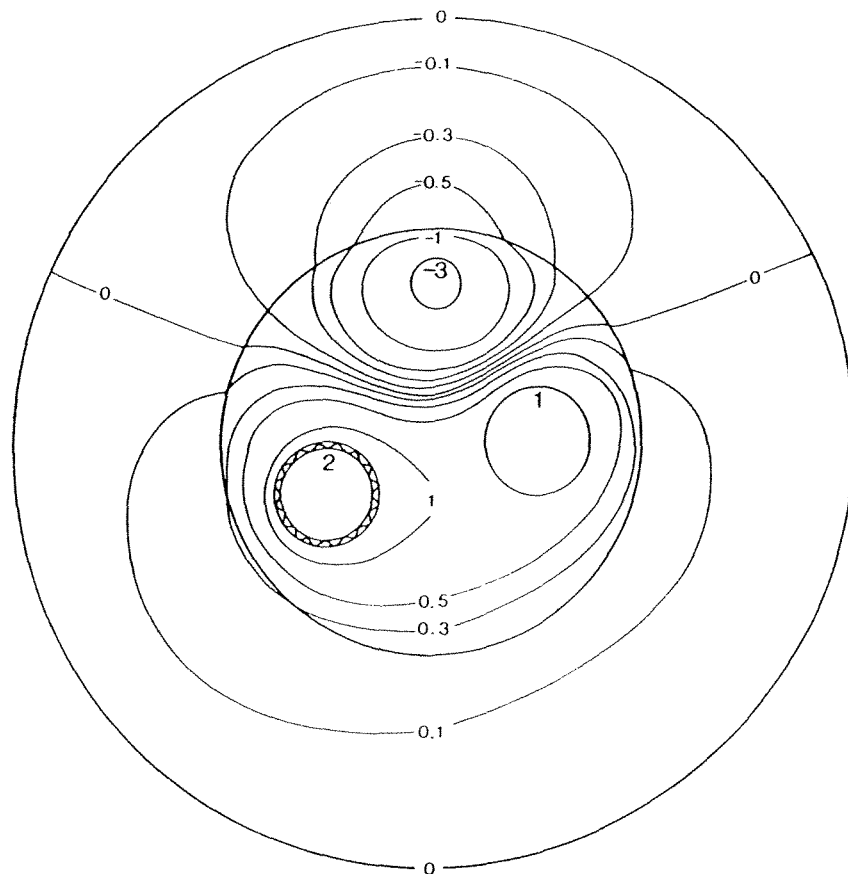


MULTIPOLE METHOD TO COMPUTE THE CONDUCTIVE HEAT FLOWS TO AND BETWEEN PIPES IN A COMPOSITE CYLINDER

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1. INTRODUCTION

A so-called multipole method to compute the conductive heat flows to and between pipes in a cylinder is presented in [1]. The steady-state heat conduction is two-dimensional in a circular region perpendicular to the pipes. The region is homogeneous. In this paper the method is extended to the case of a composite cylinder. The pipes lie in an inner circular region, which is surrounded by an annulus of different thermal conductivity.

The applications for which the method has been developed concern so-called heat extraction boreholes and certain types of heat stores in the ground. The boreholes are used for heat extraction or as heat exchangers for heat injection/extraction. The heat carrier fluid may for example flow in a U-shaped tube in the borehole. Outside the tubes the borehole may be filled with sand. The local thermal problem in and near the borehole is quite important for the heat transfer capacity of the heat exchanger. This problem is essentially a steady-state one in the region between the fluid in the pipes and a suitably chosen cylinder around the borehole. The pipes are imbedded in a composite cylinder with an internal boundary at the borehole wall. The thermal resistance between the pipes and the outer circle is of particular interest. We are also interested in the thermal resistances between the pipes. The presented method and the computer program give a very rapid and accurate way to obtain these resistances and the complete temperature field. The method is used extensively in [2].

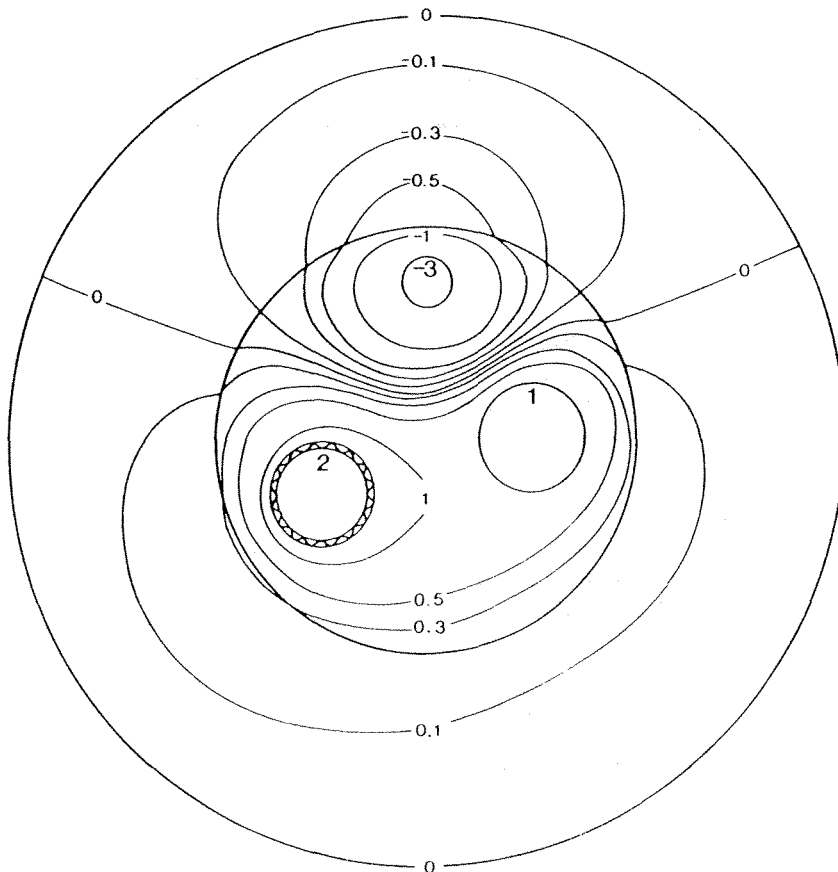


Figure 1.1. An example showing the temperature field for a case with 3 pipes. Data according to (8.12).

Figure 1.1 shows the computed temperature field for an example with three pipes. The fluid temperatures in the pipes and on the outer circle are indicated in the figure. One of the pipes has as indicated a thermal resistance layer. The complete set of data for the example is given by (8.12). The computer

time for this case is only a few seconds on a main frame computer (Norsk Data ND-500) or about 40 seconds on an IBM-PC AT-3 (10 MHz) with a 80287 math coprocessor.

2. THERMAL PROBLEM

Figure 2.1 shows the considered thermal problem. There are N pipes ($N \geq 1$), which lie within the inner circular region with the radius r_b (b = borehole). The inner circle is surrounded by an annular region of another material. The outer circle has the radius r_c .

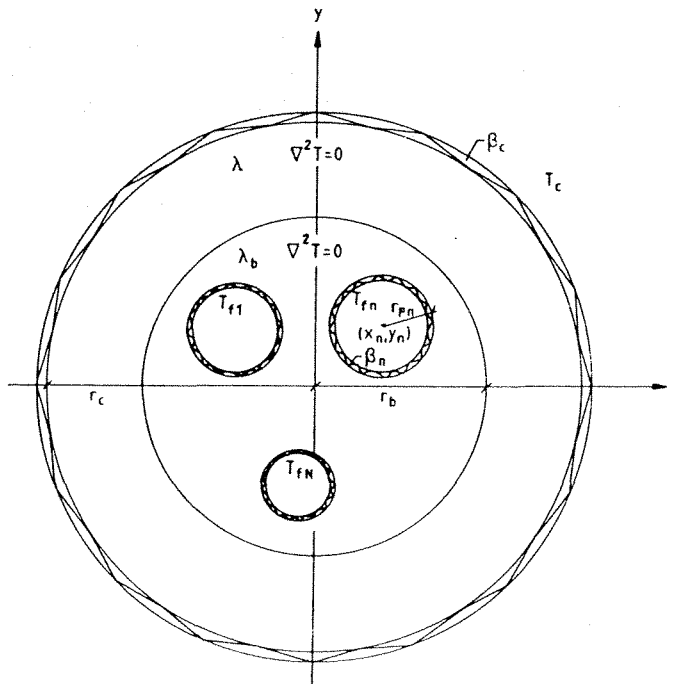


Figure 2.1 Steady-state heat conduction in a composite circular region with heat flows between the N pipes and the outer circle.

The outer radius of pipe n is r_{pn} , and its center lies at (x_n, y_n) . The fluid in pipe n has the constant temperature T_{fn} , while the temperature outside the outer circle is T_c .

The annular region, $r_b < r < r_c$, is homogeneous with the thermal conductivity λ . The inner circular region outside the pipes has the thermal conductivity λ_b . The steady-state temperature $T(x, y)$ satisfies the heat conduction equation in the annular region and in the inner circle:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (2.1)$$

Cartesian, complex and polar coordinates will be used:

$$z = x + iy = r e^{i\phi} \quad (2.2)$$

The center of pipe n is in complex coordinates:

$$z_n = x_n + iy_n \quad (2.3)$$

We will use the local polar coordinates ρ_n, ψ_n from the center of any pipe n . See Figure 2.2.

$$z - z_n = \rho_n e^{i\psi_n} \quad (2.4)$$

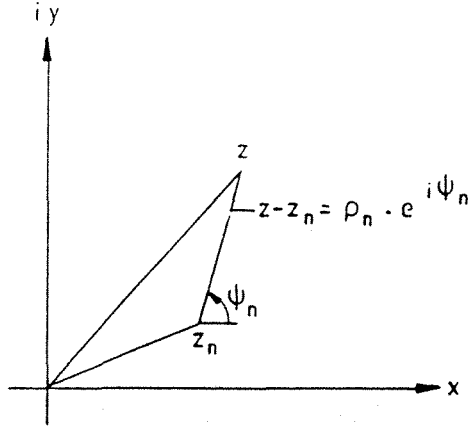


Figure 2.2. Local polar coordinates ρ_n, ψ_n from the center of pipe n .

The temperature and the radial heat flux are continuous at the inner boundary $r = r_b$:

$$T|_{r_b-0} = T|_{r_b+0}$$

$$\lambda_b \frac{\partial T}{\partial r} \Big|_{r_b-0} = \lambda \frac{\partial T}{\partial r} \Big|_{r_b+0} \quad (2.5)$$

The boundary conditions at the pipes and the outer circle $r = r_c$ are the same as in [1]. The boundary condition at pipe n is:

$$T - \beta_n r_{pn} \frac{\partial T}{\partial \rho_n} = T_{fn} \quad \begin{array}{l} \rho_n = r_{pn} \\ 0 \leq \psi_n \leq 2\pi \end{array} \quad (2.6)$$

The boundary condition at the outer circle is:

$$T + \beta_c r_c \frac{\partial T}{\partial r} = T_c \quad \begin{array}{l} r = r_c \\ 0 \leq \phi \leq 2\pi \end{array} \quad (2.7)$$

The dimensionless coefficient β_n determines the thermal resistance between the fluid in pipe n and the material just outside the pipe. This resistance is $\beta_n r_{pn} / \lambda_b$ ($\text{K}/(\text{W}/\text{m}^2)$). The corresponding thermal resistance R_{pn} ($\text{K}/(\text{W}/\text{m})$) per unit pipe length is obtained by division with the perimeter $2\pi r_{pn}$:

$$R_{pn} = \frac{\beta_n}{2\pi \lambda_b} \quad (\text{K}/(\text{W}/\text{m})) \quad (2.8)$$

The thermal resistance per unit length (in the axial direction) of the outer circle is:

$$R_{pc} = \frac{\beta_c}{2\pi \lambda} \quad (\text{K}/(\text{W}/\text{m})) \quad (2.9)$$

The thermal resistance per unit area at the outer circle is $\beta_c r_c / \lambda = 2\pi r_c R_{pc}$ ($\text{K}/(\text{W}/\text{m}^2)$). The thermal resistance coefficients β_n and β_c may take any non-negative value: $0 \leq \beta_n \leq \infty, 0 \leq \beta_c \leq \infty$. The value $\beta_n = +\infty$ means zero heat flux, $\frac{\partial T}{\partial \rho_n} = 0$. The value of T_{fn} is then redundant.

3. LINE SOURCE FOR THE COMPOSITE REGION

The solution in the previous paper [1] for pipes in a homogeneous region is based on the line source solution in complex form. Suitable derivatives gave the multipoles, which were needed for the solution. We will here in the same way use the line source for a composite region.

The thermal problem for the basic line source solution is shown in Figure 3.1. The thermal conductivity is λ_b in the circle $0 \leq r < r_b$, and λ in the infinite region outside the circle, $r_b \leq r < \infty$. There is a line source with the strength $+q_n$ (W/m) at the point (x_n, y_n) , which lies within the circle ($r_n < r_b$).

The solution to this problem is given in [2].

$0 \leq r \leq r_b$:

$$T(x, y) = \frac{q_n}{2\pi\lambda_b} \left\{ \ln \left(\frac{r_b}{\sqrt{(x-x_n)^2 + (y-y_n)^2}} \right) + \sigma \cdot \ln \left(\frac{r_b^2/r_n}{\sqrt{(x-x_n r_b^2/r_n^2)^2 + (y-y_n r_b^2/r_n^2)^2}} \right) \right\} \quad (3.1)$$

$r_b \leq r < \infty$:

$$T(x, y) = \frac{q_n}{2\pi\lambda} \left\{ (1-\sigma) \ln \left(\frac{r_b}{\sqrt{(x-x_n)^2 + (y-y_n)^2}} \right) + \sigma \cdot \ln \left(\frac{r_b}{\sqrt{x^2 + y^2}} \right) \right\} \quad (3.2)$$

We have introduced the notation:

$$\sigma = \frac{\lambda_b - \lambda}{\lambda_b + \lambda} \quad (-1 < \sigma < 1) \quad (3.3)$$

The case $\sigma = 0$, i.e. $\lambda_b = \lambda$, is the one studied in [1].

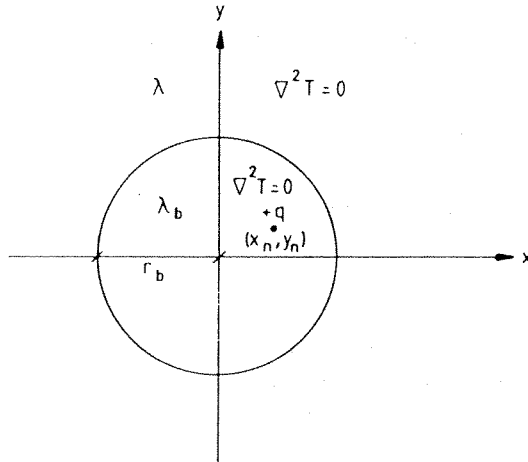


Figure 3.1. Fundamental line source problem for the composite region.

