Multisolitons, breathers, and rogue waves for the Hirota equation generated by the Darboux transformation

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MULTISOLITONS, BREATHERS, AND ROGUE WAVES FOR THE HIROTA EQUATION GENERATED BY DARBOUX TRANSFORMATION

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Abstract. The determinant representation of the n-fold Darboux transformation of the Hirota equation is given. Based on our analysis, the 1-soliton, 2-soliton and breathers are given explicitly. Further, the first order rogue wave solutions are given by Taylor expansion of the breather solutions. In particular, the explicit formula of the rogue wave has several parameters, which is more general than earlier reported results and thus provides a systematic way to tune experimentally the rogue waves by choosing different values of them.

Keywords: Darboux transformation, Hirota equation, Soliton, Breather, Rogue wave.

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1. Introduction

It is well known that the completely integrable nonlinear Schrödinger equation (NLSE)

$$iq_t + 2|q|^2 q + q_{xx} = 0 \tag{1}$$

plays an important role in many branches of physics and applied mathematics, such as nonlinear optics [1,2], plasma physics [3] and nonlinear quantum field theory [4]. Especially in nonlinear optics, the propagation of a picosecond optical pulse in an optical fiber is governed by the NLSE. After theoretical prediction of the existence of solitary waves [5] and experimental demonstration of the optical solitons [6], the research on optical soliton is more and more fascinating since it may be applied as bit rates in the next generation of optical communication system.

The NLSE has been used successfully to describe the propagation of a picosecond optical pulse. However, for the propagation of subpisecond or femtosecond pulse, the higher order effects should be taken into account and one version of higher-order nonlinear Schrödinger equation (HNLSE) is of the form

$$iq_t + \alpha_1 q_{xx} + \alpha_2 q|q|^2 + i\alpha_3 q + i\alpha_4 q_{xxx} + \alpha_5 q(|q|^2)_x + i\alpha_6 (q|q|^2)_x = 0. \tag{2}$$

This equation was first proposed by Hasegawa and Kodama [7]. Mathematically, for equation(2), many authors have obtained the following four completely integrable cases:

(1) $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_6 : \mathrm{Im}(\alpha_5) + \alpha_6 = \frac{1}{2} : 1 : 0 : 0 : 1 : 1;$

(2) $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_6 : \mathrm{Im}(\alpha_5) + \alpha_6 = \frac{1}{2} : 1 : 0 : 0 : 1 : 0;$

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(3) $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_6 : \mathrm{Im}(\alpha_5) + \alpha_6 = \frac{1}{2} : 1 : 0 : 1 : 6 : 0$, which implies the Hirota equation [8,9];

(4) $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_6 : \mathrm{Im}(\alpha_5) + \alpha_6 = \frac{1}{2} : 1 : 0 : 1 : 6 : 3$, which implies the Sasa-Satsuma equation [10,11]

by using different approaches like the Painlevé test [12], the Galilean transformation [13], the Wahlquist-Estabrook prolongation method [14]. There are multi-component extensions [15–17] of the above NLSE.

In recent years, a new wave called rogue wave attracts much attention. It was observed in many fields, such as oceanics [18–22], nonlinear optics [23–25]. Though rogue wave has caused many marine disasters, fortunately, there are already some achievements to understand this natural phenomenon. In [24], a system of extremely steep and large wave has been studied and the observation of rogue wave has been reported in an optical fiber. In [25], a mathematical solution called Peregrine soliton as a prototype of ocean rogue wave has been observed in a physical system. In [26], the authors have used an experimental set up to observe Peregrine soliton in a water wave tank.

The rogue wave of the Hirota equation is given by a very simple and powerful Darboux transformation(DT) with the help of the author's very rich empirical ideas [27]. However, there are two unusual points in this work, i.e., 1) the Lax pair does not contain spectral parameters and 2) the "seed" solution $\psi = e^{ix}$ is too special, such that its rogue wave is not universal enough. Considering the wide applicability of the Hirota equation, we shall try to find a more general form of the rogue wave of the Hirota equation by the DT [28–31] from a general "seed" solution. Specifically, we follow the AKNS procedure [32] to construct the Lax pair with spectral parameters and the corresponding Hirota equation takes the form

$$iq_t + \alpha(2|q|^2 q + q_{xx}) + i\beta(q_{xxx} + 6|q|^2 q_x) = 0, \tag{3}$$

with the choice of coefficients $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_6 : \mathrm{Im}(\alpha_5)+\alpha_6 = 1 : 2 : 0 : 1 : 6 : 0$. If letting $\alpha = 1, \beta = 0$, equation (3) reduces to equation (1). Note that equation (3) is another equivalent form of the Hirota equation [27]. This Lax pair is more convenient to construct the DT due to its parameters. Furthermore, solitons are derived from zero "seed" and breathers are derived from a periodic "seed" with a constant amplitude. At last, the rogue wave of equation (3) is given by Taylor expansion of the breather, which implies the rogue wave [18,19] of NLSE (1).

