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Multirate-Output-Feedback-Based LQ-Optimal Discrete-Time Sliding Mode Control

S. Janardhanan and Vinay Kariwala

***Abstract*—The traditional approach for sliding mode control design has been the design of a controller to achieve a predesigned sliding objective. However, not much research has been carried out on the design of the sliding surface. This note presents a technique for designing a sliding surface such that when confined to the surface, the closed-loop system has optimality in the linear quadratic sense. The paper also proposes a multirate-output-feedback-based controller that leads the system to the aforementioned optimal sliding mode.**

***Index Terms*—Optimal control, output feedback, sliding mode control, uncertain systems.**

I. Introduction

The concept of sliding mode control (SMC) has received much attention in the past few decades. SMC is a technique in which an appropriate input is provided so that the system states are confined to a desired submanifold of the state space. The concept of SMC was proposed by Emelyanov [1] and Utkin [2], who showed that sliding mode can be achieved by changing the controller structure. The system state trajectory is forced to move along a chosen manifold in the state space, called the sliding manifold, by the use of an appropriate variable structure control signal.The closed-loop behavior of the system is, thus, governed by the dynamics of the surface [3], [4]. Researchers have worked on the idea of robust optimal control in a system using the concept of variable structure control [5]–[7]. Concepts such as model following control [5] and integral sliding mode [7] have been used to design control algorithms that give robust performance. Due to the use of computers for control purpose, the concept of a digital sliding mode (DSM) controller design has also been a topic of study during the past few years [8], [9]. In the case of the DSM design, the control input is applied only at certain sampling instants and the control effort is constant over the entire sampling period.

In spite of the availability of a large volume of literature on the SMC, much less research has been carried out on the design of the sliding surface [10], [11]. Most of the existing literature suggests that the sliding surface is designed so that the closed-loop system is stable and has some desirable properties when confined to it. However, not much literature is available for identifying and correlating these desired properties with an appropriate sliding surface for discrete-time systems.

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In their paper on discrete-time variable structure control [9], Gao *et al.* have detailed a method for sliding surface design based on eigenvalue placement. Tang and Misawa [12] proposed a sliding surface design method, which is based partially on the LQR optimization and partially on the pole placement. These methods require *a priori* knowledge of the best locations of the eigenvalues for the closed-loop system, which is not realistic. Minimization of a linear quadratic performance index is a more general property that is desired in many control systems.

This note aims at designing a sliding surface such that when confined to the surface, the minimization of a specified linear quadratic cost function [7] for discrete-time systems is guaranteed. It is shown that this surface can be found by solving a matrix quadratic equation involving the state and input matrices of the system. The note also proposes a multirate-output-feedback-based controller that leads the system to the aforementioned sliding mode. The organization of the rest of the paper is as follows: Section II describes the problem statement and the quadratic performance index under consideration. The proposed sliding surface design is presented in Section III. A state feedback SMC based on a disturbance estimator is discussed in Section IV. The multirate-output-feedback-based controller is proposed in Section V. This is followed by illustration through a numerical example in Section VI and the conclusion in Section VII.

II. Problem Statement

Consider the discrete-time representation of a controllable and observable system sampled with a sampling period of τ sec

$$\begin{aligned} x(k+1) &= \Phi_\tau x(k) + \Gamma_\tau u(k) + D_\tau d(k) \\ y(k) &= Cx(k). \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d(k) \in \mathbb{R}^{n_d}$, and $y \in \mathbb{R}^p$. The following assumptions are made regarding the structure of the system.

1) *Assumption 1.* It is assumed that the disturbance affecting the system is a matched disturbance [13]. This implies $D_\tau = \Gamma_\tau T_m$ with T_m being a matrix with dimensions $m \times n_d$. In physical terms, the assumption implies that the disturbance affecting the system enters through the input channels.
2) *Assumption 2.* It is assumed, without loss of generality, that the system (1) is in the normal form [14], [15]. It should be noted that even if the original system is not in the normal form, there always exists a similarity transformation $z = Tx$ such that the system is in normal form in the z-coordinate frame.

The aim is to design the sliding surface and its corresponding controller, such that when the closed-loop system is confined to the sliding surface, the quadratic performance index [7]

$$J = \sum_{k=0}^{\infty} x^T(k)Qx(k) + u_{\text{eq}}^T(k)Ru_{\text{eq}}(k) \tag{2}$$

is minimized. Here, u_{eq} denotes the equivalent control that maintains the nominal system obtained by ignoring the disturbance in (1)

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u_{\text{eq}}(k) \tag{3}$$

on the sliding hyperplane. In (2), Q and R are positive semidefinite and positive definite weighing matrices, respectively.