The first term in (3.1) represents a line sink $+q_n$ at (x_n, y_n) in a material with the thermal conductivity λ_b . The second term is due to the fact that the thermal conductivity is λ for $r > r_b$. This term represents a line sink with the strength σq_n situated at the mirror point $(x_n r_b^2/r_n^2, y_n r_b^2/r_n^2)$. The mirror point lies on the same radius as (x_n, y_n) . The product of the distances to the center is $r_n \cdot r_b^2/r_n = r_b^2$. The temperature field (3.2) in the outer region $r \geq r_b$ consists of a line sink with the strength $q_n(1-\sigma)$ at (x_n, y_n) and another one with the remaining heat $q_n \cdot \sigma$ at the $(0, 0)$. It shall be observed that the thermal conductivity is λ in (3.2) and λ_b in (3.1).

The solution (3.1-2) is rewritten in the complex form using the following expressions:

$$\ln \left(\frac{r_b}{\sqrt{(x-x_n)^2 + (y-y_n)^2}} \right) = \text{Re} \left[\ln \left(\frac{r_b}{z-z_n} \right) \right] \quad (3.4)$$

$$\ln \left(\frac{r_b}{\sqrt{x^2 + y^2}} \right) = \text{Re} \left[\ln \left(\frac{r_b}{z} \right) \right] \quad (3.5)$$

$$\begin{aligned}
& \ln \left(\frac{r_b^2/r_n}{\sqrt{(x - x_n r_b^2/r_n^2)^2 + (y - y_n r_b^2/r_n^2)^2}} \right) \\
&= \ln \left(\frac{r_b^2/|z_n|}{|z - z_n r_b^2/r_n^2|} \right) = \ln \left(\frac{r_b^2}{|\bar{z}_n| \cdot |z - z_n r_b^2/(z_n \bar{z}_n)|} \right) \\
&= \ln \left(\frac{r_b^2}{|r_b^2 - \bar{z}_n z|} \right) = \ln \left(\frac{r_b^2}{|r_b^2 - \bar{z} z_n|} \right) = \operatorname{Re} \left[\ln \left(\frac{r_b^2}{r_b^2 - \bar{z} z_n} \right) \right]
\end{aligned} \tag{3.6}$$

In the last two lines we used that $r_n^2 = |z_n|^2 = z_n \bar{z}_n$ and $|w| = |\bar{w}|$ for any complex number w and its conjugate \bar{w} .

The temperature (3.1-2) may now be written in the following form:

$$T(x, y) = \frac{q_n}{2\pi\lambda_b} \cdot \operatorname{Re}[W_{no}] \tag{3.7}$$

The complex-valued function W_{no} is from (3.1-7) defined by:

$$\begin{cases} W_{no} = \ln \left(\frac{r_b}{z - z_n} \right) + \sigma \cdot \ln \left(\frac{r_b^2}{r_b^2 - \bar{z} z_n} \right) & r \leq r_b \\ W_{no} = (1 + \sigma) \ln \left(\frac{r_b}{z - z_n} \right) + \frac{\lambda_b}{\lambda} \sigma \cdot \ln \left(\frac{r_b}{z} \right) & r \geq r_b \end{cases} \tag{3.8-9}$$

The factor $1 + \sigma$ in (3.9) is in accordance with (3.3) equal to $\lambda_b(1 - \sigma)/\lambda$. The simplicity of the expressions (3.7-9) is noteworthy.

Let us verify that (3.7-9) is indeed the solution to the fundamental line source problem for the composite region shown in Figure 3.1. The expression (3.9) is a regular (analytic) function in the region $r > r_b$, since the sources lie within the circle. Its real part $\operatorname{Re}[W_{no}]$ satisfies the heat conduction equation $\nabla^2 T = 0$. The expression (3.8) is also a regular function in $r < r_b$ except at the point $z = z_n$. It remains to verify that the boundary conditions (2.5) at $r = r_b$ are satisfied. We have at the circle $z = r_b \cdot e^{i\phi}$:

$$\begin{aligned}
W_{no}|_{r_b-0} &= \ln \left(\frac{r_b}{r_b \cdot e^{i\phi} - z_n} \right) + \sigma \cdot \ln \left(\frac{r_b^2}{r_b^2 - r_b e^{-i\phi} \cdot z_n} \right) \\
&= \ln \left(\frac{r_b}{r_b \cdot e^{i\phi} - z_n} \right) + \sigma \cdot \ln \left(\frac{r_b \cdot e^{i\phi}}{r_b e^{i\phi} - z_n} \right)
\end{aligned} \tag{3.10}$$

$$W_{no}|_{r_b+0} = (1 + \sigma) \ln \left(\frac{r_b}{r_b \cdot e^{i\phi} - z_n} \right) + \frac{\lambda_b}{\lambda} \sigma \cdot \ln(e^{-i\phi}) \tag{3.11}$$

The difference is

$$W_{no}|_{r_b-0} - W_{no}|_{r_b+0} = \sigma \cdot i\phi - \frac{\lambda_b}{\lambda} \sigma \cdot (-i\phi)$$

,or

$$W_{no}|_{r_b-0} - W_{no}|_{r_b+0} = \frac{\lambda_b - \lambda}{\lambda} \cdot i\phi \tag{3.12}$$

This difference contains only an imaginary part. The temperature, which is given by the real part, is therefore continuous at $r = r_b$.

For the radial heat flux at $r = r_b$ we have:

$$\begin{aligned}
\lambda_b \frac{\partial W_{no}}{\partial r} \Big|_{r_b-0} &= \lambda_b \left\{ \frac{-e^{i\phi}}{r_b e^{i\phi} - z_n} + \sigma \cdot \frac{(-)(-z_n)e^{-i\phi}}{r_b^2 - r_b e^{-i\phi} z_n} \right\} \\
&= -\frac{\lambda_b}{r_b} \cdot \frac{r_b e^{i\phi} - \sigma z_n}{r_b e^{i\phi} - z_n}
\end{aligned} \tag{3.13}$$

$$\begin{aligned}\lambda \frac{\partial W_{no}}{\partial r} \Big|_{r_b+0} &= \lambda \left\{ (1+\sigma) \cdot \frac{-e^{i\phi}}{r_b e^{i\phi} - z_n} + \frac{\lambda_b}{\lambda} \sigma \cdot \frac{-e^{i\phi}}{r_b e^{i\phi}} \right\} \\ &= \dots = -\frac{\lambda_b}{r_b} \cdot \frac{r_b e^{i\phi} - \sigma z_n}{r_b e^{i\phi} - z_n}\end{aligned}\quad (3.14)$$

The heat flux, i.e. the real parts of (3.13-14), is therefore continuous at $r = r_b$. This is in fact true also for the imaginary part of W_{no} :

$$\lambda_b \frac{\partial W_{no}}{\partial r} \Big|_{r_b-0} = \lambda \frac{\partial W_{no}}{\partial r} \Big|_{r_b+0} \quad (3.15)$$

4. MULTIPOLES FOR THE COMPOSITE REGION

The multipoles of order j were in complex form given by $(z - z_n)^{-j}$ in the case of a single region, [1]. They were obtained by derivatives of the complex logarithm $\ln(z - z_n)$. Let W_{n1} be the derivative of W_{no} , (3.8-9), with respect to the complex variable z_n :

$$W_{n1} = \frac{\partial}{\partial z_n} (W_{no}) \quad (4.1)$$

We get from (3.8) and (3.9):

$$W_{n1} = \frac{1}{z - z_n} + \sigma \cdot \frac{\bar{z}}{r_b^2 - \bar{z}z_n} \quad r \leq r_b \quad (4.2)$$

$$W_{n1} = (1 + \sigma) \cdot \frac{1}{z - z_n} \quad r \geq r_b \quad (4.3)$$

The function W_{n1} is regular. Thus, the real and imaginary parts, $Re(W_{n1})$ and $Im(W_{n1})$, both satisfy $\nabla^2 T = 0$, except at the point $z = z_n$. At the boundary $r = r_b$ the function W_{no} satisfies (3.12) and (3.15). The identities are valid for any z_n , and hence for the derivative with respect to z_n . But the derivative of $\phi (= \arctan(y/x))$ in (3.12) with respect to z_n is zero. Therefore we have:

$$W_{n1}|_{r_b-0} = W_{n1}|_{r_b+0} \quad (4.4)$$

$$\lambda_b \frac{\partial W_{n1}}{\partial r} \Big|_{r_b-0} = \lambda \frac{\partial W_{n1}}{\partial r} \Big|_{r_b+0} \quad (4.5)$$

This means that W_{n1} and the radial heat flux are continuous at $r = r_b$. This is true both for $Re(W_{n1})$ and $Im(W_{n1})$.

The multipole of any order j is given by the j :th derivative of W_{no} with respect to z_n . We define W_{nj} by:

$$W_{nj} = \frac{1}{(j-1)!} \cdot \frac{\partial^j}{\partial z_n^j} (W_{no}) \quad (4.6)$$

From (3.8) and (3.9) we get the neat expressions:

$$\begin{cases} W_{nj} = \frac{1}{(z - z_n)^j} + \sigma \cdot \left(\frac{\bar{z}}{r_b^2 - \bar{z}z_n} \right)^j & r \leq r_b \\ W_{nj} = (1 + \sigma) \frac{1}{(z - z_n)^j} & r \geq r_b \end{cases} \quad (4.7-8)$$

The expressions (4.7-8) are regular functions, which implies that the real and imaginary parts satisfy $\nabla^2 T = 0$, except at $z = z_n$. The boundary conditions (4.4) and (4.5) may be derived with respect to z_n any number of times. This means that W_{nj} and the radial heat flux are continuous at $r = r_b$ for any j :

$$W_{nj}|_{r_b-0} = W_{nj}|_{r_b+0} \quad (4.9)$$

$$\lambda_b \frac{\partial W_{nj}}{\partial r} \Big|_{r_b-0} = \lambda \frac{\partial W_{nj}}{\partial r} \Big|_{r_b+0} \quad (4.10)$$

The multipole (4.7) expressed in local polar coordinates around $z = z_n$ becomes:

$$W_{nj} = \frac{1}{(\rho_n e^{i\psi_n})^j} + \sigma \cdot \left(\frac{\bar{z}_n + \rho_n e^{-i\psi_n}}{r_b^2 - r_n^2 - \rho_n e^{-i\psi_n} \cdot z_n} \right)^j \quad (4.11)$$

,or

$$W_{nj} = (\cos(j\psi_n) - i \cdot \sin(j\psi_n)) \cdot \rho_n^{-j} + \sigma \cdot \left(\frac{\bar{z}_n + \rho_n e^{-i\psi_n}}{r_b^2 - r_n^2 - \rho_n e^{-i\psi_n} \cdot z_n} \right)^j \quad (\rho_n < r_b - r_n) \quad (4.12)$$

The first part represents a pure multipole behaviour, while the second part is a correction to account for the effect of the different thermal conductivity λ for $r > r_b$. The real part of W_{nj} , $Re(W_{nj})$, gives a variation $\cos(j\psi_n)$ around the point $z = z_n$, and the imaginary part, $Im(W_{nj})$, gives a variation $\sin(j\psi_n)$. The temperature field from the general multipole of order j at $z = z_n$ is:

$$T = Re(P_{nj} \cdot W_{nj}) = Re((c_{nj} + i \cdot s_{nj})W_{nj}) = c_{nj} \cdot Re(W_{nj}) - s_{nj} \cdot Im(W_{nj}) \quad (4.13)$$

Here P_{nj} is an arbitrary complex number.

The line sink and multipoles at z_n can be used to represent an arbitrary temperature solution outside pipe n . We need a corresponding representation for the outer boundary circle $r = r_c$. Here we need a solution in the composite region inside $r = r_c$. The solution in complex form shall vary as $e^{i \cdot j \phi}$ on $r = r_c$. We have the corresponding solutions to $\nabla^2 T = 0$ in polar coordinates in $r_b < r < r_c$:

$$r^j \cdot e^{i \cdot j \phi} = z^j \quad r^{-j} e^{i \cdot j \phi} = \frac{1}{\bar{z}^j} \quad (4.14)$$

In the inner region $0 \leq r < r_b$ we can use only the first solution z^j , since the second one is infinite at $r = 0$ or $z = 0$.

Therefore we start with the following expressions:

$$\begin{aligned} W_{cj} &= A \cdot z^j & 0 \leq r \leq r_b \\ W_{cj} &= z^j + B/\bar{z}^j & r_b \leq r \leq r_c \end{aligned} \quad (4.15)$$

Continuity at $z = r_b \cdot e^{i\phi}$ requires:

$$Ar_b^j = r_b^j + B/r_b^j \quad (4.16)$$

The radial heat flux is also continuous at $z = r_b e^{i\phi}$:

$$\begin{aligned} \lambda_b \cdot \frac{\partial W_{cj}}{\partial r} \Big|_{r_b-0} &= \lambda_b A \cdot j r_b^{j-1} \cdot e^{i \cdot j \phi} = \lambda \frac{\partial W_{cj}}{\partial r} \Big|_{r_b+0} \\ &= \lambda \cdot j \cdot e^{i \cdot j \phi} \left(r_b^{j-1} - B/r_b^{j+1} \right) \end{aligned} \quad (4.17)$$

The constants A and B are determined by (4.16-17). This gives the following basic expressions:

$$\begin{cases} W_{cj} = (1 - \sigma) z^j & r \leq r_b \\ W_{cj} = z^j - \sigma \cdot (r_b^2/\bar{z})^j & r \geq r_b \end{cases} \quad (4.18)$$

We will call these solutions W_{cj} multipoles at infinity. The index c refers to the fact that they are needed at the outer boundary $r = r_c$. The multipole of order j can represent the variation $e^{i \cdot j \phi}$ or in real form any combination of $\cos(j\phi)$ and $\sin(j\phi)$ at $r = r_c$.

5. GENERAL EXPRESSION FOR THE TEMPERATURE

The general expression for the temperature field uses all the line sinks and the multipoles of all orders at the pipes and at infinity. We have the following general expression:

$$T = T_o + Re \left[\sum_{n=1}^N P_n \cdot W_{no} + \sum_{n=1}^N \sum_j P_{nj} \cdot r_{pn}^j \cdot W_{nj} + \sum_j P_{cj} r_c^{-j} W_{cj} \right] \quad (5.1)$$

Here W_{no} is given by (3.8-9), W_{nj} by (4.7-8) and W_{cj} by (4.18). The factors r_{pn}^j and r_c^{-j} are introduced for dimensional reasons. The dimension of P_n , P_{nj} , and P_{cj} is that of a temperature.

The strength of the line source q_n is in accordance with (3.7) related to P_n by:

$$P_n = \frac{q_n}{2\pi\lambda_b} \quad (5.2)$$

The multipole factors P_{nj} and P_{cj} are complex numbers, while P_n of course is real. The temperature T_o is an arbitrary constant. The summation in j runs from 1 to infinity for an exact solution, and from 1 to J in the truncated approximate solution. The multipoles are not used in the case $J = 0$.

