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MULTIVALUED WEAKLY PICARD OPERATORS ON PARTIAL METRIC SPACES

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Abstract. In the present paper, we introduce the multivalued weak contractions and the multivalued weakly Picard operators on partial metric space motivated by the metric space version of these concepts given by Berinde and Berinde [14]. Then we give Mizoguchi-Takahashi type fixed point theorem for multivalued mappings on partial metric spaces. An illustrative example is also presented.

1. Introduction

Let (X, d) be a metric space and let $CB(X)$ denote the class of all nonempty, closed and bounded subsets of X. It is well known that, $H : CB(X) \times CB(X) \to \mathbb{R}$ defined by

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\right\}$$

is a metric on $CB(X)$, which is called Hausdorff metric, where

$$D(x, B) = \inf\{d(x, y) : y \in B\}.$$

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Let $T : X \to CB(X)$ be a map, then T is called multivalued contraction if for all $x, y \in X$ there exists $\lambda \in [0, 1)$ such that

$$H(Tx, Ty) \leq \lambda d(x, y).$$

In 1969, Nadler [25] proved a fundamental fixed point theorem for multivaled maps: Every multivalued contractions on complete metric space has a fixed point.

Then, a lot of generalizations of the result of Nadler were given (See, for example [17, 30, 39]). Two important generalizations of it were given by Berinde and Berinde [14] and Mizoguchi and Takahashi [24].

In [14], Berinde and Berinde introduced the concept of multivalued weakly Picard operator as follows: (for single valued Picard and weakly Picard operators we refer to [10, 13, 27]).

Definition 1.1. Let (X, d) be a metric space and $T : X \to \mathcal{P}(X)$ (the family of all nonempty subsets of X) be a multivalued operator. T is said to be multivalued weakly Picard (MWP) operator if and only if for each $x \in X$ and any $y \in Tx$, there exists a sequence $\{x_n\}$ in X such that
(i) $x_0 = x$, $x_1 = y$,
(ii) $x_{n+1} \in Tx_n$,
(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

Then they give some examples of MWP operators such that, every Nadler type multivalued contractions [25], every Reich type multivalued contractions [31], every Rus type multivalued contractions [36] and every Petrusel type multivalued contractions [29] on complete metric space are MWP operators. Mizoguchi and Takahashi [24] proved the following fixed point theorem. This is also an example of MWP operator.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued map. Assume*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $x, y \in X$, where α is an $\mathcal{MT}$-function (that is, it satisfies $\limsup_{s \to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$). Then T is an MWP operator.

In the same paper, Berinde and Berinde [14] introduced the concepts of multivalued (δ, L)-weak contraction and multivalued (α, L)-weak contraction and proved the following nice fixed point theorems:

Theorem 1.3. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued (δ, L)-weak contraction, that is, there exist two constants*

$\delta \in (0,1)$ *and* $L \geq 0$ *such that*

$$H(Tx, Ty) \leq \delta d(x,y) + LD(y, Tx)$$

for all $x, y \in X$. *Then* T *is an MWP operator.*

Theorem 1.4. *Let* (X, d) *be a complete metric space and let* $T : X \to CB(X)$ *be a multivalued* (α, L)*-weak contraction, that is, there exist a* $\mathcal{MT}$*-function* α *and a constant* $L \geq 0$ *such that*

$$H(Tx, Ty) \leq \alpha(d(x,y))d(x,y) + LD(y, Tx)$$

for all $x, y \in X$. *Then* T *is an MWP operator.*

We can find some detailed information about the singe-valued case of (δ, L)-weak contraction and the nonlinear case of it in [11, 12, 28].

The aim of this paper is to introduce the multivalued weak contractions and multivalued weakly Picard operators on partial metric space as the parallel manner on metric space. First, we recall the concept of partial metric space and some properties. In 1992, Matthews [23] introduced the notion of a partial metric space, which is a generalization of usual metric spaces in which the self distance for any point need not be equal to zero. The partial metric space has a wide applications in many branches of mathematics as well as in the field of computer domain and semantics. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties.