III. Sliding Surface Design

Let us partition the state x of the system as $x = [\,x_1^T \quad x_2^T\,]^T$ such that $x_1 \in \mathbb{R}^{n-m}$ and $x_2 \in \mathbb{R}^m$. Due to Assumption 2,

$$x_1(k+1) = \Phi_{11}x_1(k) + \Phi_{12}x_2(k) \tag{4}$$

$$x_2(k+1) = \Phi_{21}x_1(k) + \Phi_{22}x_2(k) + \Gamma_2 u_{\text{eq}}(k). \tag{5}$$

Note that as the system is assumed to be controllable, $\Gamma_2 \in \mathbb{R}^{m\times m}$ is invertible.

Comparing the systems (1) and (4) and (5), it can be seen that with

$$u(k) = u_{\rm eq}(k) - \Gamma_2^{-1} T_m d(k) \tag{6}$$

both the systems are equivalent. As the disturbances are matched, the design of the sliding surface for the nominal system is equivalent to the sliding surface design for the system in (1).

Theorem 1: The optimal sliding function that leads to minimization of the quadratic performance index in (2) is of the form $s(k) = [\,K \quad I\,]x(k)$, where K is the solution of

$$K\Phi_{12}K + (\Phi_{22} + \Gamma_2 F_2)K - K\Phi_{11} - (\Gamma_2 F_1 + \Phi_{21}) = 0$$

with $u_{\rm eq}(k) = Fx(k) = [\,F_1 \quad F_2\,]x(k)$ being the LQR optimal state feedback control for minimizing (2).

Proof: Consider the sliding function

$$s(k) = [\,K \quad I\,]\begin{bmatrix} x_1(k) \\ x_2(k)\end{bmatrix}. \tag{7}$$

When the system states are on the sliding surface, $s(k) = 0$. Thus, $x_2(k) = -Kx_1(k)$. In order to maintain the sliding motion, $s(k+1)$ should also be zero. A simple analysis of (4) and (5) reveals that the control $u_{\rm eq}(k)$ needed to guarantee $s(k+1) = 0$, provided $s(k) = 0$, is

$$u_{\rm eq}^{\rm smc}(k) = -\Gamma_2^{-1}(K\Phi_{11} - K\Phi_{12}K + \Phi_{21} - \Phi_{22}K)x_1(k). \tag{8}$$

The optimal control input that minimizes the cost function in (2) can be derived to be [16]

$$u_{\rm eq}^{\rm opt}(k) = -\left(R + \Gamma_\tau^T P \Gamma_\tau\right)^{-1}\left(\Gamma_\tau^T P\right)x(k), \tag{9}$$

where P is the solution of the discrete algebraic Riccati equation [17]

$$P = Q + \Phi_\tau^T P \Phi_\tau - \Phi_\tau^T P \Gamma_\tau\left(R + \Gamma_\tau^T P\Gamma_\tau\right)^{-1}\Gamma_\tau^T P\Phi_\tau. \tag{10}$$

Let the optimal control in (9) be represented as

$$u_{\rm eq}^{\rm opt}(k) = [\,F_1 \quad F_2\,]\begin{bmatrix} x_1(k) \\ x_2(k)\end{bmatrix}.$$

Thus, confined to the sliding surface,

$$u_{\rm eq}^{\rm opt}(k) = (F_1 - F_2K)\,x_1(k). \tag{11}$$

For the sliding surface to be optimal, the equivalent control that maintains the sliding mode should be equal to the linear quadratic optimal control of the system when confined to the sliding surface. Equating the control expressions in (11) and (8) and noting that the equality needs to be satisfied for all $x_1(k)$, the following bilateral matrix quadratic equation is arrived at

$$K\Phi_{12}K + (\Phi_{22} + \Gamma_2 F_2)\,K - K\Phi_{11} - (\Gamma_2 F_1 + \Phi_{21}) = 0. \tag{12}$$

The optimal sliding function can then be represented as $s(k) = [\,K \quad I\,]x(k) = c^T x(k)$, where K is the solution of (12).

A brief explanation for the procedure used to solve (12) has been provided in the Appendix.

Remark 1: It is worth noting that even if the system (1) is not in the normal form, the optimal sliding function can be designed as

$$s(k) = [\,K \quad I\,]\,Tx(k)$$

where the transformation $z = Tx$ renders the system to be in the normal form in the z-coordinate frame and the sliding function $s(k) = [\,K \quad I\,]z(k)$ is optimal for the LQ weighing matrices $T^{-T}QT^{-1}$ and R.

IV. Disturbance-Estimator-Based Control

With the optimal sliding surface being determined, the sliding mode control can be easily computed using an appropriate discrete-time SMC algorithm, as proposed in [18].

To introduce the results in [18], let the bounded disturbance $d(k)$ be such that $|c_i D_\tau d(k)| \le \delta_i$ for all $d(k)$, with c_i denoting the ith row of the sliding parameter c^T. It is known that using the control proposed in [8], the system can be brought within the vicinity of the sliding surface up to an accuracy of $|s_i(k)| < \delta_i$.

