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# Multivalued self almost local contractions

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*Dedicated to the memory of Professor Ştefan Măruşter*

**Abstract.** We introduce a new class of contractive mappings: the almost local contractions, starting from the almost contractions presented by V. Berinde in [V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration* Nonlinear Analysis Forum 9 (2004) No.1, 43-53], and also from the concept of local contraction presented by Filipe Martins da Rocha and Vailakis in [V. Filipe Martins-da-Rocha, Y. Vailakis, *Existence and uniqueness of a fixed point for local contractions*, Econometrica, vol.78, No.3 (May, 2010) 1127-1141]. First of all, we present the notion of multivalued self almost contractions with many examples. The main results of this paper are given by the extension to the case of multivalued self almost local contractions.

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## 1 Almost contractions, local contractions

**Definition 1.1.** *Let* $(X, d)$ *be a metric space.* $T : X \to X$ *is called almost contraction or* $(\delta, L)$*- contraction if there exist a constant* $\delta \in (0, 1)$ *and some* $L \geq 0$ *such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(y, Tx), \forall x, y \in X \tag{1.1}$$

**Remark 1.1.** The term of almost contraction is equivalent to weak contraction, and it was first introduced by V. Berinde in [3].

**Remark 1.2.** Because of the simmetry of the distance, the almost contraction condition (1.1) includes the following dual one:

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty), \forall x, y \in X \tag{1.2}$$

obtained from (1.1) by replacing $d(Tx, Ty)$ by $d(Ty, Tx)$ and $d(x, y)$ by $d(y, x)$, and after that step, changing $x$ with $y$, and viceversa. Obviously, to prove the almost contractiveness of $T$, it is necessary to check both (1.1) and (1.2).

**Remark 1.3.** A strict contraction satisfies (1.1), with $\delta = a$ and $L = 0$, therefore is an almost contraction with a unique fixed point.

Other examples of almost contractions are given in [4], [5], [2], [3]. There are many other examples of contractive conditions which implies the almost contractiveness condition, see for example Taskovic [22], Rus [18].

We present an existence theorem 1.1, then an existence and uniqueness theorem 1.2, as they are presented in [3]. Their main merit is that they extend Banach's contraction principle and Zamfirescu's fixed point theorem ([24]). They also show us a method for approximating the fixed point, for which both a priori and a posteriori error estimates are available.

**Theorem 1.1.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ a weak (almost) contraction. Then*

1. $Fix(T) = \{x \in X : \ Tx = x\} \neq \phi$;
2. *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$ converges to some $x^* \in Fix(T)$;*
3. *The following estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2... \tag{1.3}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2... \tag{1.4}$$

*hold, where $\delta$ is the constant appearing in* (1.1).

**Theorem 1.2.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an almost contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \forall x, y \in X \tag{1.5}$$

*Then*

1. *T has a unique fixed point,i.e.,* $Fix(T) = \{x^*\}$*;*
2. *For any* $x_0 \in X$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *converges to* $x^*$*;*
3. *The a priori and a posteriori error estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2...$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2...$$

   *hold.*
4. *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta \cdot d(x_{n-1}, x^*), \quad n = 1, 2... \tag{1.6}$$

**Remark 1.4.** (i) Weak contractions represent a generous concept, due to various mappings satisfying the condition (1.1). Such examples of weak contraction were given by V. Berinde in [3], for example it was proved that:
- any Zamfirescu mapping from Theorem Z in [24] is an almost contraction;
- any quasi-contraction with $0 < h < \frac{1}{2}$ is an almost contraction;
- any Kannan mapping (in [10]) is the same kind of almost contraction
(ii) There are many other examples of contractive conditions which imply the weak contractiveness condition, see for example Taskovic [22] , Rus [18] for some of them.
(iii) Weak contractions need not have a unique fixed point, however, the weak contractions possess other important properties, amongst which we mention
a) In the class of weak contractions a method for constructing the fixed points - i.e. the Picard iteration - is always available;
b) Moreover, for this method of approximating the fixed points, both a priori and a posteriori error estimates are available. These are very important from a practical point of view, since they provide stopping criteria for the iterative process;
c) Last, but not least, the weak contractive condition (1.1) and (1.2) may easily be handled and checked in concrete applications.
(iv) The fixed point $x^*$ attained by the Picard iteration depends on the initial guess $x_0 \in X$. Therefore, the class of weak contractions provides a large class of weakly Picard operators.

