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MULTIPOLAR VISCOUS FLUIDS

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Abstract. The purpose of the paper is to develop a thermodynamic theory of constitutive equations of multipolar viscous fluids. The restrictions which the principle of material frame-indifference and the Clausius-Duhem inequality place on the constitutive equations are derived. Explicit forms of the viscous stresses are obtained for linear viscous fluids.

1. Introduction. In the classical Navier-Stokes theory, the viscosity of fluids is modelled by the dependence of the stress on the first spatial gradients of velocity. It is well known that the corresponding mathematical theory contains a number of unsolved problems; in particular, there is no adequate existence theory for compressible flows of such fluids. These problems have led the present authors to believe that a stronger mechanism of dissipation and viscosity, namely the dependence of the stress on the higher gradients of velocity, must occur in the flows of viscous fluids. In a work in progress, the authors together with A. Novotný [1] show that a satisfactory existence theory can be developed for materials of this type (see also Nečas and Růžička [2] for an analogous treatment of viscous incompressible solids).

In materials in which the higher gradients of velocity and the higher gradients of deformation influence the response, the rate of work of internal forces cannot be expected to be only the product of the usual second order stress tensor with the first gradient of velocity; instead of this, a more general expression must be assumed containing additionally the sum of products of higher order multipolar stress tensors with the higher gradients of velocity. Otherwise such materials cannot be compatible with the Clausius-Duhem inequality. The general theory of multipolar materials is due to Green and Rivlin [3, 4]. However, they consider only the constitutive equations of elastic, nonviscous materials. (See also Bucháček [5] for the theory of multipolar materials with fading memory.)

The purpose of this paper is to develop a thermodynamic theory of constitutive equations of multipolar viscous fluids within the framework of the theory of Green and Rivlin. The postulated constitutive equations express the Helmholtz free

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energy, entropy, heat flux vector, and the multipolar stress tensors as functions of the following variables: the density and its gradients up to a fixed order, gradients of velocity up to a fixed order, the temperature, and the gradient of temperature. We derive the general restrictions which the principle of material frame-indifference and the Clausius-Duhem inequality place on the constitutive functions of the fluid. Then we restrict our attention to linear fluids for which the constitutive quantities depend on the gradients of velocity and temperature linearly, with the coefficients independent of temperature and gradients of density. Using the representation theorems for isotropic linear functions, we obtain explicit forms of the viscous stresses. The corresponding scalar coefficients in front of the gradients of velocity in these expressions generalize (and include as special cases) the classical viscosities of the fluid. As in the classical case, the Clausius-Duhem inequality gives the nonnegativeness of the viscous work which, in the strengthened form, plays a crucial role in the existence theory [1, 2].

2. Balance equations and the Clausius-Duhem inequality. We refer to Green and Rivlin [3, 4] for a detailed exposition of the thermomechanics of multipolar bodies.

We use the spatial description of the processes of the fluid. The fields associated with the processes are functions of the actual position $x = (x_i)$ $(i = 1, 2, 3)$ of the points of the fluid and of time t. Let $N \geq 1$ be an integer. A thermodynamic process of a multipolar fluid R of grade N is a collection of $8 + N$ functions of position and time: $v, \theta, \rho, e, \eta, b, r, q, T^{(k)}$ $(k = 0, 1, \ldots, N-1)$, whose interpretation and tensorial nature is as follows:

- $v = (v_i)$ is the velocity field,
- θ is the field of positive absolute temperature,
- ρ is the actual density,
- e is the specific internal energy,
- η is the specific entropy,
- $b = (b_i)$ is the specific external body force,
- r is the rate of the external communication of heat to the body,
- $q = (q_i)$ is the heat flux vector,
- $T^{(k)} = (T^{(k)}_{jj_1\cdots j_k i})$ is the spatial multipolar stress tensor of order $k+2$, $k = 0, 1, \ldots, N-1$; it is assumed that $T^{(k)}$ is symmetric in the indices $j_1 \cdots j_k$. The symmetry is motivated by the fact that the tensor $T^{(k)}$ enters the basic equations only through its product with the spatial gradient of velocity which has the same type of symmetry (cf. (2.2) below).

It is also assumed that any function occurring in this paper is as many times continuously differentiable as needed to make the expressions meaningful. Each process satisfies the equations of balance of mass, energy, linear and angular momentum, and the Clausius-Duhem inequality. Their local forms read as follows:

$$\dot{\rho} + \rho v_{i,i} = 0 \quad \text{or} \quad \partial\rho/\partial t + (\rho v_i)_{,i} = 0, \tag{2.1}$$

$$\rho\left(e+\frac{1}{2}v_iv_i\right)^{\cdot} = \left(-q_i+\sum_{k=0}^{N-1} T^{(k)}_{jj_1\cdots j_k i}v_{j,j_1\cdots j_k}\right)_{,i} + \rho b_i v_i + \rho r, \tag{2.2}$$

$$\rho\dot{v}_j = T^{(0)}_{ji,i} + \rho b_j, \tag{2.3}$$

$$\rho(\varepsilon_{jkp}x_kv_p)^{\cdot} = (\varepsilon_{jkp}x_kT^{(0)}_{pi} + \varepsilon_{jkp}T^{(1)}_{pki})_{,i} + \rho\varepsilon_{jkp}x_kb_p, \tag{2.4}$$

$$\rho\dot{\eta} \geq -\left(\frac{q_i}{\theta}\right)_{,i} + \rho\frac{r}{\theta}. \tag{2.5}$$

Here a superimposed dot denotes the material time derivative, and a comma followed by an index (or several indices) denotes the partial derivative with respect to the coordinate corresponding to the index (or indices). Green and Rivlin [3, 4] show that the equations of balance of linear and angular momenta follow from the equation of balance of energy and the principle of material frame-indifference. We also note that the higher order stresses $T^{(1)}, T^{(2)}, \ldots$ do not enter the equation of balance of linear momentum while only the first two stresses $T^{(0)}, T^{(1)}$ enter the equation of balance of angular momentum.

Routine manipulations with the balance equations give the reduced forms of the balance equations:

$$\rho\dot{e} = \sum_{k=0}^{N-1}(T^{(k)}_{jj_1\cdots j_k i} + T^{(k+1)}_{jj_1\cdots j_k ip,p})v_{j,j_1\cdots j_k i} - q_{i,i} + \rho r, \tag{2.6}$$

$$\varepsilon_{ijk}(T^{(0)}_{kj} + T^{(1)}_{kjp,p}) = 0, \tag{2.7}$$

$$\rho\theta\dot{\eta} \geq \rho\dot{e} - \sum_{k=0}^{N-1}(T^{(k)}_{jj_1\cdots j_k i} + T^{(k+1)}_{jj_1\cdots j_k ip,p})v_{j,j_1\cdots j_k i} + q_i\theta_{,i}/\theta. \tag{2.8}$$

In (2.6) and in what follows we use the convention that

$$T^{(k)} = 0 \quad \text{for } k \geq N. \tag{2.9}$$

Another equivalent form of the Clausius-Duhem inequality is the dissipation inequality which reads

$$\rho\dot{\psi} \leq \sum_{k=0}^{N-1}(T^{(k)}_{jj_1\cdots j_k i} + T^{(k+1)}_{jj_1\cdots j_k ip,p})v_{j,j_1\cdots j_k i} - \rho\eta\dot{\theta} - q_i\theta_{,i}/\theta, \tag{2.10}$$

where

$$\psi = e - \theta\eta \tag{2.11}$$

is the Helmholtz free energy.

