+nx_1) + ny_0/(ny_0+ny_1) - (2nz_2+nz_3+nz_4)/(nz_2+nz_3+nz_4+nz_5))/2$.
We first find the settings $b', d'$ for which $F(b,d) = a^a b^b c^c d^d e^e f^f g^g h^h$ is maximized for our settings for $a, c, e, f, g, h$ above in terms of $b$ and $d$. With MAPLE we get $b = 0.00790328517086545, d = 0.0936429784188324$.
Then we find the settings $b'', d''$ for which $V_{3310}(b,d)/F(b,d)$ is minimized. We get $b'' = 0.00790325512918024, d'' = 0.0936943525122263$. Finally, we output that $V_{3310} \ge F(b',d') \cdot V_{3310}(b'',d'')/F(b'',d'') \ge 1.242573275 \cdot 10^5$.

□

**Lemma 25.** *For $q = 5, \tau = 2.372873/3$, $V_{349} \ge 3.209787942 \cdot 10^5$.*
*In general,*

$$V_{349} \ge 2\left(V_{035}^3 V_{134}^3 + V_{134}^3 V_{125}^3\right)^{1/3} \left(\frac{V_{044}^3}{V_{134}^3} + 1 + \frac{V_{224}^3}{2V_{134}^3}\right)^{1/3} \left(\frac{V_{134}^3}{V_{044}^3 V_{035}^3} + \frac{V_{017}^3 V_{224}^3}{V_{044}^3 V_{125}^3} + \frac{V_{026}^3 V_{134}^3}{V_{044}^3 V_{125}^3} + 1\right)^{1/3} \times$$
$$\left(\frac{V_{233}V_{044}V_{125}}{V_{224}V_{035}V_{134}}\right)^b \left(\frac{V_{233}V_{044}V_{125}}{V_{224}V_{134}V_{035}}\right)^c \left(\frac{V_{116}V_{233}V_{044}}{V_{026}V_{134}^2}\right)^g.$$

*Above, $b, c, g$ are constrained as in our framework (see the proof).*

*Proof.* $I = 3, J = 4, K = 9$, so the variables are $a = \alpha_{008}, b = \alpha_{017}, c = \alpha_{026}, d = \alpha_{035}, e = \alpha_{044}, f = \alpha_{107}, g = \alpha_{116}, h = \alpha_{125}, i = \alpha_{134}, j = \alpha_{143}$. The linear system becomes
$X_0 = a + b + c + d + e$,
$Y_0 = a + e + f + j$,
$Y_1 = b + d + g + i$,
$Z_1 = a$,
$Z_2 = b + f$,
$Z_3 = c + g + j$,
$Z_4 = d + e + h + i$.

The rank is 7 and the number of variables is 10 so we pick 3 variables, $b, c, g$, to place into $\Delta$. We solve the system:
$a = Z_1$,
$h = a + (b+f) + (c+g+j) + (d+e+h+i) - (a+e+f+j) - (b+d+g+i) - c = (Z_1+Z_2+Z_3+Z_4) - (Y_0+Y_1) - c$,
$f = (b+f) - b = Z_2 - b$,
$j = (c+g+j) - c - g = Z_3 - c - g$,

$e = (a+e+f+j) - a - f - j = Y_0 - Z_1 - Z_2 - Z_3 + b + c + g,$
$d = (a+b+c+d+e) - a - b - c - e = X_0 - Y_0 + Z_2 + Z_3 - 2b - 2c - g,$
$i = (b+d+g+i) - b - d - g = -X_0 + Y_0 + Y_1 - Z_2 - Z_3 + b + 2c.$

$nx_0 = V_{035}^3/V_{125}^3,$
$nx_1 = 1,$
$ny_0 = V_{044}^3/V_{224}^3,$
$ny_1 = V_{134}^3/V_{224}^3,$
$ny_2 = 1/2,$
$nz_1 = V_{134}^3 V_{125}^3 V_{224}^3/(V_{044}^3 V_{035}^3),$
$nz_2 = V_{017}^3 V_{224}^6/V_{044}^3,$
$nz_3 = V_{134}^3 V_{224}^3 V_{026}^3/V_{044}^3,$
$nz_4 = V_{125}^3 V_{224}^3,$

$nb = V_{233}V_{044}V_{125}/(V_{224}V_{035}V_{134}),$
$nc = V_{233}V_{044}V_{125}/(V_{224}V_{134}V_{035}),$
$ng = V_{116}V_{233}V_{044}/(V_{026}V_{134}^2).$

$$V_{349} \geq 2\left(\frac{V_{035}^3}{V_{125}^3}+1\right)^{1/3}\left(\frac{V_{044}^3}{V_{224}^3}+\frac{V_{134}^3}{V_{224}^3}+\frac{1}{2}\right)^{1/3}\left(\frac{V_{134}^3V_{125}^3V_{224}^3}{V_{044}^3V_{035}^3}+\frac{V_{017}^3V_{224}^6}{V_{044}^3}+\frac{V_{134}^3V_{224}^3V_{026}^3}{V_{044}^3}+V_{125}^3V_{224}^3\right)^{1/3}\times$$
$$\left(\frac{V_{233}V_{044}V_{125}}{V_{224}V_{035}V_{134}}\right)^b\left(\frac{V_{233}V_{044}V_{125}}{V_{224}V_{134}V_{035}}\right)^c\left(\frac{V_{116}V_{233}V_{044}}{V_{026}V_{134}^2}\right)^g.$$

We now look at the constraints on $b, c, g$:

Constraint 1: since $h = (Z_1+Z_2+Z_3+Z_4) - (Y_0+Y_1) - c = Y_2/2 - c \geq 0$, and $Y_2/2 = ny_2/(ny_0+ny_1+ny_2)$, we get
$c \leq ny_2/(ny_0+ny_1+ny_2).$

Constraint 2: since $f = Z_2 - b \geq 0$ and $Z_2 = nz_2/(nz_1+nz_2+nz_3+nz_4)$, we get
$b \leq nz_2/(nz_1+nz_2+nz_3+nz_4).$

Constraint 3: since $j = Z_3 - c - g \geq 0$ and $Z_3 = nz_3/(nz_1+nz_2+nz_3+nz_4)$, we get
$c + g \leq nz_3/(nz_1+nz_2+nz_3+nz_4).$

Constraint 4: since $e = Y_0 - Z_1 - Z_2 - Z_3 + b + c + g \geq 0$, and $Y_0 = ny_0/(ny_0+ny_1+ny_2)$ and $Z_1 = nz_1/(nz_1+nz_2+nz_3+nz_4)$, we get
$b + c + g \geq (nz_1+nz_2+nz_3)/(nz_1+nz_2+nz_3+nz_4) - ny_0/(ny_0+ny_1+ny_2).$

Constraint 5: since $d = X_0 - Y_0 + Z_2 + Z_3 - 2b - 2c - g \geq 0$ and $X_0 = nx_0/(nx_0+nx_1)$, we get
$2b + 2c + g \leq nx_0/(nx_0+nx_1) - ny_0/(ny_0+ny_1+ny_2) + (nz_2+nz_3)/(nz_1+nz_2+nz_3+nz_4).$

Constraint 6: since $i = -X_0 + Y_0 + Y_1 - Z_2 - Z_3 + b + 2c \geq 0$, we get that $b + 2c \geq nx_0/(nx_0 + nx1) - (ny_0+ny_1)/(ny+0+ny_1+ny_2) + (nz_2+nz_3)/(nz_1+nz_2+nz_3+nz_4)$.

Now, we first find the values $b' = 0.00106083709584428, c' = 0.0371057688416268, g' = 0.0507162807556667$ that maximize $F(b,c,g) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j$ under our settings for $a, d, e, f, h, i, j$ as functions of $b, c, g$ over the above 6 linear constraints. Then we find the values $b'' = 0.00105515376175534, c'' = 0.0370594417928395, g'' = 0.0505876038619197$ that minimize $V_{349}(b,c,g)/F(b,c,g)$ over the above 6 linear constraints. Finally, we conclude that $V_{349} \geq F(b',c',g') \cdot V_{349}(b'',c'',g'')/F(b'',c'',g'') \geq 3.209787942 \cdot 10^5$.

