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MULTISECTIONED PARTITIONS OF INTEGERS

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This paper presents identities on generating functions for multisectioned partitions of integers by developing in the language of partitions some powerful and essentially combinatorial techniques from the literature of principal differential ideals. D. Mead has stated in Vol. 42 of this journal that one can obtain interesting combinatorial relations by constructing different vector space bases for a subspace of a differential ring and using the fact that the cardinality of all bases is the same. The results of the present paper are of this nature.

In particular, we enumerate certain sets of ordered pairs of generalized tableaux that have a central role in Mead's paper. Tableaux were used by A. Young and others to study the structure of the symmetric groups S_n. In [3], D. Knuth used an "insertion into tableau" construction of C. Schensted to give a direct 1-to-1 correspondence between "generalized permutations" and ordered pairs of "generalized Young tableaux" having the same shape. In [5], Mead independently proved the existence of such a bijection while developing a new vector space basis for the ring of differential polynomials in n independent differential indeterminates. Mead's paper deals with principal differential ideals generated by Wronskians and used determinantal identities going back to Cayley. The ordered pairs of generalized tableaux used by Mead appear in a more general setting in the paper [1] by Doubilet, Rota, and Stein.

1. Multisectioned partitions.

Let w be a nonnegative integer. Here a *partition of w into d parts*, or of *degree d*, is a d-tuple $(p_1, \cdots p_d)$ of nonnegative integers p_k with $p_1 \leqq p_2 \leqq \cdots \leqq p_d$ and $w = p_1 + \cdots + p_d$. If $d = 0$, we agree that there is one (ideal) partition of 0 but no partition of any $w > 0$.

Let $P = (P_1, \cdots, P_n)$, where P_i is a partition of w_i of degree d_i, and let $w = w_1 + \cdots + w_n$; then P is an *n-section partition of w with signature* $D = [d_1, \cdots, d_n] = \operatorname{sig} P$ and the *weight* of P is w.

For applications to differential algebra, it is convenient to use two rowed matrices

$$(1) \qquad M = \begin{pmatrix} i_1 & i_2 \cdots i_d \\ j_1 & j_2 \cdots j_d \end{pmatrix}$$

in which $i_k \in \{1, 2, \cdots, n\}$, $j_k \in \{0, 1, \cdots\}$, $i_k \leqq i_{k+1}$ for $1 \leqq k < d$, and

$j_k \leqq j_{k+1}$ whenever $i_k = i_{k+1}$. (These are sometimes called generalized permutations (e.g., see D. E. Knuth [3], p. 710.) The M of (1) can be associated in a 1-to-1 manner with the n-section partition

$$(2) \qquad P = P(M) = (P_1, \cdots, P_n)$$

in which, for $1 \leqq h \leqq n$, P_h has as its parts the j_k with $i_k = h$.

Let $y_{ij} (i = 1, 2, \cdots, n; j = 0, 1, \cdots)$ form a denumerable set of algebraically independent indeterminates over a field F and let $R = F[y_{ij}]$ denote the ring of polynomials in the y_{ij} with coefficients in F. (Each of the sequences $\{y_{i0}, y_{i1}, y_{i2}, \cdots\}$ is said to be a differential indeterminate y_i.) Corresponding to an M as in (1), or a P as in (2), is the power product

$$(3) \qquad \pi = y_{i_1 j_1} y_{i_2 j_2} \cdots y_{i_d j_d} .$$

The signature and weight of π are defined to be the same as for the associated n-section partition P. The power products π form a basis for R as a vector space over F.