The general temperature (5.1) becomes with the explicit formulas (3.8-9), (4.7-8) and (4.18):

$r \leq r_b$:

$$T = T_o + Re \left[\sum_{n=1}^N P_n \left\{ \ln \left(\frac{r_b}{z - z_n} \right) + \sigma \cdot \ln \left(\frac{r_b^2}{r_b^2 - \bar{z}z_n} \right) \right\} + \sum_{n=1}^N \sum_{j=1}^J P_{nj} \left\{ \left(\frac{r_{pn}}{z - z_n} \right)^j + \sigma \left(\frac{r_{pn}\bar{z}}{r_b^2 - z_n\bar{z}} \right)^j \right\} + \sum_{j=1}^J P_{cj} (1 - \sigma) \left(\frac{z}{r_c} \right)^j \right] \quad (5.3)$$

$r_b \leq r \leq r_c$:

$$T = T_o + Re \left[\sum_{n=1}^N P_n \left\{ (1 + \sigma) \cdot \ln \left(\frac{r_b}{z - z_n} \right) + \frac{\lambda_b}{\lambda} \sigma \cdot \ln \left(\frac{r_b}{z} \right) \right\} + \sum_{n=1}^N \sum_{j=1}^J P_{nj} (1 + \sigma) \left(\frac{r_{pn}}{z - z_n} \right)^j + \sum_{j=1}^J P_{cj} \left\{ \left(\frac{z}{r_c} \right)^j - \sigma \cdot \left(\frac{r_b^2}{r_c \bar{z}} \right)^j \right\} \right] \quad (5.4)$$

The general expressions (5.1) or (5.3-4) satisfy the heat conduction equation (2.1) everywhere except at the points $z = z_n$ for any choice of the real constants T_o , P_n and of the complex ones, P_{nj} and P_{cj} . Each term satisfies the boundary conditions (2.5) at $r = r_b$. The constants T_o and P_{cj} are chosen so that the boundary condition (2.7) at the outer circle is fulfilled. This is done in the next chapter, while Chapter 7 deals with the boundary conditions at the pipes. We will get a set of equations for T_o , P_n , P_{nj} and P_{cj} .

The method of solution presented in this paper and in [1] is essentially to use Fourier series expansions; one for each pipe and one for the outer circle. The problem is to transform the coordinates of the different

components into polar coordinates for each pipe and for the outer circle. A basic idea in the present method is to use the complex form, which greatly facilitates the derivation of the equation system for the boundary conditions.

6. EQUATIONS FROM THE BOUNDARY CONDITION AT THE OUTER CIRCLE

The expressions (5.3) and (5.4) shall satisfy the boundary conditions (2.6-7). In the previous study [1], each type of term was analysed separately (in Chapter 5 of [1]). We will here use a slightly different approach. The temperature (5.3) is first expressed directly in the local polar coordinates of the considered pipe. Then the expression is inserted in the boundary condition (2.6), and we get an equation system that determines the line sinks P_n and the multipoles P_{nj} . This is done in the next chapter.

In this chapter we consider the boundary condition (2.7) at the outer circle $r = r_c$. Our first goal is to express the temperature (5.4) for the outer region $r_b \leq r \leq r_c$ in polar coordinates: $T = T(r, \phi)$. This is achieved by putting $z = re^{i\phi}$ in (5.4). We will need to distinguish between different powers $(e^{i\phi})^k$ or z^k in the subsequent analysis. The functions of z in (5.4) are therefore expanded in Taylor series.

For the logarithm of (5.4) we have:

$$\ln \left(\frac{r_b}{z - z_n} \right) = \ln \left(\frac{r_b}{z} \right) + \ln \left(\frac{1}{1 - z_n/z} \right) \quad (6.1)$$

We need the Taylor series:

$$\ln \left(\frac{1}{1 - s} \right) = \sum_{k=1}^{\infty} \frac{1}{k} s^k \quad |s| < 1 \quad (6.2)$$

Then we have:

$$\ln \left(\frac{r_b}{z - z_n} \right) = \ln \left(\frac{r_b}{z} \right) + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z_n}{z} \right)^k \quad |z_n| < |z| \quad (6.3)$$

We also need a Taylor expansion of the multipole term $(r_{pn}/(z - z_n))^j$ in (5.4). We have:

$$\left(\frac{r_{pn}}{z - z_n} \right)^j = \left(\frac{r_{pn}}{z} \right)^j \cdot \sum_{k'=0}^{\infty} \binom{j + k' - 1}{j - 1} \left(\frac{z_n}{z} \right)^{k'} \quad |z_n| < |z| \quad (6.4)$$

Here we have used a Taylor series given in [1]:

$$\frac{1}{(1 - s)^j} = \sum_{k=0}^{\infty} \binom{j + k - 1}{j - 1} s^k \quad |s| < 1 \quad (6.5)$$

The multipoles (6.4) are summed over j in (5.4). We need to rearrange the double sum in the following way:

$$\begin{aligned} \sum_{j=1}^{\infty} P_{nj} \left(\frac{r_{pn}}{z - z_n} \right)^j &= \sum_{j=1}^{\infty} \sum_{k'=0}^{\infty} P_{nj} \left(\frac{r_{pn}}{z} \right)^j \binom{j + k' - 1}{j - 1} \left(\frac{z_n}{z} \right)^{k'} \\ &= [j + k' = k] = \sum_{k=1}^{\infty} \sum_{j=1}^k P_{nj} \binom{k - 1}{j - 1} \frac{r_{pn}^j z_n^{k-j}}{z^k} \end{aligned} \quad (6.6)$$

The change of summation from j, k' to k, j means that j varies between 1 and k for $k = 1, 2, \dots$

We insert (6.3) and (6.6) in (5.4) and put $z = re^{i\phi}$. Consider first the terms containing $\ln(z)$. For these we have from (5.4) and (6.3):

$$\begin{aligned}
& \operatorname{Re} \left[\sum_{n=1}^N P_n \left\{ (1 + \sigma) \cdot \ln \left(\frac{r_b}{r e^{i\phi}} \right) + \frac{\lambda_b}{\lambda} \sigma \cdot \ln \left(\frac{r_b}{r e^{i\phi}} \right) \right\} \right] \\
&= \sum_{n=1}^N P_n \left(1 + \sigma + \frac{\lambda_b}{\lambda} \sigma \right) \cdot \ln \left(\frac{r_b}{r} \right) = \sum_{n=1}^N P_n \frac{\lambda_b}{\lambda} \cdot \ln \left(\frac{r_b}{r} \right)
\end{aligned} \tag{6.7}$$

In the last line we used the definition (3.3) for σ . The temperature as a function of r and ϕ is now using (5.4), (6.3), (6.6) and (6.7) with z equal to $r e^{i\phi}$:

$$\begin{aligned}
T(r, \phi) &= T_o + \sum_{n=1}^N P_n \frac{\lambda_b}{\lambda} \cdot \ln \left(\frac{r_b}{r} \right) + \operatorname{Re} \left[\sum_{n=1}^N \sum_{k=1}^{\infty} P_n (1 + \sigma) \frac{1}{k} \left(\frac{z_n}{r} \right)^k e^{-i \cdot k \phi} \right. \\
&+ \sum_{n=1}^N \sum_{k=1}^{\infty} \sum_{j=1}^k P_{nj} (1 + \sigma) \binom{k-1}{j-1} \frac{r_{pn}^j z_n^{k-j}}{r^k} \cdot e^{-i \cdot k \phi} \\
&\left. + \sum_{k=1}^{\infty} P_{ck} \left\{ \left(\frac{r}{r_c} \right)^k - \sigma \cdot \left(\frac{r_b^2}{r_c r} \right)^k \right\} e^{i \cdot k \phi} \right] \quad (r_b \leq r \leq r_c)
\end{aligned} \tag{6.8}$$

The summation index in the last line changed from j to k .

The temperature (6.8) shall satisfy the boundary condition (2.7). For the radial derivative of any power of r we have:

$$\begin{aligned}
\beta_c r_c \frac{\partial}{\partial r} (r^k) \Big|_{r=r_c} &= \beta_c r_c \cdot k \cdot r_c^{k-1} = k \beta_c \cdot r_c^k \\
\beta_c r_c \frac{\partial}{\partial r} (r^{-k}) \Big|_{r=r_c} &= -k \beta_c \cdot r_c^{-k}
\end{aligned} \tag{6.9}$$

We will separate different k , i.e. different orders in $e^{-i \cdot k \phi}$. The last line of (6.8) has a positive exponent, which is changed by complex conjugation:

$$\operatorname{Re} [P_{ck} \cdot e^{i \cdot k \phi}] = \operatorname{Re} [\bar{P}_{ck} \cdot e^{-i \cdot k \phi}] \tag{6.10}$$

The boundary condition (2.7) may now be written:

$$\begin{aligned}
T_c &= \left(T + \beta_c \frac{\partial T}{\partial r} \right) \Big|_{r=r_c} = T_o + \sum_{n=1}^N P_n \frac{\lambda_b}{\lambda} \left\{ \ln \left(\frac{r_b}{r_c} \right) + \beta_c r_c \cdot \frac{-1}{r_c} \right\} \\
&+ \sum_{k=1}^{\infty} \operatorname{Re} \left[e^{-i \cdot k \phi} \cdot \left\{ \sum_{n=1}^N P_n (1 + \sigma) \frac{1}{k} (1 - k \beta_c) \left(\frac{z_n}{r_c} \right)^k \right. \right. \\
&+ \sum_{n=1}^N \sum_{j=1}^k P_{nj} (1 + \sigma) \binom{k-1}{j-1} (1 - k \beta_c) \frac{r_{pn}^j z_n^{k-j}}{r_c^k} \\
&\left. \left. + \bar{P}_{ck} \left((1 + k \beta_c) \cdot \left(\frac{r_c}{r_c} \right)^k - \sigma (1 - k \beta_c) \left(\frac{r_b^2}{r_c \cdot r_c} \right)^k \right) \right\} \right] \quad 0 \leq \phi \leq 2\pi
\end{aligned} \tag{6.11}$$

Equation (6.11) is valid for $0 \leq \phi \leq 2\pi$. The constant part (independent of ϕ) must vanish. This determines T_o :

$$T_o = T_c + \sum_{n=1}^N P_n \frac{\lambda_b}{\lambda} \left(\ln \left(\frac{r_c}{r_b} \right) + \beta_c \right) \tag{6.12}$$

This equation relates the temperature level T_o to the temperature T_c at the outer circle and the line sources P_n .

The remaining part of (6.11) is an equation of the following type:

$$\begin{aligned}
0 &= \sum_{k=1}^{\infty} \operatorname{Re} [e^{-i \cdot k\phi} \cdot Z_k] \\
&= \sum_{k=1}^{\infty} \{ \cos(k\phi) \cdot \operatorname{Re}(Z_k) + \sin(k\phi) \cdot \operatorname{Im}(Z_k) \} \quad 0 \leq \phi \leq 2\pi
\end{aligned} \tag{6.13}$$

All coefficients before $\cos(k\phi)$ and $\sin(k\phi)$ must vanish. This means that the complex factor Z_k is zero for all k . We have the following equations:

$k = 1, 2, \dots$:

$$\begin{aligned}
P_{ck} \cdot \left\{ 1 - \sigma \frac{1 - k\beta_c}{1 + k\beta_c} \left(\frac{r_b}{r_c} \right)^{2k} \right\} \\
+ (1 + \sigma) \frac{1 - k\beta_c}{1 + k\beta_c} \cdot \left\{ \sum_{n=1}^N P_n \frac{1}{k} \left(\frac{z_n}{r_c} \right)^k + \sum_{n=1}^N \sum_{j=1}^k P_{nj} \binom{k-1}{j-1} \frac{r_{pn}^j z_n^{k-j}}{r_c^k} \right\} = 0
\end{aligned} \tag{6.14}$$

This equation relates the value of a multipole P_{ck} at infinity to the values of line sources P_n and the multipoles P_{nj} at the pipes up to order $j = k$.

7. EQUATIONS FROM THE BOUNDARY CONDITION AT THE PIPES

The boundary condition at pipe m is from (2.6):

$$T_{fm} = \left(T - \beta_m r_{pm} \frac{\partial T}{\partial \rho_m} \right) \Big|_{\rho_m=r_{pm}} \quad 0 \leq \psi_m \leq 2\pi \tag{7.1}$$

The temperature is given by (5.3). We will first express this temperature in the local polar coordinates ρ_m, ψ_m of pipe m , which are given by (2.4):

$$z - z_m = \rho_m e^{i\psi_m} \tag{7.2}$$

We want to separate different powers $(z - z_m)^k$ or $(e^{i\psi_m})^k$. We therefore expand the various terms of (5.3) in Taylor series in $z - z_m$.

We will use the following expressions.

A. $n \neq m$

$$\begin{aligned}
\ln \left(\frac{r_b}{z - z_n} \right) &= \ln \left(\frac{r_b}{z - z_m + z_m - z_n} \right) \\
&= \ln \left(\frac{r_b}{z_m - z_n} \right) + \ln \left(\frac{1}{1 - \frac{z - z_m}{z_n - z_m}} \right)
\end{aligned} \tag{7.3}$$

The second logarithm is expanded in the Taylor series (6.2).

B.

$$\begin{aligned}
\ln \left(\frac{r_b^2}{r_b^2 - z_n \bar{z}} \right) &= \ln \left(\frac{r_b^2}{r_b^2 - z_n \bar{z}_m - z_n (\bar{z} - \bar{z}_m)} \right) \\
&= \ln \left(\frac{r_b^2}{r_b^2 - z_n \bar{z}_m} \right) + \ln \left(\frac{1}{1 - \frac{z_n (\bar{z} - \bar{z}_m)}{r_b^2 - z_n \bar{z}_m}} \right)
\end{aligned} \tag{7.4}$$

The second logarithm is expanded in the Taylor series (6.2).

C. $n \neq m$

$$\left(\frac{r_{pn}}{z - z_n}\right)^j = \left(\frac{r_{pn}}{z_m - z_n}\right)^j \cdot \frac{1}{\left(1 - \frac{z - z_m}{z_n - z_m}\right)^j} \quad (7.5)$$

The second factor is expanded with the Taylor series (6.5).