□

**Lemma 26.** *When* $q = 5, \tau = 2.372873/3$, $V_{358} \geq 6.082545902 \cdot 10^5$.

*In general,*

$$V_{358} \geq 2\left(V_{035}^3 + V_{125}^3\right)^{1/3}\left(V_{035}^3 + V_{134}^3 + V_{233}^3\right)^{1/3}\left(\frac{1}{V_{035}^3} + \frac{V_{017}^3}{V_{035}^3} + \frac{V_{026}^3}{V_{035}^3} + 1 + \frac{V_{134}^3 V_{224}^3}{2V_{125}^3 V_{233}^3}\right)^{1/3} \times$$

$$\left(\frac{V_{116}V_{035}V_{224}}{V_{026}V_{134}V_{125}}\right)^h \left(\frac{V_{044}V_{125}V_{233}}{V_{224}V_{134}V_{035}}\right)^e.$$

*Here* $b, c, h, e$ *are constrained as in our framework (see the proof).*

*Proof.* $I = 3, J = 5, K = 8$, so the variables are $a = \alpha_{008}$, $b = \alpha_{017}$, $c = \alpha_{026}$, $d = \alpha_{035}$, $e = \alpha_{044}$, $f = \alpha_{053}$, $g = \alpha_{107}$, $h = \alpha_{116}$, $i = \alpha_{125}$, $j = \alpha_{134}$, $k = \alpha_{143}$, $l = \alpha_{152}$. The linear system becomes
$X_0 = a + b + c + d + e + f$,
$Y_0 = a + f + g + l$,
$Y_1 = b + e + h + k$,
$Z_0 = a$,
$Z_1 = b + g$,
$Z_2 = c + h + l$,
$Z_3 = d + f + i + k$,
$Z_4 = 2(e + j)$.

The system has rank 8 and 12 variables, so we pick 4 variables, $b, c, h, e$ to put in $\Delta$. We then solve the system:
$a = Z_0$,
$g = (b + g) - b = Z_1 - b$,
$j = (e + j) - e = Z_4/2 - e$,
$l = (c + h + l) - c - h = Z_2 - c - h$,
$f = (a + f + g + l) - a - g - l = Y_0 - Z_0 - Z_1 - Z_2 + b + c + h$,
$d = (a + b + c + d + e + f) - a - b - c - e - f = X_0 - Y_0 + Z_1 + Z_2 - 2b - 2c - e - h$,
$i = (c+d+i+j) - c - d - j = (Z_1 + Z_2 + Z_3 + Z_4/2) - (Y_0 + Y_1) - c - d - j = (Z_0 + Z_1 + Z_2 + Z_3 + Z_4/2 - Y_0 - Y_1) - c - (X_0 - Y_0 + Z_1 + Z_2 - 2b - 2c - e - h) - (Z_4/2 - e) = -X_0 - Y_1 + Z_0 + Z_3 + 2b + c + 2e + h$
$k = (b + e + h + k) - b - e - h = Y_1 - b - e - h$.

$nx_0 = V_{035}^3/V_{125}^3$,
$nx_1 = 1$,
$ny_0 = V_{035}^3/V_{233}^3$,
$ny_1 = V_{134}^3/V_{233}^3$,
$ny_2 = 1$,

$nz_0 = V_{125}^3 V_{233}^3 / V_{035}^3,$
$nz_1 = V_{233}^3 V_{017}^3 V_{125}^3 / V_{035}^3,$
$nz_2 = V_{233}^3 V_{125}^3 V_{026}^3 / V_{035}^3,$
$nz_3 = V_{125}^3 V_{233}^3,$
$nz_4 = V_{134}^3 V_{224}^3 / 2,$

$nb = 1,$
$nc = 1,$
$nh = V_{116} V_{035} V_{224} / (V_{026} V_{134} V_{125}),$
$ne = V_{044} V_{125} V_{233} / (V_{224} V_{134} V_{035}).$

$$V_{358} \geq 2 \left( \frac{V_{035}^3}{V_{125}^3} + 1 \right)^{1/3} \left( \frac{V_{035}^3}{V_{233}^3} + \frac{V_{134}^3}{V_{233}^3} + 1 \right)^{1/3} \left( \frac{V_{125}^3 V_{233}^3}{V_{035}^3} + \frac{V_{017}^3 V_{125}^3 V_{233}^3}{V_{035}^3} + \frac{V_{026}^3 V_{125}^3 V_{233}^3}{V_{035}^3} + V_{125}^3 V_{233}^3 + \frac{V_{134}^3 V_{224}^3}{2} \right)^{1/3}$$
$$\left( \frac{V_{116} V_{035} V_{224}}{V_{026} V_{134} V_{125}} \right)^h \left( \frac{V_{044} V_{125} V_{233}}{V_{224} V_{134} V_{035}} \right)^e.$$

Let's look at the constraints on $b, c, h, e$:
Constraint 1: since $g = Z_1 - b \geq 0$ and we set $Z_1 = nz_1/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$, we get
$b \leq nz_1/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 2: since $j = Z_4/2 - e \geq 0$ and we set $Z_4/2 = nz_4/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$, we get
$e \leq nz_4/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 3: since $l = Z_2 - c - h \geq 0$ and $Z_2 = nz_2/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$, we get
$c + h \leq nz_2/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 4: since $f = Y_0 - Z_0 - Z_1 - Z_2 + b + c + h \geq 0$ and $Y_0 = ny_0/(ny_0 + ny_1 + ny_2)$ and $Z_0 = nz_0/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$, we get
$b + c + h \geq (nz_0 + nz_1 + nz_2)/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4) - ny_0/(ny_0 + ny_1 + ny_2)$.

Constraint 5: since $d = X_0 - Y_0 + Z_1 + Z_2 - 2b - 2c - e - h \geq 0$ and $X_0 = nx_0/(nx_0 + nx_1)$, we get
$2b+2c+e+h \leq nx_0/(nx_0+nx_1) - ny_0/(ny_0+ny_1+ny_2) + (nz_1+nz_2)/(nz_0+nz_1+nz_2+nz_3+nz_4)$.

Constraint 6: since $i = -X_0 - Y_1 + Z_0 + Z_3 + 2b + c + 2e + h = Y_2 - Z_4/2 - X_0 + Y_0 - Z_1 - Z_2 + 2b + c + 2e + h \geq 0$, we get
$2b + c + 2e + h \geq nx_0/(nx_0 + nx_1) - (ny_0 + ny_2)/(ny_0 + ny_1 + ny_2) + (nz_1 + nz_2 + nz_4)/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)$.

Constraint 7: since $k = Y_1 - b - e - h \geq 0$, we get
$b + e + h \leq ny_1/(ny_0 + ny_1 + ny_2)$.
Assume that $\tau = 2.372873/3$. Now, we first find the setting $b' = 0.000101938664845774, c' = 0.00885151618359280, e' = 0.0974448528665440, h' = 0.00670556068057964$ that maximizes $F(b, c, h, e) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j k^k l^l$. Then we find the setting $[b'' = 0.000102396163321746, c'' = 0.00884748699884917, e'' = 0.0970349718925811, h'' = 0.00671965662663561$ that minimizes $V_{358}(b, c, h, e)/F(b, c, h, e)$ and conclude that $V_{358} \geq F(b', c', h', e') \cdot V_{358}(b'', c'', h'', e'')/F(b'', c'', h'', e'') \geq 6.082545902 \cdot 10^5$.