Let X be a nonempty set and let $p : X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$: (i) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$ (T_0-separation axiom), (ii) $p(x,x) \leq p(x,y)$ (small self-distance axiom), (iii) $p(x,y) = p(y,x)$ (symmetry), (iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (modified triangular inequality). Then p is said to be a partial metric on X. A partial metric space (for short PMS) is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then $x = y$. But if $x = y$, then $p(x,y)$ may not be 0.

At this point it seems interesting to remark the fact that partial metric spaces play an important role in constructing models in the theory of computation (see for instance [16, 18, 20], etc).

A basic example of a PMS is the pair $([0, \infty), p)$, where $p(x,y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. For another example, let I denote the set of all intervals $[a, b]$ for any real numbers $a \leq b$. Let $p : I \times I \to [0, \infty)$ be the function such that $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (I,p) is a PMS. Other examples of partial metric spaces may be found in [21, 32], etc.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls

$$\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$.

Observe that a sequence $\{x_n\}$ in a PMS (X,p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

If p is a partial metric on X, then the functions $p^s, p^w : X \times X \to [0,\infty)$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

and

$$p^w(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\}, \tag{1.1}$$

are equivalent metrics on X.

According to [23], a sequence $\{x_n\}$ in a partial metric space (X,p) converges, with respect to τ_{p^s}, to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x). \tag{1.2}$$

A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$. (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Finally, the following crucial facts are shown in [23]:

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).
(b) (X,p) is complete if and only if (X,p^s) is complete.

Matthews obtained, among other results, a partial metric version of the Banach fixed point theorem ([23, Theorem 5.3]) as follows: Let (X,p) be a complete partial metric space and let $T : X \to X$ be a contraction mapping, that is, there exists $\lambda \in [0,1)$ such that

$$p(Tx,Ty) \leq \lambda p(x,y)$$

for all $x,y \in X$. Then T has a unique fixed point $z \in X$. Moreover, $p(z,z) = 0$.

Later on, Acar et al. [1, 2], Altun et al. [4, 6, 7, 8], Karapinar and Erhan [22], Oltra and Valero [26], Romaguera [33, 34] and Valero [40], gave some generalizations of the result of Matthews. Also, Ćirić et al. [15], Samet et al. [37] and Shatanawi et al. [38] proved some common fixed point results in partial metric spaces. But, so far all of fixed point theorems have been given for single valued mappings. To prove Nadler's fixed point theorem for multivalued maps on partial metric spaces, Aydi et al.[9] introduced the concept of partial Hausdorff distance a parallel manner to that in the Hausdorff metric in their nice paper [9]. Then, they give some properties of partial Hausdorff distance, some important lemmas and a fundamental fixed point theorem for multivalued mappings. We can find some nice fixed point results for single and multivalued maps on partial metric space in [3, 19, 35].

Now we recall the concept of partial Hausdorff distance and some properties: Let (X, p) be partial metric space and $A \subseteq X$, then A is said to be bounded if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(a, a) + M$. A is closed if and only if $A = \overline{A}$, where $\overline{A}$ is the closure of A with respect to τ_p. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of (X, p). For $A \subseteq X$ and $x \in X$ define

$$P(x, A) = \inf\{p(x, a) : a \in A\}$$

and for $A, B \in CB^p(X)$, define

$$\delta_p(A, B) = \sup\{P(a, B) : a \in A\}$$

and

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Lemma 1.5. ([8]) *Let (X, p) be a partial metric space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if $P(x, A) = p(x, x)$.*

Example 1.6. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $A = [2, 3] \cup \{5\}$, then by the routine calculation we have

$$x \in \overline{A} \Leftrightarrow x \in [2, \infty)$$

that is, $\overline{A} = [2, \infty)$. Therefore, A is not closed in X. Note that all closed subsets of X are the form $[x, \infty)$, $x \in X$. Thus every nonempty closed subsets of X are not bounded, that is $CB^p(X)$ is empty.