D.

$$\begin{aligned} \left(\frac{r_{pn}\bar{z}}{r_b^2 - z_n\bar{z}}\right)^j &= \left(\frac{r_{pn}(\bar{z}_m + \bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m - z_n(\bar{z} - \bar{z}_m)}\right)^j \\ &= \left(\frac{r_{pn}}{r_b^2 - z_n\bar{z}_m}\right)^j \cdot (\bar{z}_m + \bar{z} - \bar{z}_m)^j \cdot \frac{1}{\left(1 - \frac{z_n(\bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m}\right)^j} \end{aligned} \quad (7.6)$$

The second factor is expanded as a binomial. The third factor is expanded in the series (6.5):

$$(\bar{z}_m + \bar{z} - \bar{z}_m)^j = \sum_{j'=0}^j \binom{j}{j'} (\bar{z} - \bar{z}_m)^{j'} \cdot \bar{z}_m^{j-j'} \quad (7.7)$$

$$\frac{1}{\left(1 - \frac{z_n(\bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m}\right)^j} = \sum_{k'=0}^{\infty} \binom{j+k'-1}{j-1} \left(\frac{z_n(\bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m}\right)^{k'} \quad (7.8)$$

The product of (7.7) and (7.8) is a double sum containing the powers $(\bar{z} - \bar{z}_m)^{k'+j'}$. We change the summation from k' and j' to $k = k' + j'$ and j' . The index k will vary from 0 to infinity. The values of j' run from 0 to $\min(j, k)$. We have the expansion:

$$\begin{aligned} \left(\frac{r_{pn}\bar{z}}{r_b^2 - z_n\bar{z}}\right)^j &= \sum_{k=0}^{\infty} \sum_{j'=0}^{\min(j,k)} \binom{j}{j'} \binom{j+k-j'-1}{j-1} \\ &\quad \cdot \frac{r_{pn}^j \bar{z}_m^{j-j'} z_n^{k-j'}}{(r_b^2 - z_n\bar{z}_m)^{j+k-j'}} \cdot (\bar{z} - \bar{z}_m)^k \end{aligned} \quad (7.9)$$

E.

$$\begin{aligned} \sum_{j=1}^{\infty} P_{cj}(1-\sigma) \left(\frac{z}{r_c}\right)^j &= \sum_{j=1}^{\infty} P_{cj}(1-\sigma) \left(\frac{z_m + z - z_m}{r_c}\right)^j \\ &= \sum_{j=1}^{\infty} \sum_{k=0}^j P_{cj}(1-\sigma) \binom{j}{k} \frac{z_m^{j-k} \cdot (z - z_m)^k}{r_c^j} \end{aligned} \quad (7.10)$$

The summation order is changed. Then we have:

$$\sum_{j=1}^{\infty} \sum_{k=0}^j \dots = \sum_{k=0}^{\infty} \sum_{j=\max(1,k)}^{\infty} \dots \quad (7.11)$$

The temperature (5.3) may now with the use of (7.2-11) be written in the following way:

$$\begin{aligned}
T &= T_o + P_m \ln \left(\frac{r_b}{\rho_m} \right) \\
&+ Re \left[\sum_{\substack{n=1 \\ n \neq m}}^N P_n \left\{ \ln \left(\frac{r_b}{z_m - z_n} \right) + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\rho_m}{z_n - z_m} \right)^k \cdot e^{i \cdot k \psi_m} \right\} \right. \\
&+ \sum_{n=1}^N P_n \sigma \left\{ \ln \left(\frac{r_b^2}{r_b^2 - \bar{z}_m z_n} \right) + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z_n \rho_m}{r_b^2 - \bar{z}_m z_n} \right)^k \cdot e^{-i \cdot k \psi_m} \right\} \\
&+ \sum_{j=1}^{\infty} P_{mj} \left(\frac{r_{pm}}{\rho_m} \right)^j \cdot e^{-i \cdot j \psi_m} \\
&+ \sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} P_{nj} \left(\frac{r_{pn}}{z_m - z_n} \right)^j \binom{j+k-1}{j-1} \left(\frac{\rho_m}{z_n - z_m} \right)^k \cdot e^{i \cdot k \psi_m} \\
&+ \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \sum_{j'=0}^{\min(j,k)} P_{nj} \sigma \binom{j}{j'} \binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j \bar{z}_m^{j-j'} z_n^{k-j'} \cdot \rho_m^k}{(r_b^2 - z_n \bar{z}_m)^{j+k-j'}} \cdot e^{-i \cdot k \psi_m} \\
&+ \left. \sum_{k=0}^{\infty} \sum_{j=\max(1,k)}^{\infty} P_{cj} (1-\sigma) \binom{j}{k} \frac{z_m^{j-k} \cdot \rho_m^k}{r_c^j} e^{i \cdot k \psi_m} \right] \tag{7.12}
\end{aligned}$$

The summation index on the fourth line (concerning P_{mj}) is changed to k . The dependence on ψ_m lies in the exponents $e^{i \cdot k \psi_m}$ and $e^{-i \cdot k \psi_m}$. The latter terms may be changed to the positive exponent $e^{i \cdot k \psi_m}$ by taking the complex conjugate as in (6.10). Expression (7.12) is with these modifications inserted in the boundary condition (7.1) for pipe m . The derivatives of powers of ρ_m are simple. We have as in (6.9):

$$\begin{aligned}
-\beta_m r_{pm} \frac{\partial}{\partial \rho_m} (\rho_m^k) \Big|_{\rho_m=r_{pm}} &= -k \beta_m r_{pm}^k \\
-\beta_m r_{pm} \frac{\partial}{\partial \rho_m} (\rho_m^{-k}) \Big|_{\rho_m=r_{pm}} &= k \beta_m r_{pm}^{-k} \tag{7.13}
\end{aligned}$$

We finally get the following expression for the boundary condition at pipe m :

$$\begin{aligned}
T_{fm} &= \left(T - \beta_m r_{pm} \frac{\partial T}{\partial \rho_m} \right) \Big|_{\rho_m = r_{pm}} \\
&= T_o + P_m \left\{ \ln \left(\frac{r_b}{r_{pm}} \right) - \beta_m r_{pm} \cdot \frac{-1}{r_{pm}} \right\} \\
&+ \sum_{\substack{n=1 \\ n \neq m}}^N P_n \ln \left(\frac{r_b}{r_{mn}} \right) + \sum_{n=1}^N P_n \sigma \cdot \ln \left(\frac{r_b^2}{r_b^2 - \bar{z}_m z_n} \right) \\
&+ \operatorname{Re} \left[\sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^{\infty} P_{nj} \left(\frac{r_{pn}}{z_m - z_n} \right)^j + \sum_{n=1}^N \sum_{j=1}^{\infty} P_{nj} \sigma \left(\frac{r_{pn} \bar{z}_m}{r_b^2 - z_n \bar{z}_m} \right)^j \right. \\
&+ \left. \sum_{j=1}^{\infty} P_{cj} (1 - \sigma) \left(\frac{z_m}{r_c} \right)^j \right] \\
&+ \operatorname{Re} \left[\sum_{k=1}^{\infty} e^{i \cdot k \psi_m} \left\{ \sum_{\substack{n=1 \\ n \neq m}}^N P_n (1 - k \beta_m) \frac{1}{k} \left(\frac{r_{pm}}{z_n - z_m} \right)^k \right. \right. \\
&+ \sum_{n=1}^N P_n \sigma (1 - k \beta_m) \frac{1}{k} \left(\frac{\bar{z}_n r_{pm}}{r_b^2 - z_m \bar{z}_n} \right)^k \\
&+ \bar{P}_{mk} (1 + k \beta_m) \left(\frac{r_{pm}}{r_{pm}} \right)^k \\
&+ \sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^{\infty} P_{nj} (1 - k \beta_m) \binom{j+k-1}{j-1} \left(\frac{r_{pn}}{z_m - z_n} \right)^j \left(\frac{r_{pm}}{z_n - z_m} \right)^k \\
&+ \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{j'=0}^{\min(j,k)} \bar{P}_{nj} \sigma (1 - k \beta_m) \binom{j}{j'} \binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j r_{pm}^k z_m^{j-j'} \bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \\
&+ \left. \left. \sum_{j=k}^{\infty} P_{cj} (1 - \sigma) (1 - k \beta_m) \binom{j}{k} \frac{z_m^{j-k} r_{pm}^k}{r_c^j} \right\} \right] \quad 0 \leq \psi_m \leq 2\pi
\end{aligned} \tag{7.14}$$

The first four lines on the right-hand side give the constant part; i.e. the part independent of ψ_m . The length r_{mn} denotes the distance between the centers of pipes m and n :

$$r_{mn} = |z_m - z_n| = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} \tag{7.15}$$

Equation (7.14) is valid for all ψ_m around the pipe. The last six lines of (7.14) contain the part that depends on ψ_m . It must be equal to zero. The expression is of the same type as (6.13). The complex factor for each component $e^{i \cdot k \psi_m}$ must vanish. We get the following equations, which determine the multipoles P_{nj} :

$$m = 1, \dots, N \quad ; \quad k = 1, 2, \dots:$$

$$\begin{aligned}
& P_{mk} + \frac{1 - k\beta_m}{1 + k\beta_m} \left\{ \sum_{\substack{n=1 \\ n \neq m}}^N P_n \frac{1}{k} \left(\frac{r_{pm}}{z_n - z_m} \right)^k + \sum_{n=1}^N P_n \sigma \frac{1}{k} \left(\frac{r_{pm} \bar{z}_n}{r_b^2 - z_n \bar{z}_m} \right)^k \right. \\
& + \sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^{\infty} P_{nj} \binom{j+k-1}{j-1} \left(\frac{r_{pn}}{z_m - z_n} \right)^j \left(\frac{r_{pm}}{z_n - z_m} \right)^k \\
& + \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{j'=0}^{\min(j,k)} \bar{P}_{nj} \sigma \binom{j}{j'} \binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j r_{pm}^k z_m^{j-j'} \bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \\
& \left. + \sum_{j=k}^{\infty} P_{cj} (1 - \sigma) \binom{j}{k} \frac{z_m^{j-k} r_{pm}^k}{r_c^j} \right\} = 0
\end{aligned} \tag{7.16}$$

The constant part of (7.14) must also vanish. This means that the first four lines on the right-hand side are equal to T_{jm} . We can eliminate T_o from (6.12). Then we get the following equations:

$$m = 1, \dots, N:$$

$$\begin{aligned}
T_{jm} - T_c &= \sum_{n=1}^N q_n \cdot R_{mn}^o \\
& + Re \left[\sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^{\infty} P_{nj} \left(\frac{r_{pn}}{z_m - z_n} \right)^j + \sum_{n=1}^N \sum_{j=1}^{\infty} P_{nj} \sigma \left(\frac{r_{pn} \bar{z}_m}{r_b^2 - z_n \bar{z}_m} \right)^j \right. \\
& \left. + \sum_{j=1}^{\infty} P_{cj} (1 - \sigma) \left(\frac{z_m}{r_c} \right)^j \right]
\end{aligned} \tag{7.17}$$

The first line involves on the right-hand side the line sources P_n . We have returned to q_n via (5.2).

The coefficients R_{mn}^o are given by:

$$\begin{aligned}
R_{mm}^o &= \frac{1}{2\pi\lambda_b} \left\{ \beta_m + \ln \left(\frac{r_b}{r_{pm}} \right) + \sigma \cdot \ln \left(\frac{r_b^2}{r_b^2 - r_m^2} \right) \right\} + \frac{1}{2\pi\lambda} \left\{ \ln \left(\frac{r_c}{r_b} \right) + \beta_c \right\} \quad m = 1, \dots, N \\
R_{mn}^o &= \frac{1}{2\pi\lambda_b} \left\{ \ln \left(\frac{r_b}{r_{mn}} \right) + \sigma \cdot \ln \left(\frac{r_b^2}{|r_b^2 - z_n \bar{z}_m|} \right) \right\} + \frac{1}{2\pi\lambda} \left\{ \ln \left(\frac{r_c}{r_b} \right) + \beta_c \right\} \quad m \neq n \\
& \qquad \qquad \qquad m, n = 1, \dots, N
\end{aligned} \tag{7.18}$$

8. FINAL EQUATIONS AND ITERATIVE SOLUTION

The final equations for q_n (or P_n), P_{nj} and P_{cj} are (7.17-18), (7.16) and (6.14). Multipoles of all orders are needed in an exact solution. In the numerical one we truncate the equation system and consider multipoles up to order J at each pipe and at infinity. The sine- and cosinevariation around the pipes and around the outer circle can then be satisfied up to order J only. The truncation error is discussed in the next chapter.

We have from (7.17), (7.16), (5.2) and (6.14) the following final equations:

$m = 1, \dots, N$:

$$\begin{aligned}
 T_{fm} - T_c &= \sum_{n=1}^N q_n \cdot R_{mn}^o \\
 &+ Re \left[\sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^J P_{nj} \left(\frac{r_{pn}}{z_m - z_n} \right)^j + \sum_{n=1}^N \sum_{j=1}^J P_{nj} \sigma \left(\frac{r_{pn} \bar{z}_m}{r_b^2 - z_n \bar{z}_m} \right)^j \right. \\
 &\left. + \sum_{j=1}^J P_{cj} (1 - \sigma) \left(\frac{z_m}{r_c} \right)^j \right] \quad (8.1)
 \end{aligned}$$

$m = 1, \dots, N$; $k = 1, \dots, J$:

$$\begin{aligned}
 \bar{P}_{mk} &+ \frac{1 - k\beta_m}{1 + k\beta_m} \left\{ \sum_{\substack{n=1 \\ n \neq m}}^N \frac{q_n}{2\pi\lambda_b} \frac{1}{k} \left(\frac{r_{pm}}{z_n - z_m} \right)^k + \sum_{n=1}^N \frac{q_n}{2\pi\lambda_b} \sigma \frac{1}{k} \left(\frac{r_{pm} \bar{z}_n}{r_b^2 - z_m \bar{z}_n} \right)^k \right. \\
 &+ \sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^J P_{nj} \binom{j+k-1}{j-1} \left(\frac{r_{pn}}{z_m - z_n} \right)^j \left(\frac{r_{pm}}{z_n - z_m} \right)^k \\
 &+ \sum_{n=1}^N \sum_{j=1}^J \sum_{j'=0}^{\min(j,k)} P_{nj} \sigma \binom{j}{j'} \binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j r_{pm}^k z_m^{j-j'} \bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \\
 &\left. + \sum_{j=k}^J P_{cj} (1 - \sigma) \binom{j}{k} \frac{z_m^{j-k} r_{pm}^k}{r_c^j} \right\} = 0 \quad (8.2)
 \end{aligned}$$

$k = 1, \dots, J$:

$$\begin{aligned}
 \bar{P}_{ck} &\cdot \left\{ 1 - \sigma \frac{1 - k\beta_c}{1 + k\beta_c} \left(\frac{r_b}{r_c} \right)^{2k} \right\} \\
 &+ (1 + \sigma) \frac{1 - k\beta_c}{1 + k\beta_c} \cdot \left\{ \sum_{n=1}^N \frac{q_n}{2\pi\lambda_b} \frac{1}{k} \left(\frac{z_n}{r_c} \right)^k + \sum_{n=1}^N \sum_{j=1}^k P_{nj} \binom{k-1}{j-1} \frac{r_{pn}^j z_n^{k-j}}{r_c^k} \right\} = 0 \quad (8.3)
 \end{aligned}$$

The thermal resistances R_{mn}^o are given by (7.18):

$$\begin{aligned}
 R_{mm}^o &= \frac{1}{2\pi\lambda_b} \left\{ \beta_m + \ln \left(\frac{r_b}{r_{pm}} \right) + \sigma \cdot \ln \left(\frac{r_b^2}{r_b^2 - r_m^2} \right) \right\} + \frac{1}{2\pi\lambda} \left\{ \ln \left(\frac{r_c}{r_b} \right) + \beta_c \right\} \quad m = 1, \dots, N \\
 R_{mn}^o &= \frac{1}{2\pi\lambda_b} \left\{ \ln \left(\frac{r_b}{r_{mn}} \right) + \sigma \cdot \ln \left(\frac{r_b^2}{|r_b^2 - z_n \bar{z}_m|} \right) \right\} + \frac{1}{2\pi\lambda} \left\{ \ln \left(\frac{r_c}{r_b} \right) + \beta_c \right\} \quad m \neq n \quad (8.4) \\
 & \quad \quad \quad m, n = 1, \dots, N
 \end{aligned}$$

There are N real-valued equations (8.1) and $N \cdot J + J$ complex-valued ones (8.2-3). This corresponds to the N line sources q_n and the $N \cdot J + J$ complex-valued multipoles P_{nj} and P_{cj} .