Recall, see Rus [18], [20] that an operator $T : X \to X$ is said to be a weakly Picard operator if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0 \in X$

and the limits are fixed points of $T$.
(v) Condition (1.1) implies the so called Banach orbital condition

$$d(Tx, T^2x) \leq a \cdot d(x, Tx), \forall x \in X$$

studied by various authors in the context of fixed point theorems, see for example Rus [17] and Taskovic [22].

The next theorem shows that an almost contraction is continuous at any fixed point of it, according to [1].

**Theorem 1.3.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an almost contraction. Then $T$ is continuous at $p$, for any $p \in Fix(T)$.*

**Definition 1.2.** *(see [21]) Let $T$ be a mapping on a metric space $(X, d)$. Then $T$ is called a generalized Berinde mapping if there exist a constant $r \in [0, 1)$ and a function $b$ from $X$ into $[0, \infty)$ such that*

$$d(Tx, Ty) \leq r \cdot d(x, y) + b(y) \cdot d(y, Tx), \forall x, y \in X \tag{1.7}$$

**Definition 1.3.** *Let $(X, d)$ be a metric space. Any mapping $T : X \to X$ is called Ćirić-Reich-Rus contraction if it satisfies the condition:*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot [d(x, Tx) + d(y, Ty)], \forall x, y \in X, \tag{1.8}$$

*where $\alpha, \beta \in \mathcal{R}_+$ and $\alpha + 2\beta < 1$*

**Corollary 1.4.** [15]. *Let $(X, d)$ be a metric space. Any Ćirić-Reich-Rus contraction, i.e., any mapping $T : X \to X$ satisfying the condition (1.8), represents an almost contraction.*

**Theorem 1.5.** *A mapping satisfying the contractive condition:*
*there exists $0 \leq h < \frac{1}{2}$ such that*

$$d(Tx, Ty) \leq h \cdot max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \forall x, y \in X \tag{1.9}$$

*is a weak contraction.*
*An operator satisfying (1.9) with $0 < h < 1$ is called quasi-contraction.*

**Theorem 1.6.** *Any mapping satisfying the condition: there exists $0 \leq b < 1/2$ such that*

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \forall x, y \in X \tag{1.10}$$

*is a weak contraction.*
*A mapping satisfying (1.10) is called Kannan mapping.*

A kind of dual of Kannan mapping is due to Chatterjea [8]. The new contractive condition is similar to (1.10): there exists $0 \leq c < \dfrac{1}{2}$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \forall x, y \in X, \tag{1.11}$$

**Theorem 1.7.** *Any mapping $T$ satisfying the Chatterjea contractive condition, i.e.: there exists $0 \leq c < \dfrac{1}{2}$ such that*

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \forall x, y \in X,$$

*is a weak contraction.*

**Example 1.1.** Let $T : [0, 1] \to [0, 1]$ be a mapping given by $Tx = \frac{2}{3}$ for $x \in [0, 1)$, and $T1 = 0$. Then $T$ has the following properties:
1) $T$ satisfies (1.9) with $h \in [\frac{2}{3}, 1)$, i.e. $T$ is quasi-contraction;
2) $T$ satisfies (1.1), with $\delta \geq \frac{2}{3}$ and $L \geq 0$, i.e. $T$ is also weak contraction;
3) $T$ has a unique fixed point, $x^* = \frac{2}{3}$.
4) $T$ is not continuous.

The concept of local contraction was first introduced by Martins da Rocha and Vailakis in [11] (2010), here they studied the existence and uniqueness of fixed points for the local contractions.

**Definition 1.4.** *Let $F$ be a set and let $\mathcal{D} = (d_j)_{j\in J}$ a family of semidistances defined on $F$. We let $\sigma$ be the weak topology on $F$ defined by the family $\mathcal{D}$. Let $r$ be a function from $J$ to $J$. An operator $T : F \to F$ is a local contraction with respect $(\mathcal{D}, r)$ if, for every $j$, there exists $\beta_j \in [0, 1)$ such that*

$$\forall f, g \in F, \quad d_j(Tf, Tg) \leq \beta_j d_{r(j)}(f, g)$$

## 2 Single valued self almost local contractions

We try to combine these two different type of contractive mappings: the almost and local contractions, to study their fixed points. This new type of mappings was first introduced in [23]