□

**Lemma 27.** *When* $q = 5, \tau = 2.372873/3$, $V_{367} \geq 8.305250670 \cdot 10^5$.

*In general,*

$$V_{367} \geq 2\left(\frac{V_{035}^3}{V_{125}^3}+1\right)^{1/3}\left(V_{026}^3+V_{125}^3+V_{224}^3+V_{233}^3/2\right)^{1/3}\left(V_{017}^3+V_{116}^3+V_{125}^3+V_{134}^3\right)^{1/3}\times$$

$$\left(\frac{V_{026}V_{134}V_{125}}{V_{035}V_{224}V_{116}}\right)^b\left(\frac{V_{044}V_{233}V_{125}}{V_{035}V_{134}V_{224}}\right)^d.$$

*Here* $a, b, c, d, e$ *are constrained as in our framework (see the proof).*

*Proof.* $I = 3, J = 6, K = 7$ so the variables are $a = \alpha_{017}, b = \alpha_{026}, c = \alpha_{035}, d = \alpha_{044}, e = \alpha_{053}, f = \alpha_{062}, g = \alpha_{107}, h = \alpha_{116}, i = \alpha_{125}, j = \alpha_{134}, k = \alpha_{143}, l = \alpha_{152}, m = \alpha_{161}$. The linear system is as follows.

$X_0 = a + b + c + d + e + f$,
$Y_0 = f + g + m$,
$Y_1 = a + e + h + l$,
$Y_2 = b + d + i + k$,
$Z_0 = a + g$,
$Z_1 = b + h + m$,
$Z_2 = c + f + i + l$,
$Z_3 = d + e + j + k$.

The rank is 8 and the number of variables is 13, so we pick 5 variables, $a, b, c, d, e$, and place them in $\Delta$. Now we solve the system.

$g = (a + g) - a = Z_0 - a$,
$f = (a + b + c + d + e + f) - a - b - c - d - e = X_0 - a - b - c - d - e$,
$m = (f + g + m) - f - g = Y_0 - Z_0 - X_0 + 2a + b + c + d + e$,
$j = (c + j) - c = (Z_0 + Z_1 + Z_2 + Z_3) - (Y_0 + Y_1 + Y_2) - c$,
$k = (d + e + j + k) - d - e - j = -(Z_0 + Z_1 + Z_2) + (Y_0 + Y_1 + Y_2) - d - e + c$,
$i = (b + d + i + k) - b - d - k = (Z_0 + Z_1 + Z_2) - (Y_0 + Y_1) + e - b - c$,
$l = (c + f + i + l) - c - f - i = -X_0 - (Z_0 + Z_1) + (Y_0 + Y_1) + a + 2b + c + d$,
$h = (a + e + h + l) - a - e - l = X_0 + (Z_0 + Z_1) - Y_0 - 2a - 2b - c - d - e$.

$nx_0 = V_{035}^3/V_{125}^3$,
$nx_1 = 1$,
$ny_0 = V_{026}^3/V_{233}^3$,
$ny_1 = V_{125}^3/V_{233}^3$,
$ny_2 = V_{224}^3/V_{233}^3$,
$ny_3 = 1/2$,
$nz_0 = V_{017}^3 V_{233}^3$,
$nz_1 = V_{116}^3 V_{233}^3$,
$nz_2 = V_{125}^3 V_{233}^3$,
$nz_3 = V_{134}^3 V_{233}^3$,

$na = 1$,
$nb = V_{026}V_{134}V_{125}/(V_{035}V_{224}V_{116})$,
$nc = 1$,
$nd = V_{044}V_{233}V_{125}/(V_{035}V_{134}V_{224})$,
$ne = 1$.

$$V_{367} \geq 2\left(\frac{V_{035}^3}{V_{125}^3}+1\right)^{1/3}\left(\frac{V_{026}^3}{V_{233}^3}+\frac{V_{125}^3}{V_{233}^3}+\frac{V_{224}^3}{V_{233}^3}+1/2\right)^{1/3}\left(V_{017}^3V_{233}^3+V_{116}^3V_{233}^3+V_{125}^3V_{233}^3+V_{134}^3V_{233}^3\right)^{1/3}\times$$
$$\left(\frac{V_{026}V_{134}V_{125}}{V_{035}V_{224}V_{116}}\right)^b\left(\frac{V_{044}V_{233}V_{125}}{V_{035}V_{134}V_{224}}\right)^d.$$

Now we consider the constraints on $a, b, c, d, e$.

Constraint 1: since $g = Z_0 - a \geq 0$
$a \leq nz_0/(nz_0 + nz_1 + nz_2 + nz_3)$.

Constraint 2: since $f = X_0 - a - b - c - d - e \geq 0$
$a + b + c + d + e \leq nx_0/(nx_0 + nx_1)$.

Constraint 3: since $m = Y_0 - Z_0 - X_0 + 2a + b + c + d + e \geq 0$
$2a+b+c+d+e \geq nx_0/(nx_0+nx_1)+nz_0/(nz_0+nz_1+nz_2+nz_3)-ny_0/(ny_0+ny_1+ny_2+ny_3) = C_3$.

Constraint 4: since $j = (Z_0 + Z_1 + Z_2 + Z_3) - (Y_0 + Y_1 + Y_2) - c \geq 0$
$c \leq ny_3/(ny_0 + ny_1 + ny_2 + ny_3)$.

Constraint 5: since $k = Z_3 - Y_3/2 - d - e + c \geq 0$
$d + e - c \leq nz_3/(nz_0 + nz_1 + nz_2 + nz_3) - ny_3/(ny_0 + ny_1 + ny_2 + ny_3)$.

Constraint 6: since $i = Y_2 - Z_3 + Y_3/2 + e - b - c \geq 0$
$b + c - e \leq (ny_2 + ny_3)/(ny_0 + ny_1 + ny_2 + ny_3) - nz_3/(nz_0 + nz_1 + nz_2 + nz_3)$.

Constraint 7: since $l = Z_2 + Z_3 - X_0 - Y_2 - Y_3/2 + a + 2b + c + d \geq 0$
$a + 2b + c + d \geq (ny_2 + ny_3)/(ny_0 + ny_1 + ny_2 + ny_3) + nx_0/(nx_0 + nx_1) - (nz_2 + nz_3)/(nz_0 + nz_1 + nz_2 + nz_3)$..

Constraint 8: since $h = Y_1 + Y_2 + Y_3/2 - Z_2 - Z_3 + X_0 - 2a - 2b - c - d - e \geq 0$
$2a + 2b + c + d + e \leq (ny_1 + ny_2 + ny_3)/(ny_0 + ny_1 + ny_2 + ny_3) + nx_0/(nx_0 + nx_1) - (nz_2 + nz_3)/(nz_0 + nz_1 + nz_2 + nz_3)$.

Assume that $\tau = 2.372873/3$. Now, we first find the setting $a' = 0.896351957010172 \cdot 10^{-5}, b' = 0.00188722012414286, c' = 0.0476091293800089, d' = .133911483061472, e' = 0.0188038856390314$ minimizing $F(a, b, c, d, e) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j k^k l^l m^m$ under the settings of $f, g, h, i, j, k, l, m$ in terms of $a, b, c, d, e$ and under the above 8 constraints on $a, b, c, d, e$. Then we find the setting $a'' = 0.897911747394834 \cdot 10^{-5}, b'' = 0.00189912343335534, c'' = 0.0479980389640663, d'' = .133369689001539, e'' = 0.0189441262665244$ maximizing $V_{367}(a, b, c, d, e)/F(a, b, c, d, e)$ over the same 8 constraints. Finally, we conclude that $V_{367} \geq F(a', b', c', d', e') \cdot V_{367}(a'', b'', c'', d'', e'')/F(a'', b'', c'', d'', e'') \geq 8.305250670 \cdot 10^5$. □