2. Lax pair of the Hirota equation

The Lax pair assures the complete integrability of a nonlinear system and is often used to obtain explicit solutions by DT. In this section, we use the AKNS procedure [32] to get the Lax pair with spectral parameters of Hirota equation (3).

By a similar way of the AKNS system, the Lax pair for equation (3) can be expressed as follows

$$\varphi_x = M\varphi, \varphi_t = N\varphi, \tag{4}$$

where $\varphi = (\varphi_1, \varphi_2)^T$, and

$$M = \begin{pmatrix} -i\lambda & q \\ -q^* & i\lambda \end{pmatrix},$$

$$N = \lambda^3 \begin{pmatrix} -4\beta i & 0 \\ 0 & 4\beta i \end{pmatrix} + \lambda^2 \begin{pmatrix} -2\alpha i & 4\beta q \\ -4\beta q^* & 2\alpha i \end{pmatrix} + \lambda \begin{pmatrix} 2\beta i|q|^2 & 2\beta i q_x + 2\alpha q \\ 2\beta i q_x^* - 2\alpha q^* & -2\beta i|q|^2 \end{pmatrix} +$$

$$\begin{pmatrix} i\alpha|q|^2+\beta(qq_x^*-q^*q_x) & i\alpha q_x-\beta(q_{xx}+2|q|^2q) \\ i\alpha q_x^*+\beta(q_{xx}^*+2|q|^2q^*) & -i\alpha|q|^2-\beta(qq_x^*-q^*q_x) \end{pmatrix},$$

and λ is a complex spectral parameter, "*"denotes the complex conjugate. One can verify that the compatibility condition $M_t-N_x+[M,N]=0$ gives rise to equation (3), where the bracket represents the usual matrix commutator.

3. Darboux Transformation

The DT [28–31] is an effective method to construct solutions including $n-$soliton and breather solutions. In this section, we would like to introduce a simple gauge transformation of spectral problems (4) as follows

$$\varphi^{[1]}=T\varphi. \tag{5}$$

It can transform linear problems (4) into the same type of linear problems, namely,

$$\varphi_x^{[1]}=M^{[1]}\varphi^{[1]}, \varphi_t^{[1]}=N^{[1]}\varphi^{[1]}, \tag{6}$$

where $M^{[1]},N^{[1]}$ have the same forms with M,N except that of q,q^* in the matrices M,N are replaced with $q^{[1]},q^{[1]*}$ in the matrices $M^{[1]},N^{[1]}$. It is easy to obtain the equations

$$M^{[1]}T=T_x+TM, \tag{7}$$

$$N^{[1]}T=T_t+TN. \tag{8}$$

In general, the transformation T is a polynomial of the parameter λ, according to Hirota equation (3), we can start from

$$T=\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\lambda+\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{9}$$

where a_1,b_1,c_1,d_1,a,b,c,d are all functions of the variables x and t.

From equations (7) and (9), it is easy to have

$$\begin{pmatrix} a_{1x} & b_{1x} \\ c_{1x} & d_{1x} \end{pmatrix}\lambda+\begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix}=\begin{pmatrix} c_1q^{[1]}\lambda-ia_1\lambda^2 & d_1q^{[1]}\lambda-ib_1\lambda^2 \\ ic_1\lambda^2-a_1q^{[1]*}\lambda & id_1\lambda^2-b_1q^{[1]*}\lambda \end{pmatrix}+\begin{pmatrix} cq^{[1]}-ia\lambda & dq^{[1]}-ib\lambda \\ ic\lambda-aq^{[1]*} & id\lambda-bq^{[1]*} \end{pmatrix}$$

