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MULTIVALUED GENERALIZATIONS OF THE KANNAN FIXED POINT THEOREM

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Abstract

In this paper we obtain multi-valed mapping generalizations of two recent theorems of Kikkawa and Suzuki [M. Kikkawa and T. Suzuki, Some similarity between contractions and Kannan mappings, Fixed Point Theory Appl., (2008), Article ID 649749, 1–8] and the main theorem of Enjouji et all. [Y. Enjouji, M. Nakanishi and T. Suzuki, A Generalization of Kannan's Fixed Point Theorem, Fixed Point Theory and Applications, Volume 2009, Article ID 192872, 10 pages].

1 Introduction and preliminaries

Let (X, d) be a metric space and let T be a self-mapping on X. Then T is called a Kannan mapping if there exists $a \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq ad(x, Tx) + ad(y, Ty) \tag{1}$$

for all $x, y \in X$. If T is a such that

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\}$$

for some $r \in [0, 1)$ and all $x, y \in X$, then T is called a generalized Kannan mapping. If X is complete, then every (generalized) Kannan mapping have a unique fixed point [6]. Subrahmanyam [12] proved that Kannan theorem characterizes the metric completeness of underlying spaces. It is known that the Banach theorem [1] cannot characterize the metric completeness [2].

In [7] authors generalized Kannan mappings.

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Theorem 1.1. *(Kikkawa and Suzuki [7]) Let T be a mapping on complete metric space (X, d) and let φ be a non-increasing function from $[0, 1)$ onto $(1/2, 1]$ defined by*

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \le r \le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Let $\alpha \in [0, 1/2)$ and put $r = \alpha/(1-\alpha) \in [0, 1)$. Suppose that

$$\varphi(r)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le \alpha d(x, Tx) + \alpha d(y, Ty) \tag{2}$$

for all $x, y \in X$. Then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Theorem 1.2. *(Kikkawa and Suzuki [7]) Let T be a mapping on complete metric space (X, d) and let θ be a non-increasing function from $[0, 1)$ onto $(1/2, 1]$ defined by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \dfrac{1}{2}(\sqrt{5}-1), \\ \dfrac{1-r}{r^2} & \text{if } \dfrac{1}{2}(\sqrt{5}-1) \le r \le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Suppose that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le r \max\{d(x, Tx), d(y, Ty)\} \tag{3}$$

for all $x, y \in X$. Then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

On the other side, Nadler [10] proved multi-valued extension of the Banach contraction theorem.

Theorem 1.3. *(Nadler [10]) Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Many fixed point theorems have been proved by various authors as generalizations of the Nadler's theorem (see [9], [4], [11], [3]). The following recent result [8] is a generalization of Nadler [10].

Theorem 1.4. *(Kikkawa and Suzuki [8]) Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Define a strictly decreasing function η from $[0, 1)$ onto $(1/2, 1]$ by*

$$\eta(r) = \frac{1}{1+r}$$

and assume that there exists $r \in [0,1)$ *such that*

$$\eta(r)d(x,Tx) \leq d(x,y) \text{ implies } H(Tx,Ty) \leq rd(x,y)$$

for all $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

In this paper we obtain multi-valued version of Theorem 1.2 and then as a corollary we obtain multi-valued version of Theorem 1.1.

2 Main results

Let (X,d) be a metric space. We denote by $CB(X)$ the family of all non-empty closed bounded subsets of X. Let $H(\cdot,\cdot)$ be the Hausdorff metric, i.e.,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$$

for $A, B \in CB(X)$, where $d(x,B) = \inf_{y\in B} d(x,y)$.

Now we prove our main result.

Theorem 2.1. *Define a non-increasing function* φ *from* $[0,1)$ *into* $(0,1]$ *by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \dfrac{\sqrt{5}-1}{2}, \\ 1-r, & \text{if } \dfrac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

Let (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $CB(X)$. *Assume that*

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq r\max\{d(x,Tx), d(y,Ty)\} \tag{4}$$

for all $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Proof. Let r_1 be a real number such that $0 \leq r < r_1 < 1$. Let $u_1 \in= X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, then $d(u_2,Tu_2) \leq H(Tu_1,Tu_2)$ and $\varphi = (r)d(u_1,Tu_1) \leq d(u_1,Tu_1) \leq d(u_1,u_2)$. Thus from the assumption (4),

$$d(u_2,Tu_2) \leq H(Tu_1,Tu_2) \leq r\max\{d(u_1,Tu_1), d(u_2,Tu_2)\}.$$

Hence, as $r < 1$, we have $d(u_2,Tu_2) \leq rd(u_1,u_2)$. So, there exists $u_3 \in Tu_2$ such that $d(u_2,u_3) \leq r_1 d(u_1,u_2)$. Thus, we can construct a sequence $\{u_n\}$ in X such that

$$u_{n+1} \in Tu_n \quad \text{and} \quad d(u_{n+1},u_{n+2}) \leq \; r_1 d(u_n,u_{n+1}).$$

Hence, by induction,

$$d(u_n,u_{n+1}) \leq r_1^{n-1} d(u_1,u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} r_1^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now we shall show that

$$d(z, Tx) \leq rd(x, Tx) \quad \text{for all} \quad x \in X\backslash\{z\}. \tag{5}$$

Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z, u_n) \leq (1/3)d(z, x)$ for all $n \geq n_0$. Then we have

$$\begin{aligned} \varphi(r)\, d(u_n, Tu_n) &\leq d(u_n, Tu_n) \\ &\leq d(u_n, u_{n+1}) \\ &\leq d(u_n, z) + d(u_{n+1}, z)\cdot \\ &\leq \frac{2}{3} d(x, z). \end{aligned}$$

Since

$$\begin{aligned} \frac{2}{3} d(x, z) &= d(x, z) - \frac{1}{3} d(x, z) \\ &\leq d(x, z) - d(u_n, z) \\ &\leq d(u_n, x), \end{aligned}$$

we get $\varphi(r)\, d(u_n, Tu_n) \leq d(u_n, x)$. Then from (4),

$$H(Tu_n, Tx) \leq r \max\{d(u_n, Tu_n), d(x, Tx)\}.$$

Since $u_{n+1} \in Tu_n$, then $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$. So, it follows that

$$d(u_{n+1}, Tx) \leq r \max\{d(u_n, u_{n+1}), d(x, Tx)\}$$

for all $n \in N$ with $n \geq n_0$. Letting n tends to ∞, we obtain $d(z, Tx) \leq rd(x, Tx)$. Thus we proved (5).

Now we show that $z \in Tz$. Suppose, to the contrary, that $z \notin Tz$. Consider at first the case $0 \leq r < \frac{\sqrt{5}-1}{2}$. Let $a \in Tz$. Then $a \neq z$ and so by (5), we have $d(z, Ta) \leq rd(a, Ta)$. On the other hand, since

$$\varphi(r)\, d(z, Tz) = d(z, Tz) \leq d(z, a),$$

from (4) we have

$$H(Tz, Ta) \leq r \max\{d(z, Tz), d(a, Ta)\}. \tag{6}$$

Hence we have $d(a,Ta) \le H(Tz,Ta) \le r\max\{d(z,Tz),d(a,Ta)\}$. Hence we get $d(a,Ta) \le rd(z,Tz)$. Therefore, by (5) and (6), we obtain

$$\begin{aligned} d(z,Tz) &\le d(z,Ta) + H(Ta,Tz) \\ &\le rd(a,Ta) + r\max\{d(z,Tz),d(a,Ta)\} \\ &\le rd(a,Ta) + rd(z,Tz) \\ &\le r^2 d(z,Tz) + rd(z,Tz) \\ &= \left(r^2 + r\right) d(z,Tz). \end{aligned}$$

Hence, as $r < \frac{\sqrt{5}-1}{2}$ implies $r^2 + r < 1$, we have

$$d(z,Tz) < d(z,Tz),$$

a contradiction. So we obtain $z \in Tz$.

Consider now the case $\frac{\sqrt{5}-1}{2} \le r < 1$. We first prove

$$H(Tx,Tz) \le r\max\{d(x,Tx),d(z,Tz)\} \quad \text{for all} \quad x \in X. \tag{7}$$

If $x = z$, then (7) obviously holds. So we assume $x \ne z$. Then for every $n \in N$, there exists $y_n \in Tx$ such that $d(z,y_n) \le d(z,Tx) + (1/n)d(x,z)$. We have

$$\begin{aligned} d(x,Tx) &\le d(x,y_n) \\ &\le d(x,z) + d(z,y_n) \\ &\le d(x,z) + d(z,Tx) + \frac{1}{n}d(x,z) \\ &\le d(x,z) + rd(x,Tx) + \frac{1}{n}d(x,z) \end{aligned}$$

for $n \in N$. Hence $(1-r)d(x,Tx) \le (1+1/n)d(x,z)$ holds. Letting n tend to ∞, we have $(1-r)d(x,Tx) \le d(x,z)$. Thus

$$\varphi(r)d(x,Tx) \le d(x,z).$$

From the assumption (4), we obtain (7). Therefore, as $u_{n+1} \in Tu_n$, from (7) with $x = u_n$ we have

$$\begin{aligned} d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1},Tz) \\ &\le \lim_{n\to\infty} H(Tu_n,Tz) \\ &\le \lim_{n\to\infty} r\max\{d(u_n,Tu_n),d(z,Tz)\} \\ &\le \lim_{n\to\infty} r\max\{d(u_n,u_{n+1}),d(z,Tz)\} \\ &= rd(z,Tz). \end{aligned}$$

Hence$(1-r)d(z,Tz) \le 0$, which implies $d(z,Tz) = 0$. Since Tz is closed, we obtain $z \in Tz$. This completes the proof. □

Theorem 2.1 is a multi-valued mapping generalization of the theorem 2.3 of Kikkawa and Suzuki [7] and therefore the Kannan fixed point theorem [6] for generalized Kannan mappings.

Corollary 2.1. *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Let $\alpha \in [0, 1/2)$ and put $r = 2\alpha$. Suppose that*

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad H(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \tag{8}$$

for all $x, y \in X$, where the function φ is defined as in Theorem 2.1. Then there exists $z \in X$ such that $z \in Tz$.

Corollary 2.1 is a multi-valued mapping generalization of the theorem 2.2 of Kikkawa and Suzuki [7] and therefore the well known Kannan fixed point theorem [6].

Corollary 2.1 also is a multi-valued mapping generalization of the theorem 3.1 of Y. Enjouji et all. [5], since by symmetry the inequality (3.3) in [5] implies the inequality (2) in Theorem 1.1.

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