**Definition 2.1.** *The mapping $d(x, y) : X \times X \to \mathbb{R}_+$ is said to be a pseudometric if:*

1. $d(x, y) = d(y, x)$
2. $d(x, y) \leq d(x, z) + d(z, y)$

*3.* $x = y$ *implies* $d(x, y) = 0$
*(instead of* $x = y \Leftrightarrow d(x, y) = 0$ *in the metric case)*

**Definition 2.2.** *Let* $X$ *be a set and let* $\mathcal{D} = (d_j)_{j\in J}$ *be a family of pseudometrics defined on* $X$*. We let* $\sigma$ *be the weak topology on* $X$ *defined by the family* $\mathcal{D}$*.*
*A sequence* $(x_n)_{n\in\mathbb{N}^*}$ *is said to be* $\sigma - Cauchy$ *if it is* $d_j$*-Cauchy,* $\forall j \in J$*.*
*The subset* $A$ *of* $X$ *is said to be sequentially* $\sigma$*-complete if every* $\sigma$*-Cauchy sequence in* $X$ *converges in* $X$ *for the* $\sigma$*-topology.*
*The subset* $A \subset X$ *is said to be* $\sigma$*-bounded if* $diam_j(A) \equiv sup\{d_j(x, y) : x, y \in A\}$ *is finite for every* $j \in J$*.*

**Definition 2.3.** *Let* $r$ *be a function from* $J$ *to* $J$*. An operator* $T : X \to X$ *is called an almost local contraction (ALC) with respect to (*$\mathcal{D}$*,*$r$*) if, for every* $j$*, there exist the constants* $\theta \in (0, 1)$ *and* $L \geq 0$ *such that*

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in X \tag{2.1}$$

**Remark 2.1.** The almost contractions represent a particular case of almost local contractions, by taking $(X, d)$ metric space instead of the pseudometrics $d_j$ and $d_{r(j)}$ defined on $X$. Also, to obtain the almost contractions, we take in (2.1) for $r$ the identity function, so we have $r(j) = j$.

**Definition 2.4.** *The space* $X$ *is* $\sigma$*- Hausdorff if the following condition is valid: for each pair* $x, y \in X, x \neq y$*, there exists* $j \in J$ *such that* $d_j(x, y) > 0$*.*
*If* $A$ *is a nonempty subset of* $X$*, then for each* $z$ *in* $X$*, we let*
$d_j(z, A) \equiv inf\{d_j(z, y) : y \in A\}$*.*

Theorem 2.1 is an existence fixed point theorem for almost local contractions, as they appear in [23].

**Theorem 2.1.** *Consider a function* $r : J \to J$ *and let* $T : X \to X$ *be an almost local contraction with respect to (*$\mathcal{D}$*,*$r$*). Consider a nonempty,* $\sigma$*-bounded, sequentially* $\sigma$*- complete, and* $T$*- invariant subset* $A \subset X$*. If the condition*

$$\forall j \in J, \quad \lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = 0 \tag{2.2}$$

*is satisfied, then the operator* $T$ *admits a fixed point* $x^*$ *in* $A$*.*

*Proof.* Let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

Take $x := x_{n-1}, y := x_n$ in (2.1) to obtain

$$d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n)$$

which yields

$$d_j(x_n, x_{n+1}) \le \theta \cdot d_{r(j)}(x_{n-1}, x_n), \forall j \in J \tag{2.3}$$

Using (2.1), we obtain by induction with respect to $n$:

$$d_j(x_n, x_{n+1}) \le \theta^n \cdot d_{r(j)}(x_0, x_1), \quad n = 0, 1, 2, \cdots \tag{2.4}$$

According to the triangle rule, by (2.4) we get:

$$\begin{aligned} d_j(x_n, x_{n+p}) &\le \theta^n(1 + \theta + \cdots + \theta^{p-1}) d_{r(j)}(x_0, x_1) = \\ &= \frac{\theta^n}{1-\theta}(1 - \theta^p) \cdot d_{r(j)}(x_0, x_1), \quad n, p \in \mathbb{N}, p \neq 0 \end{aligned}$$

These relations show us that the sequence $(x_n)_{n\in\mathbb{N}}$ is $d_j$- Cauchy for each $j \in J$. The subset $A$ is assumed to be sequentially $\sigma$-complete, there exists $f^*$ in $A$ such that $(T^n x)_{n\in\mathbb{N}}$ is $\sigma$- convergent to $x^*$. Besides, the sequence $(T^n x)_{n\in\mathbb{N}}$ converges for the topology $\sigma$ to $x^*$, which implies

$$\forall j \in J, \quad d_j(Tx^*, x^*) = \lim_{n\to\infty} d_j(Tx^*, T^{n+1}x).$$