$$-\begin{pmatrix} -ia_1\lambda^2-b_1q^*\lambda & a_1q\lambda+ib_1\lambda^2 \\ -ic_1\lambda^2-d_1q^*\lambda & c_1q\lambda+id_1\lambda^2 \end{pmatrix}-\begin{pmatrix} -ia\lambda-q^*b & aq+ib\lambda \\ -ic\lambda-q^*d & qc+id\lambda \end{pmatrix}. \tag{10}$$

and comparing the coefficients of $\lambda^k(k=0,1,2)$ of the above formula gives

$$b_1=c_1=0, \text{for } k=2, \tag{11}$$

$$\begin{gathered} a_{1x}=d_{1x}=0, \\ -2ib+q^{[1]}d_1-qa_1=0, 2ic-q^{[1]*}a_1+q^*d_1=0, \text{for } k=1, \end{gathered} \tag{12}$$

$$\begin{gathered} a_x=q^{[1]}c+q^*b, b_x=q^{[1]}d-qa, \\ c_x=-q^{[1]*}a+q^*d, d_x=-q^{[1]*}b-qc, \text{for } k=0. \end{gathered} \tag{13}$$

By using the calculation above, it is obvious that a_1,d_1 can be constanted and let them equal to 1 without loss of generality, so DT for equation (3) could be in the form of

$$\varphi^{[1]}=T\varphi=(\lambda I-S)\varphi, \tag{14}$$

where λ is a complex spectral parameter, I is a 2×2 identity matrix and S is a nonsingular matrix.

Substituting the expressions of $M, M^{[1]}$ and T into equation (7), and then the coefficients of λ becomes

$$\begin{pmatrix} 0 & q^{[1]} \\ -q^{[1]*} & 0 \end{pmatrix} = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} + i[S, \sigma],$$

where $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$. Therefore, the new solutions are given by

$$q^{[1]} = q - 2is_{12}, -q^{[1]*} = -q^* + 2is_{21}, \tag{15}$$

under a constraint

$$s_{12}^* = -s_{21}. \tag{16}$$

Similar to the case of the NLSE [28, 29], to obtain the explicit formula of S by the solutions of the Lax pair, we introduce

$$S = H\Lambda H^{-1}, \tag{17}$$

with

$$H = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $(f_1, f_2)^T$ is a solution of the eigenvalue equation of Lax pair (4) when $\lambda = \lambda_1$. It is useful to know that $(g_1, g_2)^T = (-f_2^*, f_1^*)^T$ is a solution of (4) when $\lambda = \lambda_1^*$. In order to satisfy the constraint of S, let $\lambda_2 = \lambda_1^*$ and $(g_1, g_2)^T = (-f_2^*, f_1^*)^T$, then

$$S = \frac{1}{\Delta} \begin{pmatrix} \lambda_1|f_1|^2 + \lambda_1^*|f_2|^2 & (\lambda_1 - \lambda_1^*)f_1f_2^* \\ (\lambda_1 - \lambda_1^*)f_1^*f_2 & \lambda_1|f_2|^2 + \lambda_1^*|f_1|^2 \end{pmatrix}, \tag{18}$$

here $\Delta = |f_1|^2 + |f_2|^2$. By a direct calculation, constraint (16) of the S can be verified. So from (15) and (18), the DT generates a new solution of the Hirota equation as

$$q^{[1]} = q - \frac{2i}{\Delta}(\lambda_1 - \lambda_1^*)f_1f_2^*. \tag{19}$$

In fact, as in the case of the NLSE [28,29,33], the DT of the Hirota equation also has determinant representation, which is convenient to get the solutions generated by the higher order transformation. Here we rewrite one-fold DT (19) in the form of determinant as

$$q^{[1]} = q - 2i\frac{S_2}{W_2} = q - 2i\frac{\begin{vmatrix} f_1 & \lambda_1 f_1 \\ g_1 & \lambda_2 g_1 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}}, \tag{20}$$

under the reductions $g_1 = -f_2^*, g_2 = f_1^*, \lambda_2 = \lambda_1^*$. For the two-fold DT, we obtain

$$q^{[2]} = q - 2i\frac{S_4}{W_4}, \tag{21}$$

where

$$S_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1^2 f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2^2 g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3^2 f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4^2 g_3 \end{vmatrix}, W_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 \end{vmatrix},$$

and under the reductions $g_1 = -f_2^*, g_2 = f_1^*, g_3 = -f_4^*, g_4 = f_3^*, \lambda_2 = \lambda_1^*, \lambda_4 = \lambda_3^*$. Similarly, the $n-$fold DT could be written as determinant form