Let $T = (T_1, \cdots, T_n)$ be an n-section partition of w with sig $T = D = [d_1, \cdots d_n]$ and let $T_i = (t_{i1}, \cdots, t_{id_i})$. If T has the properties:

(a) $d_1 \geqq d_2 \geqq \cdots \geqq d_n \geqq 0$ and

(b) $t_{ij} < t_{i+1,j}$ for $1 \leqq i < n$ and $1 \leqq j \leqq d_{j+1}$

then T is a (generalized Young) *n-tableau* with *signature* D and *weight* w. Motivated by matrix notation, one considers T_i to be the ith row of T and the transpose of $(t_{1j}, t_{2j}, \cdots, t_{rj})$, where r is the largest i with $d_i \geqq j$, to be the jth column of T.

The *content* of a sequence $a_1, a_2, \cdots, a_d$ of d nonnegative integers a_i whose largest term is m is the $(m + 1)$-tuple

$$(c_0, c_1, \cdots, c_m)$$

in which c_h is the number of values of i in $\{1, 2, \cdots, d\}$ such that $a_i = h$. The *content*, con(P), of a P as in (2), or of the associated M of (1) or π of (3), is the content of the sequence $j_1, \cdots, j_d$ on the second row of the M of (1). The content of an n-tableau is its content as an n-section partition.

2. Generating functions for partitions. Let $N(w, d)$ be the number of partitions of w with degree d. Then $N(0, 0) = 1$ and $N(w, 0) = 0$ for $w > 0$. Using the Cayley notation, let

$$(d) = (1 - x)(1 - x^2) \cdots (1 - x^d) , \quad (0) = 1 ,$$

and let $\begin{bmatrix} h + k \\ h \end{bmatrix}$ denote the generalized binomial coefficient $(h+k)/(h)(k)$. The generating function for $N(w, d)$ is well-known to be

$$\sum_{w=0}^{\infty} N(w, d)x^w = \frac{1}{(d)} = \frac{1}{(1-x)(1-x^2)\cdots(1-x^d)}.$$

Then the generating function for the number of n-section partitions of w with signature $[d_1, \cdots, d_n]$ is easily seen to be $1/(d_1)(d_2)\cdots(d_n)$.

3. Limited 2-section partitions. The lemmas of this and the following section are needed below.

LEMMA 1. *Let d, q, and w be nonnegative integers.*

(i) *For $i = 0, \cdots, d$, let S_i be the set of all 2-section partitions (Z, A) of $w - i(q+1)$ with signature $[j, i]$, $i + j = d$, and with each part z_k of Z satisfying $z_k \leqq q$. Then there is a bijection σ from the set T of partitions P of w with degree d onto the union $S = S_0 \cup S_1 \cup \cdots \cup S_d$.*

(ii) $$\frac{1}{(d)} = \sum_{i=0}^{d} \frac{(q+d-i)x^{i(q+1)}}{(q)(d-i)(i)} = \sum_{i+j=d} \begin{bmatrix} q+j \\ q \end{bmatrix} \frac{x^{i(q+1)}}{(i)}.$$

Proof. (i) Let $P = (p_1, \cdots, p_d)$ be a member of T. Let j be the largest subscript such that $p_j \leqq q$ and $i = d - j$. Subtracting $q + 1$ from each of the i parts $p_{j+1}, \cdots, p_d$ yields the 2-section partition $\sigma(P) = (Z, A)$ with $Z = (p_1, \cdots, p_j)$, and $A = (p_{j+1} - q - 1, \cdots, p_d - q - 1)$. It is readily seen that σ is the desired bijection.

(ii) The generating function for the number of partitions of w having degree $j = d - i$ with maximum part q is $\begin{bmatrix} q+j \\ q \end{bmatrix} = (q+j)(q)(j)$ (see Riordan, *An Introduction to Combinatorial Analysis*, especially p. 153, Problem 5). The generating function for partitions of w with degree i having parts greater than q is $x^{i(q+1)}/(i)$. Forming the Cauchy product of these two generating functions and summing over $i = 0, 1, \cdots, d$ yields the identity in (ii).

