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Multipole Moments in General Relativity

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Multipole moments in general relativity are defined as coefficients of a multipole expansion of appropriate potentials, as they are so in Newton's theory of gravitation. The essential point is the introduction of Fock's harmonic coordinate system in which the potentials are expanded in inverse powers of the distance from the source. First several moments are obtained for the Kerr, Tomimatsu-Sato and a class of the Weyl solutions of the Einstein equation, and then are inferred all moments for the Kerr and Weyl solutions.

§ 1. Introduction

The problem of obtaining multipole moments of a solution of the Einstein equation is the problem of interpreting the solution in terms of its Newtonian limit. In particular the knowledge of multipole moments serves to infer a possible source distribution which produces the gravitational field in question. Since the interior solutions which may be considered to describe the interior metric of the source of the Weyl,[1] Kerr[2] or Tomimatsu-Sato (T-S)[3] gravitational field have not been discovered at this stage, it is desirable to have a systematic method of obtaining multipole moments of these fields.

One of the methods was developed by Geroch using conformal Killing vectors,[4),5] and by means of this method Hansen was able to obtain multipole moments of the Kerr solution.[6] Although it is not so easy to find the necessary conformal factor, once it is found, Geroch's method allows us to calculate all moments in principle. There are also other methods of finding multipole moments, although they are in many cases not adequate to find higher moments. For example, Voorhees obtained the quadrupole moment of a Weyl solution;[7] Hernandez obtained all moments of the Kerr solutions;[8] Tomimatsu and Sato obtained the quadrupole moments of their solutions.[3]

The purpose of this paper is to present a new method of calculating multipole moments which, we hope, is complementary to Geroch's method and containing all the results obtained by the above authors. The idea is to introduce Fock's harmonic coordinate system[9] in which appropriate potentials are expanded in inverse powers of the distance from the source. This expansion will be hoped to be of the form of a multipole expansion in the Newtonian limit. Of cource, there is, a priori, no assurance that such a expansion will be just of the form of a multipole expansion. Then the main purpose of this paper is to show that this is the case by constructing explicitly such expansions.

In this paper stationary axially-symmetric gravitational fields which can be incorporated into the Ernst formalism[10] are exclusively treated. In § 2 we define the appropriate potentials which satisfy Laplace's equation in Euclidian 3-space to the first approximation. There are two potentials: One is the mass potential equivalent to the usual Newtonian potential, and the other is the angular momentum potential which is new and can be considered to describe the effects of the rotation of the source. These potentials are originally introduced by Hansen.[6] In § 3 Fock's harmonic coordinates are explained and the method of obtaining harmonic coordiantes is stated. In § 4 the results of the multipole expansion for the Kerr, Weyl and T-S solutions are given. In § 5 we discuss the meaning of harmonic coordinates.

§ 2. Definition of potentials

For a space-time with metric $g_{\alpha\beta}$ (signature: $+---$) which admits a Killing vector field t^α, Geroch has worked out a 3-dimensional formalism of the Einstein equation.[11] We define the norm and twist of t^α respectively by

$$\lambda = t^\alpha t_\alpha\,, \qquad \omega_\alpha = \varepsilon_{\alpha\beta\gamma\delta} t^\beta \nabla^\gamma t^\delta\,, \tag{2·1}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor and ∇_α is the covariant derivative with respect to $g_{\alpha\beta}$. A new metric $h_{\alpha\beta}$ is defined by

$$h_{\alpha\beta} = \lambda g_{\alpha\beta} - t_\alpha t_\beta\,. \tag{2·2}$$

This metric is formally a 4×4 matrix, but the rank of it is three, and therefore can be considered as a metric on a 3-dimensional manifold which, for the static case (i.e., for $\omega_\alpha = 0$), can be identified with a hyper-surface orthogonal to t^α. The covariant derivative with respect to $h_{\alpha\beta}$ will be denoted by D_α. (The notations $h_{\alpha\beta}$, D_α correspond to $\tilde{h}_{ab}$, $\tilde{D}_a$, not just to h_{ab}, D_a in Geroch's paper.) Then Geroch has shown that the source-free Einstein equation implies (1) the existence of a scalar field ω such that

$$\omega_\alpha = D_\alpha \omega\,, \tag{2·3}$$

and (2) the fields λ, ω are subject to the following equations:

