XLIM

UMR CNRS 6172

Département
Mathématiques-Informatique

Multivalued Exponentiation Analysis.

Part II: Recursive Exponentials

Alexandre Cabot & Alberto Seeger

Rapport de recherche n° 2006-07
Déposé le 4 avril 2006

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. (33) 5 55 45 73 23 - Fax. (33) 5 55 45 73 22
http://www.xlim.fr
http://www.unilim.fr/laco

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. (33) 5 55 45 73 23 - Fax. (33) 5 55 45 73 22
http://www.xlim.fr
http://www.unilim.fr/laco

To appear in SET-VALUED ANALYSIS

MULTIVALUED EXPONENTIATION ANALYSIS. PART II: RECURSIVE EXPONENTIALS

Alexandre Cabot and Alberto Seeger

Abstract. We continue with the exponentiation analysis of multivalued maps defined on Banach spaces. In Part I of this work we have explored the Maclaurin exponentiation technique which is based on the use of a suitable power series. Now we focus the attention on the so-called recursive exponentiation method. Recursive exponentials are specially useful when it comes to study the reachable set associated to a differential inclusion of the form $\dot{z} \in F(z)$. The definition of the recursive exponential of $F : X \rightrightarrows X$ uses as ingredient the set of trajectories associated to the discrete time system $z_{k+1} \in F(z_k)$. Although we are taking inspiration from a recent paper by Alvarez/Correa/Gajardo (2005) on the relation between continuous and discrete time evolution systems, our analysis and results go far beyond the particular context of convex processes considered by these authors.

Mathematics Subject Classifications. 26E25, 33B10, 34A60.

Key Words. Exponentiation, multivalued map, differential inclusion, discrete trajectory, reachable set, Painlevé-Kuratowski limits.

1 Introduction

1.1 From Maclaurin to Recursive Exponentials

We use the same notation and terminology as in our previous work [5]. In particular, X refers to a real Banach space equipped with a norm $|\cdot|$, and $\mathbb{B}_X$ stands for the closed unit ball in X. The vector space

$$\mathcal{L}(X) = \{A : X \to X \mid A \text{ is linear continuous}\}$$

is equipped with the operator norm $\|A\| = \sup_{|x|=1} |Ax|$. The symbols

$$\begin{aligned} D(F) &= \{x \in X \mid F(x) \neq \emptyset\}, \\ \mathrm{gr}(F) &= \{(x,y) \in X \times X \mid y \in F(x)\} \end{aligned}$$

indicate respectively the domain and the graph of a multivalued map $F : X \rightrightarrows X$.

For the sake of completeness, we recall below the concept of Maclaurin exponentiability.

Definition 1. *One says that $F : X \rightrightarrows X$ is Maclaurin exponentiable at $x \in D(F)$ if the limit*

$$[\mathrm{Exp}\, F](x) = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} F^p(x) \tag{1}$$

exists in the Painlevé-Kuratowski sense and it is a nonempty set. Maclaurin exponentiability of F simply means that (1) exists nonvacuously for every $x \in D(F)$.

The theory behind this exponentiation concept is very rich and opens the way for the discussion of numerous interesting questions. The expression (1) corresponds of course to the multivalued analogue of the classical Maclaurin series defining the exponential of a linear continuous operator.

The recursive exponentiation technique has some similarities with the Maclaurin exponentiation approach, but the spirit is not the same. The motivation behind the definition of a recursive exponential is the analysis of a discrete time evolution system of the form

$$\begin{cases} z_{k+1} \in F(z_k) & \text{for } k = 0, 1, \dots \\ z_0 = x. \end{cases} \tag{2}$$

The multivalued iteration model (2) arises in areas of applied mathematics as diverse as management of renewable resources (Rapaport/Sraidi/Terreaux [17]), modeling of economic dynamics (Rubinov/Makarov [18], Rubinov/Vladimirov [19]), and discrete time constrained control problems (Phat [13, 14]).

1.2 Finite Horizon Truncations

Before introducing the recursive exponential of F we pause in our way and present an intermediate exponentiation concept. The notion of semi-recursive exponentiation is based on the idea of truncating (2) to a finite horizon. If one stops the evolution of (2) at a finite time, say after n iterations, then one gets a picture on how the system has evolved insofar. In a finite horizon setting, one generates a *chain* $(z_0, z_1, \dots, z_n)$ according to

$$\begin{cases} z_{k+1} \in F(z_k) & \text{for } k \in \{0, 1, \dots, n-1\} \\ z_0 = x. \end{cases} \tag{3}$$

Each chain $(z_0, z_1, \dots, z_n)$ yields a corresponding average $\sum_{p=0}^{n} \frac{1}{p!} z_p$, where the term $1/p!$ is interpreted as a decay or discount factor. By considering all the possible chains one gets the set

$$S_n F(x) = \left\{ \sum_{p=0}^{n} \frac{1}{p!} z_p \;\middle|\; (z_0, z_1, \dots, z_n) \text{ satisfies } (3) \right\}.$$

Definition 2. *One says that $F : X \rightrightarrows X$ is semi-recursively exponentiable at $x \in D(F)$ if the limit*

$$[\exp^* F](x) = \lim_{n \to \infty} S_n F(x) \tag{4}$$

exists in the Painlevé-Kuratowski sense and it is a nonempty set. Semi-recursive exponentiability of F means that (4) exists nonvacuously for every $x \in D(F)$.

1.3 Infinite Horizon at Once

For distinguishing between the infinite horizon model and the finite horizon counterpart, we use the term discrete trajectory in the first case and chain in the second one. For the sake of convenience, we introduce the notation $\vec{z} = \{z_p\}_{p \geq 0}$ for any sequence in X, and refer to

$$M_F(x) = \{\vec{z} \in X^{\mathbb{N}} \mid \{z_p\}_{p \geq 0} \text{ solves } (2)\}.$$

as the set of all discrete trajectories of F emanating from x.

Once an element $\vec{z} \in M_F(x)$ has been formed, there are three possibilities concerning the behavior of the partial sum $\sum_{p=0}^{n} \frac{1}{p!} z_p$ as we let $n \to \infty$. The most favourable case occurs when the limit $\sum_{p=0}^{\infty} \frac{1}{p!} z_p$ exists. The second best situation occurs when the set of accumulation points

$$\text{accum}_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} z_p = \bigcap_{N\geq 0} \text{cl}\Big\{ \sum_{p=0}^{n} \frac{1}{p!} z_p \;\Big|\; n \geq N \Big\} \tag{5}$$

is nonempty. The worse situation occurs when there is no accumulation point at all because in such a case no asymptotic information can be drawn from the discrete trajectory $\vec{z}$. At this point we face a crucial dilemma: should we take into consideration the information provided by (5) or should we simply drop all the accumulation points which are not limit points? Both strategies are perfectly acceptable but, for simplicity in the analysis, we prefer to adopt the second one.

Definition 3. *One says that $F : X \rightrightarrows X$ is recursively exponentiable at $x \in D(F)$ if*

(i) $M_F(x) \neq \emptyset$, i.e. there is a discrete trajectory of F emanating from x, and

(ii) $\forall \vec{z} \in M_F(x)$, the limit $\sum_{p=0}^{\infty} \frac{1}{p!} z_p$ exists in $(X, |\cdot|)$.

In such a case, the set

$$[\exp F](x) = \left\{ \sum_{p=0}^{\infty} \frac{1}{p!} z_p \;\Big|\; \vec{z} \in M_F(x) \right\}$$

is called the recursive exponential of F at x. Recursive exponentiability of F simply means that (i) and (ii) hold for every $x \in D(F)$.

1.4 Reachable Sets of Convex Processes

Perhaps the best way of motivating the introduction of recursive exponentials is by bringing the recent work by Alvarez/Correa/Gajardo [1] into the discussion. These authors were concerned with the problem of constructing a smooth function $z : [0,1] \to X$ that solves the differential inclusion

$$\begin{cases} \dot{z}(t) \in F(z(t)) \quad \text{on } [0,1] \\ z(0) = x, \end{cases} \tag{6}$$

whose right-hand side $F : X \rightrightarrows X$ is a strict closed convex process defined on a Hilbert space.

Recall that a multivalued map F is said to be strict if it is nonempty-valued everywhere. That F is a closed convex process simply means that $\text{gr}(F)$ is a closed convex cone.

Theorem 1 (Alvarez/Correa/Gajardo, 2006). *Let X be a Hilbert space and $F : X \rightrightarrows X$ be a strict closed convex process. Given an arbitrary $x \in X$, consider a discrete trajectory $\vec{z} \in M_F(x)$ such that $\sum_{p=0}^{\infty} \frac{1}{p!} |z_p| < \infty$. Then, for all $t \in [0,1]$, the limit*

$$\varphi_{\vec{z}}(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} \, z_p \tag{7}$$

exists, and $\varphi_{\vec{z}} : [0,1] \to X$ is a smooth solution to the Cauchy problem (6).

The adjective "smooth" indicates that $\varphi_{\vec{z}}$ is infinitely often differentiable. Theorem 1 is a very striking result indeed. What these authors have done is providing a nice recipe for building a solution to a continuous time system like (6) by starting from a discrete trajectory of the associated system (2). By an obvious reason, a function $\varphi_{\vec{z}}(\cdot)$ as in (7) is called an exponential-type solution to the Cauchy problem (6).

The link between the collection of exponential-type solutions to (6) and the recursive exponential of F is clear: if F is recursively exponentiable at x, then

$$[\exp F](x) = \{\varphi_{\vec{z}}(1) \mid \vec{z} \in M_F(x)\}$$

can be interpreted as the set of all states that can be reached at time $t = 1$ by following an exponential-type solution to (6). In other words, $[\exp F](x)$ can be used as lower estimate for the standard reachable set

$$\mathrm{Reach}(F, x) = \{z(1) \mid z : [0, 1] \to X \text{ is absolutely continuous and solves (6)}\}$$

associated to F and the initial state x.

Theorem 1 relies heavily on the fact that $\mathrm{gr}(F)$ is a closed convex cone. However, the notion of recursive exponentiation goes far beyond this particular setting. One of the goals of this paper is exploring in detail this exponentiability concept and convincing the reader that recursive exponentials are natural and important mathematical objects.

From the experience gathered in our previous paper [5], we feel that Maclaurin exponentials are usually too large and contain more elements than is reasonable to expect. The so-called *intrinsic Maclaurin exponentials* were introduced in [5, Section 5.3] with the idea of filtering the parasitic information provided by the usual Maclaurin exponentials. It turned out that intrinsic Maclaurin exponentials throw away too much information and don't retain some essential elements that we would like to keep. Semi-recursive and recursive exponentials are sets of appropriate size and good candidates for approximating the reachable set.

2 Comparing Recursive and Semi-recursive Exponentials

Our first observation is that $[\exp F](x) \subset [\exp^* F](x)$ if both exponentials exist. More often than not, this inclusion happens to be strict. With the help of the next example one can better understand why recursive and semi-recursive exponentiation are two different concepts.

