Multivariable biorthogonal continuous–discrete Wilson and Racah polynomials

M. V. Tratnik

Published on: 01 Jul 1990 - Journal of Mathematical Physics (American Institute of Physics)

Topics: Biorthogonal polynomial, Discrete orthogonal polynomials, Hahn polynomials, Wilson polynomials and Orthogonal polynomials

MULTIVARIABLE BIORTHOGONAL CONTINUOUS–DISCRETE WILSON AND RACAH POLYNOMIALS

By

M.V. Tratnik

IMA Preprint Series # 561

August 1989

Multivariable biorthogonal continuous–discrete Wilson and Racah polynomials

M.V. Tratnik†

Institute for Mathematics and its Applications

University of Minnesota

514 Vincent Hall

206 Church Street S.E.

Minneapolis, MN 55455

Abstract

Several families of multivariable, biorthogonal, partly continuous and partly discrete, Wilson polynomials are presented. These yield limit cases which are purely continuous in some of the variables and purely discrete in the others, or purely discrete in all the variables. The latter are referred to as the multivariable biorthogonal Racah polynomials. Interesting further limit cases include the multivariable biorthogonal Hahn and dual Hahn polynomials.

---

† Permanent address: CNLS and Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545.

I. Introduction

The Wilson polynomials[1,2] are a very general family that include as special or limiting cases all the classical orthogonal polynomials and many related families. They can be expressed as the following ${}_4F_3$ hypergeometric series[3]

$$P_n(x) = (a+b)_n(a+c)_n(a+d)_n\, {}_4F_3\left(\begin{matrix} -n,\ n+a+b+c+d-1,\ a-ix,\ a+ix \\ a+b,\ a+c,\ a+d \end{matrix};1\right), \tag{1.1}$$

where a, b, c, d are complex parameters, $(\alpha)_n \equiv \Gamma(n+\alpha)/\Gamma(\alpha)$ denotes the usual Pochhammer symbol, and n is a nonnegative integer. These are polynomials in x of degree $2n$ which one can show[1] are symmetric in all four of the parametes a, b, c, d. They are associated with the following weight function

$$w(x) = \frac{\Gamma(a+ix)\Gamma(a-ix)\Gamma(b+ix)\Gamma(b-ix)\Gamma(c+ix)\Gamma(c-ix)\Gamma(d+ix)\Gamma(d-ix)}{\Gamma(2ix)\Gamma(-2ix)}, \tag{1.2}$$

and satisfy a complex orthogonality relation

$$\int_C dx\, P_n(x)\, P_m(x)\, w(x) = \lambda_n\, \delta_{nm}\ , \tag{1.3}$$

where the normalization constant λ_n is given by

$$\begin{aligned} \lambda_n = {} & 4\pi n!(n+a+b+c+d-1)_n \\ & \times \frac{\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}, \end{aligned} \tag{1.4}$$

and the contour C is deformed from the real axis so that it separates the increasing sequences of poles of the weight function from the decreasing sequences.

When the real parts of the parameters a, b, c, d are positive one can choose C to be the real axis. If in addition the parameters are real or occur in complex conjugate pairs then the polynomials and weight function are real and the latter is positive. In this case the Wilson polynomials satisfy a continuous orthogonality relation with respect to a positive measure on the real line. If the real part of one parameter is less than zero, let us say $\mathrm{Re}(a) < 0$, $\mathrm{Re}(b, c, d) > 0$ $(2a,\ a+b,\ a+c,\ a+d \neq 0, -1, -2, ...)$, then C must be deformed from the real axis to pass over the decreasing sequence of poles given by $x = -ia - ij$, $j =$

$0,1,2,\ldots$ and under the increasing sequence found at $x = ia + ij$, $j = 0,1,2,\ldots$. This contour can then be deformed back to the real axis plus closed loops about a finite number of poles. If the closed loops are then evaluated by the method of residues the inner product in (1.3) can be written as an integral over the real axis plus a finite discrete sum. In this case the Wilson polynomials satisfy a partly continuous and partly discrete orthogonality relation

$$\int_{-\infty}^{\infty} dx\, w(x) P_n(x) P_m(x) \; + \sum_{j=0}^{\mathrm{Re}(a)+j<0} \tilde{w}(j) P_n(ia+ij) P_m(ia+ij) \; = \; \lambda_n \delta_{nm} \;, \tag{1.5}$$

where the discrete part of the weight function $\tilde{w}(j)$ is given by

$$\begin{aligned}\tilde{w}(j) = (4\pi)\Gamma(a+b)\,\Gamma(a+c)\,\Gamma(a+d)\,&\frac{\Gamma(b-a)\,\Gamma(c-a)\,\Gamma(d-a)}{\Gamma(-2a)} \\ &\times \frac{(2a)_j(a+1)_j(a+b)_j(a+c)_j(a+d)_j}{(a)_j(a-b+1)_j(a-c+1)_j(a-d+1)_j\, j!} \;.\end{aligned} \tag{1.6}$$

Formula (1.5) also yields a purely discrete orthogonality relation. Take $a+b = -\Delta+\epsilon$, where Δ is a nonnegative integer, divide (1.5) by $\Gamma(a+b) = \Gamma(-\Delta+\epsilon)$, and then take the limit $\epsilon \to 0$. The continuous term vanishes because $1/\Gamma(-\Delta+\epsilon) \to 0$ but the discrete part survives leaving

$$\sum_{j=0}^{\Delta} \rho(j) P_n(ia+ij) P_m(ia+ij) = \lambda'_n \delta_{nm} \;, \qquad 0 \le n, m \le \Delta \;, \tag{1.7}$$

where the weight function and normalization constant are given by

$$\begin{aligned}\rho(j) &= \frac{(2a)_j(a+1)_j(-\Delta)_j(a+c)_j(a+d)_j}{(a)_j(2a+\Delta+1)_j(a-c+1)_j(a-d+1)_j\, j!} \;, \\ \lambda'_n &= n!(n-\Delta+c+d-1)_n(2a+1)_\Delta(1-c-d)_\Delta \\ &\qquad \times \frac{(-\Delta)_n(a+c)_n(a+d)_n(c-a-\Delta)_n(d-a-\Delta)_n(c+d)_n}{(-\Delta+c+d)_{2n}(a-c+1)_\Delta(a-d+1)_\Delta} \;.\end{aligned} \tag{1.8}$$

The orthogonality relation (1.7) is equivalent to Racah's orthogonality for what are called Racah coefficients or $6j$ symbols. Accordingly the polynomials in (1.7) are referred to in the literature as the Racah polynomials. It is customary to redefine the parameters and write these polynomials as

$$r_n(x) = (\alpha+1)_n(\gamma+1)_n(\beta+\delta+1)_n \, {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix} ; 1 \right) , \quad (1.9)$$

and then the orthogonality relation becomes

$$\sum_{x=0}^{\Delta} \frac{(\gamma+\delta+1)_x(\gamma/2+\delta/2+3/2)_x(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x}{(\gamma/2+\delta/2+1/2)_x(\gamma+\delta-\alpha+1)_x(\gamma-\beta+1)_x(\delta+1)_x\, x!} r_n(x) r_m(x) = \lambda_n \delta_{nm} , \quad (1.10)$$

$$\lambda_n = n!\,(\alpha+1)_n(\beta+1)_n(\gamma+1)_n(\alpha-\delta+1)_n(\alpha+\beta-\gamma+1)_n(\beta+\delta+1)_n$$
$$\times \frac{(n+\alpha+\beta+1)_n}{(\alpha+\beta+2)_{2n}} \frac{\Gamma(\gamma+\delta-\alpha+1)\Gamma(-\beta-\alpha-1)\Gamma(\gamma-\beta+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+2)\Gamma(-\beta)\Gamma(\gamma-\beta-\alpha)\Gamma(\delta-\alpha)} ,$$

where $\alpha+1$, $\beta+\delta+1$, or $\gamma+1 = -\Delta$, and $0 \le n, m \le \Delta$.

Two interesting limit cases are the Hahn and dual Hahn polynomials. The limit $\delta \to \infty$ with $\gamma+1 = -\Delta$ gives the Hahn polynomial orthogonality

$$\sum_{x=0}^{\Delta} \frac{(\alpha+1)_x(-\Delta)_x}{(-\Delta-\beta)_x\, x!} h_n(x) h_m(x) = \lambda_n \delta_{nm} , \quad (1.11)$$

where

$$h_n(x) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -\Delta \end{matrix} ; 1 \right) , \qquad 0 \le n \le \Delta ,$$

$$\lambda_n = \frac{(\Delta-n)!\,n!}{\Delta!} \frac{(n+\alpha+\beta+1)_\Delta(\beta+1)_n}{(\alpha+1)_n(\beta+1)_\Delta} \frac{(\Delta+n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)} . \quad (1.12)$$

Letting $\beta \to \infty$ in (1.10) with $\alpha+1 = -\Delta$ gives the dual Hahn orthogonality

$$\sum_{x=0}^{\Delta} \binom{\Delta}{x} \frac{(\gamma+\delta+1)_x(\gamma/2+\delta/2+3/2)_x(\gamma+1)_x}{(\gamma/2+\delta/2+1/2)_x(\delta+1)_x(\Delta+\gamma+\delta+2)_x} d_n(x) d_m(x) = \lambda_n \delta_{nm} , \quad (1.13)$$

with

$$d_n(x) = {}_3F_2\left(\begin{matrix} -n, -x, x+\gamma+\delta+1 \\ -\Delta, \gamma+1 \end{matrix} ; 1 \right) , \qquad 0 \le n \le \Delta ,$$

$$\lambda_n = \frac{(\Delta-n)!\,n!}{\Delta!} \frac{(\gamma+\delta+2)_\Delta}{(\gamma+1)_n(\delta+1)_{\Delta-n}} . \quad (1.14)$$

A generalization of the Wilson polynomials to p variables $x_1, x_2, \dots x_p$ is given by the following four families[4]

$$P\begin{pmatrix} x_1, x_2, \dots x_p \\ n_1, n_2, \dots n_p \end{pmatrix} \begin{vmatrix} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_p \end{vmatrix} c, d \Big) = \left[\prod_{k=1}^{p} \Gamma(n_k + a_k + b_k) \Big/ \Gamma(a_k + b_k)\right] (A+c)_N (A+d)_N$$

$$\times F^{2:2;\dots;2}_{2:1;\dots;1}\begin{pmatrix} N+A+B+c+d-1, A-iX : -n_1, a_1+ix_1; \dots; -n_p, a_p+ix_p \\ A+c, A+d : a_1+b_1; \dots; a_p+b_p \end{pmatrix}, \tag{1.15}$$

$$\overline{P}\begin{pmatrix} x_1, x_2, \dots x_p \\ n_1, n_2, \dots n_p \end{pmatrix} \begin{vmatrix} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_p \end{vmatrix} c, d \Big) = \left[\prod_{k=1}^{p} \Gamma(n_k + a_k + b_k) \Big/ \Gamma(a_k + b_k)\right] (B+c)_N (B+d)_N$$

$$\times F^{2:2;\dots;2}_{2:1;\dots;1}\begin{pmatrix} N+A+B+c+d-1, B+iX : -n_1, b_1-ix_1; \dots; -n_p, b_p-ix_p \\ B+c, B+d : a_1+b_1; \dots; a_p+b_p \end{pmatrix}, \tag{1.16}$$

$$Q\begin{pmatrix} x_1, x_2, \dots x_p \\ n_1, n_2, \dots n_p \end{pmatrix} \begin{vmatrix} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_p \end{vmatrix} c, d \Big) = \left[\prod_{k=1}^{p} \Gamma(n_k + a_k + b_k) \Big/ \Gamma(a_k + b_k)\right] (c-iX)_N (d-iX)_N$$

$$\times F^{2:2;\dots;2}_{2:1;\dots;1}\begin{pmatrix} -N-c-d+1, B+iX : -n_1, a_1+ix_1; \dots; -n_p, a_p+ix_p \\ -N-c+iX+1, -N-d+iX+1 : a_1+b_1; \dots; a_p+b_p \end{pmatrix}, \tag{1.17}$$

$$\overline{Q}\begin{pmatrix} x_1, x_2, \dots x_p \\ n_1, n_2, \dots n_p \end{pmatrix} \begin{vmatrix} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_p \end{vmatrix} c, d \Big) = \left[\prod_{k=1}^{p} \Gamma(n_k + a_k + b_k) \Big/ \Gamma(a_k + b_k)\right] (c+iX)_N (d+iX)_N$$

$$\times F^{2:2;\dots;2}_{2:1;\dots;1}\begin{pmatrix} -N-c-d+1, A-iX : -n_1, b_1-ix_1; \dots; -n_p, b_p-ix_p \\ -N-c-iX+1, -N-d-iX+1 : a_1+b_1; \dots; a_p+b_p \end{pmatrix}, \tag{1.18}$$

where $a_1, a_2, \dots a_p, b_1, b_2, \dots b_p, c, d$ are complex parameters and $F^{q:\ell_1;\dots;\ell_p}_{r:v_1;\dots;v_p}$ is the generalized multivariable Kampé de Fériet hypergeometric series[5] defined as

$$F^{q:\ell_1;\dots;\ell_p}_{r:v_1;\dots;v_p}\begin{pmatrix} \alpha_1, \dots, \alpha_q : \beta_1^{(1)}, \dots, \beta_{\ell_1}^{(1)}; \dots; \beta_1^{(p)}, \dots, \beta_{\ell_p}^{(p)}; \\ \gamma_1, \dots, \gamma_r : \xi_1^{(1)}, \dots, \xi_{v_1}^{(1)}; \dots; \xi_1^{(p)}, \dots, \xi_{v_p}^{(p)}; \end{pmatrix} z_1, z_2, \dots, z_p \Big) =$$