$$\lambda D^2 \lambda = D\lambda \cdot D\lambda - D\omega \cdot D\omega\,, \tag{2·4}$$

$$\lambda D^2 \omega = 2 D\lambda \cdot D\omega\,, \tag{2·5}$$

$$\mathcal{R}_{\alpha\beta} = (2\lambda^2)^{-1} [(D_\alpha \lambda)(D_\beta \lambda) + (D_\alpha \omega)(D_\beta \omega)], \tag{2·6}$$

where $\mathcal{R}_{\alpha\beta}$ is the Ricci tensor associated with $h_{\alpha\beta}$ and $D^2\lambda$, $D\lambda\cdot D\lambda$ are abbreviations of $D^\alpha D_\alpha \lambda$, $D^\alpha \lambda D_\alpha \lambda$.

If we introduce a complex function defined by

$$\mathcal{E} = \lambda - i\omega\,, \tag{2·7}$$

Eqs. (2·4) and (2·5) are combined into a single equation

$$(\mathrm{Re}\,\mathcal{E})\,D^2\mathcal{E}=D\mathcal{E}\cdot D\mathcal{E}\,. \tag{2·8}$$

This equation is very similar to the equation introduced originally by Ernst.[10] In fact, it can be shown that this equation is equivalent to that of Ernst for the stationary, axially-symmetric metric treated by Ernst. Therefore Geroch's formalism may be considered as the generalization of Ernst's formalism, for Geroch has only assumed the existence of a Killing vector, and not any particular form of the metric. Following Ernst, we introduce another complex function ξ (Ernst's potential) defined by

$$\mathcal{E}=\frac{\xi-1}{\xi+1}\,. \tag{2·9}$$

Equation (2·8) becomes

$$(\xi\xi^*-1)\,D^2\xi=2\xi^*D\xi\cdot D\xi\,. \tag{2·10}$$

Using (2·6) and (2·10) we can prove that the complex function

$$\varphi=-\frac{\xi}{\xi\xi^*-1}=\phi_M+i\phi_J \tag{2·11}$$

satisfies the equation

$$D^2\varphi=2\mathcal{R}\varphi\,, \tag{2·12}$$

where $\mathcal{R}$ is given by

$$\mathcal{R}=2(D\phi_M\cdot D\phi_M+D\phi_J\cdot D\phi_J)$$
$$-(1/2)\,[D(4\phi_M{}^2+4\phi_J{}^2+1)^{1/2}]\cdot[D(4\phi_M{}^2+4\phi_J{}^2+1)^{1/2}]. \tag{2·13}$$

From now on we assume that the Killing vector t^α is time-like, and the space-time is approximately Minkowskian. Then we see that Eq. (2·12) is equivalent to the usual Laplace equation in Euclidian 3-space to first order. The potentials ϕ_M, ϕ_J are originally introduced by Hansen in the form[6]

$$\phi_M=\frac{1}{4\lambda}(\lambda^2+\omega^2-1)\,,\qquad \phi_J=\frac{1}{2\lambda}\omega\,, \tag{2·14}$$

and are called Hansen's potential in this paper. The mass potential ϕ_M is approximately equal to $(g_{00}-1)/2$ and, therefore, can be identified with Newton's gravitational potential. The angular momentum potential ϕ_J is an analog of the magnetic scalar potential in electromagnetism.

§ 3. Harmonic coordinate system

A harmonic coordinate system is a coordiante system in which the metric satisfies $\partial_\alpha(\sqrt{-g}\,g^{\alpha\beta})=0$. If the source distribution is restricted to a finite region, we further require that it is Minkowskian at spatial infinity. If a solution of the

Einstein equation is given by $ds^2=g'_{\alpha\beta}du^\alpha du^\beta$ in an arbitrary coordinate system $\{u^\alpha\}$, then a harmonic coordinate system $\{x^\alpha\}$ can be obtained as a set of four solutions of the equation[9]

$$\Box x^\alpha=\frac{1}{\sqrt{-g'}}\frac{\partial}{\partial u^\mu}\left[\sqrt{-g'}\,g'^{\mu\nu}\frac{\partial x^\alpha}{\partial u^\nu}\right]=0\,. \tag{3·1}$$