4. Levi 3-section partitions. A *Levi 3-section partition* of w having signature $[f, g, h]$ is a 3-section partition (U, V, W) with $U = (u_1, u_2, \cdots, u_f)$, $V = (v_1, v_2, \cdots, v_g)$, and $W = (w_1, w_2, \cdots, w_h)$ with the property that $v_1 \geqq f$. The following lemma makes more accessible an essentially combinatorial result contained in the proof of an important theorem in differential algebra by H. Levi [4].

LEMMA 2. (i) *There is a bijection θ from the set of all 2-section partitions of w having signature $[d, e]$ onto the set of all Levi 3-section partitions of w having signature of the form $[d-h, e-h, h]$ with $0 \leqq h \leqq \mu = \min\{d, e\}$.*

(ii) $$\frac{1}{(d)(e)} = \sum_{h=0}^{\mu} \frac{x^{(d-h)(e-h)}}{(d-h)(e-h)(h)} = \sum_{\substack{h+a=d\\h+b=e}} \frac{x^{ab}}{(a)(b)(h)} \, .$$

Proof. (i) Let (B, C) be a 2-section partition of w with signature $[d, e]$ where $B = (b_1, \cdots, b_d)$, $C = (c_1, \cdots, c_e)$. The bijection $\theta(B, C) = (U, V, W)$ is constructed as follows. If $c_1 \geqq d$, we let $h = 0$, $U = B$, $V = C$ and W be the (ideal) partition of 0 with degree 0. If $c_1 < d$, let h be the largest positive integer m with $m + c_m \leqq d$. For $k = 1, \cdots, h$ let

$$i(k) = k + c_k,\; w_k = b_{i(k)} + c_k \, . \tag{4}$$

Since the b's and c's are nondecreasing, so are the w's, i.e., $W = (w_1, \cdots, w_h)$ is a partition. Let $v_j = c_{j+h}$ and $V = (v_1, \cdots, v_{e-h})$. For $1 \leqq k \leqq h$, we delete the $b_{i(k)}$ from B and let $U = (u_1, \cdots, u_{d-h})$, where the u's are the remaining b's in their same relative order. Now we let $\theta(B, C) = (U, V, W)$. It is easily seen that the first part v_1 of V is at least as large as the degree $d - h$ of U.

It remains to show that θ is a bijection. Let (U, V, W) be a Levi 3-section partition. We show below that U, V, and a part w_k of W uniquely determine the nonnegative integers $b_{i(k)}$ and c_k to be reinserted into U and V, respectively, in the process of rebuilding them into partitions B and C such that $\theta(B, C) = (U, V, W)$.

For this purpose only, we introduce the ideal values $u_0 = -1$ and $u_{f+1} = \infty$. Then each nonnegative integer w satisfies

$$u_\gamma \leqq w - \gamma \leqq u_{\gamma+1}$$

for some γ in $\{0, 1, \cdots, \min(f, w)\}$. This $\gamma = \gamma(w)$ is unique since

$$u_\gamma \leqq w - \gamma \leqq u_{\gamma+1} \leqq \cdots \leqq u_{\gamma+\delta} \leqq w - (\gamma + \delta) \leqq u_{\gamma+\delta+1} \, ,$$

with $\delta \geqq 1$, gives the contradiction $w - \gamma \leqq w - \gamma - \delta$. Now one sees from (4) that c_k must be $\gamma(w_k)$ and $b_{i(k)}$ must be $w_k - c_k$. Hence θ is a bijection.

Part (ii) merely restates part (i) in terms of generating functions.