3. Constitutive equations of multipolar viscous fluids. We postulate the constitutive equations in the following form:

$$f = f(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \nabla v, \ldots, \nabla^K v, \theta, \nabla\theta), \tag{3.1}$$

where $M, K \geq 1$ are prescribed positive integers, and f stands for any of the functions $e, \eta, \psi, q, T^{(k)}, k = 0, 1, \ldots, N-1$. The functions occurring on the right-hand side of (3.1), i.e., the functions $e, \eta, \psi, q, T^{(k)}$, are given smooth functions of their arguments defined on the natural domain given by the physical interpretation of the arguments; the functions e, η, ψ naturally satisfy the relation (2.11). The values of f on the left-hand side of (3.1) as well as the arguments on the right-hand side of (3.1) are evaluated at (x, t). The postulated form of the constitutive equations can be motivated by the general definition of fluids in terms of the symmetry group of the material. (See Noll [6], Cross [7], Samohýl [8], and Gurtin, Vianello, and Williams [9] for various particular cases.) We assume that the integers N, M, K take their least possible values. If N, M, K are so chosen, then the material is called a multipolar viscous fluid of type (N, M, K). If $M > 1$ or $K > 1$, then the material is nonsimple in the sense of Truesdell and Noll [10].

In addition to the symmetry which has already been taken into account, we shall use two general principles to restrict the form of the constitutive functions: the principle of material frame-indifference and the requirement that the Clausius-Duhem inequality be satisfied in every process. We shall take these two principles in turn and combine them with the idea of linearization with respect to the nonequilibrium parameters $\nabla v, \ldots, \nabla^K v, \nabla\theta$.

Consider a change of frame of the form

$$\overline{x}_i = Q_{ij}(t)x_j + c_i(t), \tag{3.2}$$

where $Q(t) = (Q_{ij}(t))$ is a time-dependent orthogonal matrix,

$$Q_{ij}(t)Q_{ik}(t) = \delta_{jk}, \tag{3.3}$$

and $c(t) = (c_i(t))$ is a time-dependent vector. The principle of material frame-indifference postulates that under the changes of frame (3.2) the quantities θ, ρ, e, η, r, q, and $T^{(k)}$ transform in the following way:

$$\overline{\theta} = \theta, \quad \overline{\eta} = \eta, \quad \overline{e} = e, \quad \overline{\rho} = \rho, \quad \overline{r} = r, \tag{3.4}$$

$$\overline{q}_i = Q_{ij}q_j, \tag{3.5}$$

$$\overline{T}^{(k)}_{j_1\cdots j_{k+2}} = Q_{j_1m_1}\cdots Q_{j_{k+2}m_{k+2}}T^{(k)}_{m_1\cdots m_{k+2}}. \tag{3.6}$$

The quantities on the two sides of the above equations are evaluated at the same material point; since the spatial description is used, the arguments on the left-hand side are $(\overline{x}, t)$, while the arguments on the right-hand side are (x, t). Consequently, we have the following transformation laws for the gradients of density, velocity, and temperature:

$$\overline{\rho}_{,i_1\cdots i_k} = Q_{i_1j_1}\cdots Q_{i_kj_k}\rho_{,j_1\cdots j_k}, \tag{3.7}$$

$$\overline{v}_{i,j} = Q_{il}Q_{jm}v_{l,m} + W_{ij}, \tag{3.8}$$

$$W_{ij} = \dot{Q}_{im}Q_{jm}, \qquad W_{ij} = -W_{ji}, \tag{3.9}$$

$$\overline{v}_{i,j_1\cdots j_k} = Q_{il}Q_{j_1m_1}\cdots Q_{j_km_k}v_{l,m_1\cdots m_k} \qquad (k \geq 2), \tag{3.10}$$

$$\overline{\theta}_{,i} = Q_{ij}\theta_{,j}. \tag{3.11}$$

Let us also note that in view of (3.8) and (3.9), the symmetric part $D = (D_{ij})$ of the gradient of velocity,

$$D_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \tag{3.12}$$

transforms as

$$\overline{D}_{ij} = Q_{il}Q_{jm}D_{lm}. \tag{3.13}$$

PROPOSITION 3.1. A multipolar viscous fluid of type (N, M, K) satisfies the principle of material frame-indifference if and only if the following two conditions are satisfied:

(1) The functions $e, \eta, \psi, q, T^{(k)}$ depend on the first spatial gradient of velocity only through its symmetric part D, i.e.,

$$\begin{aligned} &f(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta) \\ &\quad = f(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, D, \nabla^2 v, \dots, \nabla^K v, \theta, \nabla\theta), \end{aligned} \tag{3.14}$$

where f stands for any of the functions $e, \eta, \psi, q, T^{(k)}$.

(2) The constitutive functions $e, \eta, \psi, q, T^{(k)}$ are isotropic scalar-, vector-, or tensor-valued functions of the scalar, vector, or tensor arguments $\rho, \nabla\rho, \dots, \nabla^{M-1}\rho$, $D, \nabla^2 v, \dots, \nabla^K v, \theta, \nabla\theta$.

The proof is omitted. We only note that the fact that the response can depend on the first gradient of velocity only through its symmetric part is related with the occurrence of the skew tensor W in the transformation law (3.8) and this in turn depends on the admittance of the time-dependent rotations in the statement of the principle of material frame-indifference (cf., e.g., Truesdell [11]). Assertion (2) means that, for instance, the constitutive function $T^{(k)}$ satisfies the functional equation of the form

$$\begin{aligned} &T^{(k)}_{j_1\cdots j_{k+2}}(\overline{\rho}, \nabla\overline{\rho}, \dots, \nabla^{M-1}\overline{\rho}, \overline{D}, \nabla^2\overline{v}, \dots, \nabla^K\overline{v}, \overline{\theta}, \nabla\overline{\theta}) \\ &\quad = Q_{j_1 m_1} \cdots Q_{j_{k+2} m_{k+2}} T^{(k)}_{m_1\cdots m_{k+2}}(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, D, \nabla^2 v, \dots, \nabla^K v, \theta, \nabla\theta). \end{aligned} \tag{3.15}$$

Using this equation and a similar one for q, we shall derive explicit expressions for the functions q and $T^{(k)}$ (see Sec. 5).

4. The consequences of the Clausius-Duhem inequality. We now demand, following Coleman and Noll [12], that the Clausius-Duhem inequality be satisfied in every process compatible with the constitutive equations and the equations of balance of energy and linear momentum. These contain the external sources r and b respectively and it is essential to admit that r and b can be arbitrary functions of x, t. In view of the form of the constitutive equations this means that all possible motions and evolutions of the absolute temperature are admitted.