The equation system (8.1-4) is solved iteratively as in [1]. Let K_{mn}^o be the elements of the inverse matrix of the resistance matrix with the elements R_{mn}^o :

$$(K_{mn}^o) = (R_{mn}^o)^{-1} \quad (8.5)$$

Inversion of (8.1) gives the heat fluxes:

$$\begin{aligned} q_{m'} = & \sum_{m=1}^N K_{m'm}^o \left\{ T_{fm} - T_c \right. \\ & - Re \left[\sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^J P_{nj} \left(\frac{r_{pn}}{z_m - z_n} \right)^j + \sum_{n=1}^N \sum_{j=1}^J P_{nj} \sigma \left(\frac{r_{pn} \bar{z}_m}{r_b^2 - z_n \bar{z}_m} \right)^j \right. \\ & \left. \left. + \sum_{j=1}^J P_{cj} (1 - \sigma) \left(\frac{z_m}{r_c} \right)^j \right] \right\} \end{aligned} \quad (8.6)$$

Let q_n^ν , P_{nj}^ν and P_{cj}^ν denote the values of our variables for iteration step ν . We start for $\nu = 0$ with the values:

$$\begin{aligned} P_{nj}^o &= 0 & P_{cj}^o &= 0 \\ q_{m'}^o &= \sum_{m=1}^N K_{m'm}^o (T_{fm} - T_c) \end{aligned} \quad (8.7)$$

This means that we start without multipoles and compute the heat fluxes with (8.6).

For $\nu = 1, 2, \dots$ we use the following recursive formulas in accordance with (8.2), (8.3) and (8.6):

$$m = 1, \dots, N \quad ; \quad k = 1, \dots, J :$$

$$\begin{aligned} \bar{P}_{mk}^{\nu+1} = & -\frac{1 - k\beta_m}{1 + k\beta_m} \left\{ \sum_{\substack{n=1 \\ n \neq m}}^N \frac{q_n^\nu}{2\pi\lambda_b} \frac{1}{k} \left(\frac{r_{pm}}{z_n - z_m} \right)^k + \sum_{n=1}^N \frac{q_n^\nu}{2\pi\lambda_b} \sigma \frac{1}{k} \left(\frac{r_{pm} \bar{z}_n}{r_b^2 - z_m \bar{z}_n} \right)^k \right. \\ & + \sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^J P_{nj}^\nu \binom{j+k-1}{j-1} \left(\frac{r_{pn}}{z_m - z_n} \right)^j \left(\frac{r_{pm}}{z_n - z_m} \right)^k \\ & + \sum_{n=1}^N \sum_{j=1}^J \sum_{j'=0}^{\min(j,k)} \bar{P}_{nj}^\nu \sigma \binom{j}{j'} \binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j r_{pm}^k z_m^{j-j'} \bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \\ & \left. + \sum_{j=k}^J P_{cj}^\nu (1 - \sigma) \binom{j}{k} \frac{z_m^{j-k} r_{pm}^k}{r_c^j} \right\} \end{aligned} \quad (8.8)$$

$$k = 1, \dots, J:$$

$$\begin{aligned} \bar{P}_{ck}^{\nu+1} \cdot \left\{ 1 - \sigma \frac{1 - k\beta_c}{1 + k\beta_c} \left(\frac{r_b}{r_c} \right)^{2k} \right\} = & \\ - (1 + \sigma) \frac{1 - k\beta_c}{1 + k\beta_c} \cdot \left\{ \sum_{n=1}^N \frac{q_n^\nu}{2\pi\lambda_b} \frac{1}{k} \left(\frac{z_n}{r_c} \right)^k + \sum_{n=1}^N \sum_{j=1}^k P_{nj}^{\nu+1} \binom{k-1}{j-1} \frac{r_{pn}^j z_n^{k-j}}{r_c^k} \right\} \end{aligned} \quad (8.9)$$

$m' = 1, \dots, N$:

$$\begin{aligned}
q_{m'}^{\nu+1} = & \sum_{m=1}^N K_{m'm}^o \left\{ T_{f_m} - T_c \right. \\
& - \operatorname{Re} \left[\sum_{\substack{n=1 \\ n \neq m}}^N \sum_{j=1}^J P_{nj}^{\nu+1} \left(\frac{r_{pn}}{z_m - z_n} \right)^j + \sum_{n=1}^N \sum_{j=1}^J P_{nj}^{\nu+1} \sigma \left(\frac{r_{pn} \bar{z}_m}{r_b^2 - z_n \bar{z}_m} \right)^j \right. \\
& \left. \left. + \sum_{j=1}^J P_{cj}^{\nu+1} (1 - \sigma) \left(\frac{z_m}{r_c} \right)^j \right] \right\}
\end{aligned} \tag{8.10}$$

The iterative solution (8.7-10) is physically reasonable. Consider an iteration step ν . There is an approximate solution with line sinks and multipoles. The equations (8.1-3) are not satisfied exactly. The equations (8.8) and (8.9) mean that we change the multipoles so that the variation vanishes exactly (up to order J) at the considered boundary circle. But the difference $P_{mk}^{\nu+1} - P_{mk}^{\nu}$ will induce a multipole behaviour at the other pipes. These secondary disturbances are however damped with factors of the type:

$$\left(\frac{z_n}{r_c} \right)^k \quad ; \quad \left(\frac{r_{pm}}{r_c} \right)^{k'} \quad ; \quad \left(\frac{r_{pm}}{z_n - z_m} \right)^{k''}$$

The convergence of the iterative procedure is therefore rapid.

The iteration procedure is quite robust. It has worked without any problems for all cases that we have tested. An example is given below.

The iterations are performed until the following criterion is satisfied.

$$\begin{aligned}
\frac{|P_{nj}^{\nu+1} - P_{nj}^{\nu}|}{\max_{1 \leq k \leq j} |P_{nk}|} &< \epsilon \quad \text{for all } n \text{ and all } j (j \leq J) \text{ with } |P_{nj}| \neq 0 \\
\frac{|P_{cj}^{\nu+1} - P_{cj}^{\nu}|}{\max_{1 \leq k \leq j} |P_{cj}|} &< \epsilon \quad \text{for all } j (j \leq J) \text{ with } |P_{cj}| \neq 0
\end{aligned} \tag{8.11}$$

Here ϵ is a measure of the iteration accuracy. The multipole differences are divided by the largest multipole up to the same order j according to the expression in the denominators. Normally, this is the multipole of first order. This will mean that ϵ gives the accuracy relative to the magnitude of the strongest multipole.

Example 8.1. Three pipes with different temperatures. We take the following data:

$$\begin{array}{cccccc}
N = 3 & \lambda_b = 0.6 & \lambda = 3.6 & & & \\
r_b = 2 & r_c = 4 & \beta_c = 0 & T_c = 0 & & \\
x_1 = 1 & y_1 = 0 & \beta_1 = 0 & T_{f1} = 1 & r_{p1} = 0.5 & \\
x_2 = 0 & y_2 = 1.5 & \beta_2 = 0 & T_{f2} = -3 & r_{p2} = 0.25 & \\
x_3 = -1 & y_3 = -0.5 & \beta_3 = 0.5 & T_{f3} = 2 & r_{p3} = 0.5 &
\end{array} \tag{8.12}$$

The temperature field of this case is shown in Figure 1.1 (and on the cover).

The number of iterations in order to obtain the required accuracy ϵ is given in Table 8.1 for different J and ϵ .

	$\epsilon=10^{-4}$	10^{-5}	10^{-6}
J=1	6	7	8
2	6	7	8
3	6	8	9
5	6	8	9
10	6	8	9
15	6	8	9
20	6	8	9

Table 8.1. Required number of iterations in example 8.1.

We have found as in [1] that the value of ϵ is not critical. One may use $\epsilon = 10^{-4}$ or 10^{-5} .

9. REQUIRED NUMBER OF MULTIPOLES

The boundary conditions at the pipes and the outer circle are satisfied up to the order J in Fourier terms i.e. up to the order $\cos(J\psi_m)$ and $\sin(J\psi_m)$. The approximation becomes better, when J is increased, but the computational effort and the execution time also increase. It is an important question what value of J to choose in any particular case. One can always increase J , until the solution does not change.

The accuracy as a function of J will be studied and illustrated in this chapter with a few examples.

9.1 ERROR ON THE BOUNDARY CIRCLES

The example shown in Figure 9.1.1. is used to illustrate the error on the boundary circles.

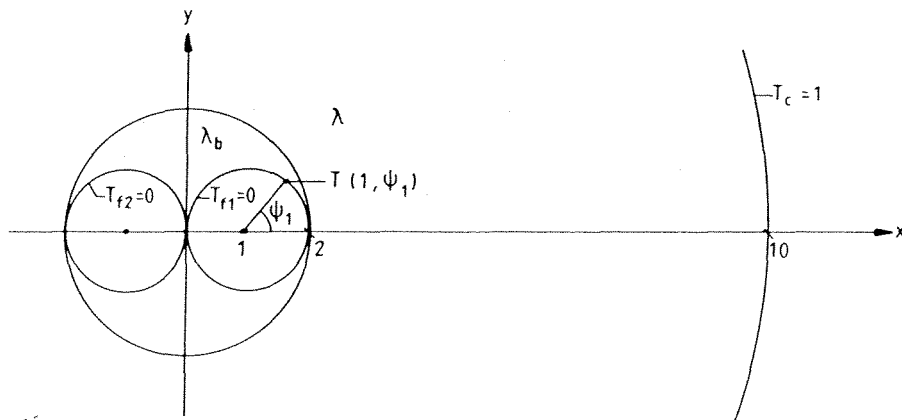


Figure 9.1.1. Example with two pipes.

The following data are used:

$$\begin{array}{llllll}
 N = 2 & \lambda_b = 1 & \lambda = 5 & \epsilon = 10^{-5} & & \\
 r_b = 2 & r_c = 10 & \beta_c = 0 & T_c = 1 & & \\
 x_1 = 1 & y_1 = 0 & \beta_1 = 0 & T_{f1} = 0 & r_{p1} = 1 & \\
 x_2 = -1 & y_2 = 0 & \beta_2 = 0 & T_{f2} = 0 & r_{p2} = 1 & \\
 \end{array} \tag{9.1.1}$$

The heat fluxes q_1 and q_2 are equal. The computed values for different J are given in Table 9.1.1. The computer model described in the following chapters is used. We see that $J = 1$ gives an error for q_1 of 2

%, and it is 0.3% for $J = 3$. The first order, $J = 1$, is sufficient in this case in order to calculate the heat fluxes.

J	0	1	2	3	4	5	10
q_1	-4.8560	-6.4986	-6.5596	-6.6044	-6.6174	-6.6206	-6.6247
$q_1(J)/q_1(10)$	0.73	0.981	0.990	0.997	0.9989	0.9994	1

Table 9.1.1. Calculated heat fluxes for example (9.1.1) for different J .

The polar coordinates of pipe 1 are ρ_1, ψ_1 . The temperature on the uninsulated pipe is $T(1, \psi_1)$. In an exact solution $T(1, \psi_1)$ is equal to $T_{f1} = 0$, but for a finite J there will be a certain error or variation around the pipe. This boundary temperature is shown in Figure 9.1.2 for $J = 10$. The calculations are made with the computer program described in Chapters 10-11. The temperature $T(1, \psi_1)$ is obtained from the general formula (5.1) with summation in j up to $J = 10$.

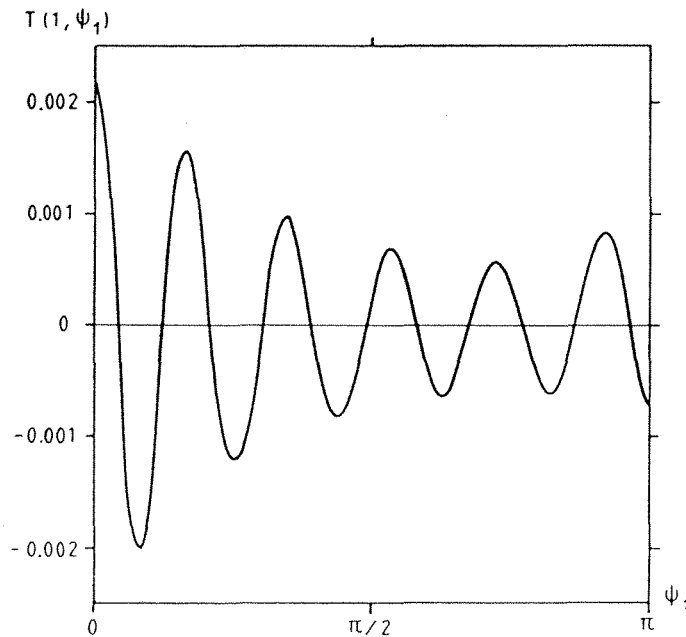


Figure 9.1.2. Temperature variation on pipe 1. Data according to (9.1.1) with $J = 10$.

The largest deviation from $T_{f1} = 0$ is 0.002. The largest error of boundary temperature is therefore 0.2% (of the temperature difference $T_c - T_{f1}$).

9.2 A THREE-PIPE PROBLEM

As a second example we take the three-pipe problem of example 8.1. The data are given by (8.12). The temperature field of this case is shown in Figure 1.1. Table 9.2 gives the calculated heat fluxes and the temperatures in a few points for different J .

J	0	1	2	3	5	10	15
q_1	3.702	3.747	3.775	3.776	3.776	3.776	3.776
q_2	-8.121	-8.644	-8.685	-8.688	-8.689	-8.689	-8.689
q_3	4.570	4.766	4.792	4.792	4.792	4.792	4.792
T(0.5,1)	-1.7209	-0.8213	-0.8090	-0.8080	-0.8079	-0.8079	-0.8079
T(0,0)	0.7851	0.7343	0.7267	0.7245	0.7243	0.7243	0.7243
T(0,-2)	1.2410	0.1452	0.1724	0.1704	0.1706	0.1706	0.1706

Table 9.2.1. Heat flows and temperatures in a few points for different J. Data according to (8.12).

The table shows that the error in heat fluxes is at most 7% for $J = 0$ and 0.8% for $J = 1$. The temperatures differ with up to a factor 2 for $J = 0$, while the largest error is 0.03 temperature units for $J = 1$. This difference decreases to 0.0024 for $J = 2$. The value $J = 1$ should be sufficient in this example.

9.3 TEST OF MAXWELL'S RECIPROCITY THEOREM

We do not have analytical solutions to compare with in more complicated cases. We will therefore use a general theorem due to J.C. Maxwell (and others). Consider a steady-state heat conduction problem. The region is bounded by a number of boundary surfaces S_1, S_2, S_3, \dots . The temperature is zero on all surfaces except one. Two cases are considered:

$$\begin{array}{llll}
 A: & T = 1 & \text{on} & S_1 \\
 B: & T = 1 & \text{on} & S_2
 \end{array}
 \quad
 \begin{array}{llll}
 T = 0 & \text{on} & S_2 & (\text{and } S_3, \dots) \\
 T = 0 & \text{on} & S_1 & (\text{and } S_3, \dots)
 \end{array}
 \quad (9.3.1)$$

Let q_2^A be the heat flux from surface 2 in case A, and q_1^B the heat flux from surface 1 in case B. The reciprocity theorem states that these two fluxes are equal:

$$q_2^A = q_1^B \quad (9.3.2)$$

We have found that the reciprocity theorem is valid for our truncated problem (8.1-3) as well. (We have not taken the trouble to try to prove this).