5. **Special $3n$-section partitions.** Let $q_1, q_2, \cdots, q_n, q_{n+1}$ be fixed nonnegative integers. Denote this sequence by Q. Let

$$\begin{aligned} &Z_i = (z_{i1}, \cdots, z_{ir_i}) \quad \text{with} \quad z_{ij} \leqq q_i \quad \text{for} \quad 1 \leqq i \leqq n+1 \, , \\ &U_i = (u_{i1}, \cdots, u_{is_i}) \quad \text{for} \quad 1 \leqq i \leqq n \, , \quad \text{and} \\ &V_i = (v_{i1}, \cdots, v_{it_i}) \quad \text{for} \quad 1 \leqq i \leqq n-1 \end{aligned}$$

be partitions of degree r_i, s_i, and t_i respectively. The *$3n$-section partition*

$$L = (Z_1, Z_2, \cdots, Z_{n+1}, U_1, \cdots, U_n, V_1, \cdots V_{n-1}) \tag{5}$$

of w is said to be *special* (with respect to Q) if $s_i \leqq v_{i1}$ for $1 \leqq i \leqq n-1$.

THEOREM 1. (i) *There is a* 1-*to*-1 *correspondence between the set* $E(w, D)$ *of all* n-*section partitions of* w *with signature* $D = [d_1, d_2, \cdots, d_n]$ *and the set* $F(w, D)$ *of all special* $3n$-*section partitions* L, *as in* (5), *of* w *with signature* $[r_1, \cdots, r_{n+1}, s_1, \cdots, s_n, t_1, \cdots, t_{n-1}]$ *satisfying the following system of conditions*:

$$\begin{cases} d_1 = s_1 + s_2 + \cdots + s_n + r_1 + r_{n+1} \\ d_2 = t_1 + s_2 + \cdots + s_n + r_2 + r_{n+1} \\ d_3 = t_2 + s_3 + \cdots + s_n + r_3 + r_{n+1} \\ \vdots \qquad\qquad\qquad \vdots \\ d_n = t_{n-1} + s_n + r_n + r_{n+1} . \end{cases} \tag{6}$$

(ii) *Let*

$$\varepsilon = \left[\sum_{i=1}^{n} (d_i - r_i)(q_i + 1) + \sum_{i=1}^{n-1} s_i t_i\right] + s_n(q_{n+1} + 1) .$$

Then

$$\frac{1}{(d_1)(d_2)\cdots(d_n)} = \sum \frac{x^{\varepsilon}}{\varepsilon(t_1)\cdots(t_{n-1})(s_1)\cdots(s_n)} \cdot \begin{bmatrix} q_1 + r_1 \\ q_1 \end{bmatrix} \cdots \begin{bmatrix} q_{n+1} + r_{n+1} \\ q_{n+1} \end{bmatrix} \tag{7}$$

where the sum is taken over all r_i, s_i, *and* t_i *satisfying system of equations in* (6).

Proof. (i) Let $P = (P_1, \cdots, P_n)$ be an n-section partition of w with signature $[d_1, \cdots, d_n]$. For $1 \leqq i \leqq n$, let $P_i = (p_{i1}, \cdots, p_{id_i})$. The σ bijection of Lemma 1(i), using q_i for q and d_i for d, applied to the P_i yields

$$\sigma(P_i) = (Z_i, A_i) , \qquad i = 1, 2, \cdots, n$$

where the 2-section partition (Z_i, A_i) has signature $[s_i, d_i - r_i]$. Let (U_1, V_1, W_1) be the result of applying the θ bijection of Lemma 2(i) to (A_1, A_2), i.e., let

$$\theta(A_1, A_2) = (U_1, V_1, W_1) .$$

Using the W_1 thus obtained, let

$$\theta(W_1, A_3) = (U_2, V_2, W_2) .$$

Further applications of θ yield

$$\theta(W_2, A_4) = (U_3, V_3, W_3)$$
$$\vdots$$
$$\theta(W_{n-2}, A_n) = (U_{n-1}, V_{n-1}, W_{n-1}) .$$

Finally, using q_{n+1} for q, apply σ to W_{n-1} yielding

$$\sigma(W_{n-1}) = (Z_{n+1}, U_n) .$$

These constructions produce the Z_i, U_i, and V_i of the desired special $3n$-section partition L. The proof of part (ii) shows the development of the system (6). The map $P \to L$ is a bijection since σ and θ are bijections.