To facilitate the statement of the restrictions which the Clausius-Duhem inequality places on the constitutive functions, we introduce the following quantities and functions:

- the equilibrium part of the multipolar stress $T^{(k,E)}$,

$$\begin{aligned} &T^{(k,E)}(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, \theta) \\ &\quad = T^{(k)}(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, 0, \dots, 0, \theta, 0) \end{aligned} \tag{4.1}$$

  $(k = 0, 1, \dots)$, as the multipolar stress corresponding to the zero values of the gradients of velocity and the zero value of the gradient of temperature;

– the viscous part of the multipolar stress $T^{(k,V)}$,

$$\begin{aligned} T^{(k,V)}(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, \nabla v, \dots, \nabla^{K} v, \theta, \nabla\theta) \\ = T^{(k)}(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, \nabla v, \dots, \nabla^{K} v, \theta, \nabla\theta) \\ - T^{(k,E)}(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, \theta), \end{aligned} \tag{4.2}$$

$k = 0, 1, \dots;$

– the variable σ is defined by

$$\sigma = \ln \rho, \tag{4.3}$$

so that the dependence of the constitutive quantities on ρ and its gradients can be replaced by the dependence on σ and its gradients.

We say that a multipolar viscous fluid R satisfies the second law of thermodynamics if the Clausius-Duhem inequality (2.5) holds in every process compatible with the constitutive equations (3.1) and the equations of balance of mass (2.1), energy (2.2), and linear momentum (2.3).

THEOREM 4.1. A multipolar viscous fluid of type (N, M, K) satisfies the second law of thermodynamics if and only if the following two conditions are satisfied in every process:

(1) The generalized Gibbs equation,

$$\rho\dot{\psi} = -\rho\eta\dot{\theta} + \sum_{k=0}^{N-1} (T^{(k,E)}_{jj_1\cdots j_k i} + T^{(k+1,E)}_{jj_1\cdots j_k ip,p}) v_{j,j_1\cdots j_k i} - \sum_{k=1}^{N-1} \frac{\partial}{\partial\theta} T^{(k,E)}_{jj_1\cdots j_k p}\theta_{,p} v_{j,j_1\cdots j_k}. \tag{4.4}$$

(2) The residual dissipation inequality,

$$\sum_{k=0}^{N-1} (T^{(k,V)}_{jj_1\cdots j_k i} + T^{(k+1,V)}_{jj_1\cdots j_k ip,p}) v_{j,j_1\cdots j_k i} + \sum_{k=1}^{N-1} \frac{\partial}{\partial\theta} T^{(k,E)}_{jj_1\cdots j_k p}\theta_{,p} v_{j,j_1\cdots j_k} - q_i\theta_{,i}/\theta \ge 0. \tag{4.5}$$

The Clausius-Duhem inequality or equivalently the dissipation inequality thus splits into the generalized Gibbs equation and the residual dissipation inequality. Further consequences of the generalized Gibbs equation will be given in Theorem 4.2, below, while the consequences of the residual dissipation inequality will be analyzed in the next section.

THEOREM 4.2. If a multipolar viscous fluid of type (N, M, K) satisfies the second law of thermodynamics then the following two assertions hold:

(1) The constitutive functions ψ, η, e are independent of the gradients of velocity and of the gradient of temperature, i.e.,

$$f(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, \nabla v, \dots, \nabla^{K} v, \theta, \nabla\theta) = f(\rho, \nabla\rho, \dots, \nabla^{M-1}\rho, \theta) \tag{4.6}$$

throughout the domain of the constitutive functions, where f stands for any of the functions ψ, η, e.

(2) The entropy relation

$$\eta = -\frac{\partial\psi}{\partial\theta} \tag{4.7}$$

and the generalized stress relations

$$\operatorname{Sym}\left(T^{(k,E)}_{jj_1\cdots j_k i} + T^{(k+1,E)}_{jj_1\cdots j_k ip,p} - \frac{\partial}{\partial\theta} T^{(k+1,E)}_{jj_1\cdots j_k ip}\theta_{,p}\right) = -\rho \operatorname{Sym}\left(\frac{\partial\psi}{\partial\sigma_{j_1\cdots j_k}}\delta_{ij}\right), \tag{4.8}$$

$k = 0, 1, \dots$, hold throughout the domain of the constitutive functions; here the constitutive functions are taken as functions of σ and its gradients, and Sym denotes the symmetrization with respect to the indices $j_1 \cdots j_k i$.

In (4.8) we have used our earlier convention that for $k \ge N$, we have set $T^{(k)} = 0$ and hence also, as a consequence of this, $T^{(k,E)} = T^{(k,V)} = 0$ (cf. (2.9), (4.1), (4.2)).

Proof of Theorems 4.1 *and* 4.2. We shall use the dissipation inequality (2.10) which is equivalent to the Clausius-Duhem inequality. We express the constitutive functions as functions of σ and its gradients, e.g.,

$$\psi = \psi(\sigma, \nabla\sigma, \dots, \nabla^{M-1}\sigma, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta), \tag{4.9}$$

etc. This form of the constitutive functions together with the definitions of the elastic and viscous parts of the stress and the continuity equation in the form

$$\dot\sigma + v_{k,k} = 0 \tag{4.10}$$

give the following explicit version of the dissipation inequality:

$$\begin{aligned}
&\rho\left(-\sum_{m=0}^{M-1}\frac{\partial\psi}{\partial\sigma_{i_1\cdots i_m}} v_{l,li_1\cdots i_m} + \sum_{n=1}^{K}\frac{\partial\psi}{\partial v_{j,j_1\cdots j_n}}\dot v_{j,j_1\cdots j_n} + \frac{\partial\psi}{\partial\theta}\dot\theta + \frac{\partial\psi}{\partial\theta_{,i}}\dot\theta_{,i}\right)\\
&\quad\le -\rho\eta\dot\theta + \sum_{k=0}^{N-1}\left(T^{(k,E)}_{jj_1\cdots j_k i} + \sum_{m=0}^{M-1}\frac{\partial T^{(k+1,E)}_{jj_1\cdots j_k ip}}{\sigma_{,i_1\cdots i_m p}}\sigma_{,i_1\cdots i_m p} + \frac{\partial}{\partial\theta}T^{(k+1,E)}_{jj_1\cdots j_k ip}\theta_{,p}\right) v_{j,j_1\cdots j_k i}\\
&\qquad + \sum_{k=0}^{N-1}\left(T^{(k,V)}_{jj_1\cdots j_k i} + \sum_{m=0}^{M-1}\frac{\partial T^{(k+1,V)}_{jj_1\cdots j_k ip}}{\partial\sigma_{,i_1\cdots i_m}}\sigma_{,i_1\cdots i_m p}\right.\\
&\qquad\qquad \left. + \sum_{n=1}^{K}\frac{\partial T^{(k+1,V)}_{jj_1\cdots j_k ip}}{\partial v_{l,l_1\cdots l_n}} v_{l,l_1\cdots l_n p} + \frac{\partial}{\partial\theta}T^{(k+1,V)}_{jj_1\cdots j_k ip}\theta_{,p} + \frac{\partial T^{(k+1,V)}_{jj_1\cdots j_k ip}}{\partial\theta_{,l}}\theta_{,lp}\right)\\
&\qquad \cdot v_{j,j_1\cdots j_k i} - q_i\theta_{,i}/\theta.
\end{aligned} \tag{4.11}$$