In [18], the authors propose an adaptive SMC algorithm of the form

$$u(k) = \begin{cases} u_{k{\rm eq}}(k), & \text{if } |u_{k{\rm eq}}(k)| \le u_0 \\ u_0 \frac{u_{k{\rm eq}}(k)}{|u_{k{\rm eq}}(k)|}, & \text{if } |u_{k{\rm eq}}(k)| > u_0 \end{cases} \tag{13}$$

$$u_{k{\rm eq}}(k) = -\left(c^T\Gamma_\tau\right)^{-1}\left(c^T\Phi_\tau x(k) + c^T D_\tau d(k)\right) \tag{14}$$

where u_0 is the maximum allowable control magnitude. In order to ensure that the system state moves toward the sliding surface, u_0 needs to satisfy the inequality

$$u_0 \ge |(c^T\Gamma_\tau)^{-1}|\,|(c^T\Phi_\tau - c^T)x(k) + c^T D_\tau d(k)|.$$

The adaptive algorithm ensures that the system always moves toward the sliding surface whenever $|u_{k{\rm eq}}(k)| > u_0$ and moves *on to* the sliding surface once the condition $|u_{k{\rm eq}}(k)| \le u_0$ is satisfied.

However, the exact implementation of this control is not possible as $d(k)$ is not a measurable quantity. Therefore, during the implementation of the algorithm, the mean value of $d(k)$ is generally used in place of $d(k)$ in the control expression. This leads to a sliding mode band of width δ_i in the case of adaptive discrete-time sliding mode.

The resultant system response is still sensitive to the disturbances present in the system and the ultimate deviation of the closed-loop system response from the sliding surface is proportional to δ_i. We propose an SMC algorithm based on the adaptive sliding mode algorithm in [18], wherein the aforementioned sensitivity is greatly reduced.

A. Modified Algorithm

The proposed algorithm is based on the concept of a disturbance estimator [19]. This algorithm reduces the width of the sliding mode band from being proportional to the magnitude of the disturbance to a smaller band proportional to the rate of change of the disturbance. The control algorithm is most efficient in cases where the disturbance is slowly varying.

Consider the system in (1). It can be seen that

$$s(k) = c^T\Phi_\tau x(k-1) + c^T\Gamma_\tau u(k-1) + c^T D_\tau d(k-1). \tag{15}$$

From (15),

$$c^T D_\tau d(k-1) = s(k) - c^T\left(\Phi_\tau x(k-1) + \Gamma_\tau u(k-1)\right).$$

If the disturbance is slowly varying, it can be said that $|d(k) - d(k-1)|$ is not significant. Hence, $d(k-1)$ would be a good estimate of $d(k)$. Thus, substituting $d(k-1)$ in place of $d(k)$ in the equivalent control expression (14), the modified control law is

$$u(k) = -(c^T\Gamma_\tau)^{-1}(c^T\Phi_\tau x(k) + c^T D_\tau d(k-1)). \tag{16}$$

When the bound u_0 is taken into consideration, (16) can be expressed as

$$u_i(k) = u_{0,i}\,{\rm sat}\left(\frac{\bar{u}_i(k)}{u_{0,i}}\right) \tag{17}$$

where u_i and $u_{0,i}$ denote the ith element of u and u_0, respectively. The disturbance estimated control $\bar{u}(k)$ can be derived to be

$$\bar{u}(k) = -(c^T\Gamma_\tau)^{-1}(c^T(\Phi_\tau + I)x(k) - c^T\Phi_\tau x(k-1)) + u(k-1) \quad (18)$$

and the vectorized saturation function is defined as [20]

$$\mathrm{sat}(\phi_i) = \begin{cases} \phi_i, & \text{if } |\phi_i| \le 1 \\ \mathrm{sgn}(\phi_i), & \text{if } |\phi_i| > 1. \end{cases}$$

Remark 2: At $k = 0$, there is no prior system information available. Hence, at this stage, the control expression in (14), with $d(0)$ assumed to be zero, is used to compute $u(0)$. The modified algorithm (17) may be used to compute $u(k)$ for $k \ge 1$.

V. Multirate Output Feedback Approach

The aforementioned control strategies are based on full-state feedback. However, entire state information may not be measured in practice. Since the output is available, output feedback can be used for controller design. Contrary to state feedback, however, complete system stabilization using static output feedback is still an open problem [21]. Using the concept of multirate output feedback [22], wherein the system output is sampled at a faster rate compared to the control input, it is possible to realize the effect of state feedback without incurring the added complexity of a dynamic controller [23].

In the case of multirate output feedback, the system states are represented in terms of past outputs and control inputs. The multirate sampling based disturbance estimator [22] estimates the disturbance affecting the system in addition to the system state.