$$q^{[n]} = q - 2i\frac{S_{2n}}{W_{2n}}, \tag{22}$$

where

$$S_{2n} = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \dots & \lambda_1^{n-1} f_1 & \lambda_1^n f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \dots & \lambda_2^{n-1} g_1 & \lambda_2^n g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 g_3 & \dots & \lambda_3^{n-1} f_3 & \lambda_3^n f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \dots & \lambda_4^{n-1} g_3 & \lambda_4^n g_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \dots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^n g_{2n-1} \end{vmatrix},$$

$$W_{2n} = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \dots & \lambda_1^{n-1} f_1 & \lambda_1^{n-1} f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \dots & \lambda_2^{n-1} g_1 & \lambda_2^{n-1} g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 g_3 & \dots & \lambda_3^{n-1} f_3 & \lambda_3^{n-1} f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \dots & \lambda_4^{n-1} g_3 & \lambda_4^{n-1} g_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \dots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^{n-1} g_{2n} \end{vmatrix}.$$

It is convenient to calculate the multi-solitons, multi-breathers, and higher order rogue waves of the Hirota equation. This result under $\alpha = 1$ and $\beta = 0$ is consistent with the corresponding determinant representation of references [28, 29, 33].

4. SOLITON AND BREATHER SOLUTION

In this section, we start from a zero "seed" solution and a periodic "seed" solution to construct new solutions(including soliton and breather solutions) by the DT obtained above, then the first order rogue wave could be obtained by Taylor expansion from the breather solution.

(1) Now let the "seed" $q = 0$ and $\lambda_1 = \xi + i\eta$, then

$$f_1 = e^{-i(\xi+i\eta)x-(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t}, f_2 = e^{i(\xi+i\eta)x+(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t}. \tag{23}$$

Taking f_1, f_2 given by equations (23) back into DT (20), we can get 1-soliton solution (See Fig.1)

$$q^{[1]}_{soliton} = 2\eta e^{2i(-\xi x-4\beta\xi^3 t-2\alpha\xi^2 t+12\beta\xi\eta^2 t+2\alpha\eta^2 t)} \sec h(-2\eta x-24\beta\eta\xi^2 t+8\beta\eta^3 t-8\alpha\eta\xi t). \tag{24}$$

(2)Let the "seed" $q = 0$ and $\lambda_1 = \xi + i\eta, \lambda_3 = \theta + i\vartheta$, by solving linear problems (4), the eigenfunctions can be obtained as follows,

$$f_1 = e^{-i(\xi+i\eta)x-(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t}, f_2 = e^{i(\xi+i\eta)x+(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t},$$

$$f_3 = e^{-i(\theta+i\vartheta)x-(4\beta i(\theta+i\vartheta)^3+2\alpha i(\theta+i\vartheta)^2)t}, f_4 = e^{i(\theta+i\vartheta)x+(4\beta i(\theta+i\vartheta)^3+2\alpha i(\theta+i\vartheta)^2)t}.$$

According to the reductions $g_1 = -f_2^*, g_2 = f_1^*, g_3 = -f_4^*, g_4 = f_3^*, \lambda_2 = \lambda_1^*, \lambda_4 = \lambda_3^*$, the 2-soliton is given explicitly by DT (21), which is plotted in Fig.2.

(3) In order to get non-trivial periodic solutions, we set "seed" $q = ce^{i\rho}$ with $\rho = ax + bt$, here a, b, c are all real constants under a condition $b = \alpha(2c^2 - a^2) +$