The values of ψ, η, q, $T^{(k,V)}$ and of their partial derivatives are evaluated at

$$(\sigma, \nabla\sigma, \dots, \nabla^{M-1}\sigma, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta) \tag{4.12}$$

while the values of $T^{(k,E)}$ and their derivatives are evaluated at

$$(\sigma, \nabla\sigma, \dots, \nabla^{M-1}\sigma, \theta). \tag{4.13}$$

The above inequality must be satisfied in every thermodynamic process. In view of the remarks made at the beginning of this section this means that the gradients of velocity, gradients of acceleration, time derivatives of temperature and its gradients, and the gradients of temperature occurring in (4.11) may be chosen in a completely arbitrary way. Observing that the inequality contains the variables

$$\dot{v}_{j,j_1\cdots j_n},\ \dot{\theta}_{,i},\ \dot{\theta} \tag{4.14}$$

linearly, and using the independence of the variables (4.14) of the remaining variables in (4.11), we see that (4.11) implies

$$\frac{\partial\psi}{\partial v_{j,j_1\cdots j_n}}=0,\quad \frac{\partial\psi}{\partial\theta_{,l}}=0,\quad \eta=-\frac{\partial\psi}{\partial\theta}. \tag{4.15}$$

In particular, the function ψ is independent of the gradients of velocity and of the gradient of temperature,

$$\psi=\psi(\sigma,\nabla\sigma,\dots,\nabla^{M-1}\sigma,\theta). \tag{4.16}$$

The entropy relation (4.15), also implies that η is a function of the variables (4.13), and finally by (2.11) this completes the proof of assertion (1) of Theorem 4.2 and of (4.7).

Inequality (4.11) now simplifies in the following way: on the left-hand side the terms containing the derivatives of ψ with respect to θ, and gradients of v and θ may be omitted and on the right-hand side the term $\rho\eta\dot{\theta}$ may be omitted. Using the independence of the gradients of velocity and temperature of the variables (4.13), we see that the gradients ∇v, $\nabla^2 v$, $\dots$, $\nabla\theta$, $\nabla^2\theta$, can be replaced systematically by $\alpha\nabla v$, $\alpha\nabla^2 v$, $\dots$, $\alpha\nabla\theta$, $\alpha\nabla^2\theta$ at a given point, where $\alpha>0$ is an arbitrary number. The inequality so obtained is written symbolically as follows:

$$\begin{aligned}
-\rho\sum_{m=0}^{M-1}&\frac{\partial\psi}{\partial\nabla^m\sigma}\alpha\nabla^{m+1}v\\
&\le\sum_{k=0}^{N-1}\left(T^{(k,E)}+\sum_{m=0}^{M-1}\frac{\partial T^{(k+1,E)}}{\partial\nabla^m\sigma}\nabla^{m+1}\sigma+\frac{\partial T^{(k+1,E)}}{\partial\theta}\alpha\nabla\theta\right)\alpha\nabla^{k+1}v\\
&\quad+\sum_{k=0}^{N-1}\left(T^{(k,V)}+\sum_{m=0}^{M-1}\frac{\partial T^{(k+1,V)}}{\partial\nabla^m\sigma}\nabla^{m+1}\sigma+\sum_{n=1}^{K}\frac{\partial T^{(k+1,V)}}{\partial\nabla^n v}\alpha\nabla^{n+1}v\right.\\
&\qquad\left.+\frac{\partial T^{(k+1,V)}}{\partial\theta}\alpha\nabla\theta+\frac{\partial T^{(k+1,V)}}{\partial\nabla\theta}\alpha\nabla^2\theta\right)\alpha\nabla^{k+1}v-q_i\alpha\theta_{,i}/\theta,
\end{aligned} \tag{4.17}$$

where the derivatives of the viscous stresses and of q are evaluated at

$$(\nabla^r\sigma,\alpha\nabla^m v,\theta,\alpha\nabla\theta), \tag{4.18}$$

and the derivatives of ψ and of the equilibrium stresses are evaluated at (4.16). The tensorial indices are omitted from (4.17); their position is analogous to that of the corresponding terms in (4.11). We now divide (4.17) by α and let α tend to 0. The arguments of the derivatives of ψ and of the equilibrium parts of the stresses remain unchanged while the arguments of the viscous parts of the stresses, their derivatives, and the arguments of q tend to $(\nabla^r\sigma, 0, \theta, 0)$. The limiting procedure thus gives

$$\begin{aligned}-\rho\sum_{m=0}^{M-1}\frac{\partial\psi}{\partial\nabla^m\sigma}\nabla^{m+1}v \le{}& \sum_{k=0}^{N-1}\left(T^{(k,E)}+\sum_{m=0}^{M-1}\frac{\partial T^{(k+1,E)}}{\partial\nabla^m\sigma}\nabla^{m+1}\sigma\right)\cdot\nabla^{k+1}v\\&+\sum_{k=0}^{N-1}\left(T^{(k,V)}+\sum_{m=0}^{M-1}\frac{\partial T^{(k+1,V)}}{\partial\nabla^m\sigma}\nabla^{m+1}\sigma\right)\cdot\nabla^{k+1}v\\&-q\cdot\nabla\theta/\theta,\end{aligned}\tag{4.19}$$

where the viscous stresses, their derivatives, and q are evaluated at $(\nabla^r\sigma, 0, \theta, 0)$. But for these values of arguments we have $T^{(k,V)} = 0$ by the very definition and hence, as a consequence, $\partial T^{(k,V)}/\partial\nabla^r\sigma = 0$ also. Furthermore, since the only term containing $\nabla\theta$ is the heat conduction term, we deduce from (4.19) that

$$q(\nabla^r\sigma, 0, \theta, 0) = 0. \tag{4.20}$$

All these facts further simplify (4.19) to

$$-\rho\sum_{m=0}^{M-1}\frac{\partial\psi}{\partial\nabla^m\sigma}\nabla^{m+1}v \le \sum_{k=0}^{N-1}\left(T^{(k,E)}+\sum_{m=0}^{M-1}\frac{\partial T^{(k+1,E)}}{\partial\nabla^m\sigma}\nabla^{m+1}\sigma\right)\nabla^{k+1}v. \tag{4.21}$$

As this inequality is linear in the gradients of velocity, relations (4.8) follow. The proof of Theorem 4.2 is complete.

Obviously, the assertions of Theorem 4.2 imply assertion (1) of Theorem 4.1, i.e., the generalized Gibbs equation. The subtraction of the generalized Gibbs relation form the full dissipation inequality (4.11) gives the residual dissipation inequality. We have thus shown that the second law of thermodynamics implies assertions (1), (2) of Theorem 4.1. Conversely, the addition of the generalized Gibbs equation with the residual dissipation inequality gives the full dissipation inequality, i.e., assertions (1), (2) imply the second law. The proof is complete.