Recall that the operator $T$ is an almost local contraction with respect to ($\mathcal{D}$,r). From that, we have

$$\forall j \in J, \quad d_j(Tx^*, x^*) \le \beta_j \lim_{n\to\infty} d_{r(j)}(x^*, T^n x).$$

The convergence for the $\sigma$- topology implies convergence for the pseudometric $d_{r(j)}$, we obtain $d_j(Tx^*, x^*) = 0$ for every $j \in J$.
This way, we prove that $Tf^* = f^*$, since $\sigma$ is Hausdorff.
So, we prove the existence of the fixed point for almost local contractions. □

**Remark 2.2.** For $T$ verifies (2.1) with $L = 0$, and $r : J \to J$ the identity function, we find the theorem of Vailakis [11] by taking $\theta = \beta_j$.

Further, for the case $d_j = d, \forall j \in J$, with $d =$ metric on $X$, we obtain the well known Banach contraction, with his unique fixed point.

**Remark 2.3.** In Theorems 2.1 and 2.3 the coefficient of contraction $\theta \in (0, 1)$ is constant, but local contractions have a coefficient of contraction $\theta_j \in [0, 1)$ which depends on $j \in J$. Our first goal is to extend the local almost contractions to the most general case of $\theta_j \in (0, 1)$.

One extends Definition 2.3 to the case of almost local contractions with variable coefficient of contraction.

**Definition 2.5.** *Let $r$ be a function from $J$ to $J$. An operator $T : X \to X$ is called almost local contraction with respect to $(\mathcal{D}, r)$ or $(\theta_j, L_j)$- contraction, if there exist a constant $\theta_j \in (0,1)$ and some $L_j \geq 0$ such that*

$$d_j(Tx, Ty) \leq \theta_j \cdot d_j(x, y) + L_j \cdot d_{r(j)}(y, Tx), \forall x, y \in X \tag{2.5}$$

**Theorem 2.2.** *With the presumptions of Theorem 2.1, if we modify the condition* (2.2) *by the following one:*

$$\forall j \in J, \quad \lim_{n\to\infty} \theta_j \theta_{r(j)} \cdots \theta_{r^n(j)} diam_{r^{n+1}(j)}(A) = 0, \tag{2.6}$$

*then the operator $T$ admits a fixed point $x^*$ in $A$.*

The next theorem represents an existence and uniqueness theorem for the almost local contractions with constant coefficient of contraction.

**Theorem 2.3.** *If to the conditions of Theorem 2.1, we add:*
*(U) for every fixed $j \in J$ there exists:*

$$\lim_{n\to\infty} (\theta + L)^n diam_{r^n(j)}(z, A) = 0, \forall x, y \in X \tag{2.7}$$

*then the fixed point $x^*$ of $T$ is unique.*

# 3 Continuity of almost local contractions

This section can be regarded as an extension of V. Berinde and M. Pacurar ([1]) analysis about the continuity of almost contractions in their fixed points. The main results are given by Theorem 3.1, which give us the answer about the continuity of local almost contractions in their fixed points.

**Theorem 3.1.** *Let $X$ be a set and $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics defined on $X$; let $T : X \to X$ be an almost local contraction satisfying condition* (2.2)*, so $T$ admits a fixed point. Then $T$ is continuous at $f$, for any $f \in Fix(T)$.*

*Proof.* The mapping $T$ is an almost local contraction, i.e. there exist the constants $\theta \in (0,1)$ and some $L \geq 0$

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in X \tag{3.1}$$

For any sequence $\{y_n\}_{n=0}^{\infty}$ in $X$ converging to $f$, we take $y := y_n, x := f$ in (3.1), and we get

$$d_j(Tf, Ty_n) \leq \theta \cdot d_j(f, y_n) + L \cdot d_{r(j)}(y_n, Tf), n = 0, 1, 2, ... \tag{3.2}$$

Using $Tf = f$, since $f$ is a fixed point of $T$, we obtain:

$$d_j(Ty_n, Tf) \leq \theta \cdot d_j(f, y_n) + L \cdot d_{r(j)}(y_n, f), n = 0, 1, 2, ... \tag{3.3}$$

Now by letting $n \to \infty$ in (3.3) we get $Ty_n \to Tf$, which shows that $T$ is continuous at $f$. The fixed point has been chosen arbitrarily, so the proof is complete. $\square$

According to Definition 2.3, the almost local contractions are defined in a subset $A \subset X$. In the case $A = X$, then an almost local contraction is actually a usual almost contraction.