**Lemma 28.** *When $q = 5, \tau = 2.372873/3$, we have $V_{448} \geq 6.908047489 \cdot 10^5$.*

*In general,*

$$V_{448} \geq \left(V_{035}^3V_{134}^3 + \frac{V_{134}^3V_{125}^3V_{233}^3}{V_{224}^3} + \frac{V_{125}^3V_{233}^3}{2}\right)^{1/3}\left(\frac{V_{134}^3V_{035}^3}{V_{125}^3V_{233}^3} + \frac{V_{134}^3}{V_{224}^3} + \frac{1}{2}\right)^{1/3}\times$$

$$\left(\frac{V_{044}^3V_{125}^3V_{233}^3}{V_{035}^6V_{134}^6}+\frac{V_{017}^3V_{224}^3}{V_{134}^3V_{035}^3}+\frac{V_{026}^3V_{224}^3}{V_{134}^3V_{035}^3}+\frac{V_{224}^3}{V_{134}^3}+\frac{V_{224}^6}{2V_{125}^3V_{233}^3}\right)^{1/3}\left(\frac{V_{044}^3V_{125}^2V_{233}^2}{V_{035}^2V_{134}^2V_{224}^2}\right)^e\left(\frac{V_{116}V_{035}V_{224}}{V_{026}V_{134}V_{125}}\right)^g.$$

*Here, $b, c, g, e$ are constrained as in our framework (see the proof).*

*Proof.* $I = J = 4$, $K = 8$ and the variables are $a = \alpha_{008}, b = \alpha_{017}, c = \alpha_{026}, d = \alpha_{035}, e = \alpha_{044}, f = \alpha_{107}, g = \alpha_{116}, h = \alpha_{125}, i = \alpha_{134}, j = \alpha_{143}, k = \alpha_{206}, l = \alpha_{215}, m = \alpha_{224}$. The linear system becomes
$X_0 = a + b + c + d + e$,
$X_1 = f + g + h + i + j$,
$Y_0 = a + e + f + j + k$,
$Y_1 = b + d + g + i + l$,
$Z_0 = a$,
$Z_1 = b + f$,
$Z_2 = c + g + k$,
$Z_3 = d + h + j + l$,
$Z_4 = 2(e + i + m)$.

The rank is 9 and the number of variables is 13 so we pick 4 variables, $b, c, e, g$, and we put them in $\Delta$. We now solve the system.
$a = Z_0$,
$f = (b + f) - b = Z_1 - b$,
$k = (c + g + k) - c - g = Z_2 - c - g$,
$d = (a + b + c + d + e) - a - b - c - e = X_0 - Z_0 - b - c - e$,
$j = (a + e + f + j + k) - a - e - f - k = Y_0 - Z_0 - Z_1 - Z_2 + b + c + g - e$,
$i = ((f+g+h+i+j)+(b+d+g+i+l)-(d+h+j+l)-f-2g-b)/2 = (X_1+Y_1-Z_3-Z_1)/2-g$,
$h = (f + g + h + i + j) - f - g - i - j = X_1/2 - Y_0 - Y_1/2 + Z_0 + Z_1/2 + Z_2 + Z_3/2 - c - g + e$,
$m = (e + i + m) - e - i = (-X_1 - Y_1 + Z_3 + Z_1)/2 + Z_4/2 + g - e$,
$l = (b + d + g + i + l) - b - d - g - i = (-X_0 - X_1/2 + Y_1/2 + Z_0 + Z_1/2 + Z_3/2) + c + e$.

$nx_0 = V_{035}^3V_{134}^3/(V_{125}^3V_{233}^3)$,
$nx_1 = V_{134}^3/V_{224}^3$,
$nx_2 = 1/2$,
$ny_0 = V_{134}^3V_{035}^3/(V_{125}^3V_{233}^3)$,
$ny_1 = V_{134}^3/V_{224}^3$,
$ny_2 = 1/2$,
$nz_0 = V_{044}^3V_{125}^6V_{233}^6/(V_{035}^6V_{134}^6)$,
$nz_1 = V_{017}^3V_{224}^3V_{125}^3V_{233}^3/(V_{134}^3V_{035}^3)$,
$nz_2 = V_{125}^3V_{233}^3V_{026}^3V_{224}^3/(V_{134}^3V_{035}^3)$,
$nz_3 = V_{125}^3V_{233}^3V_{224}^3/V_{134}^3$,
$nz_4 = V_{224}^6/(2)$,

$nb = 1$,
$nc = 1$,
$ne = V_{044}^3V_{125}^2V_{233}^2/(V_{035}^2V_{134}^2V_{224}^2)$,
$ng = V_{116}V_{035}V_{224}/(V_{026}V_{134}V_{125})$.

$$V_{448} \geq \left(\frac{V_{035}^3V_{134}^3}{V_{125}^3V_{233}^3}+\frac{V_{134}^3}{V_{224}^3}+\frac{1}{2}\right)^{1/3}\left(\frac{V_{134}^3V_{035}^3}{V_{125}^3V_{233}^3}+\frac{V_{134}^3}{V_{224}^3}+\frac{1}{2}\right)^{1/3}\times$$

$$\left(\frac{V_{044}^3V_{125}^6V_{233}^6}{V_{035}^6V_{134}^6}+\frac{V_{017}^3V_{224}^3V_{125}^3V_{233}^3}{V_{134}^3V_{035}^3}+\frac{V_{026}^3V_{224}^3V_{125}^3V_{233}^3}{V_{134}^3V_{035}^3}+\frac{V_{224}^3V_{125}^3V_{233}^3}{V_{134}^3}+\frac{V_{224}^6}{2}\right)^{1/3}\times$$
$$\left(\frac{V_{044}^3V_{125}^2V_{233}^2}{V_{035}^2V_{134}^2V_{224}^2}\right)^e\left(\frac{V_{116}V_{035}V_{224}}{V_{026}V_{134}V_{125}}\right)^g.$$

Let's look at the constraints on $b, c, g, e$.

Constraint 1: since $f = Z_1 - b \geq 0$ we get
$b \leq nz_1/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4) = C_1$.

Constraint 2: since $k = Z_2 - c - g \geq 0$ we get
$c + g \leq nz_2/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4) = C_2$.

Constraint 3: since $d = X_0 - Z_0 - b - c - e \geq 0$ we get
$b + c + e \leq nx_0/(nx_0 + nx_1 + nx_2) - nz_0/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4) = C_3$.

Constraint 4: since $j = Y_0 - Z_0 - Z_1 - Z_2 + b + c + g - e \geq 0$ we get
$b+c+g-e \geq -ny_0/(ny_0+ny_1+ny_2)+(nz_0+nz_1+nz_2)/(nz_0+nz_1+nz_2+nz_3+nz_4) = C_4$.

Constraint 5: since $i = (X_1 + Y_1 - Z_3 - Z_1)/2 - g \geq 0$ we get
$g \leq nx_1/(2(nx_0 + nx_1 + nx_2)) + ny_1/(2(ny_0 + ny_1 + ny_2)) - (nz_1 + nz_3)/(2(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)) = C_5$.

Constraint 6: since $h = X_1/2 - Y_0 - Y_1/2 + Z_0 + Z_1/2 + Z_2 + Z_3/2 - c - g + e \geq 0$ we get
$c+g-e \leq nx_1/(2(nx_0+nx_1+nx_2)) - (ny_0+ny_1/2)/(ny_0+ny_1+ny_2) + (nz_0+nz_1/2+nz_2+nz_3/2)/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4) = C_6$.

Constraint 7: since $m = (-X_1 - Y_1 + Z_3 + Z_1)/2 + Z_4/2 + g - e \geq 0$ we get
$e-g \leq -nx_1/(2(nx_0+nx_1+nx_2)) - ny_1/(2(ny_0+ny_1+ny_2)) + (nz_3/2+nz_1/2+nz_4)/(nz_0 + nz_1 + nz_2 + nz_3 + nz_4) = C_7$.

Constraint 8: since $l = (-X_0 - X_1/2 + Y_1/2 + Z_0 + Z_1/2 + Z_3/2) + c + e \geq 0$, we get
$c+e \geq (2nx_0+nx_1)/(2(nx_0+nx_1+nx_2)) - ny_1/(2(ny_0+ny_1+ny_2)) - (2nz_0+nz_1+nz_3)/(2(nz_0 + nz_1 + nz_2 + nz_3 + nz_4)) = C_8$.