Proposition 4.1. If a multipolar viscous fluid of type (N, M, K) satisfies the second law of thermodynamics and $k \ge \max\{M, K+1\}$, we have

$$\mathrm{Sym}(T^{(k)}_{jj_1\cdots j_k i} + T^{(k+1)}_{jj_1\cdots j_k ip,p}) = 0 \tag{4.22}$$

in every process. If $T^{(k)}_{jj_1\cdots j_k i}$ is symmetric in $j_1\cdots j_k i$ for $k \ge \max\{M, K+1\}$, then

$$N \le K+1, \qquad N \le M. \tag{4.23}$$

(4.22) says that the multipolar stresses $T^{(k)}$, with k exceeding the number of gradients of density and the number of gradients of velocity, essentially vanish, i.e., they vanish modulo the symmetrization indicated. These stresses vanish exactly if the mentioned symmetry holds; consequently, we have the inequalities (4.23). Thus we have essentially an upper bound for the number of nonvanishing stresses N in terms of the number of gradients of density and the number of gradients of velocity. For the specific model to be treated in the next two sections, we shall also give a lower bound for N in terms of K.

Proof. We first split the working term in the dissipation inequality (4.11) as follows:

$$\begin{aligned}\sum_{k=0}^{N-1}(T^{(k)} + \operatorname{div} T^{(k+1)}) \cdot \nabla^{k+1} v \\ = \sum_{k=0}^{N_0-1}(T^{(k+1)} + \operatorname{div} T^{(k+1)}) \cdot \nabla^{k+1} v \\ + \sum_{k=N_0}^{N-1}(T^{(k)} + \operatorname{div} T^{(k+1)}) \cdot \nabla^{k+1} v ,\end{aligned} \tag{4.24}$$

where N_0 is defined by

$$N_0 = \max\{M , K + 1\}. \tag{4.25}$$

We now insert the split expression in the dissipation inequality (4.11) and examine the dependence of various terms of the resulting inequality on the gradients of velocity of different orders. We first consider all the terms except for the second sum on the right-hand side of (4.24). We have the following dependences: $\dot{\psi}$ depends on

$$\nabla v , \dots , \nabla^{M} v , \tag{4.26}$$

$T^{(k)}$ and its divergence depend on

$$\nabla v , \dots , \nabla^{K+1} v , \tag{4.27}$$

and the working terms in the first sum in (4.24) depend explicitly on

$$\nabla v , \dots , \nabla^{N_0} v. \tag{4.28}$$

We have $N_0 \geq M$, $N_0 \geq K + 1$ and hence, except for the second term in (4.24), everything else depends on the variables in (4.28). On the other hand, the second term in (4.24) depends explicitly on

$$\nabla^{N_0+1} v , \nabla^{N_0+2} v , \dots \tag{4.29}$$

and the coefficients in front of these gradients are independent of (4.29). To summarize, the dissipation inequality depends on the variables (4.29) only through the second sum in (4.24), and the dependence is linear. Equation (4.22) then follows. The rest is obvious. The proof is complete.

We concludle this section with a proposition concerning the validity of the reduced equation of balance of angular momentum (2.7). Its simple proof is omitted.

PROPOSITION 4.2. If a multipolar viscous fluid satisfies the reduced equation of balance of angular momentum in every process compatible with the constitutive equations, then

$$\varepsilon_{ijk}\left(T^{(0,E)}_{kj} + T^{(1,E)}_{kjp,p} - \frac{\partial}{\partial\theta}T^{(1,E)}_{kjp}\theta_{,p}\right) = 0 \tag{4.30}$$

and

$$\varepsilon_{ijk}\left(T^{(0,V)}_{kj} + T^{(1,V)}_{kjp,p} + \frac{\partial}{\partial\theta}T^{(1,E)}_{kjp}\theta_{,p}\right) = 0 \tag{4.31}$$

in every process.

5. Linear viscous fluids. Starting from this section we restrict ourselves to the case when only the density and not its gradients enter the constitutive equations, i.e., $M = 1$, and to the case when the viscous stresses and the heat flux vector depend linearly on the gradients of velocity and the gradient of temperature.

PROPOSITION 5.1. If a multipolar viscous fluid of type $(N, 1, K)$ satisfies the second law of thermodynamics and the principle of material frame-indifference, then

$$T_{ij}^{(0,E)} = -p\delta_{ij}, \tag{5.1}$$

where $p = p(\rho, \theta)$ is given by

$$p = \rho^2 \frac{\partial \psi}{\partial \rho}; \tag{5.2}$$

if k is odd, then

$$T^{(k,E)} = 0; \tag{5.3}$$

if k is even and $k > 0$, then

$$T^{(k,E)} = T^{(k,E)}(\theta) \tag{5.4}$$

and

$$\operatorname{Sym} T_{jj_1\cdots j_k i}^{(k,E)} = 0, \tag{5.5}$$

where Sym denotes the symmetrization with respect to the indices $j_1 \cdots j_k i$.

Proof. We first note that the transformation law (3.15) for the total multipolar stress implies analogous laws for the equilibrium and viscous parts of the stress. Hence

$$T_{j_1\cdots j_{k+1}}^{(k,E)}(\rho, \theta) = Q_{j_1 m_1} \cdots Q_{j_{k+2} m_{k+2}} T_{m_1\cdots m_{k+2}}^{(k,E)}(\rho, \theta) \tag{5.6}$$

for every orthogonal tensor Q. Consequently, $T^{(k,E)}$ is an isotropic tensor of order $k+2$. If k is odd, then the only isotropic tensor of order $k+2$ is 0 (it is enough to set $Q_{ij} = -\delta_{ij}$ in (5.6)). This proves (5.3). Formulas (5.1) and (5.2) are a combination of the generalized stress relation (4.8) for $k = 0$ with the fact that $T^{(1,E)} = 0$ by (5.3). If k is even and $k > 0$, then the generalized stress relation (4.8) with k replaced by $k - 1$ gives

$$\operatorname{Sym}\left(T_{jj_1\cdots j_{k-1} i}^{(k-1,E)} + \frac{\partial}{\partial \rho} T_{jj_1\cdots j_{k-1} ip}^{(k,E)} \rho_{,p}\right) = 0. \tag{5.7}$$

But since $k - 1$ is odd, the first term in (5.7) vanishes by (5.3); moreover, the symmetrization is irrelevant as $T_{jj_1\cdots j_{k-1} ip}^{(k,E)}$ is symmetric in $j_1 \cdots j_{k-1} i$. Thus

$$\frac{\partial}{\partial \rho} T_{jj_1\cdots j_{k-1} ip}^{(k,E)} \rho_{,p} = 0, \tag{5.8}$$

which implies

$$\frac{\partial}{\partial \rho} T_{jj_1\cdots j_{k-1} ip}^{(k,E)} = 0 \tag{5.9}$$

and hence also (5.4). Finally, combining the generalized stress relation (4.8) with the fact that $T^{(k+1,E)} = 0$ for k even, we obtain (5.5). The proof is complete.

The analysis of the viscous part of stresses will be given only for linear fluids. A multipolar viscous fluid of type (N, M, K) is said to be linear if $M = 1$ and if for every ρ, θ the quantities

$$T^{(k,V)}(\rho, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta) \tag{5.10}$$

and

$$q(\rho, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta) \tag{5.11}$$

depend linearly on $\nabla v, \dots, \nabla^K v, \nabla\theta$.