$$= \sum_{\{j_k\}} \frac{\prod_{i=1}^{q} (\alpha_i)_J \prod_{i=1}^{\ell_1} \left(\beta_i^{(1)}\right)_{j_1} \cdots \prod_{i=1}^{\ell_p} \left(\beta_i^{(p)}\right)_{j_p}}{\prod_{i=1}^{r} (\gamma_i)_J \prod_{i=1}^{v_1} \left(\xi_i^{(1)}\right)_{j_1} \cdots \prod_{i=1}^{v_p} \left(\xi_i^{(p)}\right)_{j_p}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \cdots \frac{z_p^{j_p}}{j_p!}, \tag{1.19}$$

where $\{j_k\}$ denotes summation indices $j_1, j_2 ... j_p$ which run over all nonnegative integers and we have introduced the following shorthand notation

$$X \equiv \sum_{k=1}^{p} x_k \ , \quad N \equiv \sum_{k=1}^{p} n_k \ , \quad J \equiv \sum_{k=1}^{p} j_k \ , \quad A \equiv \sum_{k=1}^{p} a_k \ , \quad B \equiv \sum_{k=1}^{p} b_k \ , \tag{1.20}$$

and in the absence of specifying the arguments $z_1, z_2 ... z_p$ unity is to be understood. The overbars in (1.16) and (1.18) denote distinct families of polynomials and should not be confused with complex conjugation. The p–tuple of nonnegative integers $n_1, n_2, ... n_p$ labels the different polynomials whose degrees are given by $2N$ where N is defined in (1.20). These polynomials are associated with the following multivariable weight function

$$\begin{aligned} w\left(x_1, x_2, ... x_p \middle| \begin{matrix} a_1, a_2, ... a_p \\ b_1, b_2, ... b_p \end{matrix} \middle| c, d \right) &= \left[\prod_{k=1}^{p} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] \\ &\times \frac{\Gamma(A - iX)\, \Gamma(B + iX)\, \Gamma(c + iX)\, \Gamma(c - iX)\, \Gamma(d + iX)\, \Gamma(d - iX)}{\Gamma(2iX)\, \Gamma(-2iX)} \ , \end{aligned} \tag{1.21}$$

which notice is symmetric under the interchange of c and d as are all four families of polynomials. When no ambiguity arises we simply write $P_n(x)$, $\overline{P}_n(x)$, $Q_n(x)$, $\overline{Q}_n(x)$, and $w(x)$ for the polynomials and weight function, respectively.

These satisfy the following biorthogonality relations[4]

$$\begin{aligned} P_n \cdot Q_m &= \overline{P}_n \cdot \overline{Q}_m = \lambda_n \prod_{k=1}^{p} \delta_{n_k m_k} \ , \\ P_n \cdot \overline{P}_m &= Q_n \cdot \overline{Q}_m = 0 \quad \text{if} \quad N \neq M \ , \end{aligned} \tag{1.22}$$

where the normalization constant is given by

$$\begin{aligned} \lambda_n = 2(2\pi)^p &\left[\prod_{k=1}^{p} \Gamma(n_k + a_k + b_k)\; n_k! \right] \\ &\times \frac{\Gamma(N + A + c)\, \Gamma(N + A + d)\, \Gamma(N + B + c)\, \Gamma(N + B + d)\, \Gamma(N + c + d)}{(2N + A + B + c + d - 1)\, \Gamma(N + A + B + c + d - 1)} \ , \end{aligned} \tag{1.23}$$

and for positive real parts of the parameters the inner product is defined

$$\begin{aligned} P_n \cdot Q_m &\equiv \int_{-\infty}^{\infty} dx_p ... \int_{-\infty}^{\infty} dx_1\, w(x_1 ... x_p)\, P_n(x_1 ... x_p)\, Q_m(x_1 ... x_p) \ , \\ &\mathrm{Re}(a_1, a_2 ... a_p, b_1, b_2 ... b_p, c, d) > 0 \ , \end{aligned} \tag{1.24}$$

where the integration contours are simply the real axes.

In Section II we extend these purely continuous multivariable biorthogonal Wilson polynomials to several "mixed" cases where the inner product is partly continuous and partly discrete. In Section III we discuss the purely discrete family which are the multivariable biorthogonal Racah polynomials. Taking appropriate limits then yields multivariable biorthogonal Hahn and dual Hahn polynomials.

II. Multivariable mixed type inner products

The biorthogonality relations (1.22) are still valid for negative real parts of the parameters provided each of the contours are suitably deformed to separate the increasing sequences of poles of the weight function from the decreasing sequences, assuming these two sets are disjoint. However, due to the multiple integrals involved it is not always clear where the poles lie in each of the variables and what the appropriate contours are.

We consider several specific cases the first being the following parameter domain

$$\begin{gathered}\mathrm{Re}(a_1, a_2 ... a_p, b_1, b_2 ... b_p, d) > 0 \ , \qquad \mathrm{Re}(c) < 0 \ , \\ \mathrm{Re}(a_1 + c) \ , \ \mathrm{Re}(b_1 + c) > 0 \ , \quad \mathrm{Re}(c) \ , \ 2c \ , \ c + d \neq 0, -1, -2, ... \ ,\end{gathered} \tag{2.1}$$

for which the x_1 contour C_1 is deformed from the real axis to pass above the decreasing sequence of poles given by $x_1 = -X_2^p - ic - ij$, $j = 0, 1, 2, ...$ and underneath the increasing sequence found at $x_1 = -X_2^p + ic + ij$, $j = 0, 1, 2, ...$, and also above and below the remaining decreasing and increasing sequences, respectively. We have introduced the following shorthand notation to denote partial sums

$$X_j^\ell \equiv \sum_{k=j}^{\ell} x_k \ , \qquad A_j^\ell \equiv \sum_{k=j}^{\ell} a_k \ , \qquad B_j^\ell \equiv \sum_{k=j}^{\ell} b_k \ . \tag{2.2}$$

If we furthermore choose C_1 sufficiently close to (and under) $-X_2^p + ic$ and also sufficiently close to (and over) $-X_2^p - ic$ then the remaining contours $C_2 ... C_p$ can be chosen on the real axes.

Let us first demonstrate that this choice of contours leads to the norm of the weight function as given by (1.23) with $N = 0$. Making a change of variables from $x_1, x_2 ... x_p$ to

$X, x_2 \dots x_p$ and reversing the order of the integrations gives

$$\int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1\, w(x_1 \dots x_p) = \tag{2.3}$$

$$= \int_C dX \, \frac{\Gamma(A - iX)\, \Gamma(B + iX)\, \Gamma(c + iX)\, \Gamma(c - iX)\, \Gamma(d + iX)\, \Gamma(d - iX)}{\Gamma(2iX)\, \Gamma(-2iX)}$$

$$\times \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_p \, \Gamma(a_1 + iX - iX_2^p)\, \Gamma(b_1 - iX + iX_2^p) \left[\prod_{k=2}^{p} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] ,$$

where the contour C passes over the decreasing sequence of poles $X = -ic - ij,\ j = 0, 1, 2, \dots$ and under the increasing sequence $X = ic + ij,\ j = 0, 1, 2, \dots$, and also above and below the remaining decreasing and increasing sequences, respectively. To evaluate the $x_2, x_3 \dots x_p$ integrations we introduce the following single variable integral formula[3]

$$\int_{C'} dx\, \Gamma(\alpha + ix) \Gamma(\beta + ix) \Gamma(\gamma - ix) \Gamma(\delta - ix) = \tag{2.4}$$

$$= (2\pi) \frac{\Gamma(\alpha + \gamma) \Gamma(\alpha + \delta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)} ,$$

$$\alpha + \gamma, \alpha + \delta, \beta + \gamma, \beta + \delta \neq 0, -1, -2, \dots ,$$

where the contour C' separates the increasing sequences of poles of the integrand from the decreasing sequences; in the special case when $\mathrm{Re}(\alpha, \beta, \gamma, \delta) > 0$, C' can be chosen on the real axis. Returning to (2.3) and recalling that $\mathrm{Re}(a_1 + c), \mathrm{Re}(b_1 + c) > 0$, the contour C is assumed to pass sufficiently close to (and over) $-ic$ so that $\mathrm{Re}(a_1 + iX) > 0$ and also sufficiently close to (and under) $+ic$ so that $\mathrm{Re}(b_1 - iX) > 0$. In this case formula (2.4) with C' on the real axis and induction can be used to evaluate the $x_2, x_3 \dots x_p$ integrations leading to

$$\int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_p \, \Gamma(a_1 + iX - iX_2^p)\, \Gamma(b_1 - iX + iX_2^p) \left[\prod_{k=2}^{p} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] =$$

$$= (2\pi)^{p-1} \left[\prod_{k=1}^{p} \Gamma(a_k + b_k) \right] \Gamma(A + iX)\, \Gamma(B - iX) / \Gamma(A + B) , \tag{2.5}$$

and if this is substituted into (2.3) the norm of the weight function becomes

$$\int_{-\infty}^{\infty} dx_p \ldots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1\, w(x_1 \ldots x_p) =$$

$$= (2\pi)^{p-1} \left[\prod_{k=1}^{p} \Gamma(a_k + b_k) \right] [\Gamma(A+B)]^{-1} \int_C dX\, \Gamma(A+iX)\, \Gamma(A-iX) \tag{2.6}$$

$$\times \frac{\Gamma(B+iX)\, \Gamma(B-iX)\, \Gamma(c+iX)\, \Gamma(c-iX)\, \Gamma(d+iX)\, \Gamma(d-iX)}{\Gamma(2iX)\, \Gamma(-2iX)} ,$$

which is simply proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Using this result in (2.6) then yields the multivariable norm as defined in (1.23) with $N = 0$.

Next we express the multiple integral in the left of (2.6) as a finite discrete sum and integrations over the real axes. This is achieved by deforming C_1 to the real axis plus closed loops about a finite number of poles and then evaluating the latter by the method of residues. This leads to two additional terms involving discrete sums which by using (2.5) one can show are equal. The norm then becomes

$$\int_{-\infty}^{\infty} dx_p \ldots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1\, w(x_1 \ldots x_p) = \int_{-\infty}^{\infty} dx_p \ldots \int_{-\infty}^{\infty} dx_1\, w(x_1 \ldots x_p) \tag{2.7}$$

$$+ \int_{-\infty}^{\infty} dx_p \ldots \int_{-\infty}^{\infty} dx_2 \sum_{j=0}^{\mathrm{Re}(c)+j<0} \tilde{w}(j, x_2 \ldots x_p) ,$$

where all the integrals on the right are over the real axes and the "mixed" weight function in the second term is given by

$$\tilde{w}(j, x_2 \ldots x_p) = (4\pi) \Gamma(a_1 - c - j - iX_2^p)\, \Gamma(b_1 + c + j + iX_2^p) \left[\prod_{k=2}^{p} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] \tag{2.8}$$

$$\times \frac{\Gamma(A+c+j)\, \Gamma(B-c-j)\, \Gamma(2c+j)\, \Gamma(d-c-j)\, \Gamma(d+c+j)}{\Gamma(2c+2j)\, \Gamma(-2c-2j)} \frac{(-1)^j}{j!} .$$

The first purely continuous term in the right of (2.7) is that which arises for positive real parts of the parameters. The second mixed term represents the contribution arising from the negative real part of the parameter c.

The inner product of the multivariable Wilson polynomials is defined

$$P_n \cdot Q_m \equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1\, w(x_1 \dots x_p) P_n(x_1 \dots x_p) Q_m(x_1 \dots x_p) \ , \tag{2.9}$$

which in analogy with (2.7) can be written as

$$\begin{aligned} P_n \cdot Q_m \ &= \ \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_1\, w(x_1 \dots x_p) P_n(x_1 \dots x_p) Q_m(x_1 \dots x_p) \\ &+ \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \sum_{j=0}^{\mathrm{Re}(c)+j<0} \tilde{w}(j, x_2 \dots x_p) P_n(-X_2^p + ic + ij, x_2 \dots x_p) Q_m(-X_2^p + ic + ij, x_2 \dots x_p) \end{aligned} \tag{2.10}$$

and similarly for $\overline{P}_n \cdot \overline{Q}_m$, $P_n \cdot \overline{P}_m$, and $Q_n \cdot \overline{Q}_m$. Having verified the norm of the weight function it then follows by the same proof as for the purely continuous family[4] that these inner products satisfy biorthogonality relations (1.22).