(ii) Let $n = 2$. Replacing d, q, and j by d_1, q_1, and r_1 resp., and then by d_2, q_2, and r_2 resp., in Lemma 1(ii) we have

$$\frac{1}{(d_1)} = \sum_{i+r_1=d_1} \begin{bmatrix} q_1 + r_1 \\ q_1 \end{bmatrix} \frac{x^{i(q_1+1)}}{(i)} , \quad \frac{1}{(d_2)} = \sum_{k+r_2=d_2} \begin{bmatrix} q_2 + r_2 \\ q_2 \end{bmatrix} \frac{x^{k(q_2+1)}}{(k)}$$

and the product

$$(8) \qquad \frac{1}{(d_1)(d_2)} = \sum_{i+r_1=d_1} \sum_{k+r_2=d_2} \frac{x^{i(q_1+1)+k(q_2+1)}}{(i)(k)} \begin{bmatrix} q_1 + r_1 \\ q_1 \end{bmatrix} \begin{bmatrix} q_2 + r_2 \\ q_2 \end{bmatrix} .$$

Replacing d, e, h, a, and b by i, k, s, s_1 and t_1 resp. in Lemma 2(ii) yields,

$$(9) \qquad \frac{1}{(i)(k)} = \sum_{\substack{s-s_1=i \\ s+t_1=k}} \frac{x^{s_1 t_1}}{(s_1)(t_1)(s)} .$$

Substituting (9) into (8) yields

$$(10) \quad \frac{1}{(d_1)(d_2)} = \sum_{\substack{d_1=s_1+r_1+s \\ d_2=t_1+r_2+s}} \frac{x^{s_1 t_1 + (d_1-r_1)(q_1+1)+(d_2-r_2)(q_2+1)}}{(s_1)(t_1)(s)} \begin{bmatrix} q_1 + r_1 \\ q_1 \end{bmatrix} \cdot \begin{bmatrix} q_2 + r_2 \\ q_2 \end{bmatrix} .$$

Finally, apply Lemma 1(ii), using q_3 and r_3, to the factor $1/(s)$, yielding

$$\frac{1}{(s)} = \sum_{s_2+r_3=s} \frac{x^{s_2(q_3+1)}}{(s_2)} \begin{bmatrix} q_3 + r_3 \\ q_3 \end{bmatrix} .$$

Inserting this expression into (10) gives the result:

$$\frac{1}{(d_1)(d_2)} = \sum_{\substack{d_1=s_1+s_2+r_1+r_3 \\ d_2=t_1+s_2+r_2+r_3}} \frac{x^{s_1 t_1 + (d_1-r_1)(q_1+1)+(d_2-r_2)(q_2+1)+s_2(q_3+1)}}{(s_1)(s_2)(t_1)} \cdot \begin{bmatrix} q_1 + r_1 \\ q_1 \end{bmatrix} \cdot \begin{bmatrix} q_2 + r_2 \\ q_2 \end{bmatrix} \cdot \begin{bmatrix} q_3 + r_3 \\ q_3 \end{bmatrix} .$$

For general n, the procedure is similar; Lemma 1(ii) is used n times, Lemma 2(ii), $(n-1)$ times and then Lemma 1(ii) once again. This completes the proof.

If $D=[d_1, \cdots, d_n]$, let D^* denote $(0, d_1, d_2, \cdots, d_n)$. Then let $G(w, D)$ be the set of all ordered pairs of n-tableau (T, T') with $\operatorname{sig} T = \operatorname{sig} T'$, $\operatorname{con}(T') = D^*$, and w as the weight of T.

THEOREM 2. (i) *There is a 1-to-1 correspondence between the set $G(w, D)$ and the set $F(w, D)$ of all special $3n$-section partitions of w with signature $[r_1, \cdots, r_{n+1}, s_1, \cdots, s_n, t_1, \cdots, t_{n-1}]$ satisfying the system in* (6).