Example 1. Let C be a closed convex nonempty set in a Hilbert space X. Consider the multivalued map $F : X \rightrightarrows X$ given by $F(x) = x + N_C(x)$, where $N_C(x)$ denotes the normal cone to C at x. In order to compute $S_n F(x)$, take $z_0 = x \in C$ and generate $z_1, \ldots, z_n$ according to the iteration rule

$$\begin{cases} z_1 \in x + N_C(x) \\ z_2 \in z_1 + N_C(z_1) \\ \quad \vdots \\ z_n \in z_{n-1} + N_C(z_{n-1}). \end{cases}$$

Notice that $z_1 \in x + N_C(x)$ and, at the same time, $z_1 \in C$ (because $N_C(z_1)$ contains at least one element, namely, the point $z_2 - z_1$). Since $[x + N_C(x)] \cap C = \{x\}$, one deduces that $z_1 = x$. Now, by combining $z_2 \in x + N_C(x)$ and $z_2 \in C$, one obtains $z_2 = x$. One can repeat the same argument until getting $z_{n-1} = x$. The situation is somewhat different for the end-state z_n. Clearly $z_n \in x + N_C(x)$, but we don't know whether

z_n belongs to C or not. Hence,

$$S_nF(x) = \sum_{p=0}^{n-1} \frac{1}{p!}x + \frac{1}{n!}\left[x + N_C(x)\right] = \left[\sum_{p=0}^{n} \frac{1}{p!}\right] x + N_C(x).$$

By passing to the limit as $n \to \infty$, one arrives at $[\exp^* F](x) = ex + N_C(x)$ for all $x \in C$, with $e \approx 2,718...$ denoting the Neperian constant. Let us examine now what happens when the evolution system

$$\begin{cases} z_{k+1} \in z_k + N_C(z_k) & \text{for } k = 0, 1, \dots \\ z_0 = x \end{cases}$$

runs over an infinite horizon. This time one has $z_k = x$ not only for $k \in \{0, \cdots, n-1\}$, but also for $k \geq n$. So, the recursive exponential of F exists and is given by $[\exp F](x) = \{ex\}$ for all $x \in C$.

The lesson that we learn from Example 1 is that finite horizon truncations do have an important impact in the process of exponentiation. More specifically, finite horizon truncations remove a possible constraint linking the end-state z_n with an hypothetical future state z_{n+1}. Said in other words, in a semi-recursive approach one keeps memory of the past only until the truncation occurs. After that, one continues with a Painlevé-Kuratowski limiting process which is "memoryless".

Example 2 is a variant of Example 1 that helps to illustrate the following two principles:

i) Recursive exponentiability doesn't imply semi-recursive exponentiability (obviously, the second kind of exponentiability doesn't imply the first one).

ii) If a multivalued map F is semi-recursively exponentiable, it doesn't follow that its opposite $-F$ is semi-recursively exponentiable as well.

Example 2. We define G as the opposite of the map F given in Example 1, i.e. $G(x) = -\left[x + N_C(x)\right]$ for all $x \in X$. In order to simplify some computations, we ask the closed convex set C to be symmetric, i.e. $C = -C$. By proceeding as in Example 1, one can check that G is recursively exponentiable and $[\exp G](x) = \{e^{-1}x\}$ for all $x \in C$. On the other hand, one can show that

$$S_nG(x) = \left[\sum_{p=0}^{n} \frac{(-1)^p}{p!}\right] x + (-1)^n N_C(x) \qquad \forall x \in C.$$

Hence,

$$\begin{aligned} \liminf_{n\to\infty} \; S_nG(x) &= e^{-1}x + N_C(x) \cap (-N_C(x)), \\ \limsup_{n\to\infty} \; S_nG(x) &= e^{-1}x + N_C(x) \cup (-N_C(x)). \end{aligned}$$

These Painlevé-Kuratowski limits coincide if and only if $N_C(x)$ is a linear subspace. In short, G fails to be semi-recursively exponentiable at any point $x \in C$ such that $N_C(x)$ is not a linear subspace. To fix the ideas, consider the interval $C = [-1, 1]$ in the space $X = \mathbb{R}$ and the point $x = 1$. In this case $N_C(x) = \mathbb{R}_+$. Observe that the difference between $\liminf_{n\to\infty} \; S_nG(x) = \{e^{-1}x\}$ and $\limsup_{n\to\infty} \; S_nG(x) = \mathbb{R}$ is quite substantial.

3 Comparing Semi-recursive and Maclaurin Exponentials

Semi-recursive and Maclaurin exponentials coincide for a single-valued map $f : X \to X$ because

$$S_n f(x) = \sum_{p=0}^{n} \frac{1}{p!} f^p(x) \qquad \forall n \in \mathbb{N}, \forall x \in X.$$

Simple examples show that the above equality is not true for a general multivalued map $F : X \rightrightarrows X$. From the very definition of $S_n F(x)$, one sees that

$$S_n F(x) \subset \sum_{p=0}^{n} \frac{1}{p!} F^p(x) \qquad \forall n \in \mathbb{N}, \forall x \in X.$$

By passing to the lower and upper Painlevé-Kuratowski limits one gets

$$\begin{aligned} \liminf_{n\to\infty} \; S_n F(x) \quad &\subset \quad \liminf_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} F^p(x), \\ \limsup_{n\to\infty} \; S_n F(x) \quad &\subset \quad \limsup_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} F^p(x), \end{aligned}$$

for every $x \in X$. In particular, one has the inclusion $[\exp^* F](x) \subset [\mathrm{Exp}\ F](x)$ if both exponentials exist. The next example shows that this inclusion may be strict.

Example 3. Let $\Omega \subset X$ be closed and nonempty. Let $P_\Omega : X \rightrightarrows X$ be defined by $P_\Omega(x) = \mathrm{argmin}_{z\in\Omega} |z - x|$ for all $x \in X$. Let us evaluate the set

$$S_n P_\Omega(x) = \left\{ x + \sum_{p=1}^{n} \frac{1}{p!} z_p \;\middle|\; z_1 \in P_\Omega(x),\ z_2 \in P_\Omega(z_1), \ldots, z_n \in P_\Omega(z_{n-1}) \right\}.$$

Since $z_1 \in \Omega$, it is immediate that $z_1 = z_2 = \ldots = z_n$, and hence

$$S_n P_\Omega(x) = \left\{ x + \left(\sum_{p=1}^{n} \frac{1}{p!} \right) z_1 \;\middle|\; z_1 \in P_\Omega(x) \right\} = x + \left(\sum_{p=1}^{n} \frac{1}{p!} \right) P_\Omega(x).$$

By letting $n \to \infty$ one gets the semi-recursive exponential

$$[\exp^* P_\Omega](x) = x + (e - 1) P_\Omega(x).$$

On the other hand, as shown in [5, Section 4.1], the Maclaurin exponential of the projector P_Ω is given by

$$[\mathrm{Exp}\, P_\Omega](x) = x + \lim_{n\to\infty} \sum_{p=1}^{n} \frac{1}{p!} P_\Omega(x).$$

The inclusion $[\exp^* P_\Omega](x) \subset [\mathrm{Exp}\, P_\Omega](x)$ is strict, for instance, when $\Omega = \{0, 1\}$ and $x = 1/2$. In this case the difference between $[\exp^* P_\Omega](x)$ and $[\mathrm{Exp}\, P_\Omega](x)$ is quite dramatic: the semi-recursive exponential is formed by just two elements, while the Maclaurin exponential is not even countable (cf. Proposition 4 in Section 8).

3.1 A Representation Formula for S_nF

Although the semi-recursive exponential $[\exp^* F](x)$ is quite often strictly contained in the Maclaurin exponential $[\mathrm{Exp}\, F](x)$, there are special classes of multivalued maps for which both exponentials coincide. To better understand this issue, a more careful examination of the expression $S_nF(x)$ is needed.

Observe that S_nF can be viewed as a multivalued map from X to X. One clearly has $S_0F = I$, $S_1F = I + F$, and with a small extra effort one gets

$$S_2F = I + (I + \frac{1}{2!}F) \circ F.$$

In the next lemma we derive the general form of S_nF. In order not to obscure the presentation with excessive mathematical notation, we simply assume that F is nonempty-valued.

Lemma 1. *Let $F : X \rightrightarrows X$ be nonempty-valued. Then, for all integer $n \geq 1$, the map $S_nF : X \rightrightarrows X$ admits the representation formula*

$$S_nF(x) = \left[\frac{1}{0!}I + \left[\frac{1}{1!}I + \left[\frac{1}{2!}I + \ldots + \left[\frac{1}{(n-1)!}I + \frac{1}{n!}F\right] \circ F\right] \circ F \ldots\right] \circ F\right](x) \qquad \forall x \in X. \tag{8}$$

Proof. Let $x \in X$. From the definition of the set $S_nF(x)$, one has

$$\begin{aligned} S_nF(x) &= \left\{ \sum_{p=0}^{n} \frac{1}{p!} z_p \;\middle|\; (z_0, \ldots, z_n) \text{ satisfies (3)} \right\} \\ &= \bigcup_{(z_0,\ldots,z_{n-1})} \left\{ \left(\sum_{p=0}^{n-1} \frac{1}{p!} z_p \right) + \frac{1}{n!} F(z_{n-1}) \right\} \\ &= \bigcup_{(z_0,\ldots,z_{n-1})} \left\{ \left(\sum_{p=0}^{n-2} \frac{1}{p!} z_p \right) + \left[\frac{1}{(n-1)!} I + \frac{1}{n!} F \right] (z_{n-1}) \right\}, \end{aligned}$$

where both unions are taken with respect to the chains $(z_0, \ldots, z_{n-1})$ of length $n-1$ emanating from x. By pushing this development a step further, one gets

$$S_nF(x) = \bigcup_{(z_0,\ldots,z_{n-2})} \left\{ \left(\sum_{p=0}^{n-3} \frac{1}{p!} z_p \right) + \left[\frac{1}{(n-2)!} I + \left[\frac{1}{(n-1)!} I + \frac{1}{n!} F \right] \circ F \right] (z_{n-2}) \right\},$$

where the union is taken now with respect to the chains of length $n-2$. By repeating this argument several times, one ends up with the announced representation formula for S_nF. $\square$

3.2 Positive Distribution Property

We claim that (8) is exactly what we need to know in order to compare the maps S_nF and $\sum_{p=0}^{n} \frac{1}{p!} F^p$. As we shall see in a moment, this task is not too difficult after all. A key observation in this respect is that an inclusion of the form

$$\left[\alpha_0 I + \alpha_1 F + \ldots + \alpha_{n-1} F^{n-1}\right] (F(x)) \subset \alpha_0 F(x) + \alpha_1 F^2(x) + \ldots + \alpha_{n-1} F^n(x) \tag{9}$$

holds for any $F : X \rightrightarrows X$, regardless of the choice of the reference point $x \in X$, the integer $n \geq 1$, and the scalars $\alpha_0, \ldots, \alpha_{n-1}$. An equality in (9) occurs only under special circumstances.

Definition 4. *A multivalued map $F : X \rightrightarrows X$ is called positively distributive if the equality*

$$\left[\alpha_0 I + \alpha_1 F + \ldots + \alpha_{n-1} F^{n-1}\right] \circ F = \alpha_0 F + \alpha_1 F^2 + \ldots + \alpha_{n-1} F^n \tag{10}$$

holds for any integer $n \geq 1$ and any n-tuple $(\alpha_0, \ldots, \alpha_{n-1})$ of nonnegative reals.

Most multivalued maps are not positively distributive. Among the few examples of positively distributive maps one can mention:

$$\begin{cases} \text{any constant operator} : F(x) = \Omega \ \text{ for all } x \in X. \\ \text{any monotonic dilatation} : F = h(|\cdot|)\mathbb{B}_X \text{ with } h : \mathbb{R}_+ \to \mathbb{R}_+ \text{ nondecreasing.} \\ F = I + N_C \text{ with } N_C \text{ denoting the normal map of a convex set } C. \\ F = I - T_C \text{ with } T_C \text{ denoting the tangent map of a convex set } C. \end{cases}$$

Proving that a monotonic dilatation is positively distributive is not completely trivial. The details will be seen later in the proof of Proposition 2. The example involving the normal map N_C takes place in the context of a Hilbert space. Proving that $I + N_C$ and $I - T_C$ are positively distributive maps is not a trivial matter either. We just skip entering into the details to avoid excessive space consuming.