To summarize, the linear constraints on $b, c, g, e$ are

$$b \leq C_1, c+g \leq C_2, b+c+e \leq C_3, C_4 \leq b+c+g-e, g \leq C_5, c+g-e \leq C_6, e-g \leq C_7, C_8 \leq c+e.$$

Assume that $\tau = 2.372873/3$. First, we compute the setting $b' = 0.648069822559251 \cdot 10^{-4}, c' = 0.00291873112245236, e' = 0.0169690155126008, g' = 0.0106131481064985$ minimizing $F(b, c, g, e) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j k^k l^l m^m$ under the above linear constraints and using the settings of the rest of the variables in terms of $b, c, g, e$. Then we find the setting $b'' = 0.648072770234329 \cdot 10^{-4}, c'' = 0.00294155294463157, e'' = 0.0166122266309863, g'' = 0.0105674995477950$ maximizing $V_{448}(b, c, g, e)/F(b, c, g, e)$ under the linear constraints. Finally, we conclude that $V_{448} \geq F(b', c', g', e') \cdot V_{448}(b'', c'', g'', e'')/F(b'', c'', g'', e'') \geq 6.908047489 \cdot 10^5$.

□

**Lemma 29.** *For $q = 5, \tau = 2.372873/3$, we have $V_{457} \geq 1.076904071 \cdot 10^6$.*

*In general,*

$$V_{457} \geq 2\left(V_{044}^3V_{233}^3 + V_{134}^3V_{233}^3 + V_{224}^3V_{233}^3/2\right)^{1/3}\left(\frac{V_{035}^3}{V_{233}^3}+\frac{V_{134}^3}{V_{233}^3}+1\right)^{1/3}\left(\frac{V_{017}^3}{V_{134}^3}+\frac{V_{026}^3V_{125}^3}{V_{035}^3V_{224}^3}+\frac{V_{125}^3}{V_{134}^3}+1\right)^{1/3}\times$$

$$\left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{125}V_{233}}\right)^b \left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{125}V_{233}}\right)^c \left(\frac{V_{116}V_{035}V_{224}}{V_{026}V_{125}V_{134}}\right)^g.$$

*Here $a, b, c, e, g, h$ are constrained by our framework (see the proof).*

*Proof.* $I = 4, J = 5, K = 7$ and so the variables are $a = \alpha_{017}, b = \alpha_{026}, c = \alpha_{035}, d = \alpha_{044}, e = \alpha_{053}, f = \alpha_{107}, g = \alpha_{116}, h = \alpha_{125}, i = \alpha_{134}, j = \alpha_{143}, k = \alpha_{152}, l = \alpha_{206}, m = \alpha_{215}, n = \alpha_{224}$. The linear system becomes

$X_0 = a + b + c + d + e,$
$X_1 = f + g + h + i + j + k,$
$Y_0 = e + f + k + l,$
$Y_1 = a + d + g + j + m,$
$Z_0 = a + f,$
$Z_1 = b + g + l,$
$Z_2 = c + h + k + m,$
$Z_3 = d + e + i + j + n.$

The rank is 8 and the number of variables is 14 so we pick 6 variables, $a, b, c, e, g, h$, and place them in $\Delta$. We then solve the system.

$f = (a + f) - a = Z_0 - a,$
$l = (b + g + l) - b - g = Z_1 - b - g,$
$k = (e + f + k + l) - e - f - l = Y_0 - Z_0 - Z_1 + a + b + g - e,$
$d = (a + b + c + d + e) - a - b - c - e = X_0 - a - b - c - e,$
$m = (c + h + k + m) - c - h - k = -Y_0 + Z_0 + Z_1 + Z_2 - a - b - c - g - h + e,$
$j = (a + d + g + j + m) - a - d - g - m = -X_0 + Y_0 + Y_1 - Z_0 - Z_1 - Z_2 + a + 2b + 2c + h,$
$i = (f+g+h+i+j+k)-f-g-h-j-k = X_1+X_0-2Y_0-Y_1+2Z_1+Z_2+Z_0-a-2c-3b-2g+e-2h,$
$n = (d + e + i + j + n) - d - e - i - j = -X_0 - X_1 + Y_0 - Z_1 + Z_3 + a + b + c + 2g - e + h.$

$nx_0 = V_{044}^3/V_{224}^3,$
$nx_1 = V_{134}^3/V_{224}^3,$
$nx_2 = 1/2,$
$ny_0 = V_{035}^3/V_{233}^3,$
$ny_1 = V_{134}^3/V_{233}^3,$
$ny_2 = 1,$
$nz_0 = V_{017}^3 V_{233}^3 V_{224}^3/V_{134}^3,$
$nz_1 = V_{233}^3 V_{125}^3 V_{026}^3/V_{035}^3,$
$nz_2 = V_{233}^3 V_{125}^3 V_{224}^3/V_{134}^3,$
$nz_3 = V_{224}^3 V_{233}^3,$

$na = 1,$
$nb = V_{035}V_{134}V_{224}/(V_{044}V_{125}V_{233}),$
$nc = V_{035}V_{134}V_{224}/(V_{044}V_{125}V_{233}),$
$ne = 1,$
$ng = V_{116}V_{035}V_{224}/(V_{026}V_{125}V_{134}),$
$nh = 1.$

$$V_{457} \geq 2\left(\frac{V_{044}^3}{V_{224}^3}+\frac{V_{134}^3}{V_{224}^3}+1/2\right)^{1/3}\left(\frac{V_{035}^3}{V_{233}^3}+\frac{V_{134}^3}{V_{233}^3}+1\right)^{1/3}\left(\frac{V_{017}^3 V_{233}^3 V_{224}^3}{V_{134}^3}+\frac{V_{026}^3 V_{125}^3 V_{233}^3}{V_{035}^3}+\frac{V_{125}^3 V_{233}^3 V_{224}^3}{V_{134}^3}+V_{233}^3 V_{224}^3\right.$$

$$\left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{125}V_{233}}\right)^b\left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{125}V_{233}}\right)^c\left(\frac{V_{116}V_{035}V_{224}}{V_{026}V_{125}V_{134}}\right)^g.$$

Now we look at the constraints on $a, b, c, e, g, h$.

Constraint 1: since $f = Z_0 - a \geq 0$ we get
$a \leq nz_0/(nz_0 + nz_1 + nz_2 + nz_3) = C_1$.

Constraint 2: since $l = Z_1 - b - g \geq 0$ we get
$b + g \leq nz_1/(nz_0 + nz_1 + nz_2 + nz_3) = C_2$.

Constraint 3: since $k = Y_0 - Z_0 - Z_1 + a + b + g - e \geq 0$ we get
$a + b + g - e \geq -ny_0/(ny_0 + ny_1 + ny_2) + (nz_0 + nz_1)/(nz_0 + nz_1 + nz_2 + nz_3) = C_3$.

Constraint 4: since $d = X_0 - a - b - c - e \geq 0$ we get
$a + b + c + e \leq nx_0/(nx_0 + nx_1 + nx_2) = C_4$.

Constraint 5: since $m = -Y_0 + Z_0 + Z_1 + Z_2 - a - b - c - g - h + e \geq 0$ we get
$a+b+c+g+h-e \leq -ny_0/(ny_0+ny_1+ny_2)+(nz_0+nz_1+nz_2)/(nz_0+nz_1+nz_2+nz_3) = C_5$.

Constraint 6: since $j = -X_0 + Y_0 + Y_1 - Z_0 - Z_1 - Z_2 + a + 2b + 2c + h \geq 0$ we get
$a + 2b + 2c + h \geq nx_0/(nx_0 + nx_1 + nx_2) - (ny_0 + ny_1)/(ny_0 + ny_1 + ny_2) + (nz_0 + nz_1 + nz_2)/(nz_0 + nz_1 + nz_2 + nz_3) = C_6$.