(ii) *Either side of equation* (7) *is the generating function $\sum_{w=0}^{\infty} |G(w, D)| x^w$, where $|G(w, D)|$ is the number of ordered pairs in $G(w, D)$.*

Proof. C. Schensted's "insertion into tableau" procedure as developed by D. Knuth ([3], Theorem 2, p. 715) gives a 1-to-1 correspondence between the set $E(w, D)$ of all n-section partitions of w with signature D and the set $G(w, D)$. Theorem 1(i) gives a 1-to-1 correspondence between the set $E(w, D)$ and the set $F(w, D)$.

6. **Tableaux of type α and type β.** Here specific types of tableaux, as characterized by Mead in [5], are defined. Our usage of α and β is analogous to that in [2] and [4] and reverses that in [5].

For $w \geqq r(r-1)/2$, let $M(r, w)$ denote the partition $(p_1, \cdots, p_r)$ of w into r distinct nonnegative integer parts p_i that are as close to one another as possible, i.e., with $p_{i+1} - p_i \in \{1, 2\}$ for $1 \leqq i \leqq r$ and with $p_{i+1} - p_i = 2$ for at most one i.

Let T be an n-tableau, $0 \leqq k \leqq n$, and let

$$(a_1, \cdots, a_n)\,, \quad (b_1, \cdots, b_n)\,, \quad (c_1, \cdots, c_k)$$

denote transposes of columns A, B, C of T (with lengths n, n, and k). T is said to be of *type β* if it satisfies one of the following four conditions:

(1) T has a column B whose transpose is an $M(n, w)$ for some w.

(2) For some j, the $(j-1)$-st and jth columns are of the form A and B with $a_i = b_i$ for i greater than some t (which may be 0) and $(b_1, \cdots, b_t) = M(t, w)$ for some w.

(3) For some j, the jth and $(j+1)$st columns are of the form B and C such that $b_i = c_i$ for $i \leqq r$, where r is an integer with $0 \leqq r \leqq k$ and $(b_{r+1}, \cdots, b_n) = M(n-r, w)$ for some w.

(4) T has A, B, and C as the $(j-1)$st, jth, and $(j+1)$-st

columns such that $b_i = c_i$ for i less than or equal to some $r \leqq k$, $(b_{r+1}, \cdots, b_t) = M(t-r, w)$ for some t and w, and $a_i = b_i$ for $i > t$.

An n-tableau that is not of type β is said to be of *type* α. An ordered pair (T, T') of n-tableaux with $\operatorname{sig} T = \operatorname{sig} T'$ is said to be of type α or β depending on whether the type of T is α or β, respectively. Mead showed that the number of (T, T') of type β with $\operatorname{con}(T') = D^*$ and w as weight of T is the dimension of a vector subspace of $R = F[y_{ij}]$ described in §7, where we give an explicit expression for this number. (This can be put in the form of an algorithm for obtaining this dimension.)

7. Partitions of types α and β. A special $3n$-section partition L of w, is said to be of *type* α if the degree s_n of the U_n of (5) is zero, i.e., U_n is the ideal partition of 0; L is of *type* β if $s_n > 0$.

Let (C_α) denote the system of conditions (6) with the additional restriction that $s_n = 0$ and let (C_β) denote (6) with the added condition $s_n > 0$. Let $F_\alpha(w, D)$ be the subset of $F(w, D)$ consisting of the L with signature D conditioned by (C_α) and let $F_\beta(w, D)$ consist of the remaining L of $F(w, D)$. Also let $G_\alpha(w, D)$ and $G_\beta(w, D)$ denote the subsets of $G(w, D)$ of type α and type β, respectively.

Now let $N_\alpha(D, Q)$ and $N_\beta(D, Q)$ denote the expression on the right side of (7) when the sum is taken over (C_α) and (C_β), respectively.