Formula (2.10) also yields a simpler and even more interesting mixed type inner product. Take $c + d = -\Delta_1 + \epsilon$, where Δ_1 is a nonnegative integer, divide the biorthogonality relations (1.22) by $\Gamma(c + d) = \Gamma(-\Delta_1 + \epsilon)$, and then take the limit $\epsilon \to 0$. The first purely continuous term in (2.10) vanishes because $1/\Gamma(-\Delta_1 + \epsilon) \to 0$ but the second mixed term survives leaving (writing x_1 in place of j)

$$\begin{gathered} P_n^{(1)} \cdot Q_m^{(1)} \equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \sum_{x_1=0}^{\Delta_1} w^{(1)}(x_1, x_2 \dots x_p) P_n^{(1)}(x_1, x_2 \dots x_p) Q_m^{(1)}(x_1, x_2 \dots x_p) \ , \\ P_n^{(1)} \cdot Q_m^{(1)} = \overline{P}_n^{(1)} \cdot \overline{Q}_m^{(1)} = \lambda_n^{(1)} \prod_{k=1}^{p} \delta_{n_k m_k} \ , \qquad P_n^{(1)} \cdot \overline{P}_m^{(1)} = Q_n^{(1)} \cdot \overline{Q}_m^{(1)} = 0 \quad \text{if} \quad N \neq M \ , \end{gathered} \tag{2.11}$$

where the mixed weight function and normalization constant are given by

$$\begin{aligned} w^{(1)}(x_1 \dots x_p) &= \Gamma(a_1 + d + \Delta_1 - x_1 - iX_2^p)\, \Gamma(b_1 + c + x_1 + iX_2^p) \left[\prod_{k=2}^{p} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] \\ &\times \frac{\Gamma(A + c + x_1)}{\Gamma(A + c)} \frac{\Gamma(B + d + \Delta_1 - x_1)}{\Gamma(B + d)} \frac{\Gamma(1 + 2d + 2\Delta_1 - 2x_1)}{\Gamma(1 + 2d + 2\Delta_1 - x_1)} \frac{\Gamma(2d + \Delta_1 - x_1)}{\Gamma(2d + 2\Delta_1 - 2x_1)} \binom{\Delta_1}{x_1} (-1)^{x_1}, \end{aligned} \tag{2.12}$$

$$\lambda_n^{(1)} = (2\pi)^{p-1} \left[\prod_{k=1}^{p} \Gamma(n_k + a_k + b_k)\, n_k! \right] \frac{\Delta_1!}{(\Delta_1 - N)!} (-1)^N \, (A+c)_N \, (B+d)_N \qquad (2.13)$$

$$\times \frac{\Gamma(N+A+d)\,\Gamma(N+B+c)}{(2N+A+B+c+d-1)\Gamma(N+A+B+c+d-1)} ,$$

and the polynomials are defined

$$\begin{aligned}
P_n^{(1)}(x_1 ... x_p) &\equiv P_n(-X_2^p + ic + ix_1, x_2 ... x_p) , \qquad N = 0,1,2,...\infty , \\
\overline{P}_n^{(1)}(x_1 ... x_p) &\equiv \overline{P}_n(-X_2^p + ic + ix_1, x_2 ... x_p) , \qquad N = 0,1,2,...\infty , \\
Q_n^{(1)}(x_1 ... x_p) &\equiv Q_n(-X_2^p + ic + ix_1, x_2 ... x_p) , \qquad 0 \le N \le \Delta_1 , \\
\overline{Q}_n^{(1)}(x_1 ... x_p) &\equiv \overline{Q}_n(-X_2^p + ic + ix_1, x_2 ... x_p) , \qquad 0 \le N \le \Delta_1 ,
\end{aligned} \qquad (2.14)$$

with $c+d = -\Delta_1$. These biorthogonality relations can be verified independently of the limiting procedure. To calculate the norm of the weight function one uses (2.4) and induction to perform the $x_2, x_3 ... x_p$ integrations and then the following summation theorem[3]

$${}_5F_4\left(\begin{matrix} 2\alpha,\, \alpha+1,\, \alpha+\beta,\, \alpha+\gamma,\, \alpha+\delta \\ \alpha,\, \alpha-\beta+1,\, \alpha-\gamma+1,\, \alpha-\delta+1 \end{matrix} ; 1 \right) = \qquad (2.15)$$

$$= \frac{\Gamma(\alpha-\beta+1)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\alpha-\delta+1)\,\Gamma(-\alpha-\beta-\gamma-\delta+1)}{\Gamma(2\alpha+1)\,\Gamma(-\beta-\gamma+1)\,\Gamma(-\beta-\delta+1)\,\Gamma(-\gamma-\delta+1)} ,$$

$$\mathrm{Re}\,(\alpha+\beta+\gamma+\delta) < 1 ,$$

to evaluate the x_1 sum. Having calculated the norm the biorthogonality relations can then be verified in the same manner as was proved for the purely continuous family.[4] Also, some of the original restrictions in (2.1) can be removed from (2.11) leaving only

$$\mathrm{Re}(a_1, a_2 ... a_p, b_1, b_2 ... b_p, d) > 0 , \qquad \mathrm{Re}(b_1 + c) > 0 , \qquad c + d = -\Delta_1 , \qquad (2.16)$$

where recall Δ_1 is a nonnegative integer.

We consider a further generalization of (2.11)–(2.14) by also allowing some or all of the a–parameters to have negative real parts. The parameter domain in this case is defined by

$$\begin{aligned} &\mathrm{Re}(a_1, a_2 ... a_r) < 0 \ , \qquad \mathrm{Re}(a_{r+1} ... a_p, b_1 ... b_p, d) > 0 \ , \\ &\mathrm{Re}(b_1 + c) > 0 \ , \qquad \mathrm{Re}(A_1^r + d) > 0 \ , \qquad c + d = -\Delta_1 \ , \\ &\mathrm{Re}(a_2 ... a_r) \ , \ a_2 + b_2 ... a_r + b_r \neq 0, -1, -2 ... \ , \qquad r = 1, 2, ... p \ , \end{aligned} \tag{2.17}$$

for which the contours $C_2 ... C_r$, that in (2.11) were on the real axes, are here deformed to pass underneath the increasing sequences of poles $x_k = ia_k + ij_k$, $j_k = 0, 1, 2, ...$, $k = 2, 3 ... r$, while still passing above the decreasing sequences. We furthermore choose these contours sufficiently close to (and under) ia_k, $k = 2, 3 ... r$, so that $\mathrm{Re}(a_1 + d - iX_2^r) > 0$, which is always possible in light of the restriction $\mathrm{Re}(A_1^r + d) > 0$. In this case the remaining contours $C_{r+1} ... C_p$ can be chosen to lie on the real axes. The norm of the weight function is calculated in the same manner as was done for (2.11). That is, one uses (2.4) and induction to perform the $x_2 ... x_p$ integrations and then the ${}_5F_4$ summation theorem (2.15) to evaluate the x_1 sum resulting in (2.13) with $N = 0$. Having verified the norm one can then prove the biorthogonality relations (2.11) in the same manner as was done for the purely continuous family,[4] but with the more general inner product defined above or as re–expressed in (2.18).

The contour integrals can be transformed to discrete sums and real integrations by deforming $C_2 ... C_r$ to the real axes plus closed loops about a finite number of poles and then evaluating the latter by the method of residues. The inner product of the polynomials can then be schematically written as

$$\begin{aligned} P_n^{(1)} \cdot Q_m^{(1)} \equiv \int_{-\infty}^{\infty} dx_p ... \int_{-\infty}^{\infty} dx_{r+1} \left(\prod_{k=2}^{r} \left\{ \int_{-\infty}^{\infty} dx_k + (2\pi i) \sum_{j_k=0}^{\mathrm{Re}(a_k)+j_k<0} \mathrm{res}(x_k = ia_k + ij_k) \right\} \right) \\ \times \sum_{x_1=0}^{\Delta_1} w^{(1)}(x_1 ... x_p) P_n^{(1)}(x_1 ... x_p) Q_m^{(1)}(x_1 ... x_p) \ , \end{aligned} \tag{2.18}$$

and similarly for $\overline{P}_n^{(1)} \cdot \overline{Q}_m^{(1)}$, $P_n^{(1)} \cdot \overline{P}_m^{(1)}$, and $Q_n^{(1)} \cdot \overline{Q}_m^{(1)}$, where $\mathrm{res}(x_k)$ denotes the residue at x_k. The right side of (2.18) represents a multitude of mixed type terms involving integrations over the real axes and finite discrete sums.

This example which is a generalization of a limit case of (2.10) has itself an interesting limit case. Set $a_k + b_k = -\Delta_k + \epsilon$, $k = 2, 3 \ldots r$, where Δ_k are nonnegative integers, divide the biorthogonality relations by $\prod_{k=2}^{r} \Gamma(a_k + b_k) = \prod_{k=2}^{r} \Gamma(-\Delta_k + \epsilon)$, and then take the limit $\epsilon \to 0$. Since $1/\Gamma(-\Delta_k + \epsilon) \to 0$ the only term in (2.18) that survives is the one with r discrete sums leaving (writing $x_2 \ldots x_r$ in place of $j_2 \ldots j_r$ and transforming $x_1 \to \Delta_1 - x_1$)

$$P_n^{(2)} \cdot Q_m^{(2)} \equiv \int_{-\infty}^{\infty} dx_p \ldots \int_{-\infty}^{\infty} dx_{r+1} \sum_{x_r=0}^{\Delta_r} \cdots \sum_{x_2=0}^{\Delta_2} \sum_{x_1=0}^{\Delta_1} w^{(2)}(x_1 \ldots x_p) P_n^{(2)}(x_1 \ldots x_p) Q_m^{(2)}(x_1 \ldots x_p) \ , \tag{2.19}$$

$$P_n^{(2)} \cdot Q_m^{(2)} = \overline{P}_n^{(2)} \cdot \overline{Q}_m^{(2)} = \lambda_n^{(2)} \prod_{k=1}^{p} \delta_{n_k m_k} , \qquad P_n^{(2)} \cdot \overline{P}_m^{(2)} = Q_n^{(2)} \cdot \overline{Q}_m^{(2)} = 0 , \quad \text{if} \quad N \neq M ,$$

where the weight function, normalization constant, and polynomials are given by

$$\begin{aligned} w^{(2)}(x_1 \ldots x_p) = & \left[\prod_{k=1}^{r} \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^{p} \Gamma(a_k + i x_k)\, \Gamma(b_k - i x_k) \right] \Gamma(A_1^r + d + X_1^r - i X_{r+1}^p) \\ & \times \Gamma(B_1^r + c + \Delta_1^r - X_1^r + i X_{r+1}^p) \frac{\Gamma(A + c + \Delta_1 - x_1)}{\Gamma(A + c)} \frac{\Gamma(B + d + x_1)}{\Gamma(B + d)} \\ & \times \frac{\Gamma(1 + 2d + 2x_1)}{\Gamma(1 + 2d + \Delta_1 + x_1)} \frac{\Gamma(2d + x_1)}{\Gamma(2d + 2x_1)} (-1)^{\Delta_1 - x_1} , \end{aligned} \tag{2.20}$$

$$\begin{aligned} \lambda_n^{(2)} = & (2\pi)^{p-r} \Gamma(n_1 + a_1 + b_1)\, n_1! \left[\prod_{k=2}^{r} \frac{\Delta_k!}{(\Delta_k - n_k)!} n_k! (-1)^{n_k} \right] \\ & \times \left[\prod_{k=r+1}^{p} \Gamma(n_k + a_k + b_k)\, n_k! \right] \frac{\Delta_1!}{(\Delta_1 - N)!} (-1)^N (A + c)_N (B + d)_N \\ & \times \frac{\Gamma(N + A + d)\, \Gamma(N + B + c)}{(2N + A + B + c + d - 1)\, \Gamma(N + A + B + c + d - 1)} , \end{aligned} \tag{2.21}$$

$$\begin{aligned} & P_n^{(2)}(x_1 \ldots x_p) \equiv P_n(-i A_2^r - i X_1^r - X_{r+1}^p - i d, i a_2 + i x_2 \ldots i a_r + i x_r, x_{r+1} \ldots x_p) , \\ & \overline{P}_n^{(2)}(x_1 \ldots x_p) \equiv \overline{P}_n(-i A_2^r - i X_1^r - X_{r+1}^p - i d, i a_2 + i x_2 \ldots i a_r + i x_r, x_{r+1} \ldots x_p) , \\ & \qquad 0 \leq n_k \leq \Delta_k , k = 2, 3 \ldots r , \quad n_1, n_{r+1} \ldots n_p = 0, 1, 2 \ldots \infty , \end{aligned} \tag{2.22}$$

$$Q_n^{(2)}(x_1 \ldots x_p) \equiv Q_n(-iA_2^r - iX_1^r - X_{r+1}^p - id,\, ia_2 + ix_2 \ldots a_r + ix_r,\, x_{r+1} \ldots x_p)\ ,$$

$$\overline{Q}_n^{(2)}(x_1 \ldots x_p) \equiv \overline{Q}_n(-iA_2^r - iX_1^r - X_{r+1}^p - id,\, ia_2 + ix_2 \ldots a_r + ix_r,\, x_{r+1} \ldots x_p)\ ,$$

$$0 \leq n_k \leq \Delta_k\,,\ k = 2, 3 \ldots r, \quad 0 \leq N \leq \Delta_1\ ,$$

with $c + d = -\Delta_1$ and $a_k + b_k = -\Delta_k\,, k = 2, 3 \ldots r$. As an independent verification of the norm of the weight function one uses (2.4) and induction to evaluate the $x_{r+1} \ldots x_p$ integrations, the following summation theorem[6] to perform the $x_2 \ldots x_r$ sums,

$$F^{1:1;\ldots;1}_{1:0;\ldots;0}\left(\begin{matrix} \alpha\, :\, \beta^{(1)}; \ldots ; \beta^{(p)}; \\ \gamma\, :\, -; \ldots ; -; \end{matrix} 1, 1 \ldots 1 \right) = \frac{\Gamma(\gamma)\, \Gamma\big(\gamma - \alpha - \beta^{(1)} - \ldots - \beta^{(p)}\big)}{\Gamma(\gamma - \alpha)\, \Gamma\big(\gamma - \beta^{(1)} - \ldots - \beta^{(p)}\big)}\ , \qquad (2.23)$$

and then theorem (2.15) to evaluate the remaining x_1 sum, resulting in (2.21) with $N = 0$. The biorthogonality relations (2.19) can then be independently verified in the manner described for the purely continuous family.[4] Also, the restriction $\mathrm{Re}(a_2 \ldots a_r) \neq 0, -1, -2 \ldots$, is removable from (2.19)–(2.22).

Returning to the purely continuous family (1.22)–(1.24) we consider another mixed type generalization arising from the following parameter domain

$$\begin{aligned} &\mathrm{Re}(a_1, a_2 \ldots a_r) < 0\ , \quad \mathrm{Re}(a_{r+1}, a_{r+2} \ldots a_p, b_1, b_2 \ldots b_p, c, d) > 0\ , \\ &\mathrm{Re}(A_1^r + A)\ ,\ \mathrm{Re}(A_1^r + c)\ ,\ \mathrm{Re}(A_1^r + d) > 0\ , \quad r = 1, 2 \ldots p - 1\ , \qquad (2.24) \\ &\mathrm{Re}(a_1, a_2 \ldots a_r)\ ,\ a_1 + b_1\,, a_2 + b_2 \ldots a_r + b_r \neq 0, -1, -2, \ldots\ , \end{aligned}$$

for which the first r contours $C_1\,, C_2 \ldots C_r$ are deformed below the real axes to pass underneath the increasing sequences of poles $x_k = ia_k + ij_k\,,\ k = 1, 2 \ldots r\,,\ j_k = 0, 1, 2, \ldots$, while still passing above the decreasing sequences. If these are chosen sufficiently close to (and under) $ia_k\,,\ k = 1, 2 \ldots r$ then the remaining contours $C_{r+1} \ldots C_p$ can be chosen to lie on the real axes.