Constraint 7: since $i = X_1 + X_0 - 2Y_0 - Y_1 + 2Z_1 + Z_2 + Z_0 - a - 2c - 3b - 2g + e - 2h \geq 0$ we get
$a + 2c + 3b + 2g - e + 2h \leq (nx_1 + nx_0)/(nx_0 + nx_1 + nx_2) - (2ny_0 + ny_1)/(ny_0 + ny_1 + ny_2) + (2nz_1 + nz_2 + nz_0)/(nz_0 + nz_1 + nz_2 + nz_3) = C_7$.

Constraint 8: since $n = -X_0 - X_1 + Y_0 - Z_1 + Z_3 + a + b + c + 2g - e + h \geq 0$ we get
$a + b + c + 2g - e + h \geq (nx_0 + nx_1)/(nx_0 + nx_1 + nx_2) - ny_0/(ny_0 + ny_1 + ny_2) + (nz_1 - nz_3)/(nz_0 + nz_1 + nz_2 + nz_3) = C_8$.

We summarize the constraints:

$$a \leq C_1, b + g \leq C_2, C_3 \leq a + b + g - e, a + b + c + e \leq C_4, a + b + c + g + h - e \leq C_5,$$

$$C_6 \leq a + 2b + 2c + h, a + 2c + 3b + 2g - e + 2h \leq C_7, C_8 \leq a + 2b + c + 2g + h - e.$$

Assume that $\tau = 2.372873/3$. We first find the setting $a' = 0.741706133871556 \cdot 10^{-5}, b' = 0.932955091664049 \cdot 10^{-3}, c' = 0.0155807738472934, e' = 0.00300527533017264, g' = 0.00130480061845205, h' = 0.0583385585882298$ minimizing $F(a, b, c, e, g, h) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j k^k l^l m^m n^n$ given the settings of $d, f, i, j, k, l, m, n$ in terms of $a, b, c, e, g, h$ and under the above 8 linear constraints. Then we find the setting $a'' = 0.739069714604175 \cdot 10^{-5}, b'' = 0.000939491122183629, c'' = 0.0157716913296529, e'' = 0.00299817541165094, g'' = 0.0012986122878164$
0.0581626623406867 maximizing $V_{457}(a, b, c, e, g, h)/F(a, b, c, e, g, h)$ under the same linear constraints. Finally, we conclude that $V_{457} \geq F(a', b', c', e', g', h') \cdot V_{457}(a'', b'', c'', e'', g'', h'')/F(a'', b'', c'', e'', g'', h'') \geq 1.076904071 \cdot 10^6$.

□

**Lemma 30.** *For* $q = 5, \tau = 2.372873/3$*,* $V_{466} \geq 1.244977753 \cdot 10^6$.

*In general,*

$$V_{466} \geq 2\left(\frac{V_{044}^3}{V_{134}^3}+1+\frac{V_{224}^3}{2V_{134}^3}\right)^{1/3}\left(\frac{V_{116}^3V_{035}^3}{V_{125}^3V_{134}^3}+\frac{V_{125}^3}{V_{224}^3}+1+\frac{V_{233}^3}{2V_{224}^3}\right)^{1/3}\times$$

$$\left(\frac{V_{125}^3V_{134}^6V_{026}^6}{V_{116}^3V_{035}^3V_{224}^3}+V_{125}^3V_{134}^3+V_{134}^3V_{224}^3+\frac{V_{134}^3V_{233}^3}{2}\right)^{1/3}\times$$

$$\left(\frac{V_{224}V_{116}V_{035}}{V_{125}V_{134}V_{026}}\right)^a\left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{233}V_{125}}\right)^b\left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{233}V_{125}}\right)^d\left(\frac{V_{026}V_{125}V_{134}}{V_{116}V_{035}V_{224}}\right)^e\left(\frac{V_{116}^2V_{035}^2V_{224}^2}{V_{125}^2V_{134}^2V_{026}^2}\right)^f.$$

*Here $a,b,d,e,f,p$ are constrained by our framework (see the proof).*

*Proof.* $I=4$, $J=K=6$ so the variables are $a=\alpha_{026}, b=\alpha_{035}, c=\alpha_{044}, d=\alpha_{053}, e=\alpha_{062}, f=\alpha_{116}, g=\alpha_{125}, h=\alpha_{134}, i=\alpha_{143}, j=\alpha_{152}, k=\alpha_{161}, l=\alpha_{206}, m=\alpha_{215}, n=\alpha_{224}, p=\alpha_{233}$. The system becomes
$X_0=a+b+c+d+e$,
$X_1=f+g+h+i+j+k$,
$Y_0=e+k+l$,
$Y_1=d+f+j+m$,
$Y_2=a+c+g+i+n$,
$Z_0=a+f+l$,
$Z_1=b+g+k+m$,
$Z_2=c+e+h+j+n$,
$Z_3=2(d+i+p)$.

The rank is 9 and the number of variables is 15 so we pick 6 variables, $a,b,d,e,f,p$, and place them in $\Delta$. Now we solve the system. For ease of solution we let $X_2/2=(Z_0+Z_1+Z_2+Z_3/2)-(X_0+X_1)$ and $Y_3/2=(Z_0+Z_1+Z_2+Z_3/2)-(Y_0+Y_1+Y_2)$.
$c=(a+b+c+d+e)-a-b-d-e=X_0-a-b-d-e$,
$l=(a+f+l)-a-f=Z_0-a-f$,
$h=(b+h+p)-b-p=Y_3/2-b-p=(Z_0+Z_1+Z_2+Z_3/2)-(Y_0+Y_1+Y_2)-b-p$,
$i=(d+i+p)-d-p=Z_3/2-d-p$,
$k=(e+k+l)-e-l=Y_0-Z_0+a+f-e$,
$j=((d+f+j+m)-d-f+(c+e+h+j+n)-c-e-h-(l+m+n+p)+l+p)/2=(Y_1-Y_3/2-X_0-X_2/2+Z_0+Z_2+2b-2f+2p)/2=((Y_0+2Y_1+Y_2)-(Z_0+2Z_1+Z_2+Z_3)+X_1+2b-2f+2p)/2$,
$n=(c+e+h+j+n)-c-e-h-j=(Z_2-X_0-Y_3/2-Y_1+X_2/2-Z_0)/2+a+b+d+f=(-3X_0-2X_1+2Y_0+Y_1+2Y_2-Z_0+Z_2)/2+a+b+d+f$,
$m=(l+m+n+p)-l-n-p=X_2/4+X_0/2-Z_0/2-Z_2/2+Y_3/4+Y_1/2-b-d-p=(Z_0/2+Z_1+Z_2/2+Z_3/2)-X_1/2-(Y_0+Y_2)/2-b-d-p$,
$g=(b+g+k+m)-b-k-m=Z_1+Z_2/2+3Z_0/2-X_2/4-X_0/2-Y_0-Y_3/4-Y_1/2-a+d-f+e+p=(Z_0/2-Z_2/2-Z_3/2)+(X_1)/2+(-Y_0+Y_2)/2-a+d-f+e+p$.
$nx_0=V_{044}^3/V_{224}^3$,
$nx_1=V_{134}^3/V_{224}^3$,
$nx_2=1/2$,
$ny_0=V_{116}^3V_{035}^3V_{224}^3/(V_{134}^3V_{233}^3V_{125}^3)$,
$ny_1=V_{125}^3/V_{233}^3$,
$ny_2=V_{224}^3/V_{233}^3$,

$ny_3 = 1/2,$
$nz_0 = V_{134}^3 V_{233}^3 V_{026}^6 V_{125}^3/(V_{116}^3 V_{035}^3 V_{224}^3),$
$nz_1 = V_{125}^3 V_{233}^3,$
$nz_2 = V_{233}^3 V_{224}^3,$
$nz_3 = V_{233}^6/2,$