PROPOSITION 5.2. If a linear viscous fluid of type $(N, 1, K)$ satisfies the principle of material frame-indifference, then

(1) $T^{(k,V)}$ with k even depends on ρ, θ, and on the odd-order gradients of v;

$$T^{(k,V)} = T^{(k,V)}(\rho, \theta, \nabla v, \nabla^3 v, \dots, \nabla^L v); \tag{5.12}$$

(2) q and $T^{(k,V)}$ with k odd depend on ρ, θ, and on the even-order spatial gradients of v:

$$q = q(\rho, \theta, \nabla^2 v, \dots, \nabla^{L-1} v, \nabla\theta), \tag{5.13}$$

$$T^{(k,V)} = T^{(k,V)}(\rho, \theta, \nabla^2 v, \dots, \nabla^{L-1} v, \nabla\theta); \tag{5.14}$$

here L is a suitable odd number depending on K.

Proof. By linearity, the expressions for q and $T^{(k,V)}$ are sums of a number of terms depending linearly on the gradients of velocity of different orders and of a term depending linearly on $\nabla\theta$. The transformation laws for the stresses and the heat flux vector under changes of frame must be satisfied by each of these linear terms separately. Using these transformation laws with $Q_{ij} = -\delta_{ij}$ and counting the number of rotations in these transformation laws for each of the linear terms, one readily learns that only the terms with the orders of gradients of velocity indicated in assertions (1), (2) can be nonzero. The details are omitted.

THEOREM 5.1. We have the following expressions for $T^{(0,V)}$, $T^{(1,V)}$, and q in a linear viscous fluid satisfying the principle of material frame-indifference:

$$\begin{aligned} T_{ij}^{(0,V)} = {} & \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}) \\ & + \sum_{r=0}^{P}(\alpha^{(r)}\delta_{ij}\Delta^{r+1} v_{k,k} + \beta_1^{(r)}\Delta^{r+1} v_{i,j} \\ & + \beta_2^{(r)}\Delta^{r+1} v_{j,i} + \gamma^{(r)}\Delta^r v_{k,kij}), \end{aligned} \tag{5.15}$$

$$
\begin{aligned}
T_{ijk}^{(1,V)} = \sum_{r=0}^{P} (& c_1^{(r)} \delta_{ij} \Delta^{r+1} v_k + c_2^{(r)} \delta_{ij} \Delta^r v_{n,nk} \\
& + c_3^{(r)} \delta_{ik} \Delta^{r+1} v_j + c_4^{(r)} \delta_{ik} \Delta^r v_{m,mj} \\
& + c_5^{(r)} \delta_{jk} \Delta^{r+1} v_i + c_6^{(r)} \delta_{jk} \Delta^r v_{m,mi} \\
& + c_7^{(r)} \Delta^r v_{i,jk} + c_8^{(r)} \Delta^r v_{k,ij} \\
& + c_9^{(r)} \Delta^r v_{j,ki} + c_{10}^{(r)} \Delta^{r-1} v_{m,mijk}) \\
+ & c_{11} \delta_{ij} \theta_{,k} + c_{12} \delta_{ik} \theta_{,j} + c_{13} \delta_{jk} \theta_{,i},
\end{aligned}
\tag{5.16}
$$

$$
q_i = \sum_{r=0}^{P} (d_1^{(r)} \Delta^r v_{m,mi} + d_2^{(r)} \Delta^{r+1} v_i) - k\theta_{,i}. \tag{5.17}
$$

Here λ, μ, $\alpha^{(r)}$, $\beta_1^{(r)}$, $\beta_2^{(r)}$, $\gamma^{(r)}$, $c_1^{(r)}, \ldots, c_{13}^{(r)}$, $d_1^{(r)}$, $d_2^{(r)}$, and k are scalar functions of ρ, θ such that $c_{10}^{(r)}$ satisfies $c_{10}^{(0)} = 0$ and P is an appropriate integer determined by K. If the body satisfies the reduced equation of balance of angular momentum, then

$$
\beta_1^{(r)} + c_5^{(r)} + c_7^{(r)} = \beta_2^{(r)} + c_3^{(r)} + c_9^{(r)} \tag{5.18}
$$

for $r = 0, 1, \ldots, P$.

Similar but more complicated expressions can be obtained also for $T^{(k,V)}$ with $k \geq 2$.

Proof. In view of the transformation laws for $T^{(k)}$ and q under changes of frame, the coefficients in the expressions for q and $T^{(k)}$ in front of the gradients of velocity and the gradient of temperature must be isotropic tensors. Using the general forms of isotropic tensors (see Spencer [13]) and the symmetry of the gradients of v, one eventually arrives at (5.15)–(5.17). The details are omitted.

The following proposition shows that for linear viscous materials for which the scalar coefficients in the linear expressions for $T^{(k,V)}$ and q are independent of ρ, θ, the residual dissipation inequality (5.5) splits into two independent inequalities. For simplicity we make the assumption that

$$
T^{(k,E)} = 0 \tag{5.19}
$$

for $k \geq 1$, which, at least for k odd, is a consequence of the second law (cf. (5.3)–(5.5)).

PROPOSITION 5.3. Consider a linear viscous fluid such that $T^{(k,V)}$ and q are independent of ρ, θ and (5.19) holds identically. If the fluid satisfies the principle of material frame-indifference and the second law, then the following two inequalities hold:

$$\sum_{\substack{k=0\\k \text{ even}}}^{N-1} (T^{(k,V)}_{jj_1\cdots j_k i} + T^{(k+1,V)}_{jj_1\cdots j_k ip,p})v_{j,j_1\cdots j_k i} \geq 0, \tag{5.20}$$

$$\sum_{\substack{k=1\\k \text{ odd}}}^{N-1} (T^{(k,V)}_{jj_1\cdots j_k i} + T^{(k+1,V)}_{jj_1\cdots j_k ip,p})v_{j,j_1\cdots j_k i} - q_i\theta_{,i}/\theta \geq 0. \tag{5.21}$$

The expressions for q, $T^{(k,V)}$, and its divergences are linear in the gradients of velocity and in the gradients of temperature. When these linear expressions are inserted in (5.20), (5.21), these inequalities express nonnegativeness of quadratic forms. Sylvester's criterion can be used to find the inequalities which the coefficients of the linear expressions for $T^{(k,V)}$ and q must satisfy in order that (5.20), (5.21) be guaranteed. We do not pursue this possibility here.

Proof. Under the hypothesis of the proposition, the residual dissipation inequality (5.5) can be written as the sum of the left-hand sides of (5.20) and (5.21). Now the left-hand side of (5.20) depends on

$$\nabla v, \nabla^3 v, \dots, \nabla^{L+1} v, \nabla^2\theta, \tag{5.22}$$

while the left-hand side of (5.21) depends on

$$\nabla^2 v, \nabla^4 v, \dots, \nabla\theta \tag{5.23}$$

(cf. Proposition 5.2). As the sets (5.22), (5.23) are disjoint and the expressions are quadratic, the splitting of the residual inequality into (5.20), (5.21) follows. The proof is complete.