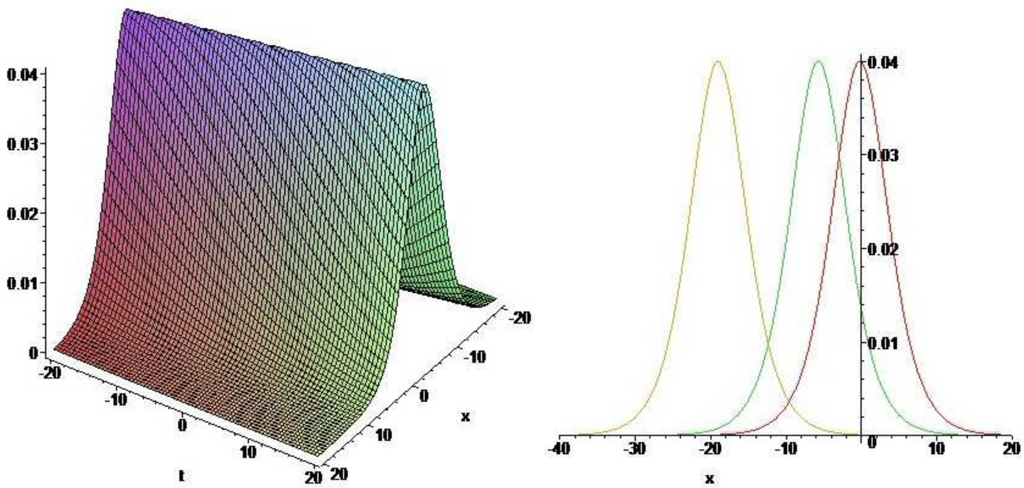

FIG. 1. (Color online)The 1-soliton solution of the Hirota equation with $\eta = 0.1, \xi = 0.05, \alpha = 1, \beta = 1$ (left) and its profiles at different times $t = 1$(red/right),$t = 30$(green/middle),$t = 100$(yellow/left).

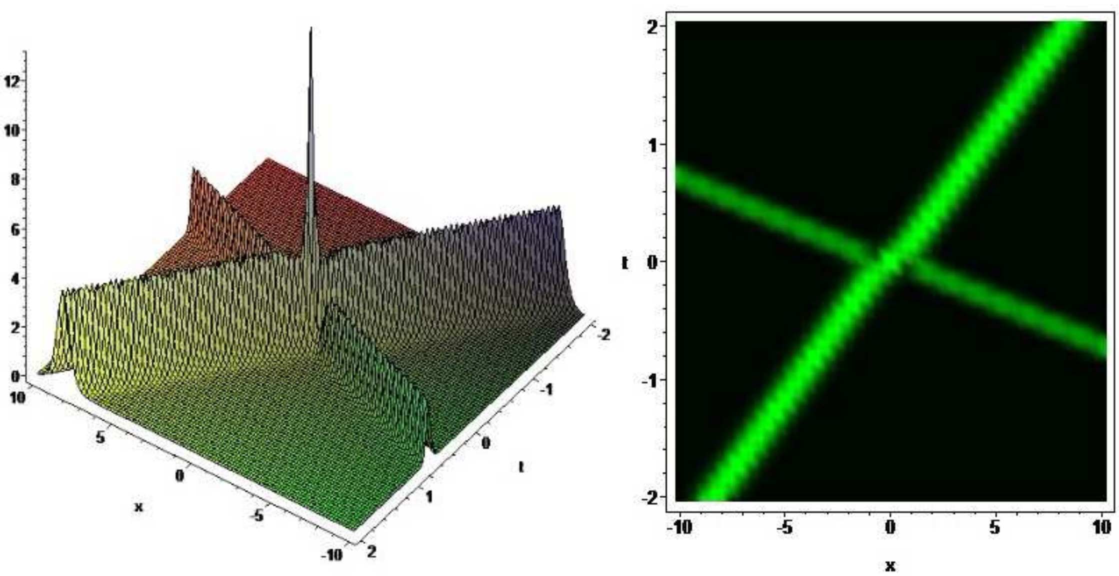

FIG. 2. (Color online)The 2-soliton solution of the Hirota equation with $\eta = 0.1, \xi = 0.8, \theta = 0, \vartheta = 1, \alpha = 1, \beta = 1$ (left) and its trajectory lines (right).

$\beta(a^3 - 6ac^2)$. The corresponding solutions of the eigenvalue equations of the Lax pair are given by

$$f_1 = ce^{i\left[(\frac{1}{2}a+c_1)x+(\frac{1}{2}b+2c_1c_2)t\right]}, f_2 = i(\frac{1}{2}a+\lambda_1+c_1)e^{i\left[(-\frac{1}{2}a+c_1)x+(-\frac{1}{2}b+2c_1c_2)t\right]}, \quad (25)$$

where

$$c_1 = \frac{1}{2}\sqrt{4c^2+4\lambda_1^2+4\lambda_1 a+a^2}, c_2 = (\alpha\lambda_1+2\beta\lambda_1^2-\frac{1}{2}a\alpha-\beta c^2+\frac{1}{2}\beta a^2-\lambda_1 a\beta).$$