Proposition 1.7. ([9]) *Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have the following:*

(1) $\delta_p(A, A) = \sup_{a \in A} p(a, a)$,

(2) $\delta_p(A, A) \leq \delta_p(A, B)$,
(3) $\delta_p(A, B) = 0$ *implies* $A \subseteq B$,
(4) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$

Example 1.8. Let $X = [0, 3]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $A = [1, 3]$ and $B = [2, 3]$, then it is easy to see that $A, B \in CB^p(X)$. For $x \in X$, we have

$$\begin{aligned} P(x, B) &= \inf\{p(x, b) : b \in B\} \\ &= \inf\{\max\{x, b\} : b \in B\} \\ &= \begin{cases} 2 & , \quad x \leq 2 \\ x & , \quad x > 2 \end{cases}, \end{aligned}$$

Therefore

$$\begin{aligned} \delta_p(A, B) &= \sup\{P(a, B) : a \in A\} \\ &= 3. \end{aligned}$$

Similarly, it is easy to see that

$$\delta_p(A, A) = \delta_p(B, B) = \delta_p(B, A) = 3.$$

Proposition 1.9. ([9]) *Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have the following:*

(1) $H_p(A, A) \leq H_p(A, B)$
(2) $H_p(A, B) = H_p(B, A)$
(3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$

Remark 1.10. We noted from Example 1.8 that $H_p(A, A) = H_p(A, B) = H_p(B, A)$, but $A \neq B$. That is, H_p is not a partial metric on $CB^p(X)$. Nevertheless, as shown in [9] we have the following property:

$$H_p(A, B) = 0 \text{ implies } A = B.$$

Also, it is easy to see that, for all $a \in A$

$$P(a, B) \leq \delta_p(A, B) \leq H_p(A, B). \tag{1.3}$$

The following lemma is very important to give fixed point results for multivalued maps on partial metric space.

Lemma 1.11. ([9]) *Let (X, p) be a partial metric space, $A, B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a, b) \leq hH_p(A, B)$.*

Lemma 1.11 can be expressed with the following version.

Lemma 1.12. *Let (X, p) be a partial metric space, $A, B \in CB^p(X)$ and $\varepsilon > 0$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a, b) \leq H_p(A, B) + \varepsilon$.*

Using the partial Hausdorff distance H_p, Aydi et al. [9] proved the following fixed point theorem for multivalued mappings.

Theorem 1.13. *Let (X, p) be a complete partial metric space. If $T : X \to CB^p(X)$ is a mapping such that*

$$H_p(Tx, Ty) \leq \delta p(x, y) \tag{1.4}$$

for all $x, y \in X$, where $\delta \in (0, 1)$. Then T has a fixed point.

The following theorem is a generalized version of Theorem 1.13, which is given by the authors of this paper.

Theorem 1.14. ([5]) *Let (X, d) be a complete partial metric space and let $T : X \to CB^p(X)$ be a multivalued map. Assume*

$$H_p(Tx, Ty) \leq \alpha(p(x, y))p(x, y)$$

for all $x, y \in X$, where α is an $\mathcal{MT}$-function. Then T has a fixed point.

2. The results

Now, we introduce the following definition:

Definition 2.1. Let (X, p) be a partial metric space and $T : X \to \mathcal{P}(X)$ be a multivalued operator. T is said to be multivalued weakly Picard (MWP) operator on (X, p) if and only if for each $x \in X$ and any $y \in Tx$, there exists a sequence $\{x_n\}$ in X such that

(i) $x_0 = x$, $x_1 = y$,
(ii) $x_{n+1} \in Tx_n$,
(iii) the sequence $\{x_n\}$ is convergent (w.r.t τ_p) and one of its limit is a fixed point of T.

The operators are mentioned in Theorem 1.13 and Theorem 1.14 are two examples of MWP operator on (X, p).

A more general class of MWP operator on a partial metric spaces will be given by Theorem 2.4 and Theorem 2.6.