Without further ado, we state below the main result concerning positively distributive maps.

Theorem 2. *Let $F : X \rightrightarrows X$ be nonempty-valued and positively distributive. Then,*

(a) $S_n F = \sum_{p=0}^{n} \frac{1}{p!} F^p$ for all integer $n \geq 0$.

(b) F is semi-recursively exponentiable if and only if F is Maclaurin exponentiable.

(c) Under the equivalent conditions stated in (b), the exponentials $\exp^* F$ *and* $\operatorname{Exp} F$ *coincide.*

Proof. In view of the distribution law (10), for every $x \in X$ one has

$$\begin{aligned} \left[\frac{1}{(n-2)!} I + \left[\frac{1}{(n-1)!} I + \frac{1}{n!} F\right] \circ F\right](x) &= \frac{1}{(n-2)!}\, x + \left[\frac{1}{(n-1)!} I + \frac{1}{n!} F\right](F(x)) \\ &= \frac{1}{(n-2)!}\, x + \frac{1}{(n-1)!} F(x) + \frac{1}{n!} F^2(x). \end{aligned}$$

In the same way one gets

$$\begin{aligned} &\left[\frac{1}{(n-3)!} I + \left[\frac{1}{(n-2)!} I + \left[\frac{1}{(n-1)!} I + \frac{1}{n!} F\right] \circ F\right] \circ F\right](x) \\ = \;& \frac{1}{(n-3)!}\, x + \left[\frac{1}{(n-2)!} I + \left[\frac{1}{(n-1)!} I + \frac{1}{n!} F\right] \circ F\right](F(x)) \\ = \;& \frac{1}{(n-3)!}\, x + \left[\frac{1}{(n-2)!} I + \frac{1}{(n-1)!} F + \frac{1}{n!} F^2\right](F(x)) \\ = \;& \frac{1}{(n-3)!}\, x + \frac{1}{(n-2)!} F(x) + \frac{1}{(n-1)!} F^2(x) + \frac{1}{n!} F^3(x), \end{aligned}$$

the last two equalities being due to the distribution law (10). By iterating the previous argument several times and recalling the representation formula (8), one ends up with the equality $S_n F(x) = \sum_{p=0}^{n} \frac{1}{p!} F^p(x)$ for all $x \in X$ and $n \in \mathbb{N}$. The parts (b) and (c) of the theorem are direct consequences of the part (a). □

What Theorem 2 essentially says is that for the class of positively distributive maps, there is no difference between exponentiating in the semi-recursive sense or exponentiating in the Maclaurin sense.

4 Existence of Semi-recursive and Recursive Exponentials

Semi-recursive exponentials are obtained as Painlevé-Kuratowski limits of sets of the form $S_n F(x)$. A monotonicity assumption like

$$S_n F(x) \subset S_{n+1} F(x) \quad \forall n \geq 1$$

would secure the existence of the limit $[\exp^* F](x)$. Unfortunately, such a monotonicity assumption is too restrictive and seldom holds in practice.

Anyhow, it is natural to ask whether there is a link between $S_n F(x)$ and $S_{n+1} F(x)$ after all. In order to answer this question, we consider a slight variant of (3) which consists in fixing both end-points of the chain:

$$\left\{ \begin{array}{l} z_{k+1} \in F(z_k) \quad \text{for } k \in \{0, 1, \ldots, n-1\} \\ \quad z_0 = x, \\ \quad z_n = y. \end{array} \right. \tag{11}$$

If one introduces the set

$$S_n F(x, y) = \left\{ \sum_{p=0}^{n} \frac{1}{p!} z_p \;\middle|\; (z_0, z_1, \ldots, z_n) \text{ satisfies (11)} \right\},$$

then one can write the Dynamic Programming Identity

$$S_{n+1} F(x) = \bigcup_{y \in F^n(x)} \left[S_n F(x, y) + \frac{1}{(n+1)!} F(y) \right]. \tag{12}$$

By using (12), or a direct argument, one can show that

$$S_{n+1} F(x) \;\subset\; S_n F(x) + \frac{1}{(n+1)!} F^{n+1}(x). \tag{13}$$

In the same vein, it is possible to derive the relation

$$S_n F(x) \;\subset\; S_{n+1} F(x) - \frac{1}{(n+1)!} F^{n+1}(x). \tag{14}$$

4.1 Convergence Radius

The inclusions (13) and (14) are at the origin of the next existence result. Theorem 3 concerns not only semi-recursive exponentials, but recursive exponentials as well. A key ingredient of this theorem is the term

$$\rho^F(x) := \sum_{p=0}^{\infty} \left[\frac{1}{p!} \sup_{v \in F^p(x)} |v| \right], \tag{15}$$

a number which can be seen as a sort of convergence radius for the multivalued power series $\sum_{p=0}^{\infty} \frac{1}{p!} F^p(x)$. General comments on the expression (15) will be given at several occasions in the sequel.

Theorem 3. *Let $F : X \rightrightarrows X$ be a nonempty-valued map and $x \in X$ be a point such that*

$$\rho^F(x) < \infty. \tag{16}$$

Then, F is both semi-recursively and recursively exponentiable at x.

Proof. Condition (16) implies that each term $r_p(x) := \sup_{v\in F^p(x)} |v|$ is finite, and therefore each set $F^p(x)$ is bounded. It follows that $\{S_nF(x)\}_{n\in\mathbb{N}}$ is a sequence of nonempty bounded sets. In fact,

$$S_nF(x) \subset \sum_{p=0}^{n} \frac{1}{p!} F^p(x) \subset \left[\sum_{p=0}^{n} \frac{r_p(x)}{p!}\right] \mathbb{B}_X \subset \rho^F(x)\mathbb{B}_X. \tag{17}$$

The combination of (13) and (14) yields the system

$$\begin{aligned} S_{n+1}F(x) \ &\subset \ S_nF(x) + \frac{r_{n+1}(x)}{(n+1)!}\, \mathbb{B}_X, \\ S_nF(x) \ &\subset \ S_{n+1}F(x) + \frac{r_{n+1}(x)}{(n+1)!}\, \mathbb{B}_X, \end{aligned}$$

which in turn produces the estimate

$$\Big|\mathrm{dist}[u, S_nF(x)] - \mathrm{dist}[u, S_{n+1}F(x)]\Big| \leq \frac{r_{n+1}(x)}{(n+1)!} \qquad \forall u \in X. \tag{18}$$

Now, pick up any $u \in X$. By using (18) and applying the triangular inequality in $(\mathbb{R}, |\cdot|)$, one gets

$$\begin{aligned} \big|\mathrm{dist}[u, S_nF(x)] - \mathrm{dist}[u, S_mF(x)]\big| \ &\leq \ \sum_{p=n}^{m-1} \big|\mathrm{dist}[u, S_pF(x)] - \mathrm{dist}[u, S_{p+1}F(x)]\big| \\ &\leq \ \sum_{p=n+1}^{m} \frac{r_p(x)}{p!} \end{aligned}$$

for all integers m, n with $m \geq n+1$. In view of (16), it follows that $\{\mathrm{dist}[u, S_nF(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, that is to say, it is convergent. Since $u \in X$ was chosen arbitrarily, we conclude that $\{S_nF(x)\}_{n\in\mathbb{N}}$ is Painlevé-Kuratowski convergent. This takes care of semi-recursive exponentiability. The existence of the recursive exponential $[\exp F](x)$ is simpler to prove. It suffices to observe that

$$\sum_{p=0}^{n} \frac{|z_p|}{p!} \leq \sum_{p=0}^{\infty} \frac{|z_p|}{p!} \leq \rho^F(x)$$

for any $\vec{z} \in M_F(x)$. □

We list below three remarks that help to put Theorem 3 in the right perspective.

Remark 1. Let $\hat{S}_n := \mathrm{cl}[S_nF(x)]$. Under the assumptions of Theorem 3, the convergence of $\{\hat{S}_n\}_{n\in\mathbb{N}}$ occurs not only in the Painlevé-Kuratowski sense, but also in a stronger sense. To be more precise, consider the space $\mathrm{CL}(X)$ of nonempty closed sets equipped with the Pompeiu-Hausdorff metric

$$\mathrm{haus}[C, D] := \sup_{u\in X} \big|\mathrm{dist}[u, C] - \mathrm{dist}[u, D]\big|.$$

While defined over $\mathrm{CL}(X)$, the metric $\mathrm{haus}[\cdot,\cdot]$ is allowed to take values in the extended positive line $[0, \infty]$. Since X is complete, the metric space $(\mathrm{CL}(X), \mathrm{haus}[\cdot,\cdot])$ is complete as well (cf. [3, Theorem 3.2.4]). Observe that by using (18) and applying the triangular inequality to $\mathrm{haus}[\cdot,\cdot]$, one gets

$$\mathrm{haus}\big[\hat{S}_n, \hat{S}_m\big] \leq \sum_{p=n}^{m-1} \mathrm{haus}\Big[\hat{S}_p, \hat{S}_{p+1}\Big] \leq \sum_{p=n+1}^{m} \frac{r_p(x)}{p!}$$

for any pair of integers m, n with $m \geq n+1$. Hence, $\{\hat{S}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathrm{CL}(X), \mathrm{haus}[\cdot,\cdot])$. The conclusion is that $\lim_{n\to\infty} \mathrm{haus}[\hat{S}_n, S] = 0$ for some nonempty closed set $S \subset X$.

Remark 2. For a nonempty-valued map $F : X \rightrightarrows X$, the convergence criterion (16) clearly implies that

$$\rho_F(x) := \sum_{p=0}^{\infty} \frac{1}{p!} \mathrm{dist}[0, F^p(x)] < \infty.$$

The latter condition is enough to secure the Maclaurin exponentiability of F at x (cf. [5, Theorem 1]). In view of (17), one gets the upper estimate $[\mathrm{Exp}\ F](x) \subset \rho^F(x)\mathbb{B}_X$.

Remark 3. The convergence criterion (16) implies also a stronger form of Maclaurin exponentiability that we call uniform Maclaurin exponentiability. A nonempty-valued map $F : X \rightrightarrows X$ is said to be *uniformly Maclaurin exponentiable* at x if the limit $\sum_{p=0}^{\infty} \frac{1}{p!} z_p$ exists in $(X, |\cdot|)$ for any sequence $\{z_p\}_{p\geq 0}$ such that

$$z_p \in F^p(x) \quad \forall p \geq 0. \tag{19}$$

In such a case, the set

$$[\mathrm{Exp}_\bullet F](x) = \left\{ \sum_{p=0}^{\infty} \frac{1}{p!} z_p \;\middle|\; \{z_p\}_{p\geq 0} \text{ satisfies (19)} \right\}$$

is called the *uniform Maclaurin exponential* of F at x. Notice that $[\exp\ F](x) \subset [\mathrm{Exp}_\bullet F](x) \subset [\mathrm{Exp}\ F](x)$. The main drawback of this exponentiability concept is that a trajectory $\{z_p\}_{p\geq 0}$ satisfying (19) is memoryless, in the sense that z_{p+1} bears no relation to the previous state z_p. The evolution model (19) is totally blind with respect to the past. As a consequence, the set $[\mathrm{Exp}_\bullet F](x)$ may be too large and contaminated with irrelevant information. Needless to say, the usual Maclaurin exponential suffers from the same defect.