Consider the system sampled with a sampling time of $\Delta = \tau/N$ sec, where N is an integer greater than the observability index [22] of the system. Let the presentation be

$$x(k+1) = \Phi x(k) + \Gamma u(k) + D_\Delta d(k)$$
$$y(k) = Cx(k) \quad (19)$$

where

$$\Phi = (\Phi_\tau)^{1/N}, \Gamma = \left(\sum_{i=0}^{N-1}\Phi\right)^{-1}\Gamma_\tau$$
$$D_\Delta = \left(\sum_{i=0}^{N-1}\Phi\right)^{-1}D_\tau.$$

Consider the system with the control input being sampled with a sampling period of τ sec and the system output being sampled every Δ sec. This multirate sampled system can be described as [24]

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + D_\tau d(k) \quad (20)$$
$$y_{k+1} = C_0 x(k) + D_0 u(k) + C_d d(k) \quad (21)$$

where y_k is the lifted output given as

$$y_k = [\, y^T((k-1)\tau) \quad y^T((k-1)\tau + \Delta) \quad \cdots \quad y^T(k\tau - \Delta)]^T,$$

and the matrices C_0, D_0, and C_d are defined as

$$C_0 = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{N-1} \end{bmatrix}, D_0 = \begin{bmatrix} 0 \\ C\Gamma \\ C\Phi\Gamma \\ \vdots \\ C\sum_{i=0}^{N-2}\Phi^i\Gamma \end{bmatrix},$$

$$C_d = \begin{bmatrix} 0 \\ CD_\Delta \\ C\Phi D_\Delta \\ \vdots \\ C\sum_{i=0}^{N-2}\Phi^i D_\Delta \end{bmatrix}.$$

From the above system description, a relationship between the available information i.e., past output samples and control input samples, and the system states and disturbance vector, can be derived in the following manner.

Due to the system being observable and N being greater than the observability index of the system, the matrix $[\, C_0 \quad C_d \,]$ would be of full rank. Thus, it can be said from (21) that

$$\begin{bmatrix} x(k) \\ d(k) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}(y_{k+1} - D_0 u(k)) \quad (22)$$

where

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = M = [\, C_0 \quad C_d \,]^\dagger$$

represents the Moore-Penrose generalized inverse of $[\, C_0 \quad C_d \,]$.

The discrete-time system (1) can be represented as

$$\begin{bmatrix} x(k+1) \\ d(k) \end{bmatrix} = \begin{bmatrix} \Phi_\tau & D_\tau \\ 0 & I \end{bmatrix}\begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} \Gamma_\tau \\ 0 \end{bmatrix} u(k). \quad (23)$$

Substituting (22) in (23) and shifting the time instant,

$$\begin{bmatrix} x(k) \\ d(k-1) \end{bmatrix} = \begin{bmatrix} \Phi_\tau M_1 + D_\tau M_2 & \Gamma_\tau - (\Phi_\tau M_1 + D_\tau M_2)D_0 \\ M_2 & -M_2 D_0 \end{bmatrix} \times \begin{bmatrix} y_k \\ u(k-1) \end{bmatrix}. \quad (24)$$

Substituting the result from (24) in the adaptive sliding mode control expression (16), the multirate sampled realization of the equivalent control would be

$$u_{\mathrm{eq}}(k) = -(c^T\Gamma_\tau)^T(c^T\Phi_\tau x(k) + c^T D_\tau d(k-1))$$
$$= F_y y_k + F_u u(k-1).$$

where

$$F_y = -(c^T\Gamma_\tau)^{-1}c^T\left(\Phi_\tau^2 M_1 + (\Phi_\tau + I)D_\tau M_2\right),$$
$$F_u = -(c^T\Gamma_\tau)^{-1}c^T\left(\Phi_\tau\Gamma_\tau - \Phi_\tau^2 M_1 D_0 - (\Phi_\tau + I)D_\tau M_2 D_0\right).$$

The multirate output feedback controller in (25) emulates the state feedback controller in (16) exactly. Thus, the closed loop response of the system using the multirate output feedback controller would be almost equal to that obtained by the state feedback controller in (16).

VI. Illustrative Example

To illustrate the proposed results, we consider the continuous-time system representation of a three cart system [25] given in Fig. 1 for the system parameters $m = 1$, $k = 2$, and $b = 3$ as

$$\dot{x} = Ax + Bu + D_c d,$$
$$y = Cx$$

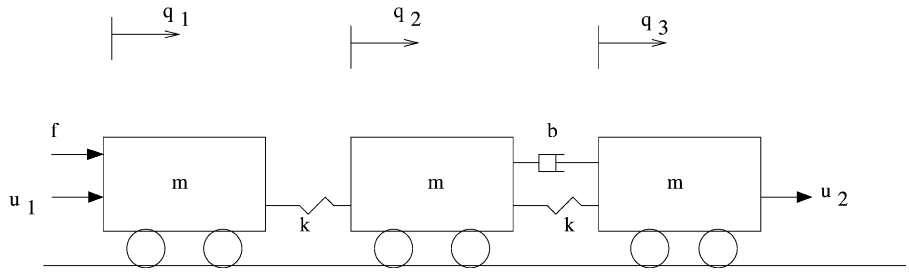

Fig. 1. Three-cart system.