By the principle of the superposition of the linear differential equation, the new eigenfunctions associated with λ_1 can be expressed by

$$F_1 = f_1 - f_2^*, F_2 = f_2 + f_1^*,$$

then we use them to get following breather solution

$$q^{[1]} = q - \frac{2i}{\Delta}(\lambda_1-\lambda_1^*)F_1F_2^* \quad (26)$$

and $\Delta = |F_1|^2 + |F_2|^2$ by DT (20). By a tedious calculation, we finally get the breather solution under $a = -2Re(\lambda_1)$ (See Fig.3)

$$q_{breather}^{[1]} = e^{i\rho}\left[c - \frac{2\eta\left(\eta\cosh(2d_2) - i\sigma\sinh(2d_2) - c\cos(2d_1)\right)}{c\cosh(2d_2) - \eta\cos(2d_1)}\right], \quad (27)$$

where

$$\lambda_1 = \xi + i\eta, a = -2\xi, \rho = ax + bt = -2\xi x + bt,$$
$$d_1 = \sigma x + (4\sigma\alpha\xi + 12\sigma\beta\xi^2 - 4\sigma\beta\eta^2 - 2\sigma^3\beta - 2\sigma\beta\eta^2)t, d_2 = (2\sigma\alpha\eta + 12\sigma\beta\xi\eta)t,$$
$$\sigma = \sqrt{\frac{-b-4\alpha\xi^2-8\beta\xi^3}{-2\alpha-12\beta\xi} - \eta^2}.$$

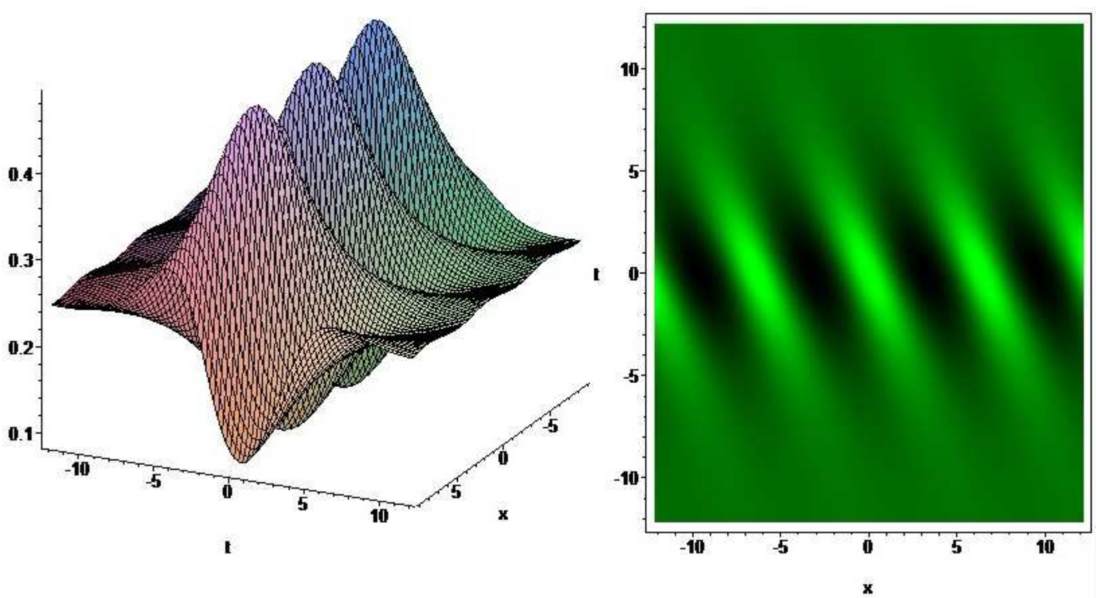

FIG. 3. (Color online)Breather solution (27) of the Hirota equation with $\alpha = 1, \beta = 1, \xi = -0.5, \eta = 0.1, b = 1$ (left) and its density plot (right).

5. ROGUE WAVE SOLUTIONS

There are at least two examples—the NLSE [18] and the derivative NLSE [34] to get rogue wave by the Taylor expansion of the breather solutions. Here we shall use this approach again to get the rogue wave of the Hirota equation from breather solution (27).