The computation of the convergence radius $\rho^F(x)$ is not always an easy matter, specially when the evaluation of the set $F^p(x)$ is already a complicated business by itself. A simple way of ensuring the finite-valuedness of the function $\rho^F(\cdot)$ is by imposing some kind of "boundedness" assumption on F. As an immediate consequence of Theorem 3 one gets:

Corollary 1. *Consider a map $F : X \rightrightarrows X$ and a nonempty set $K \subset D(F)$ such that*

i) $F(K) \subset K$,

ii) $F(K)$ is bounded.

Then, F is semi-recursively and recursively exponentiable at each point in K. In particular, if $F : X \rightrightarrows X$ is a nonempty-valued map with bounded range $F(X)$, then F is semi-recursively and recursively exponentiable.

Proof. Take $x \in K$. Under the assumptions of the corollary, every $F^p(x)$ is nonempty and contained in the bounded set $F(K)$. In particular, the convergence radius

$$\rho^F(x) \leq |x| + (e-1) \sup_{v\in F(K)} |v|$$

is finite. It suffices then to apply Theorem 3 to the restriction of F over K. ☐

4.2 Strong Affine Growth Hypothesis

Another way of ensuring the finite-valuedness of the function $\rho^F(\cdot)$ is by imposing a bound on the growth of $F(x)$ with respect to $|x|$. A map $F : X \rightrightarrows X$ is said to satisfy the *Strong Affine Growth Hypothesis* if

$$\left\{ \begin{array}{l} \text{there are nonnegative constants } a \text{ and } b \text{ such that} \\ F(x) \subset (a|x| + b)\,\mathbb{B}_X \ \text{ for all } \ x \in X. \end{array} \right. \tag{20}$$

Such a growth hypothesis appears from time to time in the literature dealing with differential inclusions. For example, Kloeden and Valero [7] use (20) in connection with the existence of weak attractors for a certain class of multivalued dynamical systems.

The class of nonempty-valued maps satisfying the Strong Affine Growth Hypothesis (20) includes

$$\left\{ \begin{array}{l} \bullet \text{ any affine-like operator } x \in X \rightrightarrows F_{A,K}(x) := Ax + K \\ \text{with } A \in \mathcal{L}(X) \text{ and } K \subset X \text{ bounded,} \\ \bullet \text{ any map } F : X \rightrightarrows X \text{ of the form } F(x) := \{Ax + b \mid (A, b) \in \Xi \times K\} \\ \text{with } \Xi \times K \text{ bounded in } \mathcal{L}(X) \times X, \\ \bullet \text{ any bounded-valued map } F : X \rightrightarrows X \text{ admitting a constant } L \in \mathbb{R}_+ \\ \text{such that } F(x) \subset F(y) + L\,|x - y|\,\mathbb{B}_X \text{ for all } x, y \in X. \end{array} \right.$$

Of course, the above three examples are not independent. We are listing them in an order of increasing generality.

In Proposition 1 and the sequel, we use the notation

$$a^{\oplus} := \sum_{p=1}^{\infty} \frac{1 + a + \ldots + a^{p-1}}{p\,!} = \left\{ \begin{array}{ll} \frac{e^a - e}{a - 1} & \text{if } a \neq 1, \\ e & \text{if } a = 1. \end{array} \right.$$

Proposition 1. *Suppose that $F : X \rightrightarrows X$ is a nonempty-valued map satisfying the Strong Affine Growth Hypothesis (20). Then, F is both semi-recursively and recursively exponentiable. Furthermore, F is Maclaurin exponentiable and the Maclaurin exponential* Exp $F : X \rightrightarrows X$ *satisfies the strong affine growth condition*

$$[\text{Exp } F](x) \subset \left(e^a|x| + ba^{\oplus}\right)\mathbb{B}_X \quad \forall x \in X. \tag{21}$$

Proof. Take any $x \in X$. Condition (20) yields the upper estimate

$$F^p(x) \subset \Big[a^p|x| + (1 + a + \ldots + a^{p-1})b\Big]\mathbb{B}_X \qquad \forall p \geq 1, \tag{22}$$

and therefore

$$\begin{aligned} \sum_{p=0}^{n} \frac{1}{p!} F^p(x) \quad &\subset \quad x + \left(\sum_{p=1}^{n} \frac{a^p|x| + (1 + a + \ldots + a^{p-1})\,b}{p\,!} \right) \mathbb{B}_X \\ &\subset \quad x + \left((e^a - 1)|x| + ba^{\oplus}\right)\mathbb{B}_X \\ &\subset \quad \left(e^a|x| + ba^{\oplus}\right)\mathbb{B}_X. \end{aligned}$$

By letting $n \to \infty$ one proves the strong affine growth condition (21) for the Maclaurin exponential Exp F. The first part of the proposition follows from Theorem 3 and the fact that $\rho^F(x) \leq e^a|x| + ba^{\oplus} < \infty$. $\square$

Remark 4. If one has to deal with a map F taking possibly empty values, then one can invoke the generalized hypothesis

$$\left\{\begin{array}{l}\text{there are a set } K \subset D(F) \text{ and nonnegative constants } a \text{ and } b \text{ such that}\\ F(K) \subset K \text{ and } F(x) \subset (a|x|+b)\,\mathbb{B}_X \ \text{ for all } \ x \in K.\end{array}\right. \tag{23}$$

Under (23) the conclusion is that F is semi-recursively and recursively exponentiable at every point in K. In contrast to Corollary 1, the set $F(K)$ in (23) doesn't need to be bounded.

Remark 5. For a nonempty-valued map F, the assumption (20) implies obviously the so-called *Weak Affine Growth Hypothesis*

$$\left\{\begin{array}{l}\text{there are nonnegative constants } a \text{ and } b \text{ such that}\\ \operatorname{dist}[0, F(x)] \le a|x| + b \ \text{ for all } \ x \in X,\end{array}\right. \tag{24}$$

which in turn implies the Maclaurin exponentiability of F (cf. [5, Theorem 2]).

The class of nonempty-valued maps satisfying the Strong Affine Growth Hypothesis (20) is stable with respect to scalar multiplication, composition, and addition. Thus, one can state semi-recursive and recursive counterparts of [5, Corollary 1], namely:

Corollary 2. *Let $F : X \rightrightarrows X$ be a nonempty-valued map satisfying (20). Then,*

(a) For all $t \in \mathbb{R}$, tF is recursively exponentiable.

(b) For all integer $m \ge 1$, F^m is recursively exponentiable.

More generally, any polynomial expression $t_0 I + t_1 F + \ldots + t_m F^m$ is recursively exponentiable. Furthermore, all the conclusions of the corollary hold when the term "recursive" is changed by "semi-recursive".

As a direct consequence of Proposition 1, one gets semi-recursive and recursive exponentiability results for positively homogeneous maps with finite outer norm.

Corollary 3. *Let $F : X \rightrightarrows X$ be a nonempty-valued positively homogeneous map such that*

$$\|F\|_{\text{out}} := \sup_{|x|\le 1} \sup_{v \in F(x)} |v|$$

is finite. Then, F is semi-recursively and recursively exponentiable. Furthermore, F is Maclaurin exponentiable and the Maclaurin exponential Exp $F : X \rightrightarrows X$ *satisfies*

$$[\text{Exp } F](x) \subset e^{\|F\|_{\text{out}}} |x|\, \mathbb{B}_X \qquad \forall x \in X.$$

Proof. F satisfies the Strong Affine Growth Hypothesis (20) with constants $a = \|F\|_{\text{out}}$ and $b = 0$. $\square$

4.3 Modulable Maps

We mention now a few additional words on the existence of recursive exponentials. Boundedness of all discrete trajectories of F emanating from x is, of course, a sufficient condition for recursive exponentiability of F at x. However, the recursive exponential may exist even if F admits unbounded discrete trajectories. What is important in fact is that each discrete trajectory should not grow too fast in norm.

One possible way of controlling the growth of a discrete trajectory, say $\vec{z} \in M_F(x)$, is by imposing a bound on $|z_p|$ that depends on the previous r terms $|z_{p-1}|, \ldots, |z_{p-r}|$. This is the idea behind the following definition.

Definition 5. *A map $F : X \rightrightarrows X$ is called modulable if there exist an integer $r \geq 1$ (called period) and positive constants $c_0, c_1, \ldots, c_r$ such that*

$$|\xi_r| \leq c_r|\xi_{r-1}| + \cdots + c_1|\xi_0| + c_0 \tag{25}$$

for all $(\xi_0, \xi_1, \ldots, \xi_r) \in X^{r+1}$ such that $\xi_1 \in F(\xi_0), \xi_2 \in F(\xi_1), \ldots, \xi_r \in F(\xi_{r-1})$.

Some general comments on Definition 5 are in order.

i) Modulability with period $r = 1$ amounts to saying that F satisfies the Strong Affine Growth Hypothesis (20).

ii) For an ordinary function $f : X \to X$, modulability with period $r = 2$ corresponds to a growth condition of the form
$$|f^2(u)| \leq c_2|f(u)| + c_1|u| + c_0 \quad \forall u \in X.$$
In the case of a multivalued map $F : X \rightrightarrows X$, one must write of course
$$|\xi_2| \leq c_2|\xi_1| + c_1|\xi_0| + c_0 \tag{26}$$
for all (ξ_0, ξ_1, ξ_2) such that $\xi_1 \in F(\xi_0)$ and $\xi_2 \in F(\xi_1)$. A map F satisfying the modulability condition (26) falls beyond the context of Proposition 1. Contrarily to the case $r = 1$, the period $r = 2$ doesn't force F to have bounded values.

iii) Modulability with period higher than 2 is also of interest but it becomes more cumbersome to check in practice.

Theorem 4. *Suppose that $F : X \rightrightarrows X$ is nonempty-valued and modulable. Then, F is recursively exponentiable.*

Proof. Let r denote the period of the modulable map F. The case $r = 1$ is covered by Proposition 1, so we assume that $r \geq 2$. Take any $x \in X$ and $\vec{z} \in M_F(x)$. The modulability condition (25) yields

$$|z_{p+r}| \leq c_r\,|z_{p+r-1}| + \ldots + c_1\,|z_p| + c_0 \qquad \forall p \geq 0.$$

By adding $|z_{p+r-1}| + \ldots + |z_{p+1}|$ to each side of the above inequality, one obtains

$$\begin{aligned} |z_{p+r}| + |z_{p+r-1}| + \ldots + |z_{p+1}| &\leq (c_r + 1)|z_{p+r-1}| + \ldots + (c_2 + 1)|z_{p+1}| + c_1\,|z_p| + c_0 \\ &\leq M\left(|z_{p+r-1}| + \ldots + |z_p|\right) + c_0, \end{aligned}$$

with $M := \max\{c_r + 1, \ldots, c_2 + 1, c_1\}$. Consider now the sequence $\{\sigma_p\}_{p\geq 0}$ given by $\sigma_p := \sum_{i=p}^{p+r-1} |z_i|$. With such a notation, the previous inequality can be rewritten as

$$\sigma_{p+1} \leq M\,\sigma_p + c_0 \qquad \forall p \geq 0,$$

from where one gets
$$\sigma_p \leq M^p\,\sigma_0 + (1 + M + \ldots + M^{p-1})\,c_0 \qquad \forall p \geq 1.$$
Since $|z_p| \leq \sigma_p$, one arrives at

$$|z_p| \leq M^p\,(|z_0| + \ldots + |z_{r-1}|) + (1 + M + \ldots + M^{p-1})\,c_0 \qquad \forall p \geq 1,$$

showing in this way that $|z_p|$ doesn't grow too fast while compared to the factorial of p. More precisely, the decay factor $1/p!$ forces the convergence of the partial sum $\sum_{p=0}^{n} \frac{1}{p!} z_p$. $\square$

The radial function $\rho^F(\cdot)$ of a nonempty-valued modulable map F is not necessarily finite-valued, so Theorems 3 and 4 are independent results. Notice that Theorem 4 says nothing about semi-recursive exponentiability.