Definition 2.2. Let (X, p) be a partial metric space and let $T : X \to \mathcal{P}(X)$ be a multivalued operator. T is said to be multivalued (δ, L)-weak contraction on (X,p) if and only if there exist two constants $\delta \in (0,1)$ and $L \geq 0$ such that

$$H_p(Tx,Ty) \leq \delta p(x,y) + LP^w(y,Tx) \tag{2.1}$$

for all $x, y \in X$, where

$$P^w(y,Tx) = \inf\{p^w(y,z) : z \in Tx\}$$

and p^w as in (1.1).

Remark 2.3. Due to the symmetry of p and H_p, in order to check that T is a multivalued (δ, L)-weak contraction on (X,p), we have also check to the dual of (2.1), that is to check that T verifies

$$H_p(Tx,Ty) \leq \delta p(x,y) + LP^w(x,Ty). \tag{2.2}$$

We can find some detailed information about the singe valued case of (δ, L)-weak contractions on a partial metric spaces and the nonlinear case of it in [4].

Theorem 2.4. *Let (X,p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a multivalued (δ, L)-weak contraction on (X,p). Then T is an MWP operator on (X,p).*

Proof. Let $q > 1$ with $q\delta < 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $H_p(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$, i.e., $x_1 \in Tx_1$, which actually means that x_1 is a fixed point of T. Let $H_p(Tx_0, Tx_1) \neq 0$. Then, form Lemma 1.11 there exists $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq qH_p(Tx_0, Tx_1).$$

Then, from (2.2),

$$\begin{aligned} p(x_1,x_2) &\leq qH_p(Tx_0,Tx_1) \\ &\leq q\left[\delta p(x_0,x_1) + LP^w(x_1,Tx_0)\right] \\ &\leq q\delta p(x_0,x_1), \end{aligned} \tag{2.3}$$

since $P^w(x_1, Tx_0) = \inf\{p^w(x_1, u) : u \in Tx_0\} = 0$. We take $h = q\delta$, then from (2.3) we have

$$p(x_1, x_2) \leq hp(x_0, x_1).$$

If $H_p(Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$, i.e., $x_2 \in Tx_2$,which actually means x_2 is a fixed point of T. Let $H_p(Tx_1, Tx_2) \neq 0$. Again by Lemma 1.11 there exists $x_2 \in Tx_1$ such that

$$p(x_2, x_3) \leq hp(x_1, x_2).$$

In this manner, we obtain an orbit $\{x_n\}$ at x_0 for T satisfying, for all $n \in \{1, 2, \cdots\}$

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n)$$

and so

$$\begin{aligned} p(x_n, x_{n+1}) &\leq hp(x_{n-1}, x_n) \\ &\leq h^2 p(x_{n-2}, x_{n-1}) \\ &\;\;\vdots \\ &\leq h^n p(x_0, x_1). \end{aligned}$$

Using the modified triangular inequality for the partial metric, for any $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\leq h^n p(x_1, x_0) + h^{n+1} p(x_1, x_0) + \cdots + h^{m-1} p(x_1, x_0) \\ &= \left[h^n + h^{n+1} + \cdots + h^{m-1}\right] p(x_1, x_0) \\ &\leq \frac{h^n}{1-h} p(x_1, x_0). \end{aligned} \tag{2.4}$$

Letting $n \to \infty$, in (2.4), we get $p(x_n, x_m) \to 0$, since $0 < h < 1$. By the definition of p^s, we get

$$p^s(x_n, x_m) \leq 2p(x_n, x_m),$$

so it is obvious that $p^s(x_n, x_m)$ tends to zero as $n, m \to \infty$, since $p(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is a complete, by the relation between the spaces (X, p) and (X, p^s) mentioned in Section 1, (X, p^s) is also complete. Therefore, the sequence $\{x_n\}$ converges to some point $z \in X$ with respect to the metric p^s, that is,

$$\lim_{n\to\infty} p^s(x_n, z) = 0.$$

And, by the equation (1.2), we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.5}$$