**Example 3.1.** Let $X = [1, n] \times [1, n] \subset \mathbb{R}^2$, $\quad T : X \to X$,

$$T(x, y) = \begin{cases} (\frac{x}{2}, \frac{y}{2}) & \text{if } (x, y) \neq (1, 0) \\ (0, 0) & \text{if } (x, y) = (1, 0) \end{cases}$$

The diameter of the subset $X = [1, n] \times [1, n] \subset \mathbb{R}^2$ is given by the diagonal line of the square with $(n - 1)$ side.
We shall use the pseudometric:

$$d_j((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \cdot e^j, \forall j \in \mathcal{Q}. \tag{3.4}$$

This is a pseudometric, but not a metric: take for example $d_j((1, 4), (1, 3)) = |1 - 1| \cdot e^j = 0$, however $(1, 4) \neq (1, 3)$
In this case, the mapping $T$ is contraction, which implies that is an almost local contraction, with the unique fixed point $x = 0, y = 0$.
According to Theorem 3.1, $T$ is continuous in $(0, 0) \in Fix(T)$, but is not continuous in $(1, 0) \in X$.

**Example 3.2.** With the presumptions of Example 3.1 and the pseudometric defined by (3.4) , we get another example for almost local contractions.
Considering $T : X \to X$,

$$T(x, y) = \begin{cases} (x, -y) & \text{if } (x, y) \neq (1, 1) \\ (0, 0) & \text{if } (x, y) = (1, 1) \end{cases}$$

$T$ is not a contraction because the contractive condition:

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y), \tag{3.5}$$

is not valid $\forall x, y \in X$, and for any $\theta \in (0, 1)$. Indeed, (3.5) is equivalent with:

$$| x_1 - x_2| \cdot e^j \leq \theta \cdot |x_1 - x_2| \cdot e^j$$

The last inequality leads us to $1 \leq \theta$, which is obviously false, considering $\theta \in (0,1)$.
However, $T$ becomes an almost local contraction if:

$$|x_1 - x_2| \cdot e^j \leq \theta \cdot |x_1 - x_2| \cdot e^j + L \cdot |x_2 - x_1| \cdot e^{\frac{j}{2}}$$

which is equivalent to : $e^{\frac{j}{2}} \leq \theta \cdot e^{\frac{j}{2}} + L$

$$(1-\theta) \cdot e^{\frac{j}{2}} \leq L$$

For $\theta = 1/3 \in (0,1)$ , $L = 1 \geq 0$ and $j < 0$ , the last inequality becomes true, i.e. $T$ is an almost local contraction with many fixed points, namely $FixT = \{(x,0) : x \in \mathbb{R}\}$.

In this case, we have:

$$\forall j \in J, \quad \lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = \lim_{n\to\infty} \left(\frac{1}{3}\right)^{n+1} \cdot (n-1)^2 = 0$$

This way, the existence of the fixed point is assured, according to condition (2.1) from Theorem 2.1
Theorem 3.1 is again valid, because the continuity of $T$ in $(0,0) \in Fix(T)$, but discontinuity in $(1,1)$, which is not a fixed point of $T$.

**Example 3.3.** Let $X$ be the set of positive functions: $X = \{f | f : [0,\infty) \to [0,\infty)\}$ and $d_j(f,g) = |f(0) - g(0)| \cdot e^j, \quad \forall f,g \in X$.

Indeed, $d_j$ is a pseudometric, but not a metric, take for example $d_j(x,x^2) = 0$, but $x \neq x^2$.

Considering the mapping $Tf = |f|, \quad \forall f \in X$, and using condition (2.1) for almost local contractions:

$$|f(0) - g(0)| \cdot e^j \leq \theta \cdot |f(0) - g(0)| \cdot e^j + L \cdot |g(0) - f(0)| \cdot e^{\frac{j}{2}}$$

which is equivalent to: $e^{j/2} \leq \theta \cdot e^{j/2} + L$
This inequality becames true if $j < 0, \quad \theta = \frac{1}{3} \in (0,1), \quad L = 3 > 0$
However, $T$ is also not a contraction, because the contractive condition (3.5) leads us again to $1 \leq \theta$. The mapping $T$ has infinite number of fixed points: $FixT = \{f \in X\}$, by taking:

$$|f(x)| = f(x), \forall f \in X, x \in [0,\infty)$$

# 4 Multivalued self almost local contractions

The term of multivalued contraction was first introduced by Nadler in [12]. The following are borrowed from Nadler [12]