We are now going to show that for linear materials the Clausius-Duhem inequality restricts the dependence of the viscous stresses $T^{(k)}$ on the gradients of order larger than $2N-1-k$ very strongly. The effect of the restriction will be shown to be so severe that the terms with the gradients of velocity of order larger than $2N-1-k$ do not contribute to the equations of balance of energy and linear momentum. (See Propositions 5.4 and 5.5 below for precise statements.)

We introduce additional notation to state the results. We shall deal exclusively with linear fluids for which the viscous stresses and the heat flux do not depend on ρ, θ. We split the viscous stresses and the heat flux into the regular and singular parts, denoted $T^{(k,R)}$, $T^{(k,S)}$, $q^{(R)}$, $q^{(S)}$ as follows:

$$\begin{aligned} T^{(k,V)} &= T^{(k,R)}(\nabla v, \nabla^2 v, \dots, \nabla^{2N-1-k} v, \nabla\theta) \\ &\quad + T^{(k,S)}(\nabla^{2N-k} v, \nabla^{2N-k+1} v, \dots), \end{aligned} \tag{5.24}$$

$k = 0, 1, \dots, N-1$,

$$\begin{aligned} q &= q^{(R)}(\nabla v, \nabla^2 v, \dots, \nabla^N v, \nabla\theta) \\ &\quad + q^{(S)}(\nabla^{N+1} v, \nabla^{N+2} v, \dots), \end{aligned} \tag{5.25}$$

so that the lower the order of the stress tensor, the bigger the number of the gradients of velocity in the regular part $T^{(k,R)}$ of the stress. In particular, the highest order regular stress $T^{(N-1,R)}$ depends on the gradients of velocity up to the order N, while the lowest order regular stress $T^{(0,R)}$ depends on the gradients of velocity up to the order $2N-1$. The singular part of the stresses depends on the complementary set of variables. The singular parts of the stresses will eventually turn out to be negligible from the point of view of the balance equations. Also note that in view of Proposition 5.2 the viscous stresses and q depend either on only the even-order gradients or on only the odd-order gradients, but this is irrelevant here.

PROPOSITION 5.4. Consider a linear viscous fluid such that $T^{(k,V)}$ and q are independent of ρ, θ and (5.19) holds identically. If the fluid satisfies the second law of thermodynamics, then the following three assertions hold:

(1) The inequality

$$\sum_{k=0}^{N-1}(T^{(k,R)}_{jj_1\cdots j_k i} + T^{(k+1,R)}_{jj_1\cdots j_k ip,p})v_{j,j_1\cdots j_k i} - q^{(R)}_i\theta_{,i}/\theta \geq 0 \tag{5.26}$$

holds in every process.

(2) We have the following relations:

$$q^{(S)}_i = 0, \tag{5.27}$$

$$\mathrm{Sym}(T^{(N-1,S)}_{jj_1\cdots j_{N-1}i}) = 0, \tag{5.28}$$

$$\mathrm{Sym}(T^{(k,S)}_{jj_1\cdots j_k i} + T^{(k+1,S)}_{jj_1\cdots j_k ip,p}) = 0, \tag{5.29}$$

$k = 0, 1, \ldots, N-2$.

(3) If $T^{(k,V)}_{jj_1\cdots j_k i}$ is symmetric in $j_1\cdots j_k i$ for every k with $1 \leq k \leq N-1$, then

$$K \leq 2N-1. \tag{5.30}$$

Assertion (1) says that the regular parts of the stresses satisfy the residual dissipation inequality similar to that satisfied by the total viscous stresses. Equations (5.27) to (5.29) of assertion (2) say that the singular part of the heat flux vanishes and that also the singular stresses vanish modulo the symmetrization. The singular stresses vanish exactly if the stresses are symmetric in the indicated indices and hence the fluid satisfies inequality (5.30). Recall, on the other hand, that we also have inequality (4.23) of Proposition 4.1. We thus have a lower and an upper bound for N in terms of K for linear viscous fluids.

Proof. Consider first the residual dissipation inequality with the following choice of the arguments:

$$\nabla v = \nabla^2 v = \cdots = \nabla^N v = 0, \tag{5.31}$$

$$\nabla^{N+1}v, \nabla^{N+2}v, \ldots, \nabla\theta \quad \text{arbitrary}. \tag{5.32}$$

The inequality then reduces to

$$(q^{(R)}_i + q^{(S)}_i)\theta_{,i} \leq 0, \tag{5.33}$$

where $q^{(R)}$ is evaluated at the velocity gradients indicated in (5.32). The linearity of the dependence of $q^{(S)}$ on the arguments $\nabla^{N+1}v, \ldots$ then gives

$$q_i^{(S)}\theta_{,i} \le 0. \tag{5.34}$$

Since $\nabla\theta$ is arbitrary and $q^{(S)}$ is independent of it, equation (5.27) follows.

Next consider the residual dissipation inequality with the following choice of the arguments: let $k \in \{0, 1, \ldots, N-1\}$ be given and set

$$\nabla^m v = 0 \quad \text{if } 1 \le m \le N,\ m \ne k+1, \tag{5.35}$$

$$\nabla^m v \text{ arbitrary if } m > N \text{ or if } m = k+1, \tag{5.36}$$

$$\nabla\theta = 0. \tag{5.37}$$

The residual inequality then reduces to

$$(T^{(k,R)} + \operatorname{div} T^{(k+1,R)} + T^{(k,S)} + \operatorname{div} T^{(k+1,S)})\nabla^{k+1}v \ge 0, \tag{5.38}$$

where $T^{(k,R)}$ and $\operatorname{div} T^{(k+1,R)}$ are evaluated at the arguments

$$(0, 0, \ldots, \nabla^{k+1}v, 0, \ldots, \nabla^{N+1}v, \ldots, \nabla^{2N-1-k}v, \nabla\theta), \tag{5.39}$$

while $T^{(k,S)}$ and $\operatorname{div} T^{(k+1,S)}$ are evaluated at the arguments

$$(\nabla^{2N-k}v, \nabla^{2N-k+1}v, \ldots). \tag{5.40}$$

These can have arbitrary values and in view of (5.24) they do not enter the regular parts of the stresses and their divergences occurring in (5.38). The linearity of the dependences on the variables (5.40) together with $T^{(N,S)} = 0$ (cf. (2.9)) gives (5.28) and (5.29). This proves assertion (2).

Assertion (1) now follows from the residual dissipation inequality and the relations of assertion (2). Also assertion (3) is an obvious consequence of assertion (2). The proof is complete.

PROPOSITION 5.5. Consider a linear viscous fluid of type $(N, 1, K)$ such that $T^{(k,V)}$ and q are independent of ρ, θ and (5.19) holds identically. If $K > 2N-1$ and if the fluid satisfies the second law of thermodynamics, then there exists another linear viscous fluid satisfying the second law of thermodynamics such that it is of type $(N, 1, K')$, $K' \le 2N-1$, and the equations of balance of linear momentum and energy of the two fluids are identical. (This means that when the constitutive functions of the two fluids are inserted in the balance equations, then the resulting differential equations for the unknown functions $v(x, t)$ and $\theta(x, t)$ are the same for the two fluids.) The new fluid is defined as follows: its constitutive functions $\psi, \eta, e, q, T^{(k,E)}$ are identical with those of the original fluid, but its constitutive functions for the viscous stresses are the regular parts of the viscous stresses of the original fluid. Hence the singular parts of the stresses of the new fluid vanish.