5 Computing Semi-recursive Exponentials

Perhaps the main drawback of semi-recursive exponentials is that their computation is in general a quite cumbersome task. In Sections 5.1 and 5.2 we present two classes of maps for which the computation of $S_n F(x)$ can be carried out without too much troubles. At the same time, we will evaluate $[\exp^* F](x)$ and see if the obtained expression coincides or not with the Maclaurin exponential.

5.1 Dilatations

Multivalued maps of the form $F = \Psi(\cdot)\mathbb{B}_X$ arise in the modeling of differential inequalities. Observe that the set $F(x) = \Psi(x)\mathbb{B}_X$ corresponds to a dilatation of the unit ball $\mathbb{B}_X$, the dilatation factor being the nonnegative number $\Psi(x)$. Under suitable assumptions on the function Ψ, an explicit formula for the Maclaurin exponential of F was derived in [5, Theorem 3].

We compute now the semi-recursive exponential of a map F having the special structure

$$F(x) = h(|x|)\mathbb{B}_X \quad \forall x \in X.$$

As we shall see next, everything boils down to evaluating an expression of the form

$$[\mathrm{Exp}\; h](s) := \sum_{p=0}^{\infty} \frac{1}{p!}\; h^p(s),$$

where h^p is understood as the p-fold composition of the function h.

Proposition 2. *Suppose that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function. Then, the map $F = h(|\cdot|)\mathbb{B}_X$ is semi-recursively exponentiable and*

$$[\exp^* F](x) = [\mathrm{Exp}\; F](x) = x + \Big([\mathrm{Exp}\; h](|x|) - |x|\Big)\mathbb{B}_X.$$

Proof. By setting $\Psi = h \circ |\cdot|$, one falls within the framework of [5, Theorem 3]. In the present situation, the function $s \in \mathbb{R}_+ \mapsto \Psi_{\max}(s) := \sup_{|w|\le s} \Psi(w)$ is just h, and the Maclaurin exponential of F takes the form

$$[\mathrm{Exp}\, F](x) = x + \left(\sum_{p=1}^{\infty} \frac{1}{p!} h^p(|x|)\right)\mathbb{B}_X = x + \Big([\mathrm{Exp}\, h](|x|) - |x|\Big)\mathbb{B}_X.$$

On the other hand, we claim that the monotonicity of h implies the positive distributivity of F. Take $n \ge 1$ and $(\alpha_0, \dots, \alpha_{n-1}) \in \mathbb{R}^n_+$. Since h is nondecreasing, one has

$$F^2(x) = \bigcup_{|y|\le h(|x|)} h(|y|)\mathbb{B}_X = h^2(|x|)\mathbb{B}_X \qquad \forall x \in X.$$

In a similar way, one gets $F^p(\cdot) = h^p(|\cdot|)\mathbb{B}_X$ for all $p \geq 2$. Therefore, for all $x \in X$, one has

$$\begin{aligned}
[\alpha_0 I + \alpha_1 F + \ldots + \alpha_{n-1}F^{n-1}](F(x)) &= \bigcup_{y \in h(|x|)\mathbb{B}_X} \left[\alpha_0 y + \alpha_1 F(y) + \ldots + \alpha_{n-1}F^{n-1}(y)\right] \\
&= \bigcup_{y \in h(|x|)\mathbb{B}_X} \left[\alpha_0 y + \left(\sum_{p=1}^{n-1} \alpha_p h^p(|y|)\right) \mathbb{B}_X\right].
\end{aligned}$$

One gets in this way

$$[\alpha_0 I + \alpha_1 F + \ldots + \alpha_{n-1}F^{n-1}](F(x)) \supset \alpha_0\, h(|x|)\mathbb{B}_X, \tag{27}$$

as well as

$$\begin{aligned}
[\alpha_0 I + \alpha_1 F + \ldots + \alpha_{n-1}F^{n-1}](F(x)) &\supset \bigcup_{y \in h(|x|)\mathbb{S}_X} \left[\alpha_0 y + \left(\sum_{p=1}^{n-1} \alpha_p h^p(|y|)\right) \mathbb{B}_X\right] \\
&\supset \alpha_0 h(|x|)\mathbb{S}_X + \left(\sum_{p=1}^{n-1} \alpha_p h^{p+1}(|x|)\right) \mathbb{B}_X, \qquad (28)
\end{aligned}$$

where $\mathbb{S}_X$ denotes the unit sphere in X. The combination of (27) and (28) yields

$$\begin{aligned}
[\alpha_0 I + \alpha_1 F + \ldots + \alpha_{n-1}F^{n-1}](F(x)) &\supset \left(\sum_{p=0}^{n-1} \alpha_p h^{p+1}(|x|)\right) \mathbb{B}_X \\
&= \alpha_0 F(x) + \ldots + \alpha_{n-1}F^n(x).
\end{aligned}$$

We have proven in this way that F is positively distributive. To complete the proof of the proposition, we just need to invoke Theorem 2. $\square$

Upward monotonicity of h is an essential assumption in Proposition 2. It is interesting to observe that downward monotonicity of h does not secure the equality between semi-recursive and Maclaurin exponentials. Example 4 serves not only to illustrate this point, but it has also a further merit: it displays a very curious link between the operation of semi-recursive exponentiation and the Fibonacci sequence $\{f_p\}_{p\geq 0}$ defined recursively by

$$\begin{cases} f_{p+2} = f_p + f_{p+1} & \text{for } p = 0, 1, 2, \ldots \\ f_0 = 0,\ f_1 = 1. \end{cases}$$

A well known property of the Fibonacci sequence is that

$$r_p := \frac{f_{p+1}}{f_p} \ \to\ \phi := \frac{1+\sqrt{5}}{2} \approx 1.618$$

as $p \to \infty$. The number ϕ is usually referred to as the Golden Ratio.

Example 4. Consider the dilatation (or rather, retraction) $F : X \rightrightarrows X$ given by

$$F(x) = \frac{1}{1+|x|}\,\mathbb{B}_X \quad \forall x \in X.$$

Computing the Maclaurin exponential $[\exp F](x)$ is an easy matter because $F^p(x) = \mathbb{B}_X$ for every $p \geq 2$. One simply gets

$$[\mathrm{Exp}\ F](x) = x + \left(\frac{1}{1+|x|} + e - 2\right)\mathbb{B}_X.$$

On the other hand, F is semi-recursively exponentiable because its range $F(X) = \mathbb{B}_X$ is a bounded set (see Corollary 1). After some simplificatory work, one passes from

$$S_nF(x) = \left\{ x + \sum_{p=1}^{n} \frac{1}{p!} z_p \;\middle|\; |z_1| \leq \frac{1}{1+|x|},\ |z_2| \leq \frac{1}{1+|z_1|}, \ldots, |z_n| \leq \frac{1}{1+|z_{n-1}|} \right\}$$

to the expression

$$S_nF(x) = x + \left(\frac{1}{1+|x|} + \gamma_n(|x|)\right)\mathbb{B}_X,$$

with

$$\gamma_n(s) := \sum_{p=2}^{n} \frac{1}{p!} \frac{f_p + s f_{p-1}}{f_{p+1} + s f_p} = \sum_{p=2}^{n} \frac{1}{p!} \frac{1 + (r_p - 1)s}{r_p + s}.$$

One ends up with the semi-recursive exponential

$$[\exp^* F](x) = x + \left(\frac{1}{1+|x|} + \gamma(|x|)\right)\mathbb{B}_X,$$

with

$$\gamma(s) := \sum_{p=2}^{\infty} \frac{1}{p!} \frac{1 + (r_p - 1)s}{r_p + s}.$$

The function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ increases from $\gamma(0) \approx 0.392$ to $\gamma(\infty) \approx 0.617 < e - 2$. So, $[\exp^* F](x)$ is strictly included in $[\mathrm{Exp}\ F](x)$. The excess of $[\mathrm{Exp}\ F](x)$ over $[\exp^* F](x)$ is the largest possible when $x = 0$, but it becomes smaller as $|x|$ increases.

5.2 Affine-like Operators

By an affine-like operator one understands a multivalued map $F_{A,K} : X \rightrightarrows X$ of the form

$$F_{A,K}(x) = Ax + K,$$

where $A : X \to X$ is a linear continuous operator and K is a nonempty set in X. The importance of this class of maps has been amply justified in the control literature, so we don't need to indulge on this matter.

It has been shown in [5, Proposition 7] that any affine-like operator is Maclaurin exponentiable. Let us now examine the semi-recursive exponentiability of $F_{A,K}$. To do this, we start by working out the general representation formula (8) in this special setting. Take any $x \in X$ and write

$$\left[\frac{1}{(n-1)!} I + \frac{1}{n!} F_{A,K}\right](x) = \left(\frac{1}{(n-1)!} I + \frac{1}{n!} A\right) x + \frac{1}{n!} K.$$

It ensues that

$$
\begin{aligned}
& \left[\frac{1}{(n-2)!}I + \left[\frac{1}{(n-1)!}I + \frac{1}{n!}F_{A,K}\right] \circ F_{A,K}\right](x) \\
= \;& \frac{1}{(n-2)!}\,x + \bigcup_{y\in Ax+K} \left[\frac{1}{(n-1)!}I + \frac{1}{n!}F_{A,K}\right](y) \\
= \;& \frac{1}{(n-2)!}\,x + \bigcup_{y\in Ax+K} \left\{\left(\frac{1}{(n-1)!}I + \frac{1}{n!}A\right)y + \frac{1}{n!}K\right\} \\
= \;& \left(\frac{1}{(n-2)!}I + \frac{1}{(n-1)!}A + \frac{1}{n!}A^2\right)x + \left(\frac{1}{(n-1)!}I + \frac{1}{n!}A\right)K + \frac{1}{n!}K.
\end{aligned}
$$

By iterating this process one sees that (8) takes the form

$$
S_nF_{A,K}(x) = \left(\frac{1}{0!}I + \frac{1}{1!}A + \ldots + \frac{1}{n!}A^n\right)x + \Psi_n(A,K), \tag{29}
$$

with

$$
\Psi_n(A,K) := \left(\frac{1}{1!}I + \frac{1}{2!}A + \ldots + \frac{1}{n!}A^{n-1}\right)K + \left(\frac{1}{2!}I + \ldots + \frac{1}{n!}A^{n-2}\right)K + \ldots + \frac{1}{n!}K.
$$

On the other hand, as shown in [5, Section 4.3], one has

$$
\sum_{p=0}^{n} \frac{1}{p!}F^p_{A,K}(x) = \left(\frac{1}{0!}I + \frac{1}{1!}A + \ldots + \frac{1}{n!}A^n\right)x + \Gamma_n(A,K), \tag{30}
$$

with

$$
\Gamma_n(A,K) := K + \frac{1}{2!}(K + AK) + \ldots + \frac{1}{n!}(K + AK + \ldots + A^{n-1}K)
$$

So, if one wishes to compare $[\exp^* F_{A,K}](x)$ and $[\operatorname{Exp} F_{A,K}](x)$, then one must study the limiting behavior of the sets $\Psi_n(A,K)$ and $\Gamma_n(A,K)$ as $n \to \infty$. This is not simple in general, but there are at least two cases in which the situation is well understood. These cases are presented in the following two corollaries.