To show that these contours give (1.23) with $N = 0$ for the norm of the weight function we begin with a change of variables from $x_1, x_2 \ldots x_p$ to $x_1, x_2 \ldots x_r, X_{r+1}^p, x_{r+2} \ldots x_p$ yielding

$$\int_{C_1} dx_1 \cdots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p\, w(x_1 \ldots x_p) = \int_{C_1} dx_1 \cdots \int_{C_r} dx_r \left[\prod_{k=1}^{r} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right]$$
$$\times \int_{-\infty}^{\infty} dX_{r+1}^p \frac{\Gamma(A - iX)\, \Gamma(B + iX)\, \Gamma(c + iX)\, \Gamma(c - iX)\, \Gamma(d + iX)\, \Gamma(d - iX)}{\Gamma(2iX)\, \Gamma(-2iX)}$$
$$\times \int_{-\infty}^{\infty} dx_{r+2} \cdots \int_{-\infty}^{\infty} dx_p\, \Gamma\left(a_{r+1} + iX_{r+1}^p - iX_{r+2}^p\right) \Gamma\left(b_{r+1} - iX_{r+1}^p + iX_{r+2}^p\right) \tag{2.25}$$
$$\times \left[\prod_{k=r+2}^{p} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] ,$$

and then the $x_{r+2} \ldots x_p$ integrations are performed by using (2.4) and induction leading to

$$\int_{C_1} dx_1 \cdots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p\, w(x_1 \ldots x_p) = (2\pi)^{p-r-1} \left[\prod_{k=r+1}^{p} \Gamma(a_k + b_k) \right]$$
$$\times \left[\Gamma(A_{r+1}^p + B_{r+1}^p) \right]^{-1} \int_{C_1} dx_1 \cdots \int_{C_r} dx_r \left[\prod_{k=1}^{r} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] \tag{2.26}$$
$$\times \int_{-\infty}^{\infty} dX_{r+1}^p\, \Gamma(A_{r+1}^p + iX_{r+1}^p)\, \Gamma(B_{r+1}^p - iX_{r+1}^p)\, \Gamma(A - iX)\, \Gamma(B + iX)$$
$$\times \frac{\Gamma(c + iX)\, \Gamma(c - iX)\, \Gamma(d + iX)\, \Gamma(d - iX)}{\Gamma(2iX)\, \Gamma(-2iX)} .$$

Another change of variables from $x_1 \ldots x_r, X_{r+1}^p$ to $X, x_1 \ldots x_r$ then transforms this into

$$\int_{C_1} dx_1 \cdots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p\, w(x_1 \ldots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A_{r+1}^p + B_{r+1}^p)} \left[\prod_{k=r+1}^{p} \Gamma(a_k + b_k) \right]$$
$$\times \int_C dX \frac{\Gamma(A - iX)\, \Gamma(B + iX)\, \Gamma(c + iX)\, \Gamma(c - iX)\, \Gamma(d + iX)\, \Gamma(d - iX)}{\Gamma(2iX)\, \Gamma(-2iX)} \tag{2.27}$$
$$\times \int_{C_1} dx_1 \cdots \int_{C_r} dx_r\, \Gamma(A_{r+1}^p + iX - iX_1^r)\, \Gamma(B_{r+1}^p - iX + iX_1^r) \left[\prod_{k=1}^{r} \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] ,$$

where C passes underneath iA_1^r. The contours $C_1 \ldots C_r$, which recall pass underneath the increasing sequences $x_k = ia_k + ij_k$, $k = 1, 2, \ldots r$, $j_k = 0, 1, 2, \ldots$, are assumed to pass sufficiently close to (and under) ia_k, $k = 1,2...r$ so that $\mathrm{Re}(A_{r+1}^p - iX_1^r) > 0$, which is possible on account of $\mathrm{Re}(A_1^r + A) > 0$. Also $\mathrm{Re}(B_{r+1}^p + iX_1^r) > 0$ since $\mathrm{Re}(b_1...b_p) > 0$ and X_1^r has negative or zero imaginary part. In this case the sequences of poles in the variable X do not cross the real axis and so C can be deformed to this axis. With X real the $x_1...x_r$ integrations can then be performed by (2.4) and induction giving

$$\int_{C_1} dx_1 \cdots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p\, w(x_1...x_p) = (2\pi)^{p-1} \left[\prod_{k=1}^{p} \Gamma(a_k + b_k)\right] [\Gamma(A+B)]^{-1}$$
$$\times \int_{-\infty}^{\infty} dX\, \Gamma(A+iX)\,\Gamma(A-iX)\,\Gamma(B+iX)\,\Gamma(B-iX) \tag{2.28}$$
$$\times \frac{\Gamma(c+iX)\,\Gamma(c-iX)\,\Gamma(d+iX)\,\Gamma(d-iX)}{\Gamma(2iX)\,\Gamma(-2iX)} \,,$$

which is simply proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Using this result in (2.28) then yields the multivariable norm (1.23) with $N = 0$. Having verified the norm the biorthogonality relations (1.22) then follow by the same proof as for the purely continuous family,[4] but with the more general inner product defined above or as re–expressed in (2.30).

As before the contour integrals in the left of (2.28) can be expressed as multiple finite sums and real integrations by deforming $C_1, C_2...C_r$ to the real axes plus closed loops about a finite number of poles and then evaluating the latter by the method of residues. The norm can then be schematically written as

$$\int_{C_1} dx_1 \cdots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p\, w(x_1...x_p) = \tag{2.29}$$
$$= \left(\prod_{k=1}^{r} \left\{ \int_{-\infty}^{\infty} dx_k + (2\pi i) \sum_{j_k=0}^{\mathrm{Re}(a_k)+j_k<0} \mathrm{res}\,(x_k = ia_k + ij_k) \right\}\right) \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p\, w(x_1...x_p) \,,$$

representing a multitude of mixed type terms involving real integrations and finite discrete sums. Accordingly the inner product of the polynomials becomes

$$P_n \cdot Q_m \equiv \left(\prod_{k=1}^{r} \left\{ \int_{-\infty}^{\infty} dx_k + (2\pi i) \sum_{j_k=0}^{\mathrm{Re}(a_k)+j_k<0} \mathrm{res}\,(x_k = ia_k + ij_k) \right\}\right) \tag{2.30}$$
$$\times \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p\, w(x_1...x_p)\, P_n(x_1...x_p)\, Q_m(x_1...x_p) \,,$$

and similarly for $\overline{P}_n \cdot \overline{Q}_m$, $P_n \cdot \overline{P}_m$, and $Q_n \cdot \overline{Q}_m$.

Formula (2.30) also yields a much simpler mixed type inner product. Set $a_k + b_k = -\Delta_k + \epsilon$, $k = 1, 2, \ldots r$, where Δ_k are nonnegative integers, divide the biorthogonality relations (1.22) by $\prod_{k=1}^r \Gamma(a_k + b_k) = \prod_{k=1}^r \Gamma(-\Delta_k + \epsilon)$, and then take the limit $\epsilon \to 0$. The only term in (2.30) that survives is the one with r discrete sums leaving (writing $x_1 \ldots x_r$ in place of $j_1 \ldots j_r$)

$$P_n^{(3)} \cdot Q_m^{(3)} \equiv \sum_{x_1=0}^{\Delta_1} \sum_{x_2=0}^{\Delta_2} \cdots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p \, w^{(3)}(x_1 \ldots x_p) P_n^{(3)}(x_1 \ldots x_p) Q_m^{(3)}(x_1 \ldots x_p) \tag{2.31}$$

$$P_n^{(3)} \cdot Q_m^{(3)} = \overline{P}_n^{(3)} \cdot \overline{Q}_m^{(3)} = \lambda_n^{(3)} \prod_{k=1}^p \delta_{n_k m_k} , \qquad P_n^{(3)} \cdot \overline{P}_m^{(3)} = Q_n^{(3)} \cdot \overline{Q}_m^{(3)} = 0 \quad \text{if} \quad N \neq M ,$$

where the mixed weight function and normalization constant are given by ($\Delta_1^r \equiv \sum_{k=1}^r \Delta_k$)

$$w^{(3)}(x_1 \ldots x_p) = \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + ix_k)\, \Gamma(b_k - ix_k) \right] \tag{2.32}$$

$$\times \Gamma(A + A_1^r + X_1^r - iX_{r+1}^p)\, \Gamma(B + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p)\, \Gamma(c + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p)$$

$$\times \Gamma(c + A_1^r + X_1^r - iX_{r+1}^p) \frac{\Gamma(d + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p)\, \Gamma(d + A_1^r + X_1^r - iX_{r+1}^p)}{\Gamma(2B_1^r + 2\Delta_1^r - 2X_1^r + 2iX_{r+1}^p)\, \Gamma(2A_1^r + 2X_1^r - 2iX_{r+1}^p)},$$

$$\lambda_n^{(3)} = 2(2\pi)^{p-r} \left[\prod_{k=1}^r \frac{\Delta_k!}{(\Delta_k - n_k)!} n_k! (-1)^{n_k} \right] \left[\prod_{k=r+1}^p \Gamma(n_k + a_k + b_k)\, n_k! \right] \tag{2.33}$$

$$\times \frac{\Gamma(N + A + c)\, \Gamma(N + A + d)\, \Gamma(N + B + c)\, \Gamma(N + B + d)\, \Gamma(N + c + d)}{(2N + A + B + c + d - 1) \Gamma(N + A + B + c + d - 1)},$$

and the polynomials are defined

$$\begin{aligned}
&P_n^{(3)}(x_1 \ldots x_p) \equiv P_n(ia_1 + ix_1 \ldots ia_r + ix_r, x_{r+1} \ldots x_p) , \\
&\overline{P}_n^{(3)}(x_1 \ldots x_p) \equiv \overline{P}_n(ia_1 + ix_1 \ldots ia_r + ix_r, x_{r+1} \ldots x_p) , \qquad 0 \le n_k \le \Delta_k,\ k = 1, 2 \ldots r , \\
&Q_n^{(3)}(x_1 \ldots x_p) \equiv Q_n(ia_1 + ix_1 \ldots ia_r + ix_r, x_{r+1} \ldots x_p) , \qquad n_{r+1} \ldots n_p = 0, 1, 2, \ldots \infty , \\
&\overline{Q}_n^{(3)}(x_1 \ldots x_p) \equiv \overline{Q}_n(ia_1 + ix_1 \ldots ia_r + ix_r, x_{r+1} \ldots x_p) ,
\end{aligned} \tag{2.34}$$

where the indicated range of the indices applies to all four families.

These results can also be verified independently of the limit. To calculate the norm of the weight function begin with a change of variables from $x_{r+1} \ldots x_p$ to $X^p_{r+1}, x_{r+2} \ldots x_p$ and use (2.4) and induction to evaluate the $x_{r+2} \ldots x_p$ integrations. This gives

$$\sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p \, w^{(3)}(x_1 \ldots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A^p_{r+1} + B^p_{r+1})} \left[\prod_{k=r+1}^{p} \Gamma(a_k + b_k) \right]$$

$$\times \sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_r=0}^{\Delta_r} \left[\prod_{k=1}^{r} \binom{\Delta_k}{x_k} \right] \int_{-\infty}^{\infty} dX^p_{r+1} \, \Gamma(A^p_{r+1} + iX^p_{r+1}) \, \Gamma(B^p_{r+1} - iX^p_{r+1}) \tag{2.35}$$

$$\times \Gamma(A + A^r_1 + X^r_1 - iX^p_{r+1}) \, \Gamma(B + B^r_1 + \Delta^r_1 - X^r_1 + iX^p_{r+1}) \, \Gamma(c + B^r_1 + \Delta^r_1 - X^r_1 + iX^p_{r+1})$$

$$\times \Gamma(c + A^r_1 + X^r_1 - iX^p_{r+1}) \, \frac{\Gamma(d + B^r_1 + \Delta^r_1 - X^r_1 + iX^p_{r+1})}{\Gamma(2B^r_1 + 2\Delta^r_1 - 2X^r_1 + 2iX^p_{r+1})} \frac{\Gamma(d + A^r_1 + X^r_1 - iX^p_{r+1})}{\Gamma(2A^r_1 + 2X^r_1 - 2iX^p_{r+1})} ,$$

which by a further change of variable from X^p_{r+1} to $Z \equiv X^p_{r+1} + i(A^r_1 + X^r_1)$ becomes

$$\sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p \, w^{(3)}(x_1 \ldots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A^p_{r+1} + B^p_{r+1})} \left[\prod_{k=r+1}^{p} \Gamma(a_k + b_k) \right]$$

$$\times \sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_r=0}^{\Delta_r} \left[\prod_{k=1}^{r} \binom{\Delta_k}{x_k} \right] \int_{C(X^r_1)} dZ \, \Gamma(A + X^r_1 + iZ) \, \Gamma(B + \Delta^r_1 - X^r_1 - iZ) \tag{2.36}$$

$$\times \frac{\Gamma(A - iZ) \, \Gamma(B + iZ) \, \Gamma(c + iZ) \, \Gamma(c - iZ) \, \Gamma(d + iZ) \, \Gamma(d - iZ)}{\Gamma(2iZ) \, \Gamma(-2iZ)} ,$$

where the contours $C(X^r_1)$, $X^r_1 = 0, 1, 2 \ldots \Delta^r_1$ run parallel to the real axis with imaginary part $i(A^r_1 + X^r_1)$. Taking into account (2.24) one finds that no poles of the integrand lie in the region bounded by and including $C(X^r_1)$ and the real axis. In this case each of the contours $C(X^r_1)$ can be deformed to the real axis and the integral can be brought outside of the multiple sums. The latter are then evaluated by theorem (2.23) resulting in

$$\sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \cdots \int_{-\infty}^{\infty} dx_p \, w^{(3)}(x_1 \ldots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A + B)} \left[\prod_{k=r+1}^{p} \Gamma(a_k + b_k) \right] \tag{2.37}$$

$$\times \int_{-\infty}^{\infty} dZ \frac{\Gamma(A+iZ) \, \Gamma(A-iZ) \, \Gamma(B+iZ) \, \Gamma(B-iZ) \, \Gamma(c+iZ) \, \Gamma(c-iZ) \, \Gamma(d+iZ) \, \Gamma(d-iZ)}{\Gamma(2iZ) \, \Gamma(-2iZ)} ,$$

which is proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Using this result in (2.37) then confirms the norm as given by (2.33) with $N = 0$. The biorthogonality relations (2.31) are then verified in the same manner as for the purely continuous family.[4] Also, the restriction $\mathrm{Re}(a_1, a_2 ... a_r) \neq 0, -1, -2, ...$ is removable from (2.31)–(2.34).