Let $V(w, D)$ be the subspace of $R = F[y_{ij}]$ generated by the π of (3) with weight w and signature D. Let W_n be the Wronskian of $y_1, \cdots, y_n$ and $[W_n]$ be the principal differential ideal generated by W_n in R. Among the ideals I dealt with by the author in [2] are a family $\mathscr{F}$ such that $I \cap V(w, D)$ has the same dimension as $[W_n] \cap V(w, D)$. One such I is the principal differential ideal generated by the $[n(n-1)/2]$th derivative of the product $y_1 y_2 \cdots y_n$; for this I the q_i introduced in §5 must be given by $0 = q_1 = q_2 = \cdots = q_n$ and $q_{n+1} = n(n-1)/2$. For all the ideals $(x_0, x_1, \cdots)$ of $\mathscr{F}$, the q_i are chosen so that $q_1 + q_2 + \cdots + q_{n+1} = n(n-1)/2$; this allows x_j to be homogeneous and isobaric with the signature and weight of the j-th derivative of W_n. With such a choice of Q, the dimension of $[W_n] \cap V(w, D)$ is the same as the number $|F_\beta(w, D)|$ of special $3n$-section partitions L of w with signature conditioned by (C_β) and also equals the number $|G_\beta(w, D)|$ of ordered pairs (T, T') of n-tableaux of type β with $\operatorname{sig} T = \operatorname{sig} T'$, $\operatorname{con} T' = D^*$, and w as the weight of T.

Thus we have:

THEOREM 3. *Let* $q_1 + q_2 + \cdots + q_{n+1} = n(n-1)/2$.

(i) *There is a* 1-*to*-1 *correspondence between the sets* $F_\alpha(w, D)$

and $G_\alpha(w, D)$ [and hence a 1-to-1 correspondence between $F_\beta(w, D)$ and $G_\beta(w, D)$].

(ii) $N_\alpha(D, Q)$ *is the generating function for the number of elements in* $F_\alpha(w, D)$ *or in* $G_\alpha(w, D)$ [and $N_\beta(D, Q)$ serves the same purpose for $F_\beta(w, D)$ or $G_\beta(w, D)$].

The fact that the q_i of Q in Theorem 3(ii) may be any non-negative integers with $q_1+q_2+\cdots+q_{n+1} = n(n-1)/2$ yields a number of identities on generating functions.

EXAMPLE. Let $n = 3$, and $D = [1, 2, 2]$. The system (C_α) in (6) becomes

$$(C_\alpha): \begin{cases} 1 = s_1 + s_2 + r_1 + r_4 \\ 2 = t_1 + s_2 + r_2 + r_4 \\ 2 = t_2 + r_3 + r_4 . \end{cases}$$

Of the 20 possible Q's that satisfy $q_1 + q_2 + q_3 + q_4 = 3$, we select $Q_1 = (0, 0, 0, 3)$ and $Q_2 = (0, 2, 0, 1)$. Since $N_\alpha(D, Q_1) = N_\alpha(D, Q_2)$ we have

$$\sum_{C_\alpha} \frac{x^{5-(r_1+r_2+r_3)+s_1t_1+s_2t_2}}{(t_1)(t_2)(s_1)(s_2)} \begin{bmatrix} 3 + r_4 \\ 3 \end{bmatrix} = \sum_{C_\alpha} \frac{x^{9-(r_1+3r_2+r_3)+s_1t_1+s_2t_2}}{(t_1)(t_2)(s_1)(s_2)} \begin{bmatrix} 2 + r_2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 + r_1 \\ 1 \end{bmatrix}$$

where the sum is taken over the 28 solutions to the system C_α.

Each side of this identity is also the generating function

$$\sum_{w=0}^{\infty} |G_\alpha(w, D)| x^w$$

for the ordered pairs (T, T') of n-tableaux of type α with $\operatorname{con}(T') = D^* = (0, 1, 2, 2)$ [and w as the weight of T].

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