In prolate spheroidal coordinates, the stationary axially-symmetric metric treated by Ernst is given by

$$ds^2=f(dt-\overline{\omega}d\phi)^2-f^{-1}\left(\frac{mp}{\delta}\right)^2\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\phi^2\right], \tag{3·2}$$

where functions f, $\overline{\omega}$, γ depend only on x and y, constants p,δ are parameters which appear in the exact solutions treated later and m is the total mass of the source. If we assume four harmonic coordinates for this metric in the form

$$x^0=t,\ \ x^1=\psi(x,y)\cos\phi,\ \ x^2=\psi(x,y)\sin\phi,\ \ x^3=\chi(x,y), \tag{3·3}$$

then Eq. (3·1) implies the following equations:

$$\frac{\partial}{\partial x}\left[(x^2-1)\frac{\partial\psi}{\partial x}\right]+\frac{\partial}{\partial y}\left[(1-y^2)\frac{\partial\psi}{\partial y}\right]=\frac{e^{2\gamma}(x^2-y^2)}{(x^2-1)(1-y^2)}\,, \tag{3·4}$$

$$\frac{\partial}{\partial x}\left[(x^2-1)\frac{\partial\chi}{\partial x}\right]+\frac{\partial}{\partial y}\left[(1-y^2)\frac{\partial\chi}{\partial y}\right]=0\,. \tag{3·5}$$

Taking into account the boundary conditions we take the solutions of these equations in the form

$$\psi=\frac{mp}{\delta}x\left[1+\sum_{n=1}^{\infty}\frac{f_n(y)}{x^n}\right]\sqrt{1-y^2}\,, \tag{3·6}$$

$$\chi=\frac{mp}{\delta}xy\,. \tag{3·7}$$

The solution χ is valid for every metric. The solution ψ is taken in this form because Eq. (3·4) generally cannot be solved in a closed form. Substituting (3·6) into (3·4), we see that functions $f_n(y)$ must satisfy

$$(1-y^2)\frac{d^2f_n}{dy^2}-4y\frac{df_n}{dy}+n(n-3)f_n=b_n(y)+(n-2)(n-3)f_{n-2}+\sum_{k=1}^{n-1}f_{n-k}b_k(y)\,, \tag{3·8}$$

where $b_k(y)$ is defined by

$$\frac{e^{2\gamma}(x^2-y^2)}{(x^2-1)(1-y^2)}=\frac{1}{1-y^2}+\sum_{k=1}^{\infty}\frac{b_k(y)}{x^k}\,. \tag{3·9}$$

Equation (3·8) is solved step by step with the condition that f_n is uniquely determined by $f_1,\cdots,f_{n-1}$. By saying "uniquely", we mean that we neglect solutions of the homogeneous equation, i.e., the equation of which the right-hand side vanishes. This corresponds physically to the assumption that harmonic coordinates are uniquely determined by the boundary conditions. For the Weyl, Kerr and T-S solutions, the function b_k is zero for k=odd and a polynomial of at most degree $k-2$ for k=even. Then Eq. (3·8) can be easily solved with the result that f_n is zero for n=odd and a polynomial of at most degree $n-2$ for n=even.

The spatial distance r and polar angle θ are respectively defined by

$$r^2=(x^1)^2+(x^2)^2+(x^3)^2=\psi^2+\chi^2\,,$$
$$\cos\theta=x^3/r=\chi/(\psi^2+\chi^2)^{1/2}\,. \tag{3·10}$$

Then it is shown that the coordinates x, y can be expanded in the form

$$\frac{\delta}{px}=\frac{m}{r}\left[1+\sum_{k=1}^{\infty}\left(\frac{m}{r}\right)^k B_k(\cos\theta)\right],$$
$$y=\cos\theta\left[1+\sum_{k=1}^{\infty}\left(\frac{m}{r}\right)^k B_k(\cos\theta)\right], \tag{3·11}$$

where $B_k(\cos\theta)$ are polynomials of $\cos\theta$ determined by the formula

$$B_n=\frac{1}{2}\left[-B_n^{(2)}+\left(\frac{p}{\delta}\right)^n\sum_{l=0}^{n}K_l^{(n)}\cos^l\theta+\sum_{k=1}^{n-1}\left(\frac{p}{\delta}\right)^k\sum_{l=1}^{k}K_l^{(k)}\cos^l\theta\sum_{s=1}^{k+l}\binom{k+l}{s}B_{n-k}^{(s)}\right], \tag{3·12}$$