III. Multivariable biorthogonal Racah polynomials

The mixed type family (2.31), which is a limit case of (2.30), requires that at least one of the a–parameters have a positive real part. We consider a further generalization where all of the a–parameters have negative real parts. Choosing $r = p - 1$ in (2.31) we then define the parameter domain as

$$\begin{gathered}
\mathrm{Re}(a_1, a_2 ... a_p) < 0 \ , \quad \mathrm{Re}(b_1, b_2 ... b_p, c, d), \ \mathrm{Re}(A_1^{p-1} + c), \ \mathrm{Re}(A_1^{p-1} + d) > 0 \ , \\
2A, \ A + B, \ A + c, \ A + d, \ \mathrm{Re}(A + A_1^{p-1}), \ \mathrm{Re}(a_p), \ a_p + b_p \neq 0, -1, -2 ... \ , \\
a_k + b_k = -\Delta_k \, , \, k = 1, 2 ... p - 1 \ ,
\end{gathered} \tag{3.1}$$

for which the x_p contour C_p, which in (2.31) was on the real axis, is here deformed to pass underneath the increasing sequence $x_p = ia_p + ij_p \, , \, j_p = 0, 1, 2 ...$ and above the decreasing sequence $x_p = -iA - iA_1^{p-1} - ij \, , \, j = 0, 1, 2 ...$ while still passing above and below the remaining decreasing and increasing sequences, respectively.

To evaluate the norm of the weight function one begins with a change of variable from x_p to $Z \equiv x_p + i(A_1^{p-1} + X_1^{p-1})$ giving

$$\begin{aligned}
\sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \int_{C_p} dx_p w^{(3)}(x_1 ... x_p) &= \sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \left[\prod_{k=1}^{p-1} \binom{\Delta_k}{x_k} \right] \int_{C(X_1^{p-1})} dZ \, \Gamma(A + X_1^{p-1} + iZ) \\
\times \Gamma(B + \Delta_1^{p-1} - X_1^{p-1} - iZ) & \frac{\Gamma(A - iZ) \, \Gamma(B + iZ) \, \Gamma(c + iZ) \, \Gamma(c - iZ) \, \Gamma(d + iZ) \, \Gamma(d - iZ)}{\Gamma(2iZ) \, \Gamma(-2iZ)} \ ,
\end{aligned} \tag{3.2}$$

and then the contours $C(X_1^{p-1}), X_1^{p-1} = 0, 1, 2 ... \Delta_1^{p-1}$ are deformed back to the real axes but with indentations to pass underneath the increasing sequence $Z = iA + ij \, , \, j = 0, 1, 2 ...$ and above the decreasing sequence $Z = -iA - ij \, , \, j = 0, 1, 2 ...$ while still passing above and below the remaining decreasing and increasing sequences, respectively. The multiple sums can then be brought inside the integral and evaluated by theorem (2.23) resulting in

$$\sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \int_{C_p} dx_p \, w^{(3)}(x_1 \ldots x_p) = \frac{\Gamma(a_p + b_p)}{\Gamma(A+B)} \int_C dZ \, \Gamma(A+iZ)\Gamma(A-iZ) \tag{3.3}$$
$$\times \frac{\Gamma(B+iZ)\Gamma(B-iZ)\Gamma(c+iZ)\Gamma(c-iZ)\Gamma(d+iZ)\Gamma(d-iZ)}{\Gamma(2iZ)\Gamma(-2iZ)},$$

which is proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Substituting this result into (3.3) then yields the norm defined by (2.33) with $r = p-1$ and $N = 0$. The biorthogonality relations (2.31) then once again follow in the same manner as was proved for the purely continuous family[4] but with the inner product defined above or as expressed in (3.5).

Deforming C_p to the real axis plus closed loops about a finite number of poles and then evaluating the latter by the method of residues allows us to write schematically

$$\int_{C_p} dx_p = \int_{-\infty}^{\infty} dx_p + (2\pi i) \sum_{j_p=0}^{\mathrm{Re}\,(a_p)+j_p<0} \mathrm{res}(x_p = ia_p + ij_p) - (2\pi i) \sum_{j=0}^{\mathrm{Re}\,(A+A_1^{p-1})+j<0} \mathrm{res}(x_p = -iA - iA_1^{p-1} - ij) \,, \tag{3.4}$$

and then the inner product of the polynomials can be expressed as

$$P_n^{(3)} \cdot Q_m^{(3)} \equiv \sum_{x_1=0}^{\Delta_1} \cdots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \Bigg\{ \int_{-\infty}^{\infty} dx_p + (2\pi i) \sum_{j_p=0}^{\mathrm{Re}\,(a_p)+j_p<0} \mathrm{res}(x_p = ia_p + ij_p) \tag{3.5}$$
$$- (2\pi i) \sum_{j=0}^{\mathrm{Re}\,(A+A_1^{p-1})+j<0} \mathrm{res}(x_p = -iA - iA_1^{p-1} - ij) \Bigg\} w^{(3)}(x_1 \ldots x_p) P_n^{(3)}(x_1 \ldots x_p) Q_m^{(3)}(x_1 \ldots x_p) \,,$$

and similarly for $P_n^{(3)} \cdot \overline{P}_m^{(3)}$, $Q_n^{(3)} \cdot \overline{Q}_m^{(3)}$, and $\overline{P}_n^{(3)} \cdot \overline{Q}_m^{(3)}$.

Formula (3.5) has a limit to a purely discrete inner product. Take $a_p + b_p = -\Delta_p + \epsilon$, where Δ_p is a nonnegative integer, divide by $\Gamma(a_p + b_p) = \Gamma(-\Delta_p + \epsilon)$ and then take the limit $\epsilon \to 0$. The two purely discrete terms survive giving (writing x_p in place of j_p)

$$\lim_{\epsilon\to 0} P_n^{(3)}\cdot Q_m^{(3)}/\Gamma(-\Delta_k+\epsilon) = (2\pi)\sum_{x_1=0}^{\Delta_1}\cdots\sum_{x_p=0}^{\Delta_p}\left[\prod_{k=1}^{p}\binom{\Delta_k}{x_k}\right]\Gamma(2A+X)\,\Gamma(B-A-X)$$

$$\times\frac{\Gamma(c+A+X)\,\Gamma(c-A-X)\,\Gamma(d+A+X)\,\Gamma(d-A-X)}{\Gamma(2A+2X)\,\Gamma(-2A-2X)}$$

$$\times P_n^{(3)}(x_1\ldots x_{p-1},ia_p+ix_p)\,Q_m^{(3)}(x_1\ldots x_{p-1},ia_p+ix_p)$$

$$(3.6)$$

$$+\ (2\pi)\sum_{x_1=0}^{\Delta_1}\cdots\sum_{x_{p-1}=0}^{\Delta_{p+1}}\sum_{j=0}^{\Delta+X_1^{p-1}}\left[\prod_{k=1}^{p-1}\binom{\Delta_k}{x_k}\right]\frac{\Delta_p!}{(\Delta+X_1^{p-1}-j)!\,(j-X_1^{p-1})!}(-1)^{\Delta+\Delta_p}$$

$$\times\Gamma(2A+j)\,\Gamma(B-A+\Delta_1^{p-1}-j)\,\Gamma(c+A-X_1^{p-1}+j)\,\Gamma(c-A+X_1^{p-1}-j)$$

$$\times\frac{\Gamma(d+A-X_1^{p-1}+j)\,\Gamma(d-A+X_1^{p-1}-j)}{\Gamma(2A-2X_1^{p-1}+2j)\,\Gamma(-2A+2X_1^{p-1}-2j)}$$

$$\times P_n^{(3)}(x_1\ldots x_{p-1},-iA-iA_1^{p-1}-ij)\,Q_m^{(3)}(x_1\ldots x_{p-1},-iA-iA_1^{p-1}-ij)\ ,$$

and in turn one can show that these two terms are equal. To demonstrate this make a change of summation index in the second term, $j\to j+X_1^{p-1}$, substitute representations (1.15) and (1.17) for $P_n(x)$ and $Q_m(x)$, and then use theorem (2.23) to evaluate the $x_1, x_2\ldots x_{p-1}$ sums. This leaves only the j sum but if (2.23) is used again with a different choice of parameters this can be re-expressed as a multiple sum that is equal to the first term in (3.6). The inner product can then be taken as twice the first term

$$\lim_{\epsilon\to 0} P_n^{(3)}\cdot Q_m^{(3)}/\Gamma(-\Delta_k+\epsilon) = (4\pi)\sum_{x_1=0}^{\Delta_1}\cdots\sum_{x_p=0}^{\Delta_p}\left[\prod_{k=1}^{p}\binom{\Delta_k}{x_k}\right]\Gamma(2A+X)\,\Gamma(B-A-X)$$

$$\times\frac{\Gamma(c+A+X)\,\Gamma(c-A-X)\,\Gamma(d+A+X)\,\Gamma(d-A-X)}{\Gamma(2A+2X)\,\Gamma(-2A-2X)} \tag{3.7}$$

$$\times P_n^{(3)}(x_1\ldots x_{p-1},ia_p+ix_p)\,Q_m^{(3)}(x_1\ldots x_{p-1},ia_p+ix_p)\ ,$$

and similarly for $\overline{P}_n^{(3)}\cdot\overline{Q}_m^{(3)}$, $P_n^{(3)}\cdot\overline{P}_m^{(3)}$, and $Q_n^{(3)}\cdot\overline{Q}_m^{(3)}$. The biorthogonality relations (2.31) then yield in this limit, with a change in notation,

$$R_n \cdot W_m \equiv \sum_{\{x_k\}} \rho(x_1 \ldots x_p) R_n(x_1 \ldots x_p) W_m(x_1 \ldots x_p) \ ,$$

$$R_n \cdot W_m = \overline{R}_n \cdot \overline{W}_m = \lambda_n \prod_{k=1}^{p} \delta_{n_k m_k} \ , \qquad R_n \cdot \overline{R}_m = W_n \cdot \overline{W}_m = 0 \ , \quad \text{if} \quad N \neq M \ , \tag{3.8}$$

where $R_n(x)$, $\overline{R}_n(x)$, $W_n(x)$, and $\overline{W}_n(x)$ are the multivariable biorthogonal Racah polynomials and $\{x_k\}$ denotes the p discrete variables $x_1, x_2 \ldots x_p$. After a customary redefinition of the parameters to $\alpha_1, \alpha_2 \ldots \alpha_p, \beta, \delta, \gamma$ ($\alpha \equiv \sum_{k=1}^{p} \alpha_k$) this family is given by

$$\rho(x_1 \ldots x_p) = \left[\prod_{k=1}^{p} \frac{\Gamma(x_k + \alpha_k + 1)}{\Gamma(\alpha_k + 1)\ x_k!} \right] \frac{(\gamma + \delta + 1)_X\ (\gamma/2 + \delta/2 + 3/2)_X}{(\gamma/2 + \delta/2 + 1/2)_X\ (\gamma + \delta - \alpha - p + 2)_X} \times \frac{(\beta + \delta + 1)_X\ (\gamma + 1)_X}{(\gamma - \beta + 1)_X\ (\delta + 1)_X} \ , \tag{3.9}$$

$$\lambda_n = \left[\prod_{k=1}^{p} n_k!\ (\alpha_k + 1)_{n_k} \right] (\beta + 1)_N (\gamma + 1)_N (\alpha - \delta + p)_N (\alpha + \beta - \gamma + p)_N (\beta + \delta + 1)_N$$
$$\times \frac{(N + \alpha + \beta + p)_N}{(\alpha + \beta + p + 1)_{2N}} \frac{\Gamma(\gamma + \delta - \alpha - p + 2)\ \Gamma(-\beta - \alpha - p)\ \Gamma(\gamma - \beta + 1)\ \Gamma(\delta + 1)}{\Gamma(\gamma + \delta + 2)\ \Gamma(-\beta)\ \Gamma(\gamma - \beta - \alpha - p + 1)\ \Gamma(\delta - \alpha - p + 1)} \ , \tag{3.10}$$

$$R_n(x_1 \ldots x_p) = \left[\prod_{k=1}^{p} (\alpha_k + 1)_{n_k} \right] (\beta + \delta + 1)_N\ (\gamma + 1)_N \times F^{2:2;\ldots;2}_{2:1;\ldots;1} \left(\begin{matrix} N + \alpha + \beta + p, X + \gamma + \delta + 1 : -n_1, -x_1; \ldots; -n_p, -x_p \\ \beta + \delta + 1, \gamma + 1 : \alpha_1 + 1; \ldots; \alpha_p + 1 \end{matrix} \right) , \tag{3.11}$$

$$\overline{R}_n(x_1 \ldots x_p) = \left[\prod_{k=1}^{p} (\alpha_k + 1)_{n_k} \right] (\alpha + \beta - \gamma + p)_N\ (\alpha - \delta + p)_N \times F^{2:2;\ldots;2}_{2:1;\ldots;1} \left(\begin{matrix} N + \alpha + \beta + p, -X + \alpha - \gamma - \delta + p - 1 : -n_1, x_1 + \alpha_1 + 1; \ldots; -n_p, x_p + \alpha_p + 1 \\ \alpha + \beta - \gamma + p, \alpha - \delta + p : \alpha_1 + 1; \ldots; \alpha_p + 1 \end{matrix} \right) , \tag{3.12}$$