Corollary 4. *Let $F : X \rightrightarrows X$ be a constant operator, i.e. there is a nonempty set $K \subset X$ such that $F(x) = K$ for all $x \in X$. Then, F is semi-recursively exponentiable and*

$$
[\exp^* F](x) = [\operatorname{Exp} F](x) = x + \lim_{n\to\infty}\left[K + \frac{1}{2!}K + \ldots + \frac{1}{n!}K\right] \qquad \forall x \in X.
$$

Proof. By taking $A = 0$ in (29) and (30), one gets

$$
S_nF(x) = \sum_{p=0}^{n} \frac{1}{p!}F^p(x) = x + \left[K + \frac{1}{2!}K + \ldots + \frac{1}{n!}K\right].
$$

The term between brackets converges in the Painlevé-Kuratowski sense, so it suffices to let $n \to \infty$. ☐

Corollary 5. *Let $K \subset X$ be a nonempty closed convex set and A be a nonnegative multiple of the identity operator, say $A = a\,I$ with $a \in \mathbb{R}_+$. Then $F_{A,K}$ is semi-recursively exponentiable and*

$$
[\exp^* F_{A,K}](x) = [\operatorname{Exp} F_{A,K}](x) = e^a x + a^{\oplus} K \qquad \forall x \in X.
$$

Proof. We come back again to formulas (29) and (30). Since the set K is convex and the coefficient a is nonnegative, we can rearrange terms so as to obtain

$$S_n F_{A,K}(x) = \sum_{p=0}^{n} \frac{1}{p!} F^p(x) = \left(\frac{1}{0!} + \ldots + \frac{a^n}{n!} \right) x + \left[\frac{1}{1!} + \frac{1+a}{2!} + \ldots + \frac{1 + \ldots + a^{n-1}}{n!} \right] K.$$

It is now a matter of passing to the limit as $n \to \infty$. The suggested formula for the Maclaurin exponential is already given in [5, Proposition 8]. $\square$

Both assumptions in Corollary 5 are essential: if either the coefficient a is negative or if the set K is not convex, then $\exp^* F_{A,K}$ may be different from $\operatorname{Exp} F_{A,K}$. This fact is illustrated with the help of the next two examples.

Example 5. Let $K \subset X$ be a nonempty closed convex set and $F : X \rightrightarrows X$ be given by $F(x) = -x + K$ for all $x \in K$. Let us rewrite formula (29) with $A = -I$:

$$S_n F(x) = \left(\sum_{p=0}^{n} \frac{(-1)^p}{p!} \right) x + \left[\frac{1}{1!} + \frac{-1}{2!} + \ldots + \frac{(-1)^{n-1}}{n!} \right] K + \ldots + \left[\frac{1}{(n-1)!} + \frac{-1}{n!} \right] K + \frac{1}{n!} K.$$

Since K is convex and each sum between brackets is positive, we can rearrange terms and get

$$S_n F(x) = \left(\sum_{p=0}^{n} \frac{(-1)^p}{p!} \right) x + \left[\frac{1}{1!} + \frac{1}{2!}(1 + (-1)) + \ldots + \frac{1}{n!} \left(1 + \ldots + (-1)^{n-1} \right) \right] K. \tag{31}$$

Observe that for every $p \in \{1, \ldots, n-1\}$

$$1 + \ldots + (-1)^{p-1} = \begin{cases} 0 & \text{if} \quad p \text{ is even,} \\ 1 & \text{if} \quad p \text{ is odd,} \end{cases}$$

so the terms corresponding to even indices in the bracket disappear. By passing to the Painlevé-Kuratowski limit in (31) one arrives at

$$[\exp^* F](x) = \left(\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \right) x + \left(\sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \right) K = e^{-1} x + (\mathrm{sh}1) K,$$

whereas the Maclaurin exponential is given by (cf. [5, Proposition 8])

$$[\operatorname{Exp} F](x) = e^{-1} x + (\mathrm{sh}1) K + \frac{\mathrm{ch}1}{2} (K - K).$$

One sees that the inclusion $[\exp^* F](x) \subset [\operatorname{Exp} F](x)$ is strict unless K satisfies the equation

$$(\mathrm{sh}1) K = (\mathrm{sh}1) K + \frac{\mathrm{ch}1}{2} (K - K).$$

The above equation holds, for instance, if K is a singleton or if K is a linear subspace, but it doesn't hold if K is a convex cone such that $K \neq -K$. The difference between $[\exp^* F](x)$ and $[\operatorname{Exp} F](x)$ is specially striking if K is a convex cone close to a ray but $K - K$ is the whole space X.

Before presenting the second example, we state an easy lemma.

Lemma 2. *Let $A : X \to X$ be a linear continuous operator and $K \subset X$ be a nonempty bounded set. Then, $F_{A,K}$ is semi-recursively exponentiable.*

Proof. $F_{A,K}$ satisfies the Strong Affine Growth Hypothesis (20) with constants $a = \|A\|$ and $b = \sup_{u \in K} |u|$. The semi-recursive exponentiability of $F_{A,K}$ is then a consequence of Proposition 1. $\square$

Example 6. Consider the nonconvex set $K = \{0, 1\}$ and the map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $F(x) = x + K$ for all $x \in \mathbb{R}$. Formula (29) takes now the form

$$S_n F(x) = \left(\sum_{p=0}^{n} \frac{1}{p!} \right) x + \left(\sum_{p=1}^{n} \frac{1}{p!} \right) \{0, 1\} + \left(\sum_{p=2}^{n} \frac{1}{p!} \right) \{0, 1\} + \ldots + \frac{1}{n!} \{0, 1\}.$$

On the other hand, Section 4.3 in [5] yields

$$\begin{aligned} \sum_{p=0}^{n} \frac{1}{p!} F^p(x) &= \left(\sum_{p=0}^{n} \frac{1}{p!} \right) x + K + \frac{1}{2!}(K + K) + \ldots + \frac{1}{n!} \underbrace{(K + K + \ldots + K)}_{n \text{ terms}} \\ &= \left(\sum_{p=0}^{n} \frac{1}{p!} \right) x + \{0, 1\} + \frac{1}{2!} \{0, 1, 2\} + \ldots + \frac{1}{n!} \{0, 1, \ldots, n\}. \end{aligned}$$

To fix the ideas, take for instance $x = 0$. One sees that $S_n F(0) \neq \sum_{p=0}^{n} \frac{1}{p!} F^p(0)$ for all $n \geq 2$. With a bit of care one can check that this inequality persists after passing to the limit. Let us consider first the evaluation of $[\operatorname{Exp} F](0)$. A simple monotonicity argument (cf. [5, Lemma 1]) shows that

$$[\operatorname{Exp} F](0) = \operatorname{cl} \left[\bigcup_{n \geq 1} \left(\sum_{p=1}^{n} \frac{1}{p!} F^p(0) \right) \right].$$

Notice that

$$F(0) = \{0, 1\}, \quad \sum_{p=1}^{2} \frac{1}{p!} F^p(0) = \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \right\}, \quad \sum_{p=1}^{3} \frac{1}{p!} F^p(0) = \left\{ 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \ldots, \frac{15}{6} \right\},$$

and, in general,

$$\sum_{p=1}^{n} \frac{1}{p!} F^p(0) = \frac{1}{n!} \left\{ 0, 1, \ldots, \ n! \sum_{p=0}^{n-1} \frac{1}{p!} \right\}.$$

The elements of the above set form a regular subdivision of the interval $\left[0, \sum_{p=0}^{n-1} \frac{1}{p!} \right]$, the step of the subdivision being $1/n!$. It follows that

$$[\operatorname{Exp} F](0) = \left[0, \sum_{p=0}^{\infty} \frac{1}{p!} \right] = [0, e].$$

Even if $K = \{0, 1\}$ is formed by just two elements, the Maclaurin exponential of F at 0 is a set which has positive Lebesgue measure! This observation leads us to think that Maclaurin exponentiation is a concept that doesn't discriminate well enough the very nature of the original data. Let us see now what happens with the semi-recursive exponential $[\exp^* F](0)$, a limit which exists in view of Lemma 2. We now get a set which has null Lebesgue measure! To check this, we proceed as follows. From (13) we know already that

$$S_{n+1}F(0) \subset S_n F(0) + \frac{1}{(n+1)!} F^{n+1}(0).$$

But

$$F^{n+1}(0) = \underbrace{K + K + \ldots + K}_{n+1 \text{ terms}} = \{0, 1, \ldots, n+1\} \subset [0, n+1].$$

Thus,

$$S_{n+1}F(0) \;\subset\; S_n F(0) + \frac{1}{(n+1)!}\,[0, n+1] \;=\; S_n F(0) + \left[0, \frac{1}{n!}\right].$$

So, for every $m \geq n+1$, one can write

$$S_m F(0) \;\subset\; S_n F(0) + \left[0, \sum_{p=n}^{m-1} \frac{1}{p!}\right] \;\subset\; S_n F(0) + \left[0, \sum_{p=n}^{\infty} \frac{1}{p!}\right]. \tag{32}$$

Notice that the set on the rightmost side of (32) is closed. Taking the Painlevé-Kuratowski limit as $m \to \infty$, one gets

$$[\exp^* F](0) \;\subset\; S_n F(0) + \left[0, \sum_{p=n}^{\infty} \frac{1}{p!}\right] \;=\; \bigcup_{z \in S_n F(0)} \left\{ z + \left[0, \sum_{p=n}^{\infty} \frac{1}{p!}\right] \right\}. \tag{33}$$

Due to its special structure, the set $S_n F(0)$ is finite and its cardinal is majorized by 2^n. Denoting by λ the Lebesgue measure on $\mathbb{R}$, we then infer from (33) that

$$\lambda\left([\exp^* F](0)\right) \;\;\leq\;\; \operatorname{card}(S_n F(0)).\left(\sum_{p=n}^{\infty} \frac{1}{p!}\right) \;\leq\; 2^n \left(\sum_{p=n}^{\infty} \frac{1}{p!}\right) \qquad \forall n \geq 1.$$

But, as shown in the proof of Proposition 4 (Section 8), one has $\lim_{n\to\infty} 2^n \left(\sum_{p=n}^{\infty} \frac{1}{p!}\right) = 0$. So, one gets $\lambda\left([\exp^* F](0)\right) = 0$ as claimed. Summarizing, this example shows not only that $[\exp^* F](0)$ is strictly included in $[\operatorname{Exp} F](0)$, but also that there is substantial difference between both exponentials.

6 Recursive Exponentiability of Bundles

This section deals with the recursive exponentiability of bundles of linear continuous operators. By this expression we mean a mutivalued map $F : X \rightrightarrows X$ of the form

$$F(x) = \{Ax : A \in \Xi\} \qquad \forall x \in X,$$

with Ξ denoting a nonempty subset in $\mathcal{L}(X)$.

Bundles arise in a natural way in the modeling of continuous and discrete time evolution processes. We mention the references [4, 6, 8, 9, 10, 11, 12, 15, 16] for more information on these mathematical objects and for discovering some of their applications. Of course, this list of references is by no means exhaustive.

Recursive exponentials of bundles are better understood if one introduces first a suitable concept of "exponential mixture" for the family Ξ.

Definition 6. *The geometric exponential mixture of $\Xi \subset \mathcal{L}(X)$ is the set $\mathcal{M}(\Xi) \subset \mathcal{L}(X)$ defined by*

$$Q \in \mathcal{M}(\Xi) \quad \Longleftrightarrow \quad Q = I + \lim_{n\to\infty} \sum_{p=1}^{n} \frac{1}{p!} A_p \circ \cdots \circ A_2 \circ A_1 \tag{34}$$

$$\text{for some sequence } \{A_p\}_{p\geq 1} \text{ in } \Xi,$$

where the limit (34) takes place in the space $(\mathcal{L}(X), \|\cdot\|)$.