where constants $K_l^{(n)}$ and functions $B_k^{(s)}$ are respectively defined by

$$(1-y^2)\,(2f_n+\sum_{k=1}^{n-1}f_{n-k}f_k)=\sum_{l=1}^{n}K_l^{(n)}y^l\,, \tag{3·13}$$

$$(\sum_{k=1}^{\infty}B_kZ^k)^s=\sum_{k=1}^{\infty}B_k^{(s)}Z^k\,. \tag{3·14}$$

§ 4. Multipole expansions

From the papers of Voorhees,[7] Ernst,[10] and Tomimatsu-Sato,[3] a class of exact solutions of the Einstein equation is summarized as follows:

(1) *a series of the Weyl solution*

$$\xi=\frac{(x+1)^\delta+(x-1)^\delta}{(x+1)^\delta-(x-1)^\delta}\,,\qquad e^{2\gamma}=\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}, \tag{4·1}$$

(2) *the Kerr solution*

$$\xi=px-iqy\,,\qquad e^{2\gamma}=\frac{p^2x^2+q^2y^2-1}{p^2(x^2-y^2)}\,, \tag{4·2}$$

(3) *the Tomimatsu-Sato (T-S) solution*

$$\xi=\frac{p^2x^4+q^2y^4-1-2ipqxy(x^2-y^2)}{2px(x^2-1)-2iqy(1-y^2)}\,,\qquad e^{2\gamma}=\frac{A}{p^4(x^2-y^2)^4}\,,$$

$$A=p^4(x^2-1)^4+q^4(1-y^2)^4$$
$$-2p^2q^2(x^2-1)(1-y^2)[2(x^2-1)^2+2(1-y^2)^2+3(x^2-1)(1-y^2)]. \quad (4\cdot3)$$

Here constants p, q are related by $p^2+q^2=1$. The T-S solution approaches the Weyl solution with $\delta=2$ as $q\to0$. The T-S solution which approach the Weyl solution with $\delta=3$ as $q\to0$ is also known and given in their paper.[3)]

From the function $e^{2\gamma}$ the harmonic coordinates are calculated by the method given in §2. From Ernst's potential ξ, Hansen's potentials (2·11) are calculated and expanded in the form

$$\phi_M=-\sum_{k=0}^{\infty}\left(\frac{\delta}{px}\right)^{2k+1}M_{2k+1}(y), \qquad \phi_J=\sum_{k=1}^{\infty}\left(\frac{\delta}{px}\right)^{2k}J_{2k}(y), \quad (4\cdot4)$$

where $M_{2k+1}(y)$ and $J_{2k}(y)$ become polynomials of y. Substituting (3·11) into (4·4), we can express ϕ_M and ϕ_J in terms of r and θ. The results are

$$\phi_M=-\sum_{k=0}^{\infty}\left(\frac{m}{r}\right)^{2k+1}C_{2k+1}, \qquad \phi_J=q\sum_{k=1}^{\infty}\left(\frac{m}{r}\right)^{2k}\overline{C}_{2k}, \quad (4\cdot5)$$

where C_{2k+1}, $\overline{C}_{2k}$ are polynomials of $\cos\theta$ given in the following:

(1) *For the Weyl solution*

$$C_1=1, \qquad C_3=-(\varDelta/3)P_2(\cos\theta)+1,$$
$$C_5=(\varDelta^2/5)P_4(\cos\theta)-\varDelta W_1(\cos\theta)+1,$$
$$C_7=-(\varDelta^3/7)P_6(\cos\theta)+\varDelta^2W_3(\cos\theta)-\varDelta W_2(\cos\theta)+1,$$
$$C_9=(\varDelta^4/9)P_8(\cos\theta)+(\text{terms of lower power in }\varDelta),$$
$$\overline{C}_2=\overline{C}_4=\cdots=0, \quad \text{where} \quad \varDelta=(\delta^2-1)/\delta^2. \quad (4\cdot6)$$