On the other hand

$$\begin{aligned} P(z, Tz) &\leq p(z, x_{n+1}) + P(x_{n+1}, Tz) \\ &\leq p(z, x_{n+1}) + H_p(Tx_n, Tz) \\ &\leq p(z, x_{n+1}) + \delta p(x_n, z) + LP^w(z, Tx_n) \\ &\leq p(z, x_{n+1}) + \delta p(x_n, z) + Lp^w(z, x_{n+1}). \end{aligned} \tag{2.6}$$

Letting $n \to \infty$, in (2.6), we obtain (note that p^s and p^w are equivalent metrics)

$$P(z, Tz) = 0.$$

Therefore, from (2.5), we obtain $p(z, z) = P(z, Tz)$. Thus, from Lemma 1.5 we have $z \in \overline{Tz} = Tz$. This completes the proof. $\square$

Now, we give a more general class of MWP operator on a partial metric spaces. For this we need the following lemma.

Lemma 2.5. ([17]) *Let $\alpha : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function, then the function $\beta : [0,\infty) \to [0,1)$ defined as $\beta(t) = \frac{1+\alpha(t)}{2}$ is also an $\mathcal{MT}$-function.*

Theorem 2.6. *Let (X,d) be a complete partial metric space and let $T : X \to CB^p(X)$ be an (α, L)-weak contraction on (X,p), that is, there exist an $\mathcal{MT}$-function α and a constant $L \geq 0$ such that*

$$H_p(Tx,Ty) \leq \alpha(p(x,y))p(x,y) + LP^w(y,Tx) \tag{2.7}$$

for all $x,y \in X$. Then T is an MWP operator on (X,p).

Proof. Define a function $\beta : [0,\infty) \to [0,1)$ as $\beta(t) = \frac{1+\alpha(t)}{2}$, then from Lemma 2.5 $\beta(t)$ is also an $\mathcal{MT}$-function. Let $x,y \in X$ and $x \neq y$ be two arbitrary points, $u \in Tx$ and $\varepsilon = \frac{1-\alpha(p(x,y))}{2}p(x,y) > 0$ (note that since $x \neq y$, then $p(x,y) > 0$), then form Lemma 1.12 we can find $v \in Ty$ such that $p(u,v) \leq H_p(Tx,Ty) + \epsilon$. Therefore, from (2.7) we have

$$\begin{aligned} p(u,v) &\leq H_p(Tx,Ty) + \frac{1-\alpha(p(x,y))}{2}p(x,y) \\ &\leq \alpha(p(x,y))p(x,y) + LP^w(y,Tx) + \frac{1-\alpha(p(x,y))}{2}p(x,y) \\ &= \frac{1+\alpha(p(x,y))}{2}p(x,y) + LP^w(y,Tx) \\ &= \beta(p(x,y))p(x,y) + LP^w(y,Tx). \end{aligned} \tag{2.8}$$

Now, fix $x_0 \in X$ and $x_1 \in Tx_0$. If $x_0 = x_1$, then x_0 is a fixed point of T and so the proof is complete. Let $x_0 \neq x_1$ then from (2.8), there exists $x_2 \in Tx_1$ such that

$$\begin{aligned} p(x_1,x_2) &\leq \beta(p(x_0,x_1))p(x_0,x_1) + LP^w(x_1,Tx_0) \\ &= \beta(p(x_0,x_1))p(x_0,x_1). \end{aligned}$$

If $x_1 = x_2$, then x_1 is a fixed point of T and so the proof is complete. Let $x_1 \neq x_2$ then from (2.8), there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} p(x_2,x_3) &\leq \beta(p(x_1,x_2))p(x_1,x_2) + LP^w(x_2,Tx_1) \\ &= \beta(p(x_1,x_2))p(x_1,x_2). \end{aligned}$$