$na = V_{224}V_{116}V_{035}/(V_{125}V_{134}V_{026}),$
$nb = V_{035}V_{134}V_{224}/(V_{044}V_{233}V_{125}),$
$nd = V_{035}V_{134}V_{224}/(V_{044}V_{233}V_{125}),$
$ne = V_{026}V_{125}V_{134}/(V_{116}V_{035}V_{224}),$
$nf = V_{116}^2 V_{035}^2 V_{224}^2/(V_{125}^2 V_{134}^2 V_{026}^2),$
$np = 1.$

$$V_{466} \geq 2\left(\frac{V_{044}^3}{V_{224}^3} + \frac{V_{134}^3}{V_{224}^3} + 1/2\right)^{1/3} \left(\frac{V_{116}^3 V_{035}^3 V_{224}^3}{V_{125}^3 V_{134}^3 V_{233}^3} + \frac{V_{125}^3}{V_{233}^3} + \frac{V_{224}^3}{V_{233}^3} + \frac{1}{2}\right)^{1/3} \times$$
$$\left(\frac{V_{134}^3 V_{233}^3 V_{026}^6 V_{125}^3}{V_{116}^3 V_{035}^3 V_{224}^3} + V_{125}^3 V_{233}^3 + V_{233}^3 V_{224}^3 + \frac{V_{233}^6}{2}\right)^{1/3} \times$$
$$\left(\frac{V_{224}V_{116}V_{035}}{V_{125}V_{134}V_{026}}\right)^a \left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{233}V_{125}}\right)^b \left(\frac{V_{035}V_{134}V_{224}}{V_{044}V_{233}V_{125}}\right)^d \left(\frac{V_{026}V_{125}V_{134}}{V_{116}V_{035}V_{224}}\right)^e \left(\frac{V_{116}^2 V_{035}^2 V_{224}^2}{V_{125}^2 V_{134}^2 V_{026}^2}\right)^f.$$

The constraints on the variables are as follows.
Constraint 1: since $c = X_0 - a - b - d - e \geq 0$,

$a + b + d + e \leq nx_0/(nx_0 + nx_1 + nx_2) = C_1,$

Constraint 2: since $l = Z_0 - a - f \geq 0$,

$a + f \leq nz_0/(nz_0 + nz_1 + nz_2 + nz_3) = C_2,$

Constraint 3: since $h = Y_3/2 - b - p \geq 0$,

$b + p \leq ny_3/(ny_0 + ny_1 + ny_2 + ny_3) = C_3,$

Constraints 4: since $i = Z_3/2 - d - p \geq 0$,

$d + p \leq nz_3/(nz_0 + nz_1 + nz_2 + nz_3) = C_4,$

Constraint 5: since $k = Y_0 - Z_0 + a + f - e \geq 0$,

$a + f - e \geq nz_0/(nz_0 + nz_1 + nz_2 + nz_3) - ny_0/(ny_0 + ny_1 + ny_2 + ny_3) = C_5,$

Constraint 6: since $j = (Y_1 - Y_3/2 - X_0 - X_2/2 + Z_0 + Z_2 + 2b - 2f + 2p)/2 \geq 0$,

$f-b-p \leq (ny_1-ny_3)/(2(ny_0+ny_1+ny_2+ny_3)) - (X_0+X_2/2)/(2(nx_0+nx_1+nx_2)) + (nz_0+nz_2)/(2(nz_0+nz_1+nz_2+nz_3)) = C_6$,

Constraint 7: since $n = (Z_2 - X_0 - Y_3/2 - Y_1 + X_2/2 - Z_0)/2 + a + b + d + f \geq 0$,

$a+b+d+f \geq (nx_0 - nx_2)/(2(nx_0+nx_1+nx_2)) + (ny_3+ny_1)/(2(ny_0+ny_1+ny_2+ny_3)) + (nz_0 - nz_2)/(2(nz_0+nz_1+nz_2+nz_3)) = C_7$,

Constraint 8: since $m = X_2/4 + X_0/2 - Z_0/2 - Z_2/2 + Y_3/4 + Y_1/2 - b - d - p \geq 0$,

$b+d+p \leq (nx_0+nx_2)/(2(nx_0+nx_1+nx_2)) - (nz_0+nz_2)/(2(nz_0+nz_1+nz_2+nz_3)) + (ny_1+ny_3)/(2(ny_0+ny_1+ny_2+ny_3)) = C_8$,

Constraint 9: since $g = -X_2/4 - X_0/2 - Y_0 - Y_3/4 - Y_1/2 + Z_1 + 3Z_0/2 + Z_2/2 + d - a - f + e + p \geq 0$,

$a+f-d-e-p \leq -(nx_0+nx_2)/(2(nx_0+nx_1+nx_2)) - (2ny_0+ny_1+ny_3)/(2(ny_0+ny_1+ny_2+ny_3)) + (3nz_0+2nz_1+nz_2)/(2(nz_0+nz_1+nz_2+nz_3)) = C_9$.

The constraints are then

$$a+b+d+e \leq C_1, a+f \leq C_2, b+p \leq C_3, d+p \leq C_4, C_5 \leq a+f-e,$$

$$f-b-p \leq C_6, C_7 \leq a+3b+d+f, b+d+p \leq C_8, a+f-d-e-p \leq C_9.$$

Assume that $\tau = 2.372873/3$. We first find the setting $a' = 0.207580266779174 \cdot 10^{-3}, b' = 0.00534699819473600, d' =$ $0.00534718758229032, e' = 0.209237614926190 \cdot 10^{-3}, f' = 0.105519087916149 \cdot 10^{-3}, p' = .297000529499192$ minimizing $F(a,b,d,e,f,p) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j k^k l^l m^m n^n p^p$ for the setting of the rest of the variables in terms of $a,b,d,e,f,p$ under the above 9 constraints. Then we find the setting $a'' = 0.207390670828811 \cdot 10^{-3}, b'' = 0.00542620227211836, d'' = 0.00542637317018223, e'' = 0.209046704420474 \cdot 10^{-3}, f'' = 0.105664774141390 \cdot 10^{-3}, p'' = .296860032116101$ maximizing $V_{466}(a,b,d,e,f,p)/F(a,b,d,e,f,p)$ under the same linear constraints. Finally, we conclude that $V_{466} \geq F(a',b',d',e',f',p') \cdot V_{466}(a'',b'',d'',e'',f'',p'')/F(a'',b'',$ $1.244977753 \cdot 10^6$.

□

**Lemma 31.** *When* $q = 5, \tau = 2.372873/3$, $V_{556} \geq 1.421276476 \cdot 10^6$.

*In general,*

$$V_{556} \geq 2\left(\frac{V_{035}^3}{V_{233}^3} + \frac{V_{134}^3}{V_{233}^3} + 1\right)^{2/3} \left(V_{026}^3 V_{233}^3 + V_{125}^3 V_{233}^3 + V_{224}^3 V_{233}^3 + \frac{V_{233}^6}{2}\right)^{1/3} \times$$

$$\left(\frac{V_{044}V_{125}V_{233}}{V_{035}V_{134}V_{224}}\right)^c \left(\frac{V_{116}V_{044}V_{233}}{V_{134}^2 V_{026}}\right)^e \left(\frac{V_{044}V_{125}V_{233}}{V_{035}V_{134}V_{224}}\right)^i.$$

*Here,* $a,b,c,e,f,h,i$ *are constrained by our framework (see the proof).*

*Proof.* Since $I = J = 5$ and $K = 6$, the variables are $a = \alpha_{026}, b = \alpha_{035}, c = \alpha_{044}, d = \alpha_{053}, e = \alpha_{116}, f = \alpha_{125}, g = \alpha_{134}, h = \alpha_{143}, i = \alpha_{152}, j = \alpha_{206}, k = \alpha_{215}, l = \alpha_{224}, m = \alpha_{233}, n = \alpha_{242}, p = \alpha_{251}$. The system is