The Taylor expansion at $\eta=\sqrt{\frac{-b-4\alpha\xi^2-8\beta\xi^3}{-2\alpha-12\beta\xi}}$ of breather solution (27) implies a general form of the first order rogue wave of the Hirota equation

$$q_{roguewave}=ke^{i(-2\xi x+bt)}(1-\frac{2k_1+2k_2+ik_3t}{k_1-k_2}), \tag{28}$$

where

$$k=\sqrt{\frac{b+4\alpha\xi^2+8\beta\xi^3}{2\alpha+12\beta\xi}},$$

$$k_1=v_1t^2+v_2xt+v_3x^2, k_2=\alpha^3+18\alpha^2\beta\xi+108\alpha\beta^2\xi^2+216\beta^3\xi^3,$$

$$\begin{aligned}k_3 \;=\;& 32\xi^2\alpha^4+864\alpha\beta^2\xi^2b+144\alpha^2\beta\xi b+13824\alpha\beta^3\xi^5+13824\beta^4\xi^6\\ &+1728\beta^3\xi^3b+8\alpha^3b+4608\alpha^2\beta^2\xi^4+640\alpha^3\beta\xi^3,\end{aligned}$$

$$\begin{aligned}v_1 \;=\;& -79872\beta^3\xi^7\alpha^2-13824\beta^3\xi^5b\alpha-832\beta\xi^3\alpha^3b-4\alpha^3b^2-22528\beta^2\xi^6\alpha^3\\ &-92160\beta^5\xi^9-216\beta^2b^2\alpha\xi^2-13824\beta^4\xi^6b-432\beta^3\xi^3b^2-3200\beta\xi^5\alpha^4\\ &-138240\beta^4\xi^8\alpha-64b\alpha^4\xi^2-24\alpha^2\xi\beta b^2-192\alpha^5\xi^4-18\beta^2b^3-4992\beta^2\xi^4\alpha^2b,\end{aligned}$$

$$\begin{aligned}v_2 \;=\;& -9216\beta^4\xi^7-144\alpha^2\beta\xi^2b-64\alpha^4\xi^3-576\beta^3\xi^4b-384\alpha\beta^2\xi^3b+12\alpha\beta b^2\\ &-10752\beta^3\xi^6\alpha-896\alpha^3\xi^4\beta-16\alpha^3\xi b-4608\alpha^2\beta^2\xi^5+72\beta^2\xi b^2,\end{aligned}$$

$$v_3=-8\alpha^3\xi^2-576\beta^3\xi^5-72\beta^2\xi^2b-2\alpha^2b-24\alpha\beta\xi b-480\alpha\beta^2\xi^4-112\alpha^2\beta\xi^3.$$

It is not difficult to verify the validity of this solution. Obviously, this form of the rogue wave $q_{roguewave}$ is more general than the known result [27] because of the appearance of several parameters related to the background and the eigenvalue of the Lax pair, and thus it also provides a possible way to tune experimentally the rogue wave by choosing different values of them. Moreover, this controllability of the rogue wave highly improves the possibility of observing it in laboratory. Set $\xi=0$ in (28), then a simple rogue wave

$$q^{[1]}_{roguewave}=e^{ibt}\frac{\sqrt{\frac{b}{2\alpha}}(-2b\alpha^2x^2+12b^2\alpha\beta xt-18b^3\beta^2t^2-4b^2\alpha^3t^2+8ib\alpha^3t+3\alpha^3)}{4b^2\alpha^3t^2+2b\alpha^2x^2-12b^2\alpha\beta xt+18b^3\beta^2t^2+\alpha^3}, \tag{29}$$

is obtained, which is plotted in Fig.4. Furthermore, above rogue wave (29) reduces to the known result given by reference [27]. Moreover, setting $\alpha=1,\beta=0$, our rogue wave (29) reduces to the simplest form

$$q^{[11]}_{roguewave}=e^{ibt}\frac{\sqrt{\frac{b}{2}}(-2bx^2-4b^2t^2+8ibt+3)}{4b^2t^2+2bx^2+1}, \tag{30}$$

which is an equivalent formula of the rogue wave [18] of NLSE (1) as expected and plotted in Fig. 5. As a final remark of this paper, we would like to stress that the higher order rogue wave of the Hirota equation can be calculated from the determinant representation (22) of the DT, which will be done in a separate paper recently.