In a test of the reciprocity theorem we use example (8.1). The data (8.12) are valid except for the boundary temperatures. We use:

$$T_c = 0 \quad T_{f1} = 0 \quad J = 5 \quad (9.3.3)$$

and

$$\text{case } A: \quad T_{f2} = 1 \quad T_{f3} = 0 \quad (9.3.3A)$$

$$\text{case } B: \quad T_{f2} = 0 \quad T_{f3} = 1 \quad (9.3.3B)$$

The calculated heat fluxes became equal with seven digits:

$$\begin{array}{l}
 q_3^A = -0.1752401 \\
 q_2^B = -0.1752401
 \end{array}
 \quad (9.3.4)$$

Another example concerns a case with a single region ($\lambda_b = \lambda$). The first five pipes of the example in section 9.3 in [1] are used. The following data are valid:

$$\begin{aligned}
 N = 5 & \quad \lambda = \lambda_b = 1 & \quad J = 5 \\
 T_c = 0 & \quad r_c = 10 & \quad \beta_c = 0 \\
 x_1 = 1 & \quad y_1 = 0 & \quad r_{p1} = 0.5 & \quad \beta_1 = 0 \\
 x_2 = 2 & \quad y_2 = 1 & \quad r_{p2} = 0.5 & \quad \beta_2 = 0 & \quad T_{f2} = 0 \\
 x_3 = 1 & \quad y_3 = 2 & \quad r_{p3} = 0.5 & \quad \beta_3 = 0 & \quad T_{f3} = 0 \\
 x_4 = 0 & \quad y_4 = 2 & \quad r_{p4} = 0.5 & \quad \beta_4 = 0 & \quad T_{f4} = 0 \\
 x_5 = 2 & \quad y_5 = -1 & \quad r_{p5} = 0.5 & \quad \beta_5 = 0.3 \\
 \\
 \text{case A :} & \quad T_{f1} = 1 & \quad T_{f5} = 0 \\
 \text{case B :} & \quad T_{f1} = 0 & \quad T_{f5} = 1
 \end{aligned} \tag{9.3.5}$$

The computed reciprocal heat fluxes became again equal with 7 digits:

$$\begin{aligned}
 q_5^A &= -1.890565 \\
 q_1^B &= -1.890565
 \end{aligned} \tag{9.3.6}$$

9.4 AN EXAMPLE WITH 15 PIPES

We use the example with 15 pipes in [1], section 9.3. The data according to (9.3.1) in [1] are valid. The corresponding temperature field is shown on the cover of [1]. The outer radius r_c is equal to 10. We take r_b equal to this value and add an outer region out to $r_c = 20$. The thermal conductivity is taken to be ten times higher than in the inner region. The data of (9.3.1) in [1] are supplemented with:

$$r_b = 10 \quad \lambda_b = 1 \quad r_c = 20 \quad \lambda = 10 \tag{9.4.1}$$

Table 9.4.1 gives the computed heat fluxes q_n , $1 \leq n \leq 15$, for different J . The values do not differ much from those in [1], (Table 9.3.1). We see that $J = 1$ is not acceptable, while $J = 5$ is quite sufficient.

	J=0	1	5	10
q_1	1.1330	2.3566	2.0842	2.0843
q_2	-0.51959	-5.4018	-4.9154	-4.9020
q_3	5.1286	6.1679	6.2540	6.2540
q_4	-11.816	-10.516	-11.340	-11.353
q_5	-2.7569	-1.0451	-1.0552	-1.0552
q_6	11.743	12.983	12.116	12.116
q_7	-6.1510	-6.7283	-6.6958	-6.6958
q_8	-5.8188	-9.0225	-9.247	-9.247
q_9	10.321	17.765	20.958	20.988
q_{10}	20.065	24.527	24.925	24.926
q_{11}	10.803	12.387	12.632	12.632
q_{12}	-13.252	-14.507	-14.727	-14.727
q_{13}	1.0835	1.7532	1.7310	1.7310
q_{14}	-7.3037	-14.710	-15.793	-15.813
q_{15}	-9.8105	-12.163	-14.086	-14.098

Table 9.4.1. Computed heat fluxes q_n for different J for a case with 15 pipes.

10. COMPUTER MODEL. EXECUTION TIMES

The computer model calculates the line sources and the multipoles up to the given order J , and it gives the corresponding temperature field. The next chapter is a manual for the computer program. The source code is given in appendix 1.

The input data of the model are:

$$\begin{aligned}
 &\lambda, \lambda_b, N, J \\
 &r_b, r_c, \beta_c, T_c \\
 &x_n, y_n, r_{pn}, \beta_n, T_{fn} \quad \text{for} \quad n = 1, \dots, N
 \end{aligned} \tag{10.1}$$

There are the following restrictions on the input variables:

$$\begin{aligned}
 \lambda > 0 & & \lambda_b > 0 & & N = 1, 2, \dots & & J = 0, 1, \dots \\
 0 < r_b < r_c & & \beta_c \geq 0 & & & & \\
 r_{pn} > 0 & & \beta_n \geq 0 & \text{for} & n = 1, \dots, N & & \\
 r_n = \sqrt{x_n^2 + y_n^2} \leq r_b - r_{pn} & \text{for} & n = 1, \dots, N & & & & \\
 r_{mn} \geq r_{pm} + r_{pn} & \text{for} & m \neq n & & & & \\
 r_{mn} = r_{pm} + r_{pn} & \text{only if} & \beta_m + \beta_n > 0 & \text{or} & T_{fm} = T_{fn} & &
 \end{aligned} \tag{10.2}$$

The condition $r_{mn} \geq r_{pm} + r_{pn}$ ensures that the pipes do not cover each other. They may touch each other, if there is a thermal insulation ($\beta_m + \beta_n > 0$) or if the temperatures T_{fm} and T_{fn} are equal, so that the heat flux between the pipes remains finite.

The first step after input and test of the restrictions (10.2) is to calculate auxiliary variables and the resistance matrix (R_{mn}^o) with the elements (8.4). The inverse matrix (K_{mn}^o) is then calculated. The initial values of line sources and multipoles for $\nu = 0$ are given by (8.7). Formulas (8.8-10) are used in successive iterations, until the ϵ -criterion (8.11) is met.

The output is the values of the line sources and multipoles and, if requested, the temperature field $T(x, y)$, which is obtained from (5.3-4) with summation in j up to J .

The steady-state heat conduction problem may be solved numerically with finite difference or finite element methods. The present method is however more rapid, and it is simpler to get a high and controlled accuracy. We will give the execution time for a number of cases in order to show this.

Consider the case with 15 pipes in section 9.4. In the first examples we use the first 5 pipes ($N = 5$), then the first 10 pipes ($N = 10$) and finally all 15 pipes. The accuracy ϵ is 10^{-4} and 10^{-5} , while J is 5 and 10. The execution times in CPU-seconds on a ND-500 computer from Norsk Data are given in Table 10.1.

		$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
N = 5	J = 5	13 s	18 s
	J = 10	140s	200 s
N = 10	J = 5	44 s	60 s
	J = 10	460 s	620 s
N = 15	J = 5	110 s	150 s
	J = 10	1200 s	1700 s

Table 10.1. Execution time in CPU-seconds on ND-500 for 5 to 15 pipes and multipole order $J = 5, 10$.

Table 10.1 shows that the CPU-time is below a few minutes even for 15 pipes, if the order of multipoles is kept below $J \leq 5$. The CPU-time for a more moderate case with $N \leq 3$, $J \leq 2$ is only a few seconds. The execution times for an IBM-PC AT-3 (10 MHz) with a 80287 math co-processor are about 60 times longer.

11. MANUAL FOR THE COMPUTER CODE

The program has been adapted to run on IBM-PC and other compatible computers. The source code is written in FORTRAN77. It has been compiled with the MS-FORTRAN77 compiler V3.3.

The executable code supplied on the disk may be used for cases with a maximum of 15 pipes. The maximum order of multipoles is 10. The size of the program is then 133K. These restrictions can be removed by changing the dimensions in the PARAMETER statement at the beginning of the main program. The maximum number of pipes is IW and the maximum order of multipoles is IJ.

The basic version of the source code is listed in Appendix 1. The current version of the program differs from the basic version according to the changes listed on the file README.DOC. This file may also contain important information about changes in this manual. Be sure to read this file before you run the program.

Files on the Disk.

A brief description of the files on the enclosed disk is given below.

_MPC.EXE	Executable code
MPC.FOR	Source code for the current version of the program
SAMPLE.DAT	Input data file for the example. See below
SAMPLE.OUT	Output file for the example. See below
MPC.BAT	Batch file
SIGNAL.COM	Program that generates a signal when the job is completed
README.DOC	Contains updates of this manual and the basic version of the source code.

Using the Program.

To run the program MPC (MultiPoleComposite), insert the disk into a drive, make that drive the default drive, and type the program name:

MPC

The program is intended for interactive use. The user is prompted for the names of an input data file (optional) and the output file. The input data may be entered interactively on the screen or read directly from a disk file. If the input data is entered interactively it may be saved on a disk file.

Input Data.

The input data must satisfy the restrictions (10.2). The program indicates any violation of these restrictions. The input data list is read in free format, i.e. the values must be separated by one or more contiguous blanks or a comma. The input data records are specified below.

- | | |
|-------------|--|
| λ_b | Thermal conductivity in the inner region, (W/mK) |
| λ | Thermal conductivity in the outer region, (W/mK) |
| N | Number of pipes |
| J | Maximal order of multipoles |
- | | |
|-----------|---|
| r_b | Radius of the inner region, (m) |
| r_c | Radius of the outer region, (m) |
| β_c | Thermal resistance coefficient at the outer circle, (-) |
| T_c | Temperature at outer circle, ($^{\circ}$ C) |

The following record must be repeated for each pipe, i.e. N times.

- | | |
|-----------|--|
| x_i | x-coordinate of the center of pipe i , (m) |
| y_i | y-coordinate of the center of pipe i , (m) |
| r_{pi} | Radius of pipe i , (m) |
| β_i | Thermal resistance coefficient of pipe i , (-) |
| T_{ji} | Fluid temperature in pipe i , ($^{\circ}$ C) |
- | | |
|------------|--|
| ϵ | Iteration accuracy. See section 8. A suitable value is 10^{-4} |
| ITMAX | Maximum number of iterations. A suitable value is 500 |

The temperatures are calculated in a rectangular grid within a rectangular region with the corners determined by XMIN, XMAX, YMIN, and YMAX. The number of grid points in the X- and Y-direction are NX and NY respectively. If a grid point lies within the circle of one of the pipes or outside the

outer boundary, then there will be no temperature calculation in that point. There is no calculation of temperatures if $NX \leq 0$ or $NY \leq 0$.

- 5. XMIN Minimum x-coordinate of rectangular region for temperature calculation, (m)
- XMAX Maximum x-coordinate of rectangular region for temperature calculation, (m)
- YMIN Minimum y-coordinate of rectangular region for temperature calculation, (m)
- YMAX Maximum y-coordinate of rectangular region for temperature calculation, (m)
- NX Number of grid points in the x-direction
- NY Number of grid points in the y-direction

The program is the same as in [1] except for the necessary modifications. The source code is given in appendix 1.

AN EXAMPLE

The example 8.1 with 3 pipes is used. The complete set of data is given by (8.12). The accuracy ϵ is set to 10^{-4} and ITMAX to 500. The corner points of the region of temperature calculation are XMIN = -2., YMIN = -2., XMAX = 2., and YMAX = 2. The number of grid points in the x-direction, NX, and the number of grid points in the y-direction, NY, are both 5. This gives the following indata list:

```

1 0.6 3.6 3 10
2 2.0 4.0 0.0 0.0
3 1.0 0.0 0.5 0.0 1.0
4 0.0 1.5 0.25 0.0 -3.0
5 -1.0 -0.5 0.5 0.5 2.0
6 1.0E-4 500
7 -2.0 2.0 -2.0 2.0 5 5

```

The output is given on the next two pages. The line sources and multipoles are denoted in the following way: $q_n = q(n)$, $P_{nk} = P(n,k)$, $P_{ck} = PC(k)$.

Correction on p.28, line 35-36:

Number of grid points along the x-axis	5
Number of grid points along the y-axis	5

```

1  MULTIPOLE METHOD - Pipes in a composite cylinder
2  =====
3
4  INPUT DATA
5
6  Input file a:sample.dat
7  Output file a:sample.out
8
9  Thermal conductivity in inner region      .600 W/(m*K)
10 Thermal conductivity in outer region     3.600 W/(m*K)
11 Number of pipes                          3
12 Order of multipoles                      10
13
14 Radius of inner region                   2.000 m
15 Radius of outer region                   4.000 m
16 Outer boundary
17   Thermal resistance coefficient          .000E+00
18   Temperature                            .000 C
19
20 * * * * *
21 Pipe *   x(n) *   y(n) *   rp(n) *   beta(n) *   Temp C *
22 * * * * *
23   1 *   1.000 *   .000 *   .500 *   .000E+00 *   1.000 *
24   2 *   .000 *   1.500 *   .250 *   .000E+00 *   -3.000 *
25   3 *  -1.000 *  -0.500 *   .500 *   .500E+00 *   2.000 *
26
27 Iteration accuracy                       .10E-03
28 Maximum number of iterations             500
29
30 TEMPERATURE OUTPUT: Definition of rectangular area
31   Minimum x-value                       -2.000
32   Maximum x-value                       2.000
33   Minimum y-value                       -2.000
34   Maximum y-value                       2.000
35   Number of grid points along x-axis    4
36   Number of grid points along y-axis    4
37
38 *****
39
40 Pipe Initial values for q(n) (Order of multipole = 0)
41   1   .3701710E+01
42   2  -.8120926E+01
43   3   .4570385E+01
44
45 Number of iterations:                   6
46
47
48 Pipe      q(n)
49   1   .3775983E+01
50   2  -.8688749E+01
51   3   .4791657E+01
52
53 Pipe      Order      P(n,k)
54   1         1   .85094E-01 .26170E+00
55   1         2  -.30722E-01 -.72742E-01
56   1         3   .18398E-01 .80464E-02
57   1         4  -.27537E-02 .11994E-02

```

58	1	5	-.12744E-04	-.50169E-03
59	1	6	.14905E-03	.25332E-04
60	1	7	-.32846E-04	.30497E-04
61	1	8	.43431E-06	-.11177E-04
62	1	9	.17959E-05	.15383E-05
63	1	10	-.46264E-06	.19077E-06
64	2	1	-.19320E-01	-.20056E+00
65	2	2	.50384E-01	.47747E-02
66	2	3	.57549E-03	.63122E-02
67	2	4	-.91680E-03	-.21108E-04
68	2	5	.16576E-05	-.20578E-03
69	2	6	.33463E-04	-.14848E-05
70	2	7	-.18595E-06	.70751E-05
71	2	8	-.13949E-05	.40628E-07
72	2	9	.14387E-08	-.28067E-06
73	2	10	.59501E-07	-.37228E-09
74	3	1	-.60168E-01	.10666E-01
75	3	2	.00000E+00	.00000E+00
76	3	3	.18779E-02	.12627E-02
77	3	4	.23297E-03	.46381E-03
78	3	5	-.93677E-04	.14950E-03
79	3	6	-.21800E-04	.14709E-05
80	3	7	-.41423E-05	-.31219E-05
81	3	8	.39756E-06	-.36033E-06
82	3	9	.12760E-07	.21779E-07
83	3	10	-.28358E-07	.25599E-07