**Definition 4.1.** *Let $(X,d)$ be a metric space, we shall denote the family of all nonempty bounded and closed subsets of $X$ with $\mathcal{CB}(X)$.*
*For $A, B \subset X$, we consider*
*$D(A,B) = inf\{d(a,b) : a \in A, b \in B\}$, the distance between $A$ and $B$,*
*$\delta(A,B) = sup\{d(a,b) : a \in A, b \in B\}$, the diameter of $A$ and $B$,*
*$H(A,B) = max\,\{sup\{D(a,B) : a \in A\}, sup\{D(b,A) : b \in B\}\}$, the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$ induced by $d$.*

We know that $\mathcal{CB}(X)$ form a metric space with the Pompeiu-Hausdorff distance function $H$. It is also known,that if $(X,d)$ is a complete metric space then ($\mathcal{CB}(X)$,$H$) is a complete metric space, too (Rus [19]).

Let $\mathcal{P}(X)$ be the family of all nonempty subsets of $X$ and let $T : X \to \mathcal{P}(X)$ be a multi-valued mapping. An element $x \in X$ with $x \in T(x)$ is called a fixed point of $T$. We shall denote $Fix(T)$ the set of all fixed points of $T$, i.e.,

$$Fix(T) = \{x \in X : x \in T(x)\}$$

Let $f : X \to X$ be a single-valued map and $T : X \to \mathcal{CB}(X)$ be a multivalued map .

1. A point $x \in X$ is a fixed point of $f$ (resp. $T$) if $x = fx$ (resp. $x \in Tx$). The set of all fixed point of $f$ (resp. $T$) is denoted by $F(f)$, (resp. $F(T)$).
2. A point $x \in X$ is a coincidence point of $f$ and $T$ if $fx \in Tx$. The set of all coincidence points of $f$ and $T$ will be denoted by $\mathcal{C}(f,T)$
3. A point $x \in X$ is a common fixed point of $f$ and $T$ if $x = fx \in Tx$. The set of all common fixed points of $f$ and $T$ is denoted by $F(f,T)$

The following lemma can be found in Rus [19]. It is useful for the next theorem.

**Lemma 4.1.** *Let $(X,d)$ be a metric space, let $A, B \subset X$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that*

$$d(a,b) \leq qH(A,B) \tag{4.1}$$

**Definition 4.2.** *Let $(X,d)$ be a metric space and $T : X \to \mathcal{P}(X)$ be a multi-valued operator. $T$ is said to be a multi-valued weak contraction or a multi-valued $(\theta, L)$-weak contraction if there exist two constants $\theta \in (0,1), L \geq 0$ such that*

$$H(Tx,Ty) \leq \theta \cdot d(x,y) + L \cdot D(y,Tx), \forall x,y \in X \tag{4.2}$$

**Remark 4.1.** Because of the simmetry of the distance $d$ and $H$, the almost contraction condition (4.2) includes the following dual one:

$$H(Tx, Ty) \leq \theta \cdot d(x, y) + L \cdot D(x, Ty), \forall x, y \in X \tag{4.3}$$

Obviously, to prove the almost contractiveness of $T$, it is necessary to check both (4.2) and (4.3).

**Theorem 4.2.** *(Berinde V., Berinde M. [6]) Let $(X, d)$ be a metric space and $T : X \to \mathcal{P}(X)$ be a $(\theta, L)$-weak contraction. Then*
*(1) $Fix(T) \neq \emptyset$*
*(2) for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of $T$ at the point $x_0$ that converges to a fixed point $u$ of $T$, for which the following estimates hold:*

$$d(x_n, u) \leq \frac{h^n}{1-h} d(x_0, x_1), \quad n = 0, 1, 2... \tag{4.4}$$

$$d(x_n, u) \leq \frac{h}{1-h} d(x_{n-1}, x_n), \quad n = 1, 2... \tag{4.5}$$

*for a certain constant $h < 1$.*

## 5 Main Results

We shall use the assumptions from the definition of almost local contractions and we make the following notations:
$D_j(A, B) = inf\{d_j(a, b) : a \in A, b \in B\}$,
$\delta_j(A, B) = sup\{d_j(a, b) : a \in A, b \in B\}$,
$H_j(A, B) = max\{sup\{D_j(a, B) : a \in A\}, sup\{D_j(b, A) : b \in B\}\}$ ,
the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$ induced by $d_j$.