By continuining this way, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ (we can suppose its consecutive terms are different, otherwise the proof is complete) and

$$p(x_{n+1},x_{n+2}) \leq \beta(p(x_n,x_{n+1}))p(x_n,x_{n+1}),$$

for all $n \in \mathbb{N}$. Since $\beta(t) < 1$ for all $t \in [0, \infty)$ then $\{p(x_n, x_{n+1})\}$ is a nonincreasing sequence of real numbers. Hence $\{p(x_n, x_{n+1})\}$ converges to some $\lambda \geq 0$. Since β is an $\mathcal{MT}$-function, then $\limsup_{s \to t^+} \beta(s) < 1$ and $\beta(\lambda) < 1$. Therefore, there exists $r \in [0, 1)$ and $\varepsilon > 0$ such that $\beta(s) \leq r$ for all $s \in [\lambda, \lambda + \varepsilon)$. Since $p(x_n, x_{n+1}) \downarrow \lambda$, we can take $k_0 \in \mathbb{N}$ such that $\lambda \leq p(x_n, x_{n+1}) < \lambda + \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq k_0$. Since

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &\leq \beta(p(x_n, x_{n+1}))p(x_n, x_{n+1}) \\ &\leq rp(x_n, x_{n+1}), \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq k_0$, then we have

$$\begin{aligned} \sum_{n=1}^{\infty} p(x_n, x_{n+1}) &\leq \sum_{n=1}^{k_0} p(x_n, x_{n+1}) + \sum_{n=k_0+1}^{\infty} p(x_n, x_{n+1}) \\ &= \sum_{n=1}^{k_0} p(x_n, x_{n+1}) + \sum_{n=k_0}^{\infty} p(x_{n+1}, x_{n+2}) \\ &\leq \sum_{n=1}^{k_0} p(x_n, x_{n+1}) + \sum_{n=k_0}^{\infty} rp(x_n, x_{n+1}) \\ &\leq \sum_{n=1}^{k} p(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n p(x_{k_0}, x_{k_0+1}) \\ &< \infty. \end{aligned}$$

Then for $m, n \in \mathbb{N}$ with $m > n$, by omitting the negative terms in modified triangular inequality we obtain

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} p(x_i, x_{i+1}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore we have $\lim_{n \to \infty} p(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in (X, p) . Since (X, p) is a complete partial metric space, by the relation between the spaces (X, p) and (X, p^s) mentioned in Section 1, (X, p^s) is also complete metric space. Therefore, the sequence $\{x_n\}$ converges to same point $z \in X$ with respect to the metric p^s, that is,

$$\lim_{n \to \infty} p^s(x_n, z) = 0.$$

And, by the equation (1.2), we have

$$p(z,z)=\lim_{n\to\infty}p(x_n,z)=\lim_{n,m\to\infty}p(x_n,x_m)=0. \tag{2.9}$$

Since

$$\begin{aligned}
P(z,Tz) &\leq p(z,x_{n+1})+P(x_{n+1},Tz)\\
&\leq p(z,x_{n+1})+H_p(Tx_n,Tz)\\
&\leq p(z,x_{n+1})+\alpha(p(x_n,z))p(x_n,z)+LP^w(z,Tx_n)\\
&\leq p(z,x_{n+1})+\alpha(p(x_n,z))p(x_n,z)+Lp^w(z,x_{n+1})\\
&\leq p(z,x_{n+1})+p(x_n,z)+Lp^w(z,x_{n+1}), \qquad (2.10)
\end{aligned}$$

then letting $n\to\infty$, in (2.10), we get $P(z,Tz)=0$. Therefore, from (2.9), we obtain $p(z,z)=P(z,Tz)$. Thus, from Lemma 1.5 we have $z\in\overline{Tz}=Tz$. This completes the proof. □

Now, we give an illustrative example. This example shows that our result is a proper generalization of the result of Aydi et al.