$$W_n(x_1 \ldots x_p) = \left[\prod_{k=1}^{p} (\alpha_k + 1)_{n_k} \right] (X + \beta + \delta + 1)_N\ (X + \gamma + 1)_N \times F^{2:2;\ldots;2}_{2:1;\ldots;1} \left(\begin{matrix} -N - \beta, -X + \alpha - \gamma - \delta + p - 1 : -n_1, -x_1; \ldots; -n_p, -x_p \\ -N - X - \beta - \delta, -N - X - \gamma : \alpha_1 + 1; \ldots; \alpha_p + 1 \end{matrix} \right) , \tag{3.13}$$

$$\overline{W}_n(x_1 ... x_p) = \left[\prod_{k=1}^{p} (\alpha_k + 1)_{n_k} \right] (-X + \beta - \gamma)_N \, (-X - \delta)_N \tag{3.14}$$

$$\times F^{2:2;...;2}_{2:1;...;1} \begin{pmatrix} -N - \beta, X + \gamma + \delta + 1 : -n_1, x_1 + \alpha_1 + 1; ...; -n_p, x_p + \alpha_p + 1 \\ -N + X + \gamma - \beta + 1, -N + X + \delta + 1 : \alpha_1 + 1; ...; \alpha_p + 1 \end{pmatrix},$$

which according to the present derivation satisfy relations (3.8) when $\alpha_k + 1 = -\Delta_k$, $k = 1, 2 ... p$, and the $\{x_k\}$ sum is over the region $0 \le x_k \le \Delta_k$, $k = 1, 2 ... p$. As an independent evaluation of the norm of the weight function one uses theorem (2.23) to perform the $x_1 ... x_{p-1}$ summations at constant X and then (2.15) to evaluate the X sum resulting in (3.10) with $N = 0$. The biorthogonality relations (3.8) can then be independently verified in the same manner as for the purely continuous family[4] apart from a redefinition of the parameters. In this way one finds that (3.8) are also valid for $\beta + \delta + 1$ or $\gamma + 1 = -\Delta_{p+1}$, where Δ_{p+1} is another nonnegative integer, and in this case the $\{x_k\}$ sum is over the region $0 \le X \le \Delta_{p+1}$. Another possibility is to have a combination of $\beta + \delta + 1$ or $\gamma + 1 = -\Delta_{p+1}$ and only a subset of the α–parameters satisfying $\alpha_k + 1 = -\Delta_k$. These different possibilities and the corresponding regions of the $\{x_k\}$ sums are summarized below

$$\begin{aligned} &\alpha_k + 1 = -\Delta_k, \, k = 1, 2 ... p, && 0 \le x_k \le \Delta_k, \\ &\beta + \delta + 1 \text{ or } \gamma + 1 = -\Delta_{p+1}, && 0 \le X \le \Delta_{p+1}, \\ &\beta + \delta + 1 \text{ or } \gamma + 1 = -\Delta_{p+1} \text{ and } \alpha_k + 1 = -\Delta_k, \, k \in S \subseteq (1, 2 ... p), \\ & && (0 \le x_k \le \Delta_k) \cap (0 \le X \le \Delta_{p+1}). \end{aligned} \tag{3.15}$$

In the special case of a single variable all four families of polynomials (3.11)–(3.14) reduce, through a transformation formula satisfied by the ${}_4F_3(1)$ hypergeometric function, to (1.9) and the biorthogonality relations (3.8) reduce to the single orthogonality relation (1.10).

Alternatively one could have chosen the inner product to be twice the second term in (3.6) which leads to a different but within a change of variables equivalent family.

Another inequivalent multivariable generalization of the Racah polynomials has been studied by Gustafson.[7] These are closely related to the so called $U(n)$ multivariable hypergeometric series introduced by Holman, Biedenharn, and Louck[8] and Holman,[9] and which have been q-extended by Milne.[10−15] Gustafson's polynomials are associated with a different weight function than (3.9) and are orthogonal as opposed to biorthogonal. The difference in these two families is a reflection of the distinct hypergeometric series to which they are related, the Kampé de Fériet series (1.19) in the present case and the $U(n)$ series in Gustafson's case.

IV. Multivariable biorthogonal Hahn and dual Hahn polynomials

In analogy with the single variable case there is an interesting limit to a family of multivariable Hahn polynomials. These are obtained upon dividing the Racah polynomials by δ^N and taking the limit $\delta \to \infty$. The first two families $R_n(x)$ and $\overline{R}_n(x)$ limit to the same Hahn polynomials $H_n(x)$ while $W_n(x)$ and $\overline{W}_n(x)$ limit to the same biorthogonal counterparts $\overline{H}_n(x)$,

$$\begin{aligned}
\lim_{\delta\to\infty} \delta^{-N} R_n(x) &= \lim_{\delta\to\infty} \delta^{-N} \overline{R}_n(x) = \left[\prod_{k=1}^{p} (\alpha_k + 1)_{n_k}\right] (\gamma+1)_N \, H_n(x) \;, \\
\lim_{\delta\to\infty} \delta^{-N} W_n(x) &= \lim_{\delta\to\infty} \delta^{-N} \overline{W}_n(x) = \left[\prod_{k=1}^{p} (\alpha_k + 1)_{n_k}\right] (\gamma+1)_N \, \overline{H}_n(x) \;, \qquad (4.1) \\
\lim_{\delta\to\infty} \rho(x) &= \left[\prod_{k=1}^{p} \frac{\Gamma(x_k+\alpha_k+1)}{\Gamma(\alpha_k+1)\; x_k!}\right] \frac{(\gamma+1)_X}{(\gamma-\beta+1)_X} \;,
\end{aligned}$$

where $H_n(x)$ and $\overline{H}_n(x)$ are given by

$$\begin{aligned}
H_n(x) &= F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} N+\alpha+\beta+p \,:\, -n_1, -x_1; \ldots; -n_p, -x_p \\ \gamma+1 \,:\, \alpha_1+1; \ldots; \alpha_p+1 \end{matrix}\right) \;, \\
&= (-1)^N \frac{(\alpha+\beta-\gamma+p)_N}{(\gamma+1)_N} F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} N+\alpha+\beta+p \,:\, -n_1, x_1+\alpha_1+1; \ldots; -n_p, x_p+\alpha_p+1 \\ \alpha+\beta-\gamma+p \,:\, \alpha_1+1; \ldots; \alpha_p+1 \end{matrix}\right) , \qquad (4.2) \\
\overline{H}_n(x) &= \frac{(X+\gamma+1)_N}{(\gamma+1)_N} F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} -N-\beta \,:\, -n_1, -x_1; \ldots; -n_p, -x_p \\ -N-X-\gamma \,:\, \alpha_1+1; \ldots; \alpha_p+1 \end{matrix}\right) \;, \\
&= (-1)^N \frac{(-X+\beta-\gamma)_N}{(\gamma+1)_N} F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} -N-\beta \,:\, -n_1, x_1+\alpha_1+1; \ldots; -n_p, x_p+\alpha_p+1 \\ -N+X+\gamma-\beta+1 \,:\, \alpha_1+1; \ldots; \alpha_p+1 \end{matrix}\right) .
\end{aligned}$$

These satisfy the biorthogonality relations

$$H_n \cdot \overline{H}_m \equiv \sum_{\{x_k\}} \left[\prod_{k=1}^{p} \frac{\Gamma(x_k + \alpha_k + 1)}{\Gamma(\alpha_k + 1)\ x_k!} \right] \frac{(\gamma+1)_X}{(\gamma-\beta+1)_X} H_n(x) \overline{H}_m(x) \ ,$$

$$H_n \cdot \overline{H}_m = \lambda_n \prod_{k=1}^{p} \delta_{n_k m_k} \ , \qquad H_n \cdot H_m = \overline{H}_n \cdot \overline{H}_m = 0 \quad \text{if} \quad N \neq M \ , \tag{4.3}$$

$$\lambda_n = (-1)^N \left[\prod_{k=1}^{p} n_k!/(\alpha_k+1)_{n_k} \right] (\alpha+\beta-\gamma+p)_N \frac{(\beta+1)_N\ (N+\alpha+\beta+p)_N}{(\gamma+1)_N\ (\alpha+\beta+p+1)_{2N}} \times \frac{\Gamma(-\beta-\alpha-p)\Gamma(\gamma-\beta+1)}{\Gamma(-\beta)\Gamma(\gamma-\beta-\alpha-p+1)} \ ,$$

where the $\{x_k\}$ sum is over one of the regions

$$\begin{aligned} &\alpha_k + 1 = -\Delta_k,\ k = 1, 2 \ldots p \ , && 0 \leq x_k \leq \Delta_k \ , \\ &\gamma + 1 = -\Delta_{p+1} \ , && 0 \leq X \leq \Delta_{p+1} \ , \\ &\gamma + 1 = -\Delta_{p+1} \text{ and } \alpha_k + 1 = -\Delta_k,\ k \in S \subseteq (1, 2 \ldots p) \ , && \\ & && (0 \leq x_k \leq \Delta_k) \cap (0 \leq X \leq \Delta_{p+1}) \ . \end{aligned} \tag{4.4}$$

These polynomials have already been discussed in detail[16] for the specific case $\gamma + 1 = -\Delta_{p+1}$ and the equivalence of each pair of representations in (4.2) has been demonstrated. Among other interesting properties these polynomials possess discrete Rodrigues formulas.

Another important limit not yet studied are the multivariable biorthogonal dual Hahn polynomials. These result upon dividing the Racah polynomials by β^N and taking the limit $\beta \to \infty$. In this case $R_n(x)$ and $\overline{W}_n(x)$ limit to the same dual Hahn family $D_n(x)$ while $\overline{R}_n(x)$ and $W_n(x)$ limit to the same biorthogonal counterparts $\overline{D}_n(x)$,

$$\lim_{\beta\to\infty} \beta^{-N} R_n(x) = \lim_{\beta\to\infty} \beta^{-N} \overline{W}_n(x) = \left[\prod_{k=1}^{p} (\alpha_k+1)_{n_k} \right] (\gamma+1)_N D_n(x) \ ,$$

$$\lim_{\beta\to\infty} \beta^{-N} \overline{R}_n(x) = \lim_{\beta\to\infty} \beta^{-N} W_n(x) = \left[\prod_{k=1}^{p} (\alpha_k+1)_{n_k} \right] (\gamma+1)_N \overline{D}_n(x) \ , \tag{4.5}$$

$$\lim_{\beta\to\infty} \rho(x) = \left[\prod_{k=1}^{p} \frac{\Gamma(x_k+\alpha_k+1)}{\Gamma(\alpha_k+1)\ x_k!} \right] \frac{(\gamma+\delta+1)_X (\gamma/2+\delta/2+3/2)_X (\gamma+1)_X}{(\gamma/2+\delta/2+1/2)_X (\gamma+\delta-\alpha-p+2)_X (\delta+1)_X} (-1)^X,$$

where $D_n(x)$ and $\overline{D}_n(x)$ are given by

$$
\begin{aligned}
D_n(x) &= F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} X+\gamma+\delta+1 : -n_1, -x_1; \ldots; -n_p, -x_p \\ \gamma+1 : \alpha_1+1; \ldots; \alpha_p+1 \end{matrix} \right) \\
&= \frac{(-X-\delta)_N}{(\gamma+1)_N} F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} X+\gamma+\delta+1 : -n_1, x_1+\alpha_1+1; \ldots; -n_p, x_p+\alpha_p+1 \\ -N+X+\delta+1 : \alpha_1+1; \ldots; \alpha_p+1 \end{matrix} \right) ,
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
\overline{D}_n(x) &= \frac{(\alpha-\delta+p)_N}{(\gamma+1)_N} F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} -X+\alpha-\gamma-\delta+p-1 : -n_1, x_1+\alpha_1+1; \ldots; -n_p, x_p+\alpha_p+1 \\ \alpha-\delta+p : \alpha_1+1; \ldots; \alpha_p+1 \end{matrix} \right) \\
&= \frac{(X+\gamma+1)_N}{(\gamma+1)_N} F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} -X+\alpha-\gamma-\delta+p-1 : -n_1, -x_1; \ldots; -n_p, -x_p \\ -N-X-\gamma : \alpha_1+1; \ldots; \alpha_p+1 \end{matrix} \right) .
\end{aligned}
$$

The four biorthogonality relations satisfied by the Racah polynomials, in this limit, imply the single relation

$$
\begin{aligned}
&\sum_{\{x_k\}} \left[\prod_{k=1}^{p} \frac{\Gamma(x_k+\alpha_k+1)}{\Gamma(\alpha_k+1)\, x_k!} \right] \frac{(\gamma+\delta+1)_X (\gamma/2+\delta/2+3/2)_X (\gamma+1)_X (-1)^X}{(\gamma/2+\delta/2+1/2)_X (\gamma+\delta-\alpha-p+2)_X (\delta+1)_X} D_n(x) \overline{D}_m(x) \\
&= \left[\prod_{k=1}^{p} n_k!/(\alpha_k+1)_{n_k} \right] \frac{(\alpha-\delta+p)_N}{(\gamma+1)_N} \frac{\Gamma(\gamma+\delta-\alpha-p+2)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+2)\Gamma(\delta-\alpha-p+1)} \prod_{k=1}^{p} \delta_{n_k m_k} ,
\end{aligned}
\tag{4.7}
$$

where the $\{x_k\}$ sum is again over one of the regions given in (4.4).