**Remark 5.1.** From the definition of $D_j$, we have the following result: if $D_j(a, B) = 0$, then $a \in B$.

**Definition 5.1.** *Let $r$ be a function from $J$ to $J$. An operator $T : X \to \mathcal{P}(X)$ is called a multivalued almost local contraction (ALC) with respect to $(\mathcal{D}, r)$ if, for every $j \in J$, there exist the constants $\theta \in (0, 1)$ and $L \geq 0$ such that*

$$H_j(Tx, Ty) \leq \theta \cdot d_j(x, y) + L \cdot D_{r(j)}(y, Tx), \forall x, y \in X \tag{5.1}$$

**Lemma 5.1.** *Let $X$ be a set and let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics defined on $X$. We let $\sigma$ be the weak topology on $X$ defined by the family $\mathcal{D}$. Let $A, B \subset X$ and $q > 1$.*
*Then, for every $j \in J$ and $a \in A$, there exists $b \in B$ such that*

$$d_j(a, b) \leq qH_j(A, B) \tag{5.2}$$

*Proof.* If $H_j(A,B) = 0$, then for every $a \in A$, we have:

$$H_j(A,B) \geq D_j(a,B) \quad \Rightarrow D_j(a,B) = 0$$

From that, we conclude: there exist $b \in B$ such that $d_j(a,b) = 0$.
The inequality (5.2) is valid, i.e., $0 \leq 0$.
If $H_j(A,B) > 0$, then let us denote

$$\varepsilon = (h^{-1} - 1)H(A,B) > 0 \tag{5.3}$$

Using the definition of $H_j(A,B)$ and $D_j(a,B)$, we conclude that for any $\varepsilon > 0$ there exists $b \in B$ such that

$$d_j(a,b) \leq qD_j(a,B) + \varepsilon \leq H_j(A,B) + \varepsilon \tag{5.4}$$

Combining (5.3) and (5.4), we get (5.2). □

**Theorem 5.2.** *With the assumptions of Definition 5.1, let $T : X \to \mathcal{P}(X)$ be a multivalued ALC. Then we have:*
*(1) $Fix(T) \neq \emptyset$*
*(2) for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of $T$ at the point $x_0$ that converges to a fixed point $u$ of $T$, for which the following estimates hold:*

$$d_j(x_n,u) \leq \frac{h^n}{1-h} d_j(x_0,x_1), \quad n = 0,1,2... \tag{5.5}$$

$$d_j(x_n,u) \leq \frac{h}{1-h} d_j(x_{n-1},x_n), \quad n = 1,2... \tag{5.6}$$

*for a certain constant $h < 1$.*

*Proof.* We consider $q > 1$, let $x_0 \in X$ and $x_1 \in Tx_0$. If $H_j(Tx_0,Tx_1) = 0$, that means from the definition of $D_j$ and $H_j$:

$$0 = H_j(Tx_0,Tx_1) \geq D_j(x_1,Tx_1) \tag{5.7}$$

and that is possible only if $D_j(x_1,Tx_1) = 0$, from here, we conclude $x_1 \in Tx_1$, which leads us to the conclusion $Fix(T) \neq \emptyset$.
Let $H_j(Tx_0,Tx_1) \neq 0$. According to Lemma 5.1, there exists $x_2 \in Tx_1$ such that

$$d_j(x_1,x_2) \leq qH_j(Tx_0,Tx_1) \tag{5.8}$$

By (5.1) we have

$$d_j(x_1,x_2) \leq q[\theta \cdot d_j(x_0,x_1) + L \cdot D_{r(j)}(x_1,Tx_0)] = q\theta \cdot d_j(x_0,x_1).$$

since $x_1 \in Tx_0$ , $D_{r(j)}(x_1, Tx_0) = 0$.
We take $q > 1$ such that

$$h = q\theta < 1$$

and we obtain $d_j(x_1, x_2) < h \cdot d_j(x_0, x_1)$.
If $H_j(Tx_1, Tx_2) = 0$ then $D_j(x_2, Tx_2) = 0$, that means $x_2 \in Tx_2$ using Remark 5.1.
Let $H_j(Tx_1, Tx_2) \neq 0$. Again, using Lemma 5.1, there exists $x_3 \in Tx_2$ such that

$$d_j(x_2, x_3) \leq qh \cdot d_j(x_1, x_2) \tag{5.9}$$

This way, we obtain an orbit $\{x_n\}_{n=0}^{\infty}$ of $T$ at the point $x_0$ satisfying

$$d_j(x_n, x_{n+1}) \leq h \cdot d_j(x_{n-1}, x_n), \quad n = 1, 2, ... \tag{5.10}$$