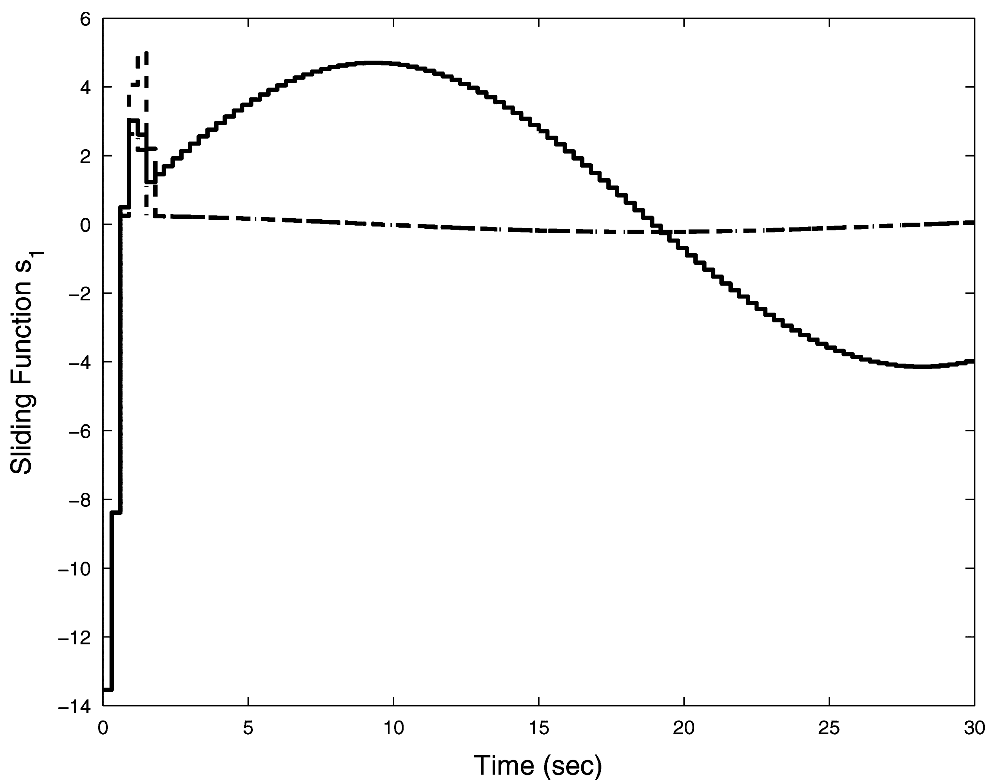

Fig. 2. Comparison of the evolution of sliding function s_1 using adaptive SMC (solid), disturbance estimator (dash–dot), and multirate output feedback (dash).

where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -4 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 & -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and where the state vector $x = [\,q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2 \quad q_3 \quad \dot{q}_3\,]^T$ is composed of the cart positions and velocities, the input vector $u = [\,u_1 \quad u_2\,]^T$ represents the force applied on the system, $d = f$ represents the disturbance and the output vector composed of the cart positions is $y = [\,q_1 \quad q_2 \quad q_3\,]^T$.

The system is sampled with $\tau = 0.3$ sec and $\Delta = 0.1$ sec. The control is required to optimize the quadratic performance index with $Q = \text{diag}\,(60, 60, 60, 60, 60, 60)$ and $R = \text{diag}\,(70, 70)$. Using the procedure described in Section III, the optimal sliding function parameter can be determined to be

$$c^T = \begin{bmatrix} 9.3723 & 2.0382 & -8.1185 & -0.7194 & -0.5966 & 0.7101 \\ 11.6105 & -1.8242 & -10.8459 & 2.2365 & 1.3770 & 4.0308 \end{bmatrix}.$$

For the simulation purpose, an initial system state of $X_0 = [1 \quad 2 \quad 2 \quad 3 \quad 15 \quad 0.5]^T$ and a slowly varying disturbance signal of $d(k) = 5\sin(k/20)\exp(-k/500)$ is used. The maximum allowable control magnitude is taken as $u_0 = (10, 10)$. However, since no information is available for the multirate controller, an estimated initial state of $X_e = [\,1 \quad 0 \quad 2 \quad 0 \quad 15 \quad 0\,]^T$ is used to generate the multirate output feedback control signal for $k = 0$.