Example 2.7. Let $X=[0,1]$ and

$$p(x,y)=\begin{cases}1+\max\{x,y\}\ , & x\neq y\\ 0\ , & x=y\end{cases}.$$

It is clear that p is a partial metric and (X,p) is complete. On the other hand, for all $x\in X$, $B_p(x,\frac{1}{2})=\{x\}$, that is, τ_p is discrete topolgy on X and so all subsets of X are both open and closed. Also, all subsets of X are bounded. Define $T:X\to CB^p(X)$ by

$$Tx=\begin{cases}\{x^3,x^2\}\ , & x\in[\frac{1}{3},\frac{1}{2})\\ \{0\}\ , & x\in[0,\frac{1}{3})\cup[\frac{1}{2},1)\\ \{1\}\ , & x=1\end{cases}.$$

Now we show that (2.1) and (2.2) are satisfied for $\delta=\frac{5}{6}$ and $L=1$. In order to show both (2.1) and (2.2), it is sufficient to show that

$$H_p(Tx,Ty)\leq\delta p(x,y)+L\min\{P^w(y,Tx),P^w(x,Ty)\}$$

for all $x,y\in X$. Consider the following six cases:
Case 1. Let $x=y$, then

$$H_p(Tx,Ty)=H_p(Tx,Tx)=\sup\{p(a,a):a\in Tx\}=0$$

and so the result is clear. Therefore we will assume $x\neq y$ in the following cases.

Case 2. Let $x, y \in [0, \frac{1}{3}) \cup [\frac{1}{2}, 1)$, then $H_p(Tx, Ty) = H_p(\{0\}, \{0\}) = 0$ and so the result is clear.

Case 3. Let $x, y \in [\frac{1}{3}, \frac{1}{2})$, then

$$\begin{aligned} H_p(Tx, Ty) &= H_p(\{x^3, x^2\}, \{y^3, y^2\}) \\ &\leq \frac{5}{6}[1 + \max\{x, y\}] = \frac{5}{6}p(x, y) \\ &\leq \frac{5}{6}p(x, y) + \min\{P^w(y, Tx), P^w(x, Ty)\}. \end{aligned}$$

Case 4. Let $x \in [0, \frac{1}{3}) \cup [\frac{1}{2}, 1)$ and $y = 1$, then

$$\begin{aligned} H_p(Tx, Ty) &= H_p(\{0\}, \{1\}) \\ &= 2 < \frac{5}{3} + 2 \\ &= \frac{5}{6}(1 + \max\{x, 1\}) + \min\{1 + \max\{1, 0\}, 1 + \max\{x, 1\}\} \\ &= \frac{5}{6}p(x, y) + \min\{P(y, Tx), P(x, Ty)\} \\ &= \frac{5}{6}p(x, y) + \min\{P^w(y, Tx), P^w(x, Ty)\}. \end{aligned}$$

Case 5. Let $x \in [\frac{1}{3}, \frac{1}{2})$ and $y = 1$, then

$$\begin{aligned} H_p(Tx, Ty) &= H_p(\{x^3, x^2\}, \{1\}) \\ &= 2 < \frac{5}{3} + 2 \\ &= \frac{5}{6}(1 + \max\{x, 1\}) \\ &\quad + \min\left\{\inf_{a \in \{x^3, x^2\}}\{1 + \max\{1, a\}\}, 1 + \max\{x, 1\}\right\} \\ &= \frac{5}{6}p(x, y) + \min\{P(y, Tx), P(x, Ty)\} \\ &= \frac{5}{6}p(x, y) + \min\{P^w(y, Tx), P^w(x, Ty)\}. \end{aligned}$$

Case 6. Let $x \in [\frac{1}{3}, \frac{1}{2})$ and $y \in [0, \frac{1}{3}) \cup [\frac{1}{2}, 1)$, then

$$\begin{aligned} H_p(Tx, Ty) &= H_p(\{x^3, x^2\}, \{0\}) \\ &= 1 + x^2 \leq \frac{5}{6}(1 + x) \\ &\leq \frac{5}{6}[1 + \max\{x, y\}] = \frac{5}{6}p(x, y) \\ &\leq \frac{5}{6}p(x, y) + \min\{P^w(y, Tx), P^w(x, Ty)\}. \end{aligned}$$

Therefore all conditions of Theorem 2.4 are satisfied, so T has a fixed point. Note that since

$$H_p(T0, T1) = H_p(\{0\}, \{1\}) = 2 = p(0, 1),$$

then the condition (1.4) of Theorem 1.13 is not satisfied and so the result of Aydi et al is not applicable to this example.

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