If the boundary conditions of the fluid are formulated in terms of the stresses, they may be different for the two fluids.

Proof. It is clear from (5.27)–(5.29) that the new fluid has the power of the viscous stresses identical to that of the original fluid, i.e.,

$$\sum_{k=0}^{N-1}(T^{(k,R)} + \operatorname{div} T^{(k+1,R)}) \cdot \nabla^{k+1} v = \sum_{k=0}^{N-1}(T^{(k,V)} + \operatorname{div} T^{(k+1,V)}) \cdot \nabla^{k+1} v. \tag{5.41}$$

Since the other quantities of the two fluids are identical by the definition, it is then clear that the reduced equation of balance of energy and the dissipation inequalities of the two fluids are identical. Hence, in particular, the new fluid satisfies the second law of thermodynamics.

We shall now prove that the equations of balance of linear momentum of the two fluids coincide. These two equations can differ only in the term $T^{(0,S)}_{jp,p}$ and our task is to prove that

$$T^{(0,S)}_{jp,p} = 0. \tag{5.42}$$

But using (5.29) with $k = 0, 1, \dots, N-2$, this term can be rewritten in the following way:

$$\begin{aligned} T^{(0,S)}_{jj_1,j_1} &= -(T^{(1,S)}_{jj_1j_2,j_2})_{,j_1} = -(T^{(1,S)}_{jj_1j_2})_{,j_1j_2} \\ &= +(T^{(2,S)}_{jj_1j_2j_3,j_3})_{,j_1j_2} = +T^{(2,S)}_{jj_1j_2j_3,j_1j_2j_3} \\ &= \cdots = (-1)^{N-1} T^{(N-1,S)}_{jj_1\cdots j_N,j_1\cdots j_N}. \end{aligned} \tag{5.43}$$

We have omitted the symmetrization symbols in the above computation because of the interchangeability of the partial derivatives occurring above. To complete the proof of (5.42), it is enough to note that the last term in (5.43) vanishes in view of (5.28). Hence the equations of balance of linear momentum of the two fluids coincide. As we have already proved, the reduced equations of balance of energy also coincide. Since the full equation of balance of energy is a consequence of these two, we see that the full equations of balance of energy of the two fluids also coincide. The proof is complete.

6. Dipolar fluids. Dipolar materials provide the lowest order example $(N = 2)$ of a genuine multipolar material. We shall restrict here to linear dipolar fluids. It is assumed that the fluid satisfies the principle of material frame-indifference and the second law. In addition to the notation introduced in (5.24) and (5.25) concerning the regular and singular parts of the stresses, we also write

$$T^{(0,R)} = T^{(0,1)}(\nabla v) + T^{(0,3)}(\nabla^3 v), \tag{6.1}$$

$$T^{(1,R)} = T^{(1,2)}(\nabla^2 v) + T^{(1,\theta)}(\nabla\theta), \tag{6.2}$$

$$q^{(R)} = q^{(\theta)}(\nabla\theta) + q^{(2)}(\nabla^2 v), \tag{6.3}$$

to split the dependences of the regular parts of the viscous stresses on the gradients of velocity of different orders.

THEOREM 6.1. If a dipolar linear fluid of type $(2, 1, K)$ satisfies the principle of material frame-indifference and the second law of thermodynamics and if $T^{(0,V)}$

and $T^{(1,V)}$ are independent of ρ and θ, then

$$T_{ij}^{(0,1)} v_{i,j} \geq 0, \tag{6.4}$$

$$T_{ij}^{(0,3)} = -T_{ijp,p}^{(1,2)}, \tag{6.5}$$

$$T_{ijp,p}^{(1,\theta)} = 0, \tag{6.6}$$

$$(T_{ijk}^{(1,2)} + T_{ijk}^{(1,\theta)}) v_{i,jk} + (q_i^{(2)} + q_i^{(\theta)}) \theta_{,i}/\theta \geq 0, \tag{6.7}$$

$$q_i^{(S)} = 0, \tag{6.8}$$

$$T_{ijk}^{(1,S)} = -T_{ikj}^{(1,S)}, \tag{6.9}$$

$$T_{ij}^{(0,S)} = -T_{ijp,p}^{(1,S)}. \tag{6.10}$$

The proof of this proposition is similar to that of Proposition 5.4 and is omitted.

Propositions 5.4(3) and 5.5 show that for dipolar fluids the response depends most typically on the gradients of velocity up to order 3 (i.e., $K = 3$) and we conclude this section with explicit expressions for the viscous stresses and the heat flux in materials of type $(2, 1, 3)$.

PROPOSITION 6.1. If a linear viscous fluid of type $(2, 1, 3)$ satisfies the principle of material frame-indifference and the second law of thermodynamics, then

$$\begin{aligned} T_{ij}^{(0,V)} &= \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}) \\ &\quad + \alpha\Delta v_{k,k}\delta_{ij} + \beta_1\Delta_{i,j} + \beta_2\Delta v_{j,i} + \gamma v_{k,kij}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} T_{ijk}^{(1,V)} &= c_1\delta_{ij}\Delta v_k + c_2\delta_{ij}v_{m,mk} \\ &\quad + c_3\delta_{ik}\Delta v_j + c_4\delta_{ik}v_{m,mj} \\ &\quad + c_5\delta_{jk}\Delta v_i + c_6\delta_{jk}v_{m,mi} \\ &\quad + c_7 v_{i,jk} + c_8 v_{k,ij} + c_9 v_{j,ki} \\ &\quad + c_{10}\delta_{ij}\theta_{,k} + c_{11}\delta_{ik}\theta_{,j} + c_{12}\delta_{jk}\theta_{,i}, \end{aligned} \tag{6.12}$$

and

$$q_i = -k\theta_{,i} + d_1 v_{m,mi} + d_2\Delta v_i, \tag{6.13}$$

where $\lambda, \mu, \alpha, \beta_1, \beta_2, \gamma, c_1, \ldots, c_{12}, k, d_1, d_2$ are real-valued functions of ρ and θ. If these functions are independent of ρ, θ, then the following relations hold:

$$\alpha + c_1 + c_2 = 0, \tag{6.14}$$

$$\beta_1 + c_5 + c_7 = 0, \tag{6.15}$$

$$\beta_2 + c_3 + c_9 = 0, \tag{6.16}$$

$$\gamma + c_4 + c_6 + c_8 = 0, \tag{6.17}$$

$$c_{10} = c_{11} + c_{12} = 0. \tag{6.18}$$

If the body satisfies the reduced equation of balance of angular momentum, then

$$\beta_1 + c_5 + c_7 = \beta_2 + c_3 + c_9. \tag{6.19}$$

Proof. Formulas (6.11)–(6.13) and (6.19) are just specializations of the corresponding general formulas of Theorem 5.1. Equations (6.14)–(6.18) follow from Eqs. (6.5), (6.6) of Theorem 6.1. The details are omitted.

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