84

85	Order	PC(k)		
86	1	.15866E-01	-.24283E+00	
87	2	-.63526E-01	.12837E-01	
88	3	-.10585E-02	.97228E-02	
89	4	.32898E-02	.67845E-03	
90	5	-.14272E-03	-.11445E-02	
91	6	-.32200E-03	.14539E-04	
92	7	-.11403E-04	.11161E-03	
93	8	.38088E-04	-.51801E-06	
94	9	-.34440E-06	-.12229E-04	
95	10	-.43815E-05	-.15054E-06	

96

97 *****

98 TEMPERATURES (Deg C)

99

100	x	y	Temp
101			
102	-2.00000	-2.00000	.14027
103	-2.00000	-1.00000	.26004
104	-2.00000	.00000	.20526
105	-2.00000	1.00000	-.00343
106	-2.00000	2.00000	-.09997
107	-1.00000	-2.00000	.20032
108	-1.00000	-1.00000	1.37087
109	-1.00000	.00000	1.31407
110	-1.00000	1.00000	-.31126
111	-1.00000	2.00000	-.35746
112	.00000	-2.00000	.17064
113	.00000	-1.00000	.82908
114	.00000	.00000	.72430

115	.00000	1.00000	-1.55455
116	.00000	2.00000	-.83812
117	1.00000	-2.00000	.12871
118	1.00000	-1.00000	.55357
119	1.00000	1.00000	-.06704
120	1.00000	2.00000	-.33614
121	2.00000	-2.00000	.08151
122	2.00000	-1.00000	.14928
123	2.00000	.00000	.18674
124	2.00000	1.00000	.03184
125	2.00000	2.00000	-.08219

REFERENCES

1. J. Claesson and J. Bennet, Multipole Method to Compute the Conductive Heat Flows between Pipes in a Cylinder; Notes on Heat Transfer 2-1987, Dep. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.
2. J. Claesson and G. Hellström , Thermal Resistances to and between Pipes in a Composite Cylinder, Notes on Heat Transfer 1987, Dep. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987. (To be published)

APPENDIX 1. LISTING OF COMPUTER CODE

```

1 C-----
2 C          PROGRAM MPC
3 C          Multipole method to compute the heat flows
4 C          and temperatures to and between insulated
5 C          circular pipes in a composite cylinder with
6 C          two concentric regions with different
7 C          thermal conductivities
8 C
9 C Authors: Johan Bennet, Johan Claesson, Goran Hellstrom
10 C          Departments of Building Technology and Mathematical Physics
11 C          Lund Institute of Technology, Box 118, S-221 00 Lund, Sweden
12 C Date:    1987-03-25
13 C Reference: Notes on Heat Transfer 3 - 1987
14 C-----
15     PARAMETER (IW=15,IJ=10,IW2=IW*IW)
16     COMPLEX*8 Z(IW),P(IW,IJ),P2(IW,IJ),PC(IJ),ZRC(IW,IJ),TERM
17     + ,RPZMN(IW,IW,IJ),RPMZN(IW,IW,IJ),PMK,PRB,ZPR,RPZ(IW,IJ)
18     DIMENSION RP(IW),BETA(IW),TF(IW),QBEG(IW)
19     + ,Q(IW),RRR(IW),QM(IW)
20     DOUBLE PRECISION RKO(IW,IW),RKOVEC(IW2),MC(IW),MD(IW)
21     CHARACTER*16 READFI,OUTFI
22     CHARACTER ANS
23     LOGICAL INFIL,OK
24     DATA INFIL/.FALSE./
25 c***** OPEN INPUT AND OUTPUT FILES *****
26     WRITE(*,441)
27     WRITE(*,*)
28     2 WRITE(*,*)'Input data from file ? (Y/N)'
29     READ(*,5) ANS
30     IF((ANS.EQ.'Y').OR.(ANS.EQ.'y')) THEN
31         WRITE(*,*) 'Name of input file ?'
32         READ(*,10) READFI
33         INQUIRE(FILE=READFI,EXIST=OK)
34         IF (OK) THEN
35             OPEN(UNIT=5,FILE=READFI)
36             INFIL=.TRUE.
37         ELSE
38             WRITE(*,*)' *** ERROR *** ',READFI,' File not found'
39             GO TO 2
40         ENDIF
41     ENDIF
42     WRITE(*,*) 'Name of output file ?'
43     READ(*,10) OUTFI
44     INQUIRE(FILE=OUTFI,EXIST=OK)
45     IF (OK) THEN
46         OPEN(UNIT=6,FILE=OUTFI)
47     ELSE
48         OPEN(UNIT=6,FILE=OUTFI,STATUS='NEW')
49     ENDIF
50     5 FORMAT(A1)
51     10 FORMAT(A16)
52 c***** READ INPUT DATA *****
53     IF(INFIL) THEN
54         READ(5,*) RLAMB,RLAM,NW,J

```

```

55     READ(5,*) RB,RC,BETAC,TC
56     DO 15 M=1,NW
57         READ(5,*) ZRE,ZIM,RP(M),BETA(M),TF(M)
58         Z(M)=CMPLX(ZRE,ZIM)
59     15 CONTINUE
60     READ(5,*) EPS,ITMAX
61     READ(5,*) XMIN,XMAX,YMIN,YMAX,NX,NY
62 ELSE
63     WRITE(*,*)'Thermal conductivity in the inner region ?'
64     READ(*,*) RLAMB
65     WRITE(*,*)'Thermal conductivity in the outer region ?'
66     READ(*,*) RLAM
67     WRITE(*,*)'Number of pipes ?'
68     READ(*,*) NW
69     WRITE(*,*)'Order of multipoles ?'
70     READ(*,*) J
71     WRITE(*,*)'Radius of inner region ?'
72     READ(*,*) RB
73     WRITE(*,*)'Radius of outer region ?'
74     READ(*,*) RC
75     WRITE(*,*)
76     WRITE(*,*)'OUTER BOUNDARY'
77     WRITE(*,*)' Thermal resistance coefficient ?'
78     READ(*,*) BETAC
79     WRITE(*,*)' Temperature ?'
80     READ(*,*) TC
81     DO 16 M=1,NW
82         WRITE(*,443) M
83         WRITE(*,*)' x-value ?'
84         READ(*,*) ZRE
85         WRITE(*,*)' y-value ?'
86         READ(*,*) ZIM
87         Z(M)=CMPLX(ZRE,ZIM)
88         WRITE(*,*)' Pipe radius ?'
89         READ(*,*) RP(M)
90         WRITE(*,*)' Thermal resistance coefficient ?'
91         READ(*,*) BETA(M)
92         WRITE(*,*)' Temperature ?'
93         READ(*,*) TF(M)
94     16 CONTINUE
95     WRITE(*,*)
96     WRITE(*,*)'Iteration accuracy ?'
97     READ(*,*) EPS
98     WRITE(*,*)'Maximum number of iterations ?'
99     READ(*,*) ITMAX
100    WRITE(*,*)
101    WRITE(*,*)'TEMPERATURE OUTPUT: Define rectangular area'
102    WRITE(*,*)' Minimum x-value ?'
103    READ(*,*) XMIN
104    WRITE(*,*)' Maximum x-value ?'
105    READ(*,*) XMAX
106    WRITE(*,*)' Minimum y-value ?'
107    READ(*,*) YMIN
108    WRITE(*,*)' Maximum y-value ?'
109    READ(*,*) YMAX
110    WRITE(*,*)' Number of grid points along the x-axis ?'
111    READ(*,*) NX

```

```

112     WRITE(*,*) ' Number of grid points along the y-axis ?'
113     READ(*,*) NY
114     WRITE(*,*)
115     IF(.NOT.INFIL) THEN
116         WRITE(*,*) 'Save input data on file ? (Y/N)'
117         READ(*,5) ANS
118         IF((ANS.EQ.'Y').OR.(ANS.EQ.'y')) THEN
119             WRITE(*,*) 'Name of input file ?'
120             READ(*,10) READFI
121             INQUIRE(FILE=READFI,EXIST=OK)
122             IF (OK) THEN
123                 OPEN(UNIT=5,FILE=READFI)
124             ELSE
125                 OPEN(UNIT=5,FILE=READFI,STATUS='NEW')
126             ENDIF
127             WRITE(5,701) RLAMB,RLAM,NW,J
128             WRITE(5,702) RB,RC,BETAC,TC
129             DO 17 M=1,NW
130                 ZRE=REAL(Z(M))
131                 ZIM=AIMAG(Z(M))
132                 WRITE(5,702) ZRE,ZIM,RP(M),BETA(M),TF(M)
133             17 CONTINUE
134             WRITE(5,703) EPS,ITMAX
135             WRITE(5,704) XMIN,XMAX,YMIN,YMAX,NX,NY
136             CLOSE(UNIT=5)
137         ENDIF
138     ENDIF
139     ENDIF
140 C***** CONSISTENCY TESTS OF INPUT DATA *****
141     IF(RLAMB.LE.0.) STOP ' RLAMB < 0.'
142     IF(RLAM.LE.0.) STOP ' RLAM < 0.'
143     IF(NW.LT.1) STOP ' NW < 1'
144     IF(J.LT.0) STOP ' J < 0'
145     IF(RB.LE.0.) STOP ' RB < 0.'
146     IF(RC.LE.0.) STOP ' RC < 0.'
147     IF(BETAC.LT.0.) STOP ' BETAC < 0.'
148     DO 20 N=1,NW
149         IF(RB.LT.CABS(Z(N))+RP(N)-1.E-6) STOP ' RB < R(N)+RP(N)'
150         IF(BETA(N).LT.0.) STOP ' BETA(N) < 0'
151         IF(RP(N).LE.0.) STOP ' RP(N) < 0.'
152         DO 20 M=1,NW
153             IF(M.NE.N) THEN
154                 IF(CABS(Z(N)-Z(M)).LT.RP(N)+RP(M)-1.E-6)
155                 + STOP ' RMN < RP(N)+RP(M)'
156                 IF(CABS(Z(N)-Z(M))-RP(N)-RP(M).LE.1.E-6) THEN
157                     IF(BETA(M)+BETA(N).EQ.0..AND.TF(N).NE.TF(M))
158                     +STOP ' RMN=RP(M)+RP(N) AND BETA(M)+BETA(N)=0. AND TF(M) NE TF(N)'
159                 END IF
160             END IF
161         20 CONTINUE
162         IF(EPS.LE.0.) STOP ' EPS LE 0.'
163 C***** LISTING OF INPUT DATA *****
164     WRITE(*,*)
165     WRITE(*,441)
166     WRITE(*,450) RLAMB,RLAM,NW,J
167     WRITE(*,460) RB,RC,BETAC,TC
168     WRITE(*,470) (M,Z(M),RP(M),BETA(M),TF(M),M=1,NW)

```

```

169 WRITE(*,480) EPS,ITMAX
170 WRITE(*,490) XMIN,XMAX,YMIN,YMAX,NX,NY
171 WRITE(6,441)
172 WRITE(6,442) READFI,OUTFI
173 WRITE(6,450) RLAMB,RLAM,NW,J
174 WRITE(6,460) RB,RC,BETAC,TC
175 WRITE(6,470) (M,Z(M),RP(M),BETA(M),TF(M),M=1,NW)
176 WRITE(6,480) EPS,ITMAX
177 WRITE(6,490) XMIN,XMAX,YMIN,YMAX,NX,NY
178 WRITE(6,410)
179 C***** SET CONSTANTS *****
180 IIT=0
181 PI=3.14159265
182 PILAMB=1./(2.*PI*RLAMB)
183 PILAM=1./(2.*PI*RLAM)
184 SIGMA=(RLAMB-RLAM)/(RLAMB+RLAM)
185 ALBETC=BETAC+ALOG(RC/RB)
186 C***** CALCULATION OF MATRICE RKO *****
187 C***** AND AUXILIARY ARRAYS *****
188 DO 60 M=1,NW
189 RBM=RB**2/(RB**2-CABS(Z(M))**2)
190 RKO(M,M)=PILAMB*(ALOG(RB/RP(M))+BETA(M)+SIGMA*ALOG(RBM))
191 + +PILAM*ALBETC
192 IF(J.GE.1) THEN
193 DO 30 K=1,J
194 ZRC(M,K)=(0.,0.)
195 RPZ(M,K)=(0.,0.)
196 RPMZN(M,M,K)=(RP(M)*CONJG(Z(M))*RBM/RB**2)**K
197 IF(CABS(Z(M)).NE.0.) THEN
198 ZRC(M,K)=(Z(M)/RC)**K
199 RPZ(M,K)=(RP(M)/Z(M))**K
200 END IF
201 30 CONTINUE
202 END IF
203 DO 50 N=1,NW
204 IF(M.NE.N) THEN
205 PMK=Z(N)-Z(M)
206 RMN=CABS(PMK)
207 RBM=RB**2/CABS(RB**2-Z(N)*CONJG(Z(M)))
208 PRB=RP(M)*CONJG(Z(N))/(RB**2-CONJG(Z(N))*Z(M))
209 RKO(M,N)=PILAMB*(ALOG(RB/RMN)+SIGMA*ALOG(RBM))+
210 + PILAM*ALBETC
211 IF(J.GE.1) THEN
212 DO 40 K=1,J
213 RPZMN(M,N,K)=(RP(M)/PMK)**K
214 RPMZN(M,N,K)=PRB**K
215 40 CONTINUE
216 END IF
217 END IF
218 50 CONTINUE
219 60 CONTINUE
220 C***** CONVERT MATRIX RKO TO VECTOR RKOVEC *****
221 K=0
222 DO 64 JJ=1,NW
223 DO 62 I=1,NW
224 K=K+1
225 RKOVEC(K)=RKO(I,JJ)

```

```

226 62 CONTINUE
227 64 CONTINUE
228 C***** INVERSION OF MATRIX RKO *****
229 CALL INV1(RKOVEC,MC,MD,NW,DET)
230 C***** CONVERT VECTOR RKOVEC TO MATRIX RKO
231 K=0
232 DO 68 JJ=1,NW
233 DO 66 I=1,NW
234 K=K+1
235 RKO(I, JJ)=RKOVEC(K)
236 66 CONTINUE
237 68 CONTINUE
238 C***** INITIAL VALUES OF ENERGY FLOWS *****
239 C***** AND MULTIPOLES *****
240 DO 80 M=1,NW
241 QBEG(M)=0.
242 DO 70 N=1,NW
243 QBEG(M)=QBEG(M)+RKO(M, N)*(TF(N)-TC)
244 70 CONTINUE
245 Q(M)=QBEG(M)
246 80 CONTINUE
247 WRITE(*,430)(N,Q(N),N=1,NW)
248 WRITE(6,430)(N,Q(N),N=1,NW)
249 IF(J.EQ.0) GO TO 280
250 DO 90 M=1,NW
251 DO 90 K=1,J
252 P(M,K)=(0.,0.)
253 90 CONTINUE
254 DO 100 K=1,J
255 PC(K)=(0.,0.)
256 100 CONTINUE
257 C***** START OF ITERATION LOOP *****
258 WRITE(*,495) EPS
259 DO 270 IIT=1,ITMAX
260 EPSMAX=0.
261 C***** MULTIPOLES AT THE PIPES *****
262 DO 160 M=1,NW
263 PMMAX=0.
264 DO 150 K=1,J
265 PMK=(0.,0.)
266 DO 130 N=1,NW
267 PRB=1./(RB**2-CONJG(Z(N))*Z(M))
268 KFAK=1
269 DO 120 JJ=1,J
270 IF(N.NE.M) PMK=PMK+P(N, JJ)*RPZMN(N, M, JJ)
271 + *RPZMN(M, N, K)*KFAK
272 JPEND=MINO(JJ, K)
273 KFAK1=KFAK
274 KFAK2=1
275 DO 110 JPRIM=0,JPEND
276 JJPRIM=JJ-JPRIM
277 KJPRIM=K-JPRIM
278 TERM=(1.,0.)
279 IF(JJPRIM.GE.1) TERM=CONJG(RPMZN(N, M, JJPRIM))
280 IF(KJPRIM.GE.1) TERM=TERM*RPMZN(M, N, KJPRIM)
281 PMK=PMK+CONJG(P(N, JJ))*SIGMA*TERM*(RP(M)*RP(N)
282 + *PRB)**JPRIM*KFAK1*KFAK2