To demonstrate the equivalence of the two representations of $D_n(x)$ we first use (2.23) do deduce the identity

$$
\frac{\Gamma(X+\delta+1)}{\Gamma(X+\delta+1-N+J)} = \sum_{\{\ell_k\}} \left[\prod_{k=1}^{p} \binom{n_k - j_k}{\ell_k} \right] \frac{\Gamma(N-L+X+\delta+\gamma+1)}{\Gamma(J+X+\delta+\gamma+1)} \frac{\Gamma(L-N-\gamma)}{\Gamma(-N-\gamma)} ,
\tag{4.8}
$$

where j_k denotes the summation indices in the Kampé de Fériet hypergeometric series (1.19). This is substituted into the second expression for $D_n(x)$, the $\{j_k\}$ and $\{\ell_k\}$ sums are interchanged, and (2.23) is used again to evaluate the former. Reversing the remaining $\{\ell_k\}$ sums by transforming $\ell_k \to n_k - \ell_k$ then yields the first expression for $D_n(x)$. For the $\overline{D}_n(x)$ representations one begins instead with the following identity also deduced from (2.23)

$$\frac{\Gamma(N+\alpha-\delta+p)}{\Gamma(J+\alpha-\delta+p)}=\sum_{\{\ell_k\}}\left[\prod_{k=1}^{p}\binom{n_k-j_k}{\ell_k}\right]\frac{\Gamma(N-L+\alpha+p-X-\delta-\gamma-1)}{\Gamma(J+\alpha+p-X-\delta-\gamma-1)} \times\frac{\Gamma(L+X+\gamma+1)}{\Gamma(X+\gamma+1)}\ . \tag{4.9}$$

Substituting this into the first expression for $\overline{D}_n(x)$ and proceeding as before then yields the second representation.

There are also mixed type counterparts to these Hahn and dual Hahn polynomials which are obtained as limit cases of the mixed type Wilson polynomials. In (2.19)–(2.22) put

$$\begin{aligned} &a_k=a_k'+\tfrac{1}{2}iw_k\,,\quad b_k=b_k'-\tfrac{1}{2}iw_k\,,\quad k=1,2\ldots p\ ,\\ &x_k=x_k'\quad k=1,2\ldots r\ ,\qquad x_k=x_k'-\tfrac{1}{2}w_k\,,\quad k=r+1\ldots p\ ,\\ &c=c'+\tfrac{1}{2}iW\,,\quad d=d'-\tfrac{1}{2}iW\,,\quad W\equiv\sum_{k=1}^{p}w_k\ , \end{aligned} \tag{4.10}$$

divide the polynomials by $(iW)^N$ and then take the limit $W\to\infty$ (and drop the primes). The mixed Wilson families $P_n^{(2)}(x)$ and $\overline{P}_n^{(2)}(x)$ both limit to the same mixed Hahn polynomials while $Q_n^{(2)}(x)$ and $\overline{Q}_n^{(2)}(x)$ limit to the same biorthogonal counterparts

$$\begin{aligned} &\lim_{W\to\infty}(iW)^{-N}P_n^{(2)}(x)=\lim_{W\to\infty}(iW)^{-N}\overline{P}_n^{(2)}(x)=H_n^{(2)}(x)\ ,\\ &\lim_{W\to\infty}(iW)^{-N}Q_n^{(2)}(x)=\lim_{W\to\infty}(iW)^{-N}\overline{Q}_n^{(2)}(x)=\overline{H}_n^{(2)}(x)\ ,\\ &\lim_{W\to\infty}w^{(2)}(x)=\left[\prod_{k=1}^{r}\binom{\Delta_k}{x_k}\right]\left[\prod_{k=r+1}^{p}\Gamma(a_k+ix_k)\Gamma(b_k-ix_k)\right]\\ &\qquad\times\Gamma(A_1^r+d+X_1^r-iX_{r+1}^p)\Gamma(B_1^r+c+\Delta_1^r-X_1^r+iX_{r+1}^p)\ , \end{aligned} \tag{4.11}$$

where $H_n^{(2)}(x)$ and $\overline{H}_n^{(2)}(x)$ are given by

$$H_n^{(2)}(x)=\left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](A+d)_N\,F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix}N+A+B+c+d-1\,:\,-n_1\,,\,A_1^r+d+X_1^r-iX_{r+1}^p\,;\,-n_2\,,\,-x_2\,;\ldots;\,-n_r\,,\,-x_r\,;\,-n_{r+1}\,,\,a_{r+1}+ix_{r+1}\,;\ldots;\,-n_p\,,\,a_p+ix_p\\ A+d\,:\,a_1+b_1\,;\,-\Delta_2\,;\ldots;\,-\Delta_r\,;\,a_{r+1}+b_{r+1}\,;\ldots;\,a_p+b_p\end{matrix}\right)$$

$$= (-1)^N \left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](B+c)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} N+A+B+c+d-1 : -n_1, B_1^r+c+\Delta_1^r \\ B+c : a_1+b_1; \end{matrix}\right.$$

$$\left.\begin{matrix} -X_1^r+iX_{r+1}^p; -n_2, -\Delta_2+x_2; \ldots; -n_r, -\Delta_r+x_r; -n_{r+1}, b_{r+1}-ix_{r+1}; \ldots; -n_p, b_p-ix_p \\ -\Delta_2; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right), \tag{4.12}$$

$$\overline{H}_n^{(2)}(x) = (-1)^N \left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](-\Delta_1+x_1)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} \Delta_1-N+1 : -n_1, B_1^r+c+\Delta_1^r \\ \Delta_1-N-x_1+1 : a_1+b_1; \end{matrix}\right.$$

$$\left.\begin{matrix} -X_1^r+iX_{r+1}^p; -n_2, -\Delta_2+x_2; \ldots; -n_r, -\Delta_r+x_r; -n_{r+1}, b_{r+1}-ix_{r+1}; \ldots; -n_p, b_p-ix_p \\ -\Delta_2; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right)$$

$$= \left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](-x_1)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} \Delta_1-N+1 : -n_1, A_1^r+d+X_1^r-iX_{r+1}^p; \\ -N+x_1+1 : a_1+b_1; \end{matrix}\right.$$

$$\left.\begin{matrix} -n_2, -x_2; \ldots; -n_r, -x_r; -n_{r+1}, a_{r+1}+ix_{r+1}; \ldots; -n_p, a_p+ix_p \\ -\Delta_2; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right),$$

with $c+d=-\Delta_1$ and $a_k+b_k=-\Delta_k$, $k=2,3\ldots r$. These satisfy

$$H_n^{(2)}\cdot\overline{H}_m^{(2)} \equiv \int_{-\infty}^{\infty}dx_p\ldots\int_{-\infty}^{\infty}dx_{r+1}\sum_{x_r=0}^{\Delta_r}\cdots\sum_{x_1=0}^{\Delta_1}\left[\prod_{k=1}^{r}\binom{\Delta_k}{x_k}\right]\left[\prod_{k=r+1}^{p}\Gamma(a_k+ix_k)\Gamma(b_k-ix_k)\right]$$

$$\times\Gamma(A_1^r+d+X_1^r-iX_{r+1}^p)\Gamma(B_1^r+c+\Delta_1^r-X_1^r+iX_{r+1}^p)H_n^{(2)}(x)\overline{H}_m^{(2)}(x),$$

$$H_n^{(2)}\cdot\overline{H}_m^{(2)} = \lambda_n^{(2)}\prod_{k=1}^{p}\delta_{n_k m_k}, \qquad H_n^{(2)}\cdot H_m^{(2)} = \overline{H}_n^{(2)}\cdot\overline{H}_m^{(2)} = 0 \quad \text{if} \quad N\neq M, \tag{4.13}$$

$$\lambda_n^{(2)} = (2\pi)^{p-r}\Gamma(n_1+a_1+b_1)\,n_1!\left[\prod_{k=2}^{r}(-\Delta_k)_{n_k}\,n_k!\right]\left[\prod_{k=r+1}^{p}\Gamma(n_k+a_k+b_k)\,n_k!\right]$$

$$\times\frac{\Delta_1!}{(\Delta_1-N)!}\frac{\Gamma(N+A+d)\Gamma(N+B+c)}{(2N+A+B+c+d-1)\Gamma(N+A+B+c+d-1)},$$

and the equivalence of each pair of representations in (4.12) follows from (4.2) upon a redefinition of the parameters and a change of variables.

Another mixed type Hahn family is obtained from (2.31)–(2.34) using the same limiting procedure. That is, transform the parameters and variables as in (4.10) and then take the limit $W\to\infty$. This yields,

$$\lim_{W\to\infty}(iW)^{-N}P_n^{(3)}(x) = \lim_{W\to\infty}(iW)^{-N}\overline{P}_n^{(3)}(x) = H_n^{(3)}(x) \ ,$$

$$\lim_{W\to\infty}(iW)^{-N}Q_n^{(3)}(x) = \lim_{W\to\infty}(iW)^{-N}\overline{Q}_n^{(3)}(x) = \overline{H}_n^{(3)}(x) \ , \tag{4.14}$$

$$\lim_{W\to\infty} w^{(3)}(x)/\Gamma(A'+c'+iW)\Gamma(B'+d'-iW) = \left[\prod_{k=1}^{r}\binom{\Delta_k}{x_k}\right]\left[\prod_{k=r+1}^{p}\Gamma(a_k+ix_k)\Gamma(b_k-ix_k)\right]$$
$$\times\Gamma(A_1^r+d+X_1^r-iX_{r+1}^p)\Gamma(B_1^r+c+\Delta_1^r-X_1^r+iX_{r+1}^p) \ ,$$

where the polynomials are given by

$$H_n^{(3)}(x) = \left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](A+d)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} N+A+B+c+d-1 : -n_1, -x_1; \\ A+d : -\Delta_1; \end{matrix}\right.$$
$$\left.\begin{matrix} -n_2, -x_2; \ldots; -n_r, -x_r; -n_{r+1}, a_{r+1}+ix_{r+1}; \ldots; -n_p, a_p+ix_p \\ -\Delta_2; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right)$$

$$= (-1)^N\left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](B+c)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} N+A+B+c+d-1 : -n_1, -\Delta_1+x_1; \\ B+c : -\Delta_1; \end{matrix}\right.$$
$$\left.\begin{matrix} -n_2, -\Delta_2+x_2; \ldots; -n_r, -\Delta_r+x_r; -n_{r+1}, b_{r+1}-ix_{r+1}; \ldots; -n_p, b_p-ix_p \\ -\Delta_2; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right), \tag{4.15}$$

$$\overline{H}_n^{(3)}(x) = \left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](d+A_1^r+X_1^r-iX_{r+1}^p)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} -N-c-d+1 : \\ -N-d-A_1^r-X_1^r \end{matrix}\right.$$
$$\left.\begin{matrix} -n_1, -x_1; \ldots; -n_r, -x_r; -n_{r+1}, a_{r+1}+ix_{r+1}; \ldots; -n_p, a_p+ix_p \\ +iX_{r+1}^p+1 : -\Delta_1; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right)$$

$$= (-1)^N\left[\prod_{k=1}^{p}(a_k+b_k)_{n_k}\right](c-A_1^r-X_1^r+iX_{r+1}^p)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} -N-c-d+1 : \\ -N-c+A_1^r+X_1^r \end{matrix}\right.$$
$$\left.\begin{matrix} -n_1, -\Delta_1+x_1; \ldots; -n_r, -\Delta_r+x_r; -n_{r+1}, b_{r+1}-ix_{r+1}; \ldots; -n_p, b_p-ix_p \\ -iX_{r+1}^p+1 : -\Delta_1; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right),$$

with $a_k+b_k = -\Delta_k$, $k = 1, 2 \ldots r$, and these satisfy

$$H_n^{(3)} \cdot \overline{H}_m^{(3)} \equiv \int_{-\infty}^{\infty} dx_p \cdots \int_{-\infty}^{\infty} dx_{r+1} \sum_{x_r=0}^{\Delta_r} \cdots \sum_{x_1=0}^{\Delta_1} \left[\prod_{k=1}^{r} \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^{p} \Gamma(a_k + ix_k)\Gamma(b_k - ix_k) \right]$$

$$\times \Gamma(A_1^r + d + X_1^r - iX_{r+1}^p)\Gamma(B_1^r + c + \Delta_1^r - X_1^r + iX_{r+1}^p) H_n^{(3)}(x)\overline{H}_m^{(3)}(x) ,$$

$$H_n^{(3)} \cdot \overline{H}_m^{(3)} = \lambda_n^{(3)} \prod_{k=1}^{p} \delta_{n_k m_k} , \qquad H_n^{(3)} \cdot H_m^{(3)} = \overline{H}_n^{(3)} \cdot \overline{H}_m^{(3)} = 0 \quad \text{if} \quad N \neq M , \tag{4.16}$$

$$\lambda_n^{(3)} = (2\pi)^{p-r} \left[\prod_{k=1}^{r} (-\Delta_k)_{n_k} \, n_k! \right] \left[\prod_{k=r+1}^{p} \Gamma(n_k + a_k + b_k) \, n_k! \right] (-1)^N$$

$$\times \frac{\Gamma(N + A + d)\Gamma(N + B + c)\Gamma(N + c + d)}{(2N + A + B + c + d - 1)\Gamma(N + A + B + c + d - 1)} .$$

Notice that the weight function in (4.16) is equivalent to that of the first mixed type Hahn family (4.13); the two families of polynomials are however distinct. The equivalence of each pair of representations in (4.15) again follows from (4.2) upon a redefinition of the parameters and a change of variables.