(2) *For the Kerr solution*

$$C_1=1, \qquad C_3=-q^2P_2(\cos\theta)+1,$$
$$C_5=q^4P_4(\cos\theta)-q^2K_1(\cos\theta)+1,$$
$$C_7=-q^6P_6(\cos\theta)+q^4K_5(\cos\theta)-q^2K_2(\cos\theta)+1,$$
$$C_9=q^8P_8(\cos\theta)+(\text{terms of lower power in }q^2),$$
$$\overline{C}_2=P_1(\cos\theta), \qquad \overline{C}_4=-q^2P_3(\cos\theta)+\cos\theta,$$
$$\overline{C}_6=q^4P_5(\cos\theta)-q^2K_3(\cos\theta)+\cos\theta,$$
$$\overline{C}_8=-q^6P_7(\cos\theta)+q^4K_6(\cos\theta)-q^2K_4(\cos\theta)+\cos\theta,$$
$$\overline{C}_{10}=q^8P_9(\cos\theta)+(\text{terms of lower power in }q^2). \quad (4\cdot7)$$

(3) *For the T-S solution*

$$C_1=1, \qquad C_3=-\left(\frac{1}{3}\varDelta p^2+q^2\right)P_2(\cos\theta)+1,$$

$$C_5=\left(\frac{1}{5}\Delta^2p^4+\frac{16}{15}\Delta p^2q^2+q^4\right)P_4(\cos\theta)-\Delta p^2W_1(\cos\theta)-q^2K_1(\cos\theta)+1\,,$$

$$C_7=-\left(\frac{1}{7}\Delta^3p^6+\frac{331}{315}\Delta^2p^4q^2+\frac{199}{105}\Delta p^2q^4+q^6\right)P_6(\cos\theta)+\Delta^2p^4W_3(\cos\theta)$$
$$+\Delta p^2q^2T_2(\cos\theta)+q^4K_5(\cos\theta)-\Delta p^2W_2(\cos\theta)-q^2K_2(\cos\theta)+1\,,$$

$$\bar{C}_2=P_1(\cos\theta)\,,\qquad \bar{C}_4=-\left(\frac{2}{3}\Delta p^2+q^2\right)P_3(\cos\theta)+\cos\theta\,,$$

$$\bar{C}_6=\left(\frac{23}{45}\Delta^2p^4+\frac{22}{15}\Delta p^2q^2+q^4\right)P_5(\cos\theta)-\Delta p^2T_1(\cos\theta)-q^2K_3(\cos\theta)+\cos\theta\,. \tag{4·8}$$

In these expression, $P_n(Z)$ denotes the n-th Legendre polynomial and $W_n(Z)$, $K_n(Z)$, $T_n(Z)$ are polynomials of Z given in the following:

$$W_1=(60)^{-1}(5Z^4+90Z^2-31)\,,$$
$$W_2=(420)^{-1}(14Z^6+105Z^4+1050Z^2-373)\,,$$
$$W_3=(2520)^{-1}(1197Z^6+7770Z^4-7175Z^2+856)\,,$$
$$K_1=2^{-1}(7Z^2-3)\,,\ K_2=9^{-1}(49Z^2-22)\,,$$
$$K_3=2^{-1}(9Z^3-5Z)\,,\qquad K_4=3^{-1}(19Z^3-10Z)\,,$$
$$K_5=8^{-1}(107Z^4-98Z^2+15)\,,$$
$$K_6=8^{-1}(151Z^5-162Z^3+35Z)\,,$$
$$T_1=(30)^{-1}(9Z^5+80Z^3-45Z)\,,$$
$$T_2=(840)^{-1}(875Z^6+245Z^4+525Z^2+1539)\,. \tag{4·9}$$

The expansion for the T-S solutions are determined with the presupposition that C_{2k+1}, $\bar{C}_{2k}$ will be homogeneous polynomials of Δp^2 and q^2, and then by using the information from the expansions of the known solutions. Although these expressions are ascertained only for $\delta=2, 3$ (i.e., for $\Delta=3/4$, $8/9$), we hope that they are valid for arbitrary values of δ, if, of course, T-S solutions with arbitrary values of δ exist.