The term "geometric" in Definition 6 doesn't have a special meaning. It is used mainly for distinguishing $\mathcal{M}(\Xi)$ from the exponential mixture in the sense of Amri/Seeger [2], the latter being a concept adapted to forward exponentiation.

Theorem 5. *Let $F : X \rightrightarrows X$ be the bundle associated to a nonempty bounded set $\Xi \subset \mathcal{L}(X)$. Then, F is recursively exponentiable and $\exp F$ is the bundle associated to $\mathcal{M}(\Xi)$, i.e.*

$$[\exp F](x) = \{Qx \mid Q \in \mathcal{M}(\Xi)\} \qquad \forall x \in X.$$

Furthermore, one has the estimates

(a) $\operatorname{dist}[0, (\exp F)(x)] \leq \operatorname{dist}[0, \mathcal{M}(\Xi)]\, |x|$.

(b) $[\exp F](x) \subset e^{\operatorname{mod}(\Xi)} |x| \mathbb{B}_{\mathcal{L}(X)}$ *with* $\operatorname{mod}(\Xi) := \sup_{A\in\Xi} \|A\|$.

Proof. Take $x \in X$. The recursive exponentiability of F at x follows from Proposition 1 and the boundedness of Ξ. In order to prove the inclusion

$$\{Qx \mid Q \in \mathcal{M}(\Xi)\} \subset [\exp F](x),$$

take $y = Qx$ with $Q \in \mathcal{M}(\Xi)$. If one represents Q as in (34), then

$$\begin{aligned} y &= \left[I + \lim_{n\to\infty} \sum_{p=1}^{n} \frac{1}{p!} A_p \circ \cdots \circ A_2 \circ A_1 \right] x \\ &= x + \lim_{n\to\infty} \left[\sum_{p=1}^{n} \frac{1}{p!} A_p \circ \cdots \circ A_2 \circ A_1 x \right] \\ &= \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} z_p, \end{aligned}$$

with $z_0 = x$ and $z_p = A_p \circ \cdots \circ A_2 \circ A_1 x$ for every $p \geq 1$. Since $\{z_p\}_{p\geq 0}$ is a discrete trajectory of F emanating from x, the limit y belongs to $[\exp F](x)$. Conversely, take an arbitrary $y \in [\exp F](x)$ and represent it in the form $y = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} z_p$, with $z_0 = x$ and $z_{p+1} \in F(z_p)$ for every $p \geq 0$. Given the specific structure of F, for each integer $p \geq 1$ there is an operator $A_p \in \Xi$ such that $z_{p+1} = A_p z_p$. Hence,

$$z_p = A_p \circ \cdots \circ A_2 \circ A_1 x \qquad \forall p \geq 1.$$

We now use the fact that Ξ is bounded, i.e. $\operatorname{mod}(\Xi) < \infty$. Since the space $(\mathcal{L}(X), \|\cdot\|)$ is complete and

$$\sum_{p=1}^{n} \frac{1}{p!} \|A_p \circ \cdots \circ A_2 \circ A_1\| \leq \sum_{p=1}^{n} \frac{[\operatorname{mod}(\Xi)]^p}{p!} \leq e^{\operatorname{mod}(\Xi)} - 1$$

for all $n \geq 1$, we deduce that

$$Q_n := I + \sum_{p=1}^{n} \frac{1}{p!} A_p \circ \cdots \circ A_2 \circ A_1$$

converges to some $Q \in \mathcal{M}(\Xi)$. So,

$$y = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} z_p = \lim_{n\to\infty} (Q_n x) = \left(\lim_{n\to\infty} Q_n\right) x = Qx,$$

as we wanted to prove. $\square$

Corollary 6. *Let $F : X \rightrightarrows X$ be the bundle associated to a nonempty bounded set $\Xi \subset \mathcal{L}(X)$. Then, for all $t \in \mathbb{R}$, the map tF is recursively exponentiable and*

$$[\exp(tF)](x) = \left\{ \left[I + \sum_{p=1}^{\infty} \frac{t^p}{p!} A_p \circ \cdots \circ A_2 \circ A_1 \right] x \;\middle|\; \{A_p\}_{p\geq 1} \textit{ in } \Xi \right\} \qquad \forall x \in X.$$

Proof. This result is a direct application of Theorem 5 and the fact that tF is the bundle associated to the bounded set $t\,\Xi := \{tA \mid A \in \Xi\}$. $\square$

Computing geometric exponential mixtures is not always an easy task. For getting a better grasp of the meaning of $\mathcal{M}(\Xi)$, let us try to identify some particular elements in this set. First of all, it should be clear that

$$e^A \in \mathcal{M}(\Xi) \qquad \forall A \in \Xi.$$

To see this, take in (34) the sequence $\{A_p\}_{p\geq 1}$ given $A_p = A$ for every $p \geq 1$. Instead of considering a constant sequence, one can also alternate between two or more elements taken from Ξ.

Example 7. Suppose that $\Xi \subset \mathcal{L}(\mathbb{R}^d)$ contains in particular the linear maps (or matrices) B and C. If one chooses

$$A_p = \begin{cases} B & \text{if} \quad p \text{ is odd,} \\ C & \text{if} \quad p \text{ is even,} \end{cases} \tag{35}$$

then one produces the limit

$$Q = I + B + \frac{1}{2!}CB + \frac{1}{3!}BCB + \frac{1}{4!}CBCB + \cdots$$

After a short rearrangement, one arrives at

$$Q = \left[I + \frac{1}{2!}(CB) + \frac{1}{4!}(CB)^2 + \cdots \right] + B \left[I + \frac{1}{3!}(CB) + \frac{1}{5!}(CB)^2 + \cdots \right].$$

Notice that if CB is symmetric and positive definite, then CB admits an invertible square root $\sqrt{CB}$ and

$$Q = \mathrm{ch}(\sqrt{CB}) + B \left[\sqrt{CB} \right]^{-1} \mathrm{sh}(\sqrt{CB}).$$

More elaborate limits are obtained by considering a sequence $\{A_p\}_{p \geq 1}$ whose alternation pattern is not as simple as in (35).

7 More on Infinitesimal Generator Formulas

Recall that any linear continuous operator $A : X \to X$ admits the "infinitesimal generator" representation

$$Ax = \lim_{t \to 0^+} \frac{e^{tA}x - x}{t} \qquad \forall x \in X,$$

where the limit is taken in the Banach space $(\mathcal{L}(X), \|\cdot\|)$. The above formula admits an interesting extension to a multivalued setting if exponentiation is understood in the Maclaurin sense. As shown in the reference [5, Theorem 4], only a very mild assumption is needed in order to obtain

$$\lim_{t \to 0^+} \frac{U_F(t)(x) - x}{t} = \mathrm{cl}[F(x)], \tag{36}$$

where the limit is taken in the Painlevé-Kuratowski sense and $U_F(t)(x) := [\mathrm{Exp}(tF)](x)$. The next proposition shows that, with the same assumption as in [5, Theorem 4], it is possible to write the multivalued infinitesimal generator formula (36) for recursive and semi-recursive exponentials as well.

Proposition 3. *Consider a nonempty-valued map $F : X \rightrightarrows X$ satisfying the following regularity requirement at the origin:*

$$\lim_{w \to 0} \sup_{y \in F(w)} |y| = 0. \tag{37}$$

Let $x \in X$ be a point such that $F(x)$ is bounded. Then,

(a) there exists $t_* > 0$ such that, for every $t \in]0, t_*[$, the map tF is both semi-recursively and recursively exponentiable at x.

(b) the multivalued infinitesimal generator formula (36) holds whatever the sense of exponentiation is taken, be it semi-recursive or recursive.

Proof. Take a real $M > 0$ such that $F(x) \subset M\mathbb{B}_X$. As can be seen from the proof of Theorem 4 in [5], under the assumption (37) it is possible to find a positive real t_* such that

$$(tF)^p(x) \subset M\mathbb{B}_X \quad \forall p \geq 1, \ \forall t \in]0, t_*[. \tag{38}$$

The upper estimate (38) leads immediately to the convergence criterion

$$\rho^{tF}(x) := \sum_{p=0}^{\infty} \left[\frac{1}{p!} \sup_{v \in (tF)^p(x)} |v| \right] \leq |x| + (e-1)M < \infty.$$

So, Theorem 3 takes care of the part (a). In order to prove (b), recall that the multivalued infinitesimal generator formula holds for Maclaurin exponentials. Since

$$[\exp(tF)](x) \subset [\exp^*(tF)](x) \subset [\mathrm{Exp}(tF)](x),$$

we just need to check that

$$F(x) \subset \liminf_{t \to 0^+} \frac{[\exp(tF)](x) - x}{t}.$$

Take any sequence $\{t_k\}_{k \in \mathbb{N}}$ converging to 0^+. We must show that

$$F(x) \subset \liminf_{k \to \infty} \Delta(t_k), \tag{39}$$

where $\Delta(t) := t^{-1} \{[\exp(tF)](x) - x\}$. Pick up any $y \in F(x)$. Consider, for each $k \in \mathbb{N}$, the sequence $\{z_{k,p}\}_{p \geq 0}$ defined recursively by

$$\begin{cases} z_{k,p+1} \in t_k F(z_{k,p}) \quad \text{for } p = 1, 2, \ldots \\ z_{k,1} = t_k \, y \\ z_{k,0} = x. \end{cases}$$

Such a choice of $z_{k,1}$ is crucial. For k large enough, the map $t_k F$ is recursively exponentiable at x and

$$x + t_k y + \sum_{p=2}^{\infty} \frac{1}{p!} z_{k,p} \in [\exp(t_k F)](x).$$

Thus, $y + w_k \in \Delta(t_k)$, with

$$w_k := \frac{1}{t_k} \sum_{p=2}^{\infty} \frac{1}{p!} z_{k,p} \in \frac{1}{t_k} \sum_{p=2}^{\infty} \frac{1}{p!} (t_k F)^p(x).$$

To complete the proof of (39), it remains to show that $w_k \to 0$ as $k \to \infty$. This part is a bit delicate since one must rely on a result that is sharper than (38). By examining again the proof of Theorem 4 in [5], one sees that

$$\frac{1}{t} \sum_{p=2}^{\infty} \frac{1}{p!} (tF)^p(x) \subset \rho(t) \mathbb{B}_X,$$

with $\rho(t) \to 0$ as $t \to 0^+$. It is posssible, in fact, to derive an explicit formula for the function $\rho(\cdot)$ but there is no need to elaborate on this detail. It is enough to observe that $|w_k| \leq \rho(t_k)$, and therefore $\{w_k\}_{k \in \mathbb{N}} \to 0$ as desired. □

Remark 6. The conclusions (a) and (b) in Proposition 3 are also valid for uniform Maclaurin exponentials. The multivalued infinitesimal generator formula (36) holds for any exponential $U_F(t)(x)$ that is sandwiched between the recursive exponential $[\exp(tF)](x)$ and the Maclaurin exponential $[\mathrm{Exp}(tF)](x)$. The two sides of the sandwich can be seen as two extremal concepts of exponentiation.