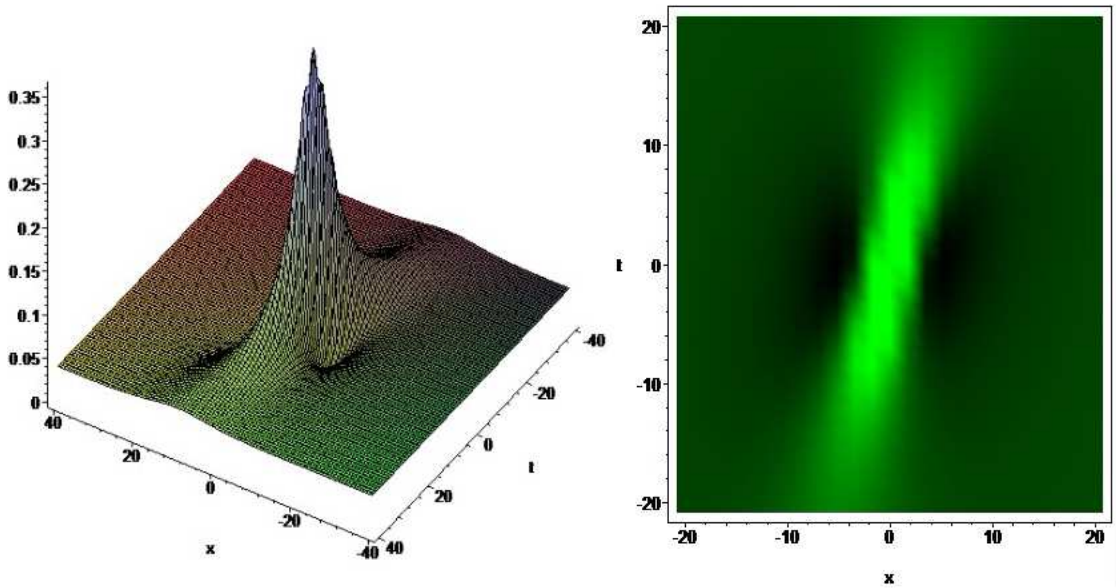

FIG. 4. (Color online)Rogue wave (29) of the Hirota equation with $\alpha = 1, \beta = 1, b = 0.08$ (left) and its density plot (right).

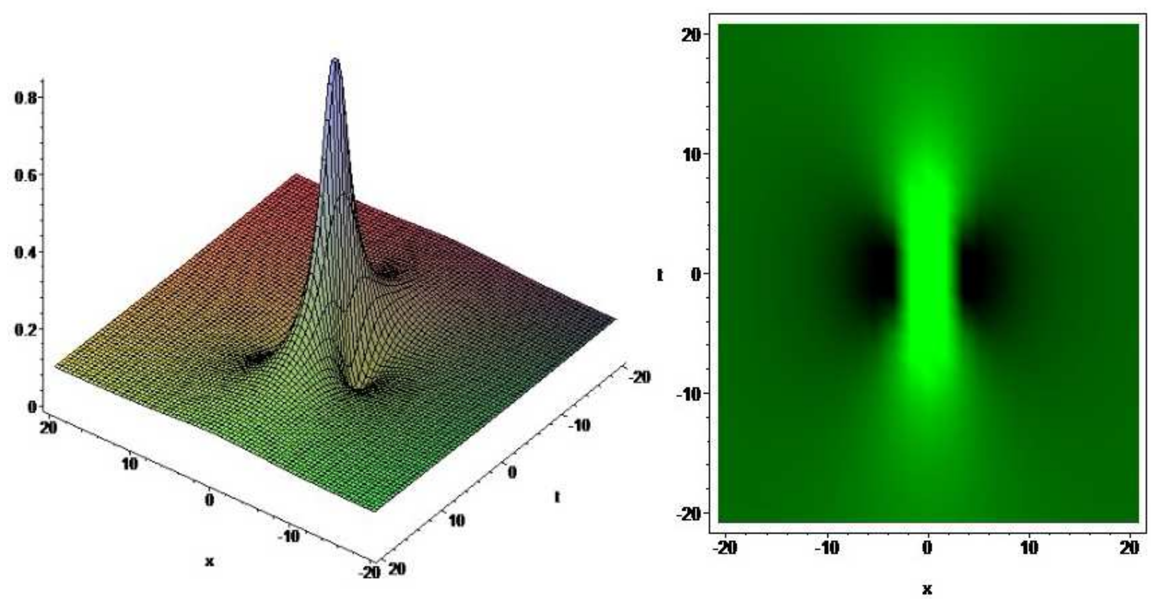

FIG. 5. (Color online)Rogue wave (30) of NLSE (1) with $b = 0.2$ (left) and its density plot (right).

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