Now we consider the Newtonian limit of the above expansions. There are two cases: One is the static case in which $q=0$. This case corresponds to the Weyl solution. In this case taking the limit $m\to 0$ holding $b(=m\Delta^{1/2})$ finite and neglecting terms of higher order in m, we obtain

$$\phi_{M,J}=\sum_{l=0}^{\infty}Q^l_{M,J}\frac{P_l(\cos\theta)}{r^{l+1}}\,, \tag{4·10}$$

where $Q^l_{M,J}$ is given by

$$Q_M{}^{2k}=(-1)^{k+1}(2k+1)^{-1}mb^{2k}\,,\quad Q_M{}^{2k+1}=0\,,$$

$$Q_J^{2k}=Q_J^{2k+1}=0\,. \tag{4·11}$$

The other is the Kinnersley-Kelley limit.[12] Following Kinnersley and Kelley we make the replacement $p\to i\hat{p}$. Then we take the limit $m\to 0$ holding a $(=mq)$ finite. Using the relation $-\hat{p}^2+q^2=1$ and neglecting terms of higher order in m, we obtain a similar expansion to Eq. (4·10) with multipole moments given in the following:

For the Kerr solution,

$$Q_M^{2k}=(-1)^{k+1}ma^{2k}\,,\quad Q_M^{2k+1}=0\,,$$
$$Q_J^{2k}=0\,,\quad Q_J^{2k+1}=(-1)^k ma^{2k+1}\,. \tag{4·12}$$

For the T-S solution,

$$Q_M^0=-m\,,\quad Q_M^2=ma^2\left(-\frac{1}{3}\Delta+1\right),$$
$$Q_M^4=-ma^4\left(\frac{1}{5}\Delta^2-\frac{16}{15}\Delta+1\right),\quad Q_M^6=ma^6\left(-\frac{1}{7}\Delta^3+\frac{331}{315}\Delta^2-\frac{199}{105}\Delta+1\right),$$
$$Q_J^1=ma\,,\quad Q_J^3=-ma^3\left(-\frac{2}{3}\Delta+1\right),$$
$$Q_J^5=ma^5\left(\frac{23}{45}\Delta^2-\frac{22}{15}\Delta+1\right),\quad Q_M^{2k+1}=Q_J^{2k}=0\,. \tag{4·13}$$

The quadrupole moment of the Weyl solution agree with that obtained by Voorhees.[7] The results for the Kerr solution agree with that of Hernandez and Hansen. The Newtonian potential having these moments was previously discussed by Keres[13] and Israel.[14] The moments of the T-S solution are the same as that given by Kinnersley and Kelley for common values of δ and l.

Numerical factors appearing in the moments of the T-S solution are rather unfamiliar and give no clue as to the general formula. We only point out that these factors also appear in the expansion (4·4) for the Weyl solution. In fact, for the Weyl solution the first several coefficients M_k are given by

$$M_1=1\,,\quad M_3=-\frac{1}{3}\Delta+1\,,\quad M_5=\frac{1}{5}\Delta^2-\frac{16}{15}\Delta+1\,,$$
$$M_7=-\frac{1}{7}\Delta^3+\frac{331}{315}\Delta^2-\frac{199}{105}\Delta+1\,,$$
$$M_9=\frac{1}{9}\Delta^4-\frac{2896}{2835}\Delta^3+\frac{844}{315}\Delta^2-\frac{872}{315}\Delta+1\,. \tag{4·14}$$

The correspondence between these coefficients and the mass moments Q_M^l of the T-S solution is clear. If this correspondence is true the next moment willl be

$$Q_M^8=-ma^8\left(\frac{1}{9}\Delta^4-\frac{2896}{2835}\Delta^3+\frac{844}{315}\Delta^2-\frac{872}{315}\Delta+1\right). \tag{4·15}$$

§ 5. Discussion

In the last section we have shown that if Hansen's potentials are expanded in the inverse powers of the distance from the source, the resulting expansions are of the form of multipole expansions in the Newtonian limit. Then the question arises as to whether Hansen's potentials or harmonic coordinates have any particular meaning concerning the multipole expansion. The answer is that since we are only concerned with the Newtonian order, i.e., the order m, other potentials and other coordinates which differ from Hansen's potentials and harmonic coordinates only in higher order in m are also permissible and will give the same results. For example, in a previous paper[15] the multipole expansion of the potential $(g_{00}-1)/2$ in harmonic coordinates are obtained with the same results as that given in this paper. The coordinate systems used by Voorhees[7] or Tomimatsu-Sato[3] or Kinnersley-Kelley[12] are approximately equivalent to harmonic ones in the above sense. This is the reason why their results agree with ours. However, the calculation is simple for Hansen's potentials and our method of using harmonic coordinates is somewhat general and straightforward in the sense that harmonic coordinates are well-defined for every solution and require no particular technique in choosing the coordinate systems which admit multipole expansions.

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