By (5.10), we inductively obtain

$$d_j(x_n, x_{n+1}) \leq h^n d_j(x_0, x_1) \tag{5.11}$$

and, respectively,

$$d_j(x_{n+k}, x_{n+k+1}) \leq h^{k+1} d_j(x_{n-1}, x_n), \quad k \in \mathbb{N} \tag{5.12}$$

Using the inequality (5.11), we obtain

$$d_j(x_n, x_{n+p}) \leq \frac{h^n(1 - h^p)}{1 - h} d_j(x_0, x_1), \quad n, p \in \mathbb{N} \tag{5.13}$$

Recall $0 < h < 1$, conditions (5.12),(5.13) show us that the sequence $(x_n)_{n\in\mathbb{N}}$ is $d_j$-Cauchy for each $j$, which shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. That means $\{x_n\}_{n=0}^{\infty}$ is convergent with the limit $u$:

$$u = \lim_{n\to\infty} x_n \tag{5.14}$$

After simple computations, we get:

$$D_{r(j)}(u, Tu) \leq D_{r(j)}(u, x_{n+1}) + D_{r(j)}(x_{n+1}, Tu) \leq d_{r(j)}(u, x_{n+1}) + H_{r(j)}(Tx_n, Tu)$$

which by (5.1) yields

$$D_{r(j)}(u, Tu) \leq d_{r(j)}(u, x_{n+1}) + \theta d_{r(j)}(x_n, u) + L \cdot D_{r(j)}(u, Tx_n) \tag{5.15}$$

Letting $n \to \infty$ and using the fact that $x_{n+1} \in Tx_n$ implies by (5.14), $D_{r(j)}(u, Tx_n) \to 0$, as $n \to \infty$. We get

$$D_{r(j)}(u, Tu) = 0$$

Since $Tu$ is closed, this implies $u \in Tu$.
We let $p \to \infty$ in (5.13) to obtain (5.5). Using (5.12), we get

$$d(x_n, x_{n+p}) \leq \frac{h(1-h^p)}{1-h} d(x_{n-1}, x_n), \quad p \in \mathbb{N}, n \geq 1 \tag{5.16}$$

and letting $p \to \infty$ in (5.16), we obtain (5.6). The proof is complete. □

The next theorem shows that any multivalued ALC is continuous at the fixed point.

**Theorem 5.3.** *With the assumptions of Definition 5.1, let $T : X \to \mathcal{P}(X)$ be a multivalued ALC, i.e., a mapping for which there exists the constants $\theta \in (0,1)$ and $L \geq 0$ such that, for every $j \in J$, the next inequality is valid:*

$$H_j(Tx, Ty) \leq \theta \cdot d_j(x, y) + L \cdot D_{r(j)}(y, Tx), \forall x, y \in X \tag{5.17}$$

*Then $Fix(T) \neq \emptyset$ and for any $p \in Fix(T)$, $T$ is continuous at $p$.*

*Proof.* The first part of the conclusion follows by Theorem 5.2.
Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in $X$ converging to the fixed point $p$. Then by taking $y := y_n$ and $x := p$ in the multivalued ALC condition (5.17), we get

$$d_j(Tp, Ty_n) \leq \delta \cdot d_j(p, y_n) + L \cdot D_{r(j)}(y_n, Tp), n = 0, 1, 2, ... \tag{5.18}$$

Using the definition of $D_{r(j)}(y_n, Tp)$, we know that is the smallest distance between $y_n$ and any element from $Tp$, take for example $p \in Tp$. Now, we have the following inequalities:

$$D_{r(j)}(y_n, Tp) \leq D_j(y_n, Tp) \leq d_j(y_n, p)$$

By replacing $D_{r(j)}(y_n, Tp)$ from (5.18) with $d_j(y_n, p)$, we get:

$$d_j(Ty_n, Tp) \leq (\delta + L) \cdot d_j(y_n, p), n = 0, 1, 2, ... \tag{5.19}$$

Now, by letting $n \to \infty$ in (5.19) we get $Ty_n \to Tp$ as $n \to \infty$, that means: $T$ is continuous at $p$. □

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