8 Exponentiation and Cantor-like Sets

We end this work with a few remarks on the structure of Maclaurin and semi-recursive exponentials in a very special setting. We want to convince the reader that both exponentials can have a very fancy form, even if F is a constant map on $\mathbb{R}$ given, for instance, by

$$F(x) = \{0, 1\} \quad \forall x \in \mathbb{R}. \tag{40}$$

This map F is positively distributive, and therefore the exponentials Exp F and $\exp^* F$ coincide. In fact,

$$[\mathrm{Exp}\ F](x) - x = [\exp^* F](x) - x = \sum_{p=1}^{\infty} \frac{1}{p!}\{0, 1\},$$

where the Painlevé-Kuratowski limit on the right-hand side is a set lying between $\{0, e-1\}$ and $[0, e-1]$. This set deserves a closer examination.

We will examine, more generally, a Painlevé-Kuratowski limit of the form $\sum_{p=1}^{\infty} \mu_p\ \{0, 1\}$, with $\{\mu_p\}_{p\geq 1}$ being a sequence of positive scalars such that $\sum_{p=1}^{\infty} \mu_p < \infty$.

Lemma 3. *Let* $\mu = \{\mu_p\}_{p\geq 1}$ *be a sequence of positive scalars such that*

$$\forall n \geq 1, \qquad \sum_{p=n+1}^{\infty} \mu_p < \mu_n. \tag{41}$$

Then, $\sum_{p=1}^{\infty} \mu_p\{0, 1\}$ *is a noncountable set.*

Proof. We introduce the function $\Phi_\mu : \{0, 1\}^{\mathbb{N}} \to \mathbb{R}$ defined by $\Phi_\mu(y) = \sum_{p=1}^{\infty} \mu_p\, y_p$. We claim that Φ_μ is an increasing function when the set $\{0, 1\}^{\mathbb{N}}$ is endowed with the *lexicographic order* $\preceq$. Consider $y = \{y_p\}_{p\geq 1} \in \{0, 1\}^{\mathbb{N}}$ and $v = \{v_p\}_{p\geq 1} \in \{0, 1\}^{\mathbb{N}}$ with $y \preceq v$ and $y \neq v$. Define $p_0 = \min\{p \in \mathbb{N} \mid y_p \neq v_p\}$. We then have $y_{p_0} = 0$ and $v_{p_0} = 1$, so that

$$\Phi_\mu(v) - \Phi_\mu(y) = \mu_{p_0} + \sum_{p=p_0+1}^{\infty} \mu_p\, (v_p - y_p).$$

Taking into account the assumption (41), one has

$$\left| \sum_{p=p_0+1}^{\infty} \mu_p\, (v_p - y_p) \right| \leq \sum_{p=p_0+1}^{\infty} \mu_p < \mu_{p_0}.$$

We infer that $\Phi_\mu(v) - \Phi_\mu(y) > 0$, ending the proof of our claim. Now, since $\{\Phi_\mu(y)\ |y \in \{0, 1\}^{\mathbb{N}}\}$ is included in $\sum_{p=1}^{\infty} \mu_p\{0, 1\}$ and Φ_μ is a one-to-one function, one can write

$$\mathrm{card}\left(\sum_{p=1}^{\infty} \mu_p\{0, 1\} \right) \geq \mathrm{card}\left(\{0, 1\}^{\mathbb{N}} \right) = 2^{\aleph_0}.$$

This completes the proof of the lemma. $\square$

We now prove that the Lebesgue measure of the set $\sum_{p=1}^{\infty} \mu_p\{0,1\}$ is equal to zero if the sequence $\mu = \{\mu_p\}_{p\geq 1}$ is chosen in a suitable way.

Lemma 4. *Let $\mu = \{\mu_p\}_{p\geq 1}$ be a sequence of positive scalars such that*

$$\lim_{n\to\infty} 2^n \sum_{p=n+1}^{\infty} \mu_p = 0. \tag{42}$$

Then, the set $\sum_{p=1}^{\infty} \mu_p\{0,1\}$ has null Lebesgue measure.

Proof. Denote by λ the Lebesgue measure on $\mathbb{R}$ and write

$$E_n = \sum_{p=n+1}^{\infty} \mu_p\{0,1\}, \quad \forall n \geq 0.$$

Observe that the set $E_0 = \sum_{p=1}^{\infty} \mu_p\{0,1\}$ can be decomposed as $E_0 = E_1 \cup (\mu_1 + E_1)$. We deduce by using a recurrence argument that for every $n \geq 1$,

$$E_0 = E_n \bigcup \left(\bigcup_{1\leq i\leq n} \mu_i + E_n \right) \bigcup \left(\bigcup_{1\leq i<j\leq n} \mu_i + \mu_j + E_n \right) \ldots \bigcup (\mu_1 + \mu_2 + \ldots + \mu_n + E_n)\,. \tag{43}$$

Since $E_n \subset \left[0, \sum_{p=n+1}^{\infty} \mu_p\right]$, we infer from (43) that E_0 is included in the union of 2^n intervals, each of them having a length less than or equal to $\sum_{p=n+1}^{\infty} \mu_p$. We deduce that

$$\forall n \geq 1, \quad \lambda(E_0) \leq 2^n \sum_{p=n+1}^{\infty} \mu_p.$$

The assumption (42) leads then to the desired conclusion. $\square$

We now combine Lemmas 3 and 4 in order to get the following result:

Proposition 4. *The set $\sum_{p=1}^{\infty} \frac{1}{p\,!}\{0,1\}$ is noncountable and has null Lebesgue measure.*

Proof. For the noncountability result, it suffices to check that

$$\forall n \geq 1, \quad \sum_{p=n+1}^{\infty} \frac{1}{p!} < \frac{1}{n!}.$$

This inequality can be shown as follows

$$\begin{aligned} \sum_{p=n+1}^{\infty} \frac{1}{p!} &= \frac{1}{(n+1)!} \left[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \ldots \right] \\ &\leq \frac{1}{(n+1)!} \sum_{p=0}^{\infty} \frac{1}{(n+2)^p} = \frac{1}{(n+1)!}\, \frac{n+2}{n+1} < \frac{1}{n!}\,. \end{aligned}$$

As a by-product one gets

$$\lim_{n\to\infty} 2^n \sum_{p=n+1}^{\infty} \frac{1}{p!} \leq \lim_{n\to\infty} \frac{2^n}{n!} = 0,$$

which is what we need for obtaining the second part of the proposition. □

Proposition 4 is telling us that the operation of Maclaurin exponentiation of a gentle operator like (40) leads to an exponential which is noncountable and with null Lebesgue measure.

Remark 7. A similar result as in Proposition 4 can be obtained for the limit $\sum_{p=1}^{\infty} \frac{1}{a^p}\{0,1\}$ with $a \in]2, \infty[$. The case $a = 3$ leads to the well known properties of the triadic Cantor set. The case $a = 2$ does not fall into this category of examples. For this border case, both assumptions (41) and (42) fail.

9 Conclusions

The classical Maclaurin series defining the exponential of a linear continuous operator $A : X \to X$ makes sense also in the case of a multivalued map $F : X \rightrightarrows X$. Although the Maclaurin exponentiation approach seems natural, it is not necessarily the most clever way of handling exponentiability issues in a multivalued context.

The chief advantage of Maclaurin exponentials is that they do exist under very mild assumptions on the map F. Unfortunately, Maclaurin exponentials are sets which are usually too large and this is a major problem.

Recursive exponentials are sets of smaller size and reflect better the intuitive idea that we have about the exponentiation operation. Semi-recursive exponentiation is somehow a compromise between Maclaurin and recursive exponentiation.

In Table 1 we summarize the situation concerning existential issues. We also indicate whether the corresponding exponential is bounded or not. The expression "may not be bounded" means that we have found an example showing that unboundedness is possible. Similarly, the expression "may not exist" means that nonexistence is possible. The only unclear item is the semi-recursive exponentiability of modulable maps.

Assumption on nonempty-valued map F	Maclaurin	Semi-recursive	Recursive
$\rho^F(\cdot)$ finite-valued	existence and boundedness	existence and boundedness	existence and boundedness
$\rho_F(\cdot)$ finite-valued	existence, may not be bounded	may not exist	may not exist
modulability	existence, may not be bounded	unclear	existence, may not be bounded

Table 1. Existence and boundedness results

References

[1] F. Alvarez, R. Correa, and P. Gajardo. Inner estimation of the eigenvalue set and exponential series solutions to differential inclusions. *J. Convex Analysis* 12 (2005), 1–11.

[2] A. Amri and A. Seeger. Exponentiating a bundle of linear operators. *Set-Valued Analysis*, 2006, in press.

[3] G. Beer. *Topologies on Closed and Closed Convex Sets.* Kluwer Acad. Publisher, Dordrecht, 1993.

[4] A.V. Bogatyrev and E.S. Pyatnitskii. Construction of piecewise-quadratic Lyapunov functions for nonlinear control systems (Russian). *Avtomat. i Telemekh.* 1987, no. 10, 30-38.

[5] A. Cabot and A. Seeger. Multivalued exponentiation analysis. Part I: Maclaurin exponentials. Submitted to this journal.

[6] R. Cominetti and R. Correa. Sur une dérivée du second ordre en analyse non différentiable. *C.R. Acad. Sc. Paris*, t. 303, Série I, (1986), 861-864.

[7] P.E. Kloeden and J. Valero. Attractors of weakly asymptotically compact set-valued dynamical systems. *Set-Valued Analysis* 13 (2005), 381-404.

[8] J.B. Hiriart-Urruty. Characterization of the plenary hull of the generalized Jacobian matrix. *Math. Programming Study* 17 (1982), 1-12.

[9] A.D. Ioffe. Nonsmooth analysis: differential calculus of nondifferentiable mappings. *Trans. Amer. Math. Soc.*, 266 (1981), 1-56.

[10] A.P. Molchanov and E.S. Pyatnitskii. Lyapunov functions that determine necessary and sufficient conditions for the stability of linear differential inclusions (Russian). *Nauka Sibirsk. Otdel.*, Novosibirsk, 323 (1987), 52-61.

[11] A.P. Molchanov and Y.S. Pyatnitskiy. Stability criteria for selector-linear differential inclusions. *Soviet Math. Dokl.* 36 (1988), 421-424.

[12] A.P. Molchanov and Y.S. Pyatnitskiy. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems Control Lett.* 13 (1989), 59-64.

[13] V.N. Phat. Weak asymptotic stabilizability of discrete-time systems given by set-valued operators. *J. Math. Anal. Appl.* 202 (1996), 363-378.

[14] V.N. Phat. *Constrained Control Problems of Discret Processes*, World Scientific Publishing Co., Singapore, 1996.

[15] E.S. Pyatnitskii and L.B. Rapoport. Boundary of the domain of asymptotic stability of selector-linear differential inclusions and the existence of periodic solutions. *Soviet Math. Dokl.* 44 (1992), 785-790.

[16] E.S. Pyatnitskii and L.B. Rapoport. Criteria of asymptotic stability of differential inclusions and periodic motions of time-varying nonlinear control systems. *IEEE Trans. Circuits and Systems*, 43 (1996), 219-229.

[17] A. Rapaport, S. Sraidi and J.P. Terreaux. Optimality of greedy and sustainable policies in the management of renewable resources. *Optimal Control Appl. Methods* 24 (2003), 23-44.

[18] A.M. Rubinov and V.L. Makarov, *Mathematical Theory of Economic Dynamics and Equilibria.* Springer, New York, 1977.

[19] A. Rubinov and A. Vladimirov. Dynamics of positive multiconvex relations. *J. Convex Analysis* 8 (2001), 387-399.

A. Cabot
Université de Limoges, Laboratoire LACO,
123 avenue Albert Thomas, 87060 Limoges, France
alexandre.cabot@unilim.fr

A. Seeger
Université d'Avignon, Departement de Mathématiques,
33, rue Louis Pasteur, 84000 Avignon, France
alberto.seeger@univ-avignon.fr