Figs. 2–5 provide a comparison of the system responses using adaptive sliding mode [18] and the proposed state- and multirate-output-feedback-based control techniques. Figs. 2 and 3 show the evolution of the sliding functions with the application of the control laws discussed in Sections IV and Sections V. It can be seen from Fig. 2 that the

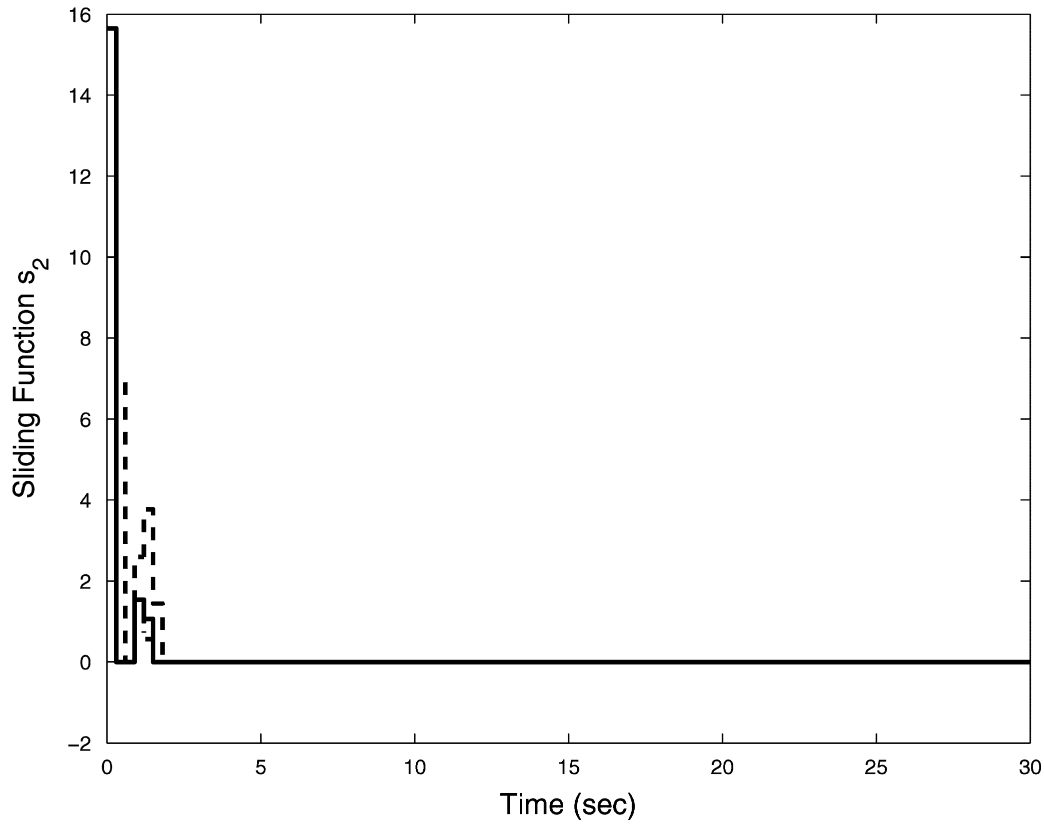

Fig. 3. Comparison of the evolution of sliding function s_2 using adaptive SMC (solid), disturbance estimator (dash–dot), and multirate output feedback (dash).