```

```

283             IF (JPRIM.NE.JPEND) THEN
284                 KFAK1=KFAK1*KJPRIM/(K+JJ-1-JPRIM)
285                 KFAK2=KFAK2*JJPRIM/(JPRIM+1)
286             END IF
287 110         CONTINUE
288             KFAK=KFAK*(K+JJ)/JJ
289 120         CONTINUE
290             IF (N.NE.M) PMK=PMK+Q(N)*PILAMB*RPZMN(M,N,K)/K
291             PMK=PMK+Q(N)*PILAMB*SIGMA*RPMZN(M,N,K)/K
292 130         CONTINUE
293             KFAK=1
294             DO 140 JJ=K,J
295                 PMK=PMK+PC(JJ)*(1.-SIGMA)*RPZ(M,K)*ZRC(M,JJ)*KFAK
296                 KFAK=KFAK*(JJ+1)/(JJ+1-K)
297 140         CONTINUE
298             PMK=CONJG(PMK)*(BETA(M)*K-1.)/(BETA(M)*K+1.)
299             PMMAX=AMAX1(CABS(PMK),PMMAX)
300             IF (CABS(PMK).GT.1.E-7) EPSMAX=
301 +             AMAX1(EPSMAX,CABS(PMK-P(M,K))/PMMAX)
302             P2(M,K)=PMK
303 150         CONTINUE
304 160         CONTINUE
305 C***** NEW MULTIPOLES ARE ASSIGNED *****
306             DO 170 M=1,NW
307                 DO 170 K=1,J
308                     P(M,K)=P2(M,K)
309 170         CONTINUE
310 C***** CALCULATION OF MULTIPOLES *****
311 C***** AT THE OUTER CIRCLE *****
312             PMMAX=0.
313             DO 200 K=1,J
314                 PMK=(0.,0.)
315                 DO 190 M=1,NW
316                     KFAK=1
317                     DO 180 JJ=1,K
318                         PMK=PMK+P(M,JJ)*ZRC(M,K)*RPZ(M,JJ)*KFAK
319                         KFAK=KFAK*(K-JJ)/JJ
320 180         CONTINUE
321                 PMK=PMK+ZRC(M,K)*Q(M)*PILAMB/K
322 190         CONTINUE
323                 XX=(1.-BETAC*K)/(1.+BETAC*K)
324                 PMK=CONJG(PMK)*XX*(SIGMA+1.)/(SIGMA*XX*(RB/RC)**(2*K)-1.)
325                 PMMAX=AMAX1(CABS(PMK),PMMAX)
326                 IF (CABS(PMK).GT.1.E-7) EPSMAX=
327 +                 AMAX1(EPSMAX,CABS(PMK-PC(K))/PMMAX)
328                 PC(K)=PMK
329 200         CONTINUE
330 C***** CALCULATION OF NEW ENERGY FLOWS *****
331             DO 240 M=1,NW
332                 QQQ=0.
333                 DO 220 N=1,NW
334                     DO 210 JJ=1,J
335                         IF (M.NE.N) QQQ=QQQ+REAL(RPZMN(N,M,JJ)*P(N,JJ))
336                         QQQ=QQQ+SIGMA*REAL(P(N,JJ)*RPMZN(N,M,JJ))
337 210         CONTINUE
338 220         CONTINUE
339             DO 230 JJ=1,J

```

```

340          QQQ=QQQ+(1.-SIGMA)*REAL(PC(JJ)*ZRC(M,JJ))
341  230      CONTINUE
342          QM(M)=QQQ
343  240      CONTINUE
344          DO 260 M=1,NW
345            QQQQ=0.
346            DO 250 N=1,NW
347              QQQQ=QM(N)*RKO(M,N)+QQQQ
348  250      CONTINUE
349            Q(M)=QBEG(M)-QQQQ
350  260      CONTINUE
351 C***** TEST WHETHER THE ACCURACY *****
352 C***** CONDITION IS FULFILLED *****
353          WRITE(*,500) IIT,EPSSMAX
354          IF(EPSSMAX.LT.EPS) GO TO 280
355  270      CONTINUE
356 C***** OUTPUT OF ENERGY FLOWS *****
357 C***** AND MULTIPOLES *****
358  280 WRITE(6,370) IIT
359          WRITE(*,370) IIT
360          WRITE(6,380) (N,Q(N),N=1,NW)
361          WRITE(*,380) (N,Q(N),N=1,NW)
362          IF(J.GE.1) THEN
363            WRITE(6,390) ((N,K,P(N,K),K=1,J),N=1,NW)
364            WRITE(6,420) (K,PC(K),K=1,J)
365          END IF
366          WRITE(6,440)
367          WRITE(6,395)
368 C***** CALCULATION AND OUTPUT *****
369 C***** OF TEMPERATURES *****
370          DO 360 MPR=0,NX
371          DO 360 NPR=0,NY
372            ZPR=CMPLX((XMAX-XMIN)*MPR/NX+XMIN,(YMAX-YMIN)*NPR/NY+YMIN)
373 C          DO 360 MPR=0,100
374 C          ZPR=(1.,0.)+CEXP(CMPLX(0.,MPR*PI/100.))
375          DO 290 N=1,NW
376            IF(CABS(ZPR-Z(N)).LT.RP(N)-1.E-5) GO TO 360
377  290      CONTINUE
378            IF(CABS(ZPR).GT.RC) GO TO 360
379            PMK=(0.,0.)
380            IF(CABS(ZPR).LE.RB) THEN
381 C***** CALCULATION OF TEMPERATURES *****
382 C***** WHEN ZPR IS LESS THEN RB *****
383              DO 310 N=1,NW
384                IF(J.GE.1) THEN
385                  PRB=1./(RB**2-Z(N)*CONJG(ZPR))
386                  DO 300 K=1,J
387                    PMK=PMK+P(N,K)*((RP(N)/(ZPR-Z(N)))**K+
388 +                      SIGMA*(RP(N)*CONJG(ZPR)*PRB)**K)
389  300      CONTINUE
390                END IF
391                RON=CABS(ZPR-Z(N))
392                PMK=PMK+Q(N)*(PILAM*ALBETC+
393 +                  PILAMB*(ALOG(RB/RON)+SIGMA*ALOG(RB**2*CABS(PR))))
394  310      CONTINUE
395                IF(J.GE.1) THEN
396                  DO 320 K=1,J

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397             PMK=PMK+(1.-SIGMA)*PC(K)*(ZPR/RC)**K
398 320         CONTINUE
399             END IF
400             ELSE
401 C***** CALCULATION OF TEMPERATURES *****
402 C***** WHEN IS ZPR GREATER THEN RB *****
403             DO 340 N=1,NW
404                 IF(J.GE.1) THEN
405                     DO 330 K=1,J
406                         PMK=PMK+P(N,K)*(1.+SIGMA)*(RP(N)/(ZPR-Z(N)))**K
407 330         CONTINUE
408                     END IF
409                     RON=CABS(ZPR-Z(N))
410                     PMK=PMK+Q(N)*(PILAM*(ALBETC+SIGMA*
411 +                 ALOG(RB/CABS(ZPR)))+PILAMB*(1.+SIGMA)*ALOG(RB/RON))
412 340         CONTINUE
413                 IF(J.GE.1) THEN
414                     DO 350 K=1,J
415                         PMK=PMK+PC(K)*((ZPR/RC)**K-SIGMA*
416 +                 (RB**2/(RC*CONJG(ZPR)))**K)
417 350         CONTINUE
418                 END IF
419             END IF
420             TPR=REAL(PMK)+TC
421             X=REAL(ZPR)
422             Y=AIMAG(ZPR)
423             WRITE(6,400) X,Y,TPR
424 360         CONTINUE
425             WRITE(*,510) OUTFI
426 370 FORMAT(/' Number of iterations: ',I4/)
427 380 FORMAT(/' Pipe      q(n)'/ (I5,3X,E14.7))
428 390 FORMAT(/' Pipe',4X,'Order',11X,'P(n,k)'/ (I5,I8,3X,2E12.5))
429 395 FORMAT(/8X,'x',13X,'y',12X,'Temp',/)
430 400 FORMAT(2X,F10.5,4X,F10.5,4X,F10.5)
431 410 FORMAT(/,75('*'))
432 420 FORMAT(/' Order',13X,'PC(k)'/ (I5,4X,2E12.5))
433 430 FORMAT(/' Pipe Initial values for q(n) (Order of multipole = 0)'
434 +         / (I5,E17.7))
435 440 FORMAT(/1X,75('*')/ ' TEMPERATURES (Deg C)')
436 441 FORMAT(' MULTIPOLE METHOD - Pipes in a composite cylinder'/,
437 + 1X,48(1H=),//, ' INPUT DATA')
438 442 FORMAT(/, ' Input file ',A16,/, ' Output file ',A16,/)
439 443 FORMAT(/, ' Pipe n=',I2)
440 450 FORMAT(' Thermal conductivity in inner region',
441 + 3X,F10.3, ' W/(m*K)'/ ' Thermal conductivity in outer region',
442 + 3X,F10.3, ' W/(m*K)'/ ' Number of pipes',8X,I10/
443 + ' Order of multipoles',4X,I10)
444 460 FORMAT(/' Radius of inner region',9X,F10.3, ' m'/
445 + ' Radius of outer region',9X,F10.3, ' m'/
446 + ' Outer boundary',/,
447 + ' Thermal resistance coefficient ',2X,E10.3,/,
448 + ' Temperature',23X,F10.3, ' C')
449 470 FORMAT(/1X,34('*')/
450 + ' Pipe * x(n) * y(n) * rp(n) * beta(n)',
451 + ' * Temp C */1X,34('*')/
452 + (I5, ' ',3(F9.3, ' '),E10.3, ' ',F9.3, ' '))
453 480 FORMAT(/' Iteration accuracy',13X,E10.2/

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454 + ' Maximum number of iterations',2X,I10)
455 490 FORMAT(/' TEMPERATURE OUTPUT: Definition of rectangular area',/,
456 + ' Minimum x-value',10X,F10.3/
457 + ' Maximum x-value',10X,F10.3/
458 + ' Minimum y-value',10X,F10.3/
459 + ' Maximum y-value',10X,F10.3/
460 + ' Number of grid points along x-axis',I11/
461 + ' Number of grid points along y-axis',I11)
462 495 FORMAT(/' Starting iterations Goal: Iteration accuracy ',E10.2,/)
463 500 FORMAT(' Number of iterations',I4,' Iteration accuracy ',E10.2)
464 510 FORMAT(//' Job ended. Output written to ',A16)
465 701 FORMAT(1X,2(E12.5,2X),2I5)
466 702 FORMAT(1X,5(E12.5,2X))
467 703 FORMAT(1X,E12.5,I7)
468 704 FORMAT(1X,4(E12.5,2X),2I5)
469 END
470 SUBROUTINE INV1(B,MC,MD,N,D)
471 C-----
472 C Calculate inverse of a matrix
473 C
474 C Author: Anders Peterson
475 C Date: 1982-05-28
476 C-----
477 IMPLICIT REAL*8 (A-H,O-Z)
478 DIMENSION B(1),MC(1),MD(1)
479 C
480 DATA C1/1.0D0/
481 DATA C2/1.0D50/
482 C
483 D=C1
484 NK=-N
485 DO 80 K=1,N
486 NK=NK+N
487 MC(K)=K
488 MD(K)=K
489 KK=NK+K
490 BIGA=B(KK)
491 DO 20 J=K,N
492 IZ=N*(J-1)
493 DO 20 I=K,N
494 IJ=IZ+I
495 R1=B(IJ)
496 IF(R1.LT.0.0D0) R1=-R1
497 R2=BIGA
498 IF(R2.LT.0.0D0) R2=-R2
499 R2=R2-R1
500 10 IF(R2.GT.0.0D0) GO TO 20
501 BIGA=B(IJ)
502 MC(K)=I
503 MD(K)=J
504 20 CONTINUE
505 J=MC(K)
506 IF(J-K.LE.0) GO TO 35
507 KI=K-N
508 DO 30 I=1,N
509 KI=KI+N
510 HOLD=-B(KI)

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511      JI=KI-K+J
512      B(KI)=B(JI)
513  30  B(JI) =HOLD
514  35  I=MD(K)
515      IF(I-K.LE.0) GO TO 45
516      JP=N*(I-1)
517      DO 40 J=1,N
518      JK=NK+J
519      JI=JP+J
520      HOLD=-B(JK)
521      B(JK)=B(JI)
522  40  B(JI) =HOLD
523  45  IF(ABS(BIGA).GT.0.0D0) GO TO 48
524  46  D=0.0
525      RETURN
526  48  DO 55 I=1,N
527      IF(I-K.EQ.0) GO TO 55
528      IK=NK+I
529      B(IK)=B(IK)/(-BIGA)
530  55  CONTINUE
531      DO 65 I=1,N
532      IK=NK+I
533      HOLD=B(IK)
534      IJ=I-N
535      DO 65 J=1,N
536      IJ=IJ+N
537      IF(I-K.EQ.0) GO TO 65
538      IF(J-K.EQ.0) GO TO 65
539      KJ=IJ-I+K
540      B(IJ)=HOLD*B(KJ)+B(IJ)
541  65  CONTINUE
542      KJ=K-N
543      DO 75 J=1,N
544      KJ=KJ+N
545      IF(J-K.EQ.0) GO TO 75
546      B(KJ)=B(KJ)/BIGA
547  75  CONTINUE
548 C
549 C      UNDEFINED VALUE OF D IF ABS(D).GT.C2
550 C
551      IF(D.LT.C2 .AND.D.GT.-C2) D=D*BIGA
552 C
553      B(KK)=C1/BIGA
554  80  CONTINUE
555      K=N
556  100 K=(K-1)
557      IF(K.LE.0) GO TO 150
558      I=MC(K)
559      IF(I-K.LE.0) GO TO 120
560  108 JQ=N*(K-1)
561      JR=N*(I-1)
562      DO 110 J=1,N
563      JK=JQ+J
564      HOLD=B(JK)
565      JI=JR+J
566      B(JK)=-B(JI)
567  110 B(JI) =HOLD

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568 120 J=MD(K)
569     IF(J-K.LE.0) GO TO 100
570 125 KI=K-N
571     DO 130 I=1,N
572         KI=KI+N
573         HOLD=B(KI)
574         JI=KI-K+J
575         B(KI)=-B(JI)
576 130 B(JI) =HOLD
577     GO TO 100
578 150 CONTINUE
579     RETURN
580     ENDæ
```