$X_0 = a + b + c + d,$
$X_1 = e + f + g + h + i,$
$Y_0 = d + i + j + p,$
$Y_1 = c + e + h + k + n,$
$Z_0 = a + e + j,$
$Z_1 = b + f + k + p,$
$Z_2 = c + g + i + l + n,$
$Z_3 = 2(d + h + m).$

The rank is 8 and there are 15 variables, so we pick 7 variables, $a, b, c, e, f, h, i$, and place them into $\Delta$. We then solve the system:

$d = (a + b + c + d) - a - b - c = X_0 - a - b - c,$
$j = (a + e + j) - a - e = Z_0 - a - e,$
$m = (d + h + m) - (a + b + c + d) + a + b + c - h = Z_3/2 - X_0 + a + b + c - h,$
$p = (d+i+j+p) - (a+b+c+d) - (a+e+j) + 2a+b+c-i+e = Y_0 - X_0 - Z_0 + 2a+b+c-i+e,$
$k = (b + f + k + p) - (d + i + j + p) + (a + b + c + d) + (a + e + j) - 2b - f - 2a - c + i - e = Z_1 - Y_0 + X_0 + Z_0 - 2b - f - 2a - c + i - e,$
$n = (c+e+h+k+n) - (b+f+k+p) + (d+i+j+p) - (a+b+c+d) - (a+e+j) - h + 2b + f + 2a - i = Y_1 - Z_1 + Y_0 - X_0 - Z_0 - h + 2b + f + 2a - i,$
$g = (e + f + g + h + i) - e - f - h - i = X_1 - e - f - h - i,$
$l = (c+g+i+l+n) - (e+f+g+h+i) - (c+e+h+k+n) + (b+f+k+p) - (d+i+j+p) + (a+b+c+d) + (a+e+j) - c+e+2h+i-2b-2a = Z_2 - X_1 - Y_1 + Z_1 - Y_0 + X_0 + Z_0 - c + e + 2h + i - 2b - 2a.$

$nx_0 = V_{035}^3/V_{233}^3,$
$nx_1 = V_{134}^3/V_{233}^3,$
$nx_2 = 1,$
$ny_0 = V_{035}^3/V_{233}^3,$
$ny_1 = V_{134}^3/V_{233}^3,$
$ny_2 = 1,$
$nz_0 = V_{026}^3 V_{233}^3,$
$nz_1 = V_{125}^3 V_{233}^3,$
$nz_2 = V_{224}^3 V_{233}^3,$
$nz_3 = V_{233}^6/2,$

$na = 1,$
$nb = 1,$
$nc = V_{044}V_{125}V_{233}/(V_{035}V_{134}V_{224}),$
$ne = V_{116}V_{044}V_{233}/(V_{134}^2 V_{026}),$
$nf = 1,$
$nh = 1,$
$ni = V_{044}V_{125}V_{233}/(V_{035}V_{134}V_{224}).$

Finally,

$$V_{556} \geq 2\left(\frac{V_{035}^3}{V_{233}^3} + \frac{V_{134}^3}{V_{233}^3} + 1\right)^{2/3}\left(V_{026}^3V_{233}^3 + V_{125}^3V_{233}^3 + V_{224}^3V_{233}^3 + \frac{V_{233}^6}{2}\right)^{1/3} \times$$

$$\left(\frac{V_{044}V_{125}V_{233}}{V_{035}V_{134}V_{224}}\right)^c \left(\frac{V_{116}V_{044}V_{233}}{V_{134}^2 V_{026}}\right)^e \left(\frac{V_{044}V_{125}V_{233}}{V_{035}V_{134}V_{224}}\right)^i.$$

Now let's consider the constraints:

Constraint 1: $d = X_0 - a - b - c \geq 0$, and so
$a + b + c \leq nx_0/(nx_0 + nx_1 + nx_2) = C_1$,

Constraint 2: $j = Z_0 - a - e \geq 0$, and so
$a + e \leq nz_0/(nz_0 + nz_1 + nz_2 + nz_3) = C_2$,

Constraint 3: $m = Z_3/2 - X_0 + a + b + c - h \geq 0$, and so
$h - a - b - c \leq nz_3/(nz_0 + nz_1 + nz_2 + nz_3) - nx_0/(nx_0 + nx_1 + nx_2) = C_3$,

Constraint 4: $p = Y_0 - X_0 - Z_0 + 2a + b + c - i + e \geq 0$, and so
$i - 2a - b - c - e \leq ny_0/(ny_0 + ny_1 + ny_2) - nx_0/(nx_0 + nx_1 + nx_2) - nz_0/(nz_0 + nz_1 + nz_2 + nz_3) = C_4$,

Constraint 5: $k = Z_1 - Y_0 + X_0 + Z_0 - 2b - f - 2a - c + i - e \geq 0$, and so
$2a + 2b + c + e + f - i \leq nx_0/(nx_0 + nx_1 + nx_2) - ny_0/(ny_0 + ny_1 + ny_2) + (nz_0 + nz_1)/(nz_0 + nz_1 + nz_2 + nz_3) = C_5$,

Constraint 6: $n = Y_1 + Y_0 - X_0 - Z_0 - Z_1 - h + 2b + f + 2a - i \geq 0$, and so
$h + i - 2a - 2b - f \leq (ny_0 + ny_1)/(ny_0 + ny_1 + ny_2) - nx_0/(nx_0 + nx_1 + nx_2) - (nz_0 + nz_1)/(nz_0 + nz_1 + nz_2 + nz_3) = C_6$,

Constraint 7: $g = X_1 - e - f - h - i \geq 0$, and so
$e + f + h + i \leq nx_1/(nx_0 + nx_1 + nx_2) = C_7$,

Constraint 8: $l = X_0 - X_1 - Y_1 - Y_0 + Z_0 + Z_1 + Z_2 - c + e + 2h + i - 2b - 2a$, and so
$2a + 2b + c - e - 2h - i \leq (nx_0 - nx_1)/(nx_0 + nx_1 + nx_2) - (ny_0 + ny_1)/(ny_0 + ny_1 + ny_2) + (nz_0 + nz_1 + nz_2)/(nz_0 + nz_1 + nz_2 + nz_3) = C_8$.

Summarizing, the constraints are

$$a + b + c \leq C_1, a + e \leq C_2, h - a - b - c \leq C_3, i - 2a - b - c - e \leq C_4,$$

$$2a + 2b + c + e + f - i \leq C_5, h + i - 2a - 2b - f \leq C_6, e + f + h + i \leq C_7, 2a + 2b + c - e - 2h - i \leq C_8.$$

Assume that $\tau = 2.372873/3$. We first find the setting $a' = 0.5629953120669663 \cdot 10^{-4}, b' = 0.00122955027040296, c' = 0.00354992813988773, e' = 0.207509360036833 \cdot 10^{-3}, f' = 0.0122589343738679, h' = 0.0618610336237278, i' = 0.00354992819549840$ minimizing $F(a, b, c, e, f, h, i) = a^a b^b c^c d^d e^e f^f g^g h^h i^i j^j k^k l^l m^m n^n p^p$ for the setting of the rest of the variables in terms of $a, b, c, e, f, h, i$ and under the above 8 linear constraints. We then find the setting $a'' = 0.576154034796757 \cdot 10^{-4}, b'' = 0.00124475629509324, c'' = 0.00351600913306182, e'' = 0.204880209163357 \cdot 10^{-3}, f'' = 0.0122439331719230, h'' = 0.0618847419702464, i'' = 0.00351595877192500$ maximizing $V_{556}(a, b, c, e, f, h, i)/F(a, b, c, e, f, h, i)$ under the above linear constraints. We then conclude that $V_{556} \geq F(a', b', c', e', f', h', i') \cdot V_{556}(a'', b'', c'', e'', f'', h'', i'')/F(a'', b'', c'', e'', f'', h'', i'') \geq 1.421276476 \cdot 10^6$. $\square$

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