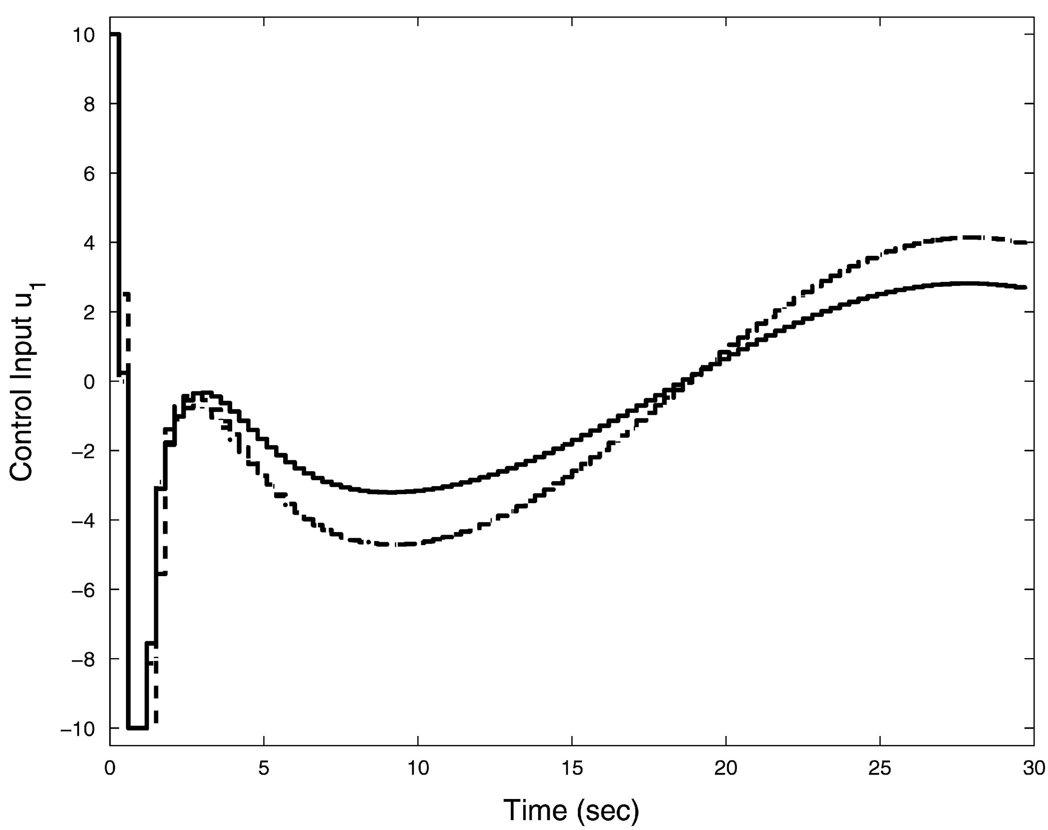

Fig. 4. Comparative plots of control input u_1 using adaptive SMC (solid), disturbance estimator (dash–dot), and multirate output feedback (dash).

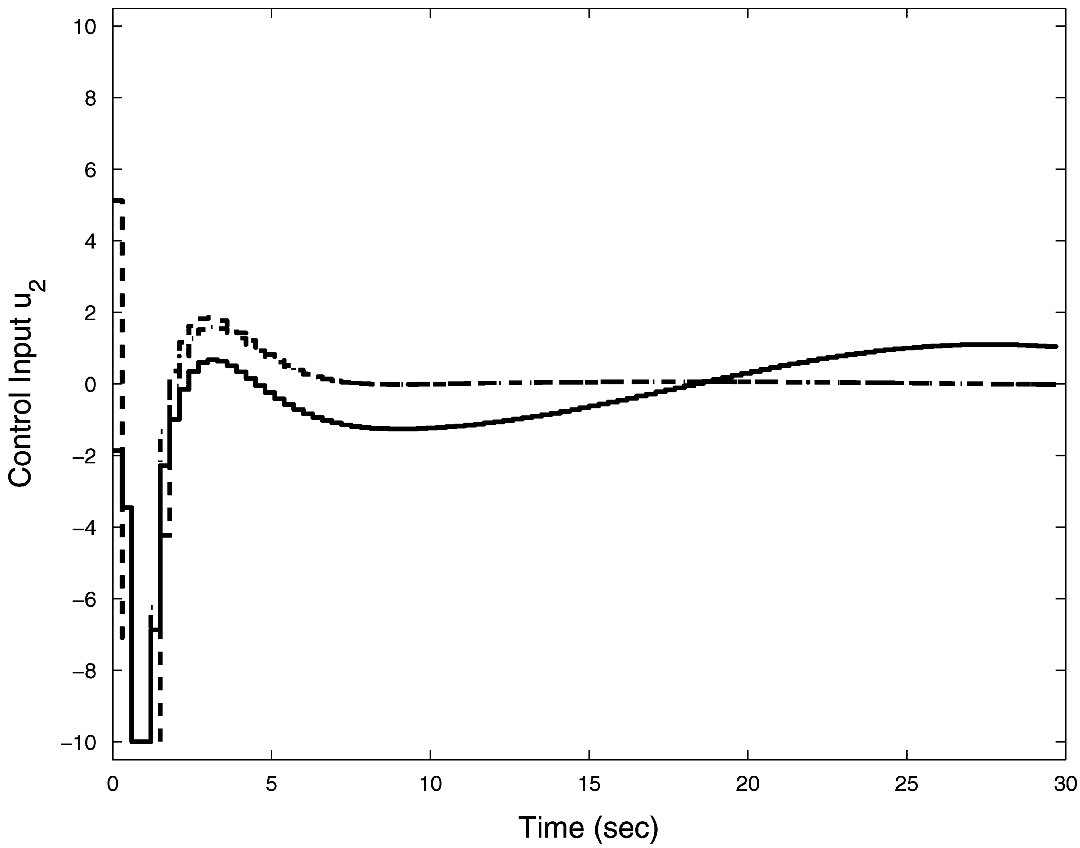

Fig. 5. Comparative plots of control input u_2 using adaptive SMC (solid), disturbance estimator (dash–dot), and multirate output feedback (dash).

adaptive SMC is not able to nullify the disturbance as efficiently as the proposed state- or multirate output feedback control laws. The plots also show that the responses for the modified-state feedback and the multirate output feedback laws are almost identical, indicating that the output feedback control is able to emulate the state feedback control. It is clear from Fig. 3 that the initial response of the multirate control is different from that of the other state-feedback-based controls. However, the output feedback control quickly recovers and ultimately gives a response that is almost equal to the state feedback case. The plots of the control inputs are given in Figs. 4 and 5. It can be observed that the control signals for the modified state feedback control and the multirate control are similar. A sinusoidal component can be clearly seen in the proposed control input in Fig. 4. This component helps in counteracting the disturbance and prevents the disturbance from affecting other parts of the system. The control input u_2 in Fig. 5 is affected indirectly by the disturbance in case of the adaptive controller. On the other hand, there is no effect of the disturbance on u_2 while using the proposed controller. This is because u_1 is effectively used to counteract the disturbance.

VII. Conclusions

A procedure for designing an optimal sliding surface for discrete-time systems has been proposed in this paper. The designed sliding surface leads to the minimization of a linear quadratic performance index, when the system is in sliding mode. The paper also proposes a modified state feedback technique to achieve the aforementioned sliding mode in discrete-time systems with matched disturbances. A disturbance-estimator-based output feedback controller is also presented in the paper. The proposed control laws are validated through a example of a three cart mechanical system. The simulation results satisfactorily confirm the validity of the proposed control strategies.

The numerical example shows that the multirate output feedback control emulates the behavior of the state feedback control in spite of the unavailability of full state information of the system. It may be noted here that the proposed algorithm would be effective against any matched and bounded uncertainty present in the system, irrespective of its source being an external disturbance or uncertainty in system dynamics.

Appendix

Solving Matrix Quadratic Equations

Equation (12) is a matrix quadratic equation which has the following general representation

$$\Omega(X) = \Psi_{00} + \Psi_{01}X + X\Psi_{10} + X\Psi_{11}X = 0 \tag{25}$$

where X is the unknown matrix-valued variable.

Let E be a perturbation matrix with the same dimension as X. Now,

$$\begin{aligned}\Omega(X+E) &= \Omega(X) + ((\Psi_{01} + X\Psi_{11})E + E(\Psi_{10} + \Psi_{11}X)) \\ &\quad + E\Psi_{11}E \\ &= \Omega(X) + D_X(E) + E\Psi_{11}E.\end{aligned} \tag{26}$$

Here $D_X(E)$ is the Fréchet derivative of Ω at X in the direction E.

A modified Newton's method [26] can be used to solve (25). In Newton's method, the second-order term of E in (26) is dropped and the equation is solved iteratively. E_i is defined as the solution of

$$(\Psi_{01} + \Psi_{11}X_i)E_i + E_i(\Psi_{10} + X_i\Psi_{11}) = -\Omega(X_i) \tag{27}$$

in each step and X_{i+1} is set as $X_{i+1} = X_i + E_i$ until $\|\Omega(X_i)\|$ becomes less than a specified tolerance value. It may be noted that (27) is a Sylvester equation for E_i.

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Unknown Input Observers for Switched Nonlinear Discrete Time Descriptor Systems

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***Abstract*—In this paper, a linear matrix inequality technique for the state estimation of discrete-time, nonlinear switched descriptor systems is developed. The considered systems are composed of linear and nonlinear parts. An observer giving a perfect unknown input decoupled state estimation is proposed. Sufficient conditions of global convergence of observers are proposed. Numerical examples are given to illustrate this method.**

***Index Terms*—Hybrid systems, polyquadratic stability, switched descriptor systems, unknown input (UI) observers.**

I. INTRODUCTION

Switched control and/or observer systems have recently received much attention. Switched systems belong to a special class of hybrid systems. They are defined by a collection of dynamical (linear and/or nonlinear) subsystems together with a switching rule that specifies the switching between these subsystems. A survey on basic problems in switched system stability and design is available in [26] (see the references therein). Many such problems occur in practice: power converter systems where the switching signal is determined by pulse with pulsewidth modulation (PWM) and gain scheduling control systems are examples among many others. One can study the existence of a switching rule that ensures the stability of the switched system. One can assume that the switching sequence is not known *a priori*, and look for stability results under arbitrary switching sequences. On the one hand, most of the contributions in this field deal with stability analysis and control synthesis [7], [18]. On the other hand, unknown input observers (UIOs) have been widely studied for nonsingular systems [9], [29], singular systems [6], [10], [16], nonlinear descriptor systems [17], and recently, for switched nonsingular systems [20]. Nevertheless, there is no result extending the method mentioned in [20] to the general representation of switched nonlinear descriptor systems, although many practical systems can be described by them [2], and their fault diagnosis may be based on UIO design [21].

As mentioned in [32], there are generally two broad approaches for a nonlinear observer design. In the first approach, the objective is to find a coordinate transformation so that the state-estimation error dynamics are linear in the new coordinates, and then, linear techniques can be performed [13], [14], [30]. Necessary and sufficient conditions have been established [19], [30] for the existence of such a coordinate transformation. The second approach does not need the transformation, and the observer design is directly based on the original sys-

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