The analogous mixed type dual Hahn family is obtained from (2.31)–(2.34) in the limit $d \to \infty$. In this case $P_n^{(3)}(x)$ and $\overline{Q}_n^{(3)}(x)$ limit to the same dual Hahn polynomials while $\overline{P}_n^{(3)}(x)$ and $Q_n^{(3)}(x)$ limit to the same biorthogonal counterparts

$$\lim_{d \to \infty} d^{-N} P_n^{(3)}(x) = \lim_{d \to \infty} d^{-N} \overline{Q}_n^{(3)}(x) = D_n^{(3)}(x) ,$$

$$\lim_{d \to \infty} d^{-N} \overline{P}_n^{(3)}(x) = \lim_{d \to \infty} d^{-N} Q_n^{(3)}(x) = \overline{D}_n^{(3)}(x) , \tag{4.17}$$

$$\lim_{d \to \infty} \frac{w^{(3)}(x)}{\Gamma^2(d)} = \rho^{(3)}(x) ,$$

where

$$\rho^{(3)}(x) = \left[\prod_{k=1}^{r} \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^{p} \Gamma(a_k + ix_k) \, \Gamma(b_k - ix_k) \right] \Gamma(A + A_1^r + X_1^r - iX_{r+1}^p)$$

$$\times \Gamma(B + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \, \frac{\Gamma(c + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \, \Gamma(c + A_1^r + X_1^r - iX_{r+1}^p)}{\Gamma(2B_1^r + 2\Delta_1^r - 2X_1^r + 2iX_{r+1}^p) \, \Gamma(2A_1^r + 2X_1^r - 2iX_{r+1}^p)},$$

$$D_n^{(3)}(x) = \left[\prod_{k=1}^{r}(-\Delta_k)_{n_k}\right]\left[\prod_{k=r+1}^{p}(a_k+b_k)_{n_k}\right](A+c)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} A+A_1^r+X_1^r-iX_{r+1}^p : \\ A+c : \end{matrix}\right.$$

$$\left.\begin{matrix} -n_1, -x_1; \ldots; -n_r, -x_r; -n_{r+1}, a_{r+1}+ix_{r+1}; \ldots; -n_p, a_p+ix_p \\ -\Delta_1; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right)$$

$$= \left[\prod_{k=1}^{r}(-\Delta_k)_{n_k}\right]\left[\prod_{k=r+1}^{p}(a_k+b_k)_{n_k}\right](c-A_1^r-X_1^r+iX_{r+1}^p)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} A+A_1^r+X_1^r \\ -N-c+A_1^r \end{matrix}\right.$$

$$\left.\begin{matrix} -iX_{r+1}^p : -n_1, -\Delta_1+x_1; \ldots; -n_r, -\Delta_r+x_r; -n_{r+1}, b_{r+1}-ix_{r+1}; \ldots; -n_p, b_p-ix_p \\ +X_1^r-iX_{r+1}^p+1 : -\Delta_1; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right),$$

(4.18)

$$\overline{D}_n^{(3)}(x) = \left[\prod_{k=1}^{r}(-\Delta_k)_{n_k}\right]\left[\prod_{k=r+1}^{p}(a_k+b_k)_{n_k}\right](B+c)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} B-A_1^r-X_1^r+iX_{r+1}^p : \\ B+c : \end{matrix}\right.$$

$$\left.\begin{matrix} -n_1, -\Delta_1+x_1; \ldots; -n_r, -\Delta_r+x_r; -n_{r+1}, b_{r+1}-ix_{r+1}; \ldots; -n_p, b_p-ix_p \\ -\Delta_1; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right)$$

$$= \left[\prod_{k=1}^{r}(-\Delta_k)_{n_k}\right]\left[\prod_{k=r+1}^{p}(a_k+b_k)_{n_k}\right](c+A_1^r+X_1^r-iX_{r+1}^p)_N F^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} B-A_1^r-X_1^r \\ -N-c-A_1^r \end{matrix}\right.$$

$$\left.\begin{matrix} +iX_{r+1}^p : -n_1, -x_1; \ldots; -n_r, -x_r; \ldots; -n_{r+1}, a_{r+1}+ix_{r+1}; \ldots; -n_p, a_p+ix_p \\ -X_1^r+iX_{r+1}^p+1 : -\Delta_1; \ldots; -\Delta_r; a_{r+1}+b_{r+1}; \ldots; a_p+b_p \end{matrix}\right),$$

with $a_k+b_k = -\Delta_k$, $k = 1, 2 \ldots r$. These satisfy the biorthogonality relation

$$\int_{-\infty}^{\infty} dx_p \cdots \int_{-\infty}^{\infty} dx_{r+1} \sum_{x_r=0}^{\Delta_r} \cdots \sum_{x_1=0}^{\Delta_1} \rho^{(3)}(x) D_n^{(3)}(x) \overline{D}_m^{(3)}(x) = 2(2\pi)^{p-r}\left[\prod_{k=1}^{r}(-\Delta_k)_{n_k}\, n_k!\right] \tag{4.19}$$

$$\times \left[\prod_{k=r+1}^{p}\Gamma(n_k+a_k+b_k)\, n_k!\right]\Gamma(N+A+c)\Gamma(N+B+c)\prod_{k=1}^{p}\delta_{n_k m_k},$$

and the equivalence of each pair of representations in (4.18) follows from (4.6) upon a redefinition of the parameters and a change of variables.

The purely continuous multivariable biorthogonal Hahn[17] and dual Hahn[4] polynomials are also known.

Acknowledgments

I would like to thank E.G. Kalnins, W. Miller, Jr., and D. Stanton for stimulating conversations. This research was supported in part by the Institute of Mathematics and its Applications (IMA) with funds provided by the National Science Foundation. I also thank the staff of the IMA for their warm hospitality during my long term visit when part of this work was done. This research was also supported by the Natural Sciences and Engineering Research Council of Canada and by the United States Department of Energy.

References

[1] J.A. Wilson, SIAM J. Math. Anal. **11**, 690 (1980).

[2] R. Askey and J. Wilson, SIAM J. Math. Anal. **13**, 651 (1982).

[3] W.N. Bailey, *Generalized Hypergeometric Series*, Cambridge Press, London (1935).

[4] M.V. Tratnik, J. Math. Phys. to appear.

[5] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood, New York, 1985, p.38.

[6] P. Appell and J. Kampé de Fériet, *Fonction Hypergéométriques et Hypersphériques*, Gauthier–Villars, Paris (1926).

[7] R.A. Gustafson, SIAM J. Math. Anal. **18**, 495 (1987).

[8] W.J. Holman III, L.C. Biedenharn, and J.D. Louck, SIAM J. Math. Anal. **7**, 529 (1976).

[9] W.J. Holman III, SIAM J. Math. Anal. **11**, 523 (1980).

[10] S.C. Milne, Adv. in Math. **58**, 1 (1985).

[11] S.C. Milne, J. Math. Anal. Appl. **122**, 223 (1987).

[12] S.C. Milne, Adv. in Math. **57**, 34 (1985).

[13] S.C. Milne, Adv. in Math. **57**, 14 (1985).

[14] S.C. Milne, Adv. in Math. **57**, 71 (1985).

[15] S.C. Milne, J. Math. Anal. Appl. **118**, 263 (1986).

[16] M.V. Tratnik, J. Math. Phys. **30**, 627 (1989).

[17] M.V. Tratnik, J. Math. Phys. **29**, 1529 (1988).

Recent IMA Preprints

#	Author/s	Title

495 **Lucas Hsu, Niky Kamran and Peter J. Olver**, Equivalence of Higher Order Lagrangians II. The Cartan Form for Particle Lagrangians

496 **D.J. Kaup and Peter J. Olver**, Quantization of BiHamiltonian Systems

497 **Metin Arik, Fahrünisa Neyzi, Yavuz Nutku, Peter J. Olver and John M. Verosky** Multi-Hamiltonian Structure of the Born-Infeld Equation

498 **David H. Wagner**, Detonation Waves and Deflagration Waves in the One Dimensional ZND Model for High Mach Number Combustion

499 **Jerrold R. Griggs and Daniel J. Kleitman**, Minimum Cutsets for an Element of a Boolean Lattice

500 **Dieter Jungnickel**, On Affine Difference Sets

501 **Pierre Leroux**, Reduced Matrices and q-log Concavity Properties of q-Stirling Numbers

502 **A. Narain and Y. Kizilyalli**, The Flow of Pure Vapor Undergoing Film Condensation Between Parallel Plates

503 **Donald A. French**, On the Convergence of Finite Element Approximations of a Relaxed Variational Problem

504 **Yisong Yang**, Computation, Dimensionality, and Zero Dissipation Limit of the Ginzburg-Landau Wave Equation

505 **Jürgen Sprekels**, One-Dimensional Thermomechanical Phase Transitions with Non-Convex Potentials of Ginzburg-Landau Type

506 **Yisong Yang**, A Note On Nonabelian Vortices

507 **Yisong Yang**, On the Abelian Higgs Models with Sources

508 **Chjan. C. Lim**, Existence of Kam Tori in the Phase Space of Vortex Systems

509 **John Weiss**, Bäcklund Transformations and the Painlevé Property

510 **Pu Fu-cho and D.H. Sattinger**, The Yang-Baxter Equation for Integrable Systems

511 **E. Bruce Pitman and David G. Schaeffer**, Instability and Ill-Posedness in Granular Flow

512 **Brian A. Coomes**, Polynomial Flows on $\mathbb{C}^{n*}$

513 **Bernardo Cockburn, Suchung Hou and Chi-Wang Shu**, The Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws IV: The Multidimensional Case

514 **Peter J. Olver**, Invariant Theory, Equivalence Problems, and the Calculus of Variations

515 **Daniel D. Joseph and Thomas S. Lundgren** with an appendix by **R. Jackson and D.A. Saville**, Ensemble Averaged and Mixture Theory Equations

516 **P. Singh, Ph. Caussignac, A. Fortes, D.D. Joseph and T. Lundgren**, Stability of Periodic Arrays of Cylinders Across the Stream by Direct Simulation

517 **Daniel D. Joseph**, Generalization of the Foscolo-Gibilaro Analysis of Dynamic Waves

518 **A. Narain and D.D. Joseph**, Note on the Balance of Energy at a Phase Change Interface

519 **Daniel D. Joseph**, Remarks on inertial radii, persistent normal stresses, secondary motions , and non-elastic extensional viscosities

520 **D. D. Joseph**, Mathematical Problems Associated with the Elasticity of Liquids

521 **Henry C. Simpson and Scott J. Spector**, Some Necessary Conditions at an Internal Boundary for Minimizers in Finite Elasticity

522 **Peter Gritzmann and Victor Klee**, On the 0-1 Maximization of Positive Definite Quadratic Forms

523 **Fu-Cho Pu and D.H. Sattinger**, The Yang-Baxter Equations and Differential Identities

524 **Avner Friedman and Fernando Reitich**, A Hyperbolic Inverse Problem Arising in the Evolution of Combustion Aerosol

525 **E.G. Kalnins, Raphael D. Levine and Willard Miller, Jr.**, Conformal Symmetries and Generalized Recurrences for Heat and Schrödinger Equations in One Spatial Dimension

526 **Wang Jinghua and Gerald Warnecke**, On Entropy Consistency of Large Time Step Godunov and Glimm Schemes

527 **C. Guillopé and J.C. Saut**, Existence Results for the Flow of Viscoelastic Fluids with a Differential Constitutive Law

528 **H.L. Bodlaender, P. Gritzmann, V. Klee and J. Van Leeuwen** Computational Complexity of Norm-Maximization

529 **Li Ta-tsien (Li Da-qian) and Yu Xin**, Life-Span of Classical Solutions to Fully Nonlinear Wave Equations

530 **Jong-Shenq Guo**, A Variational Inequality Associated with a Lubrication Problem

531 **Jong-Shenq Guo**, On the Semilinear Elliptic Equation $\Delta w - \frac{1}{2}y \cdot \nabla w + \lambda w - w^{-\beta} = 0$ in R^n

532 **Andrew E. Yagle**, Inversion of the Bloch transform in magnetic resonance imaging using asymmetric two-component inverse scattering

Recent IMA Preprints (Continued)

#	Author/s	Title

533 **Bei Hu**, A Fiber Tapering Problem
534 **Peter J. Olver**, Canonical Variables for BiHamiltonian Systems
535 **Michael Renardy**, A Well-Posed Boundary Value Problem for Supercritical Flow of Viscoelastic Fluids of Maxwell Type
536 **Michael Renardy**, Ill-Posedness Resulting from Slip As a Possible Explanation of Melt Fracture
537 **Michael Renardy**, Compatibility Conditions at Corners Between Walls and Inflow Boundaries for Fluids of Maxwell Type
538 **Rolf Rees**, The Spectrum of Restricted Resolvable Designs with $r = 2$
539 **D. Lewis and J.C. Simo**, Nonlinear stability of rotating pseudo-rigid bodies
540 **Robert Hardt and David Kinderlehrer**, Variational Principles with Linear Growth
541 **San Yih Lin and Yisong Yang**, Computation of Superconductivity in Thin Films
542 **A. Narain**, Pressure Driven Flow of Pure Vapor Undergoing Laminar Film Condensation Between Parallel Plates
543 **P.J. Vassiliou**, On Local Equivalence for Vector Field Systems
544 **Brian A. Coomes**, On Conditions Sufficient for Injectivity of Maps
545 **Yanchun Zhao**, A Class of Global Smooth Solutions of the One Dimensional Gas Dynamics System
546 **H. Holden, L. Holden and N.H. Risebro**, Some Qualitative Properties of 2×2 Systems of Conservation Laws of Mixed Type
547 **M. Slemrod**, Dynamics of Measured Valued Solutions to a Backward-Forward Heat Equation
548 **Avner Friedman and Jürgen Sprekels**, Steady States of Austenitic–Martensitic–Domains in the Ginzburg–Landau Theory of Shape Memory Alloys
549 **Avner Friedman and Bei Hu**, Degenerate Hamilton–Jacobi–Bellman Equations in a Bounded Domain
550 **E.G. Kalnins, Willard Miller, Jr., and M.V. Tratnik**, Families of Orthogonal and Biorthogonal Polynomials on the N-Sphere
551 **Heinrich Freistühler**, On Compact Linear Degeneracy
552 **Matthew Witten**, Quantifying the Concepts of Rate and Acceleration/Deceleration of Aging
553 **J.P. Albert and J.L. Bona**, Total Positivity and the Stability of Internal Waves in Stratified Fluids of Finite Depth
554 **Brian Coomes and Victor Zurkowski**, Linearization of Polynomial Flows and Spectra of Derivations
555 **Yuriko Renardy**, A Couette-Poiseuille Flow of Two Fluids in a Channel
556 **Michael Renardy**, Short wave instabilities resulting from memory slip
557 **Daniel D. Joseph and Michael Renardy**, Stokes' first problem for linear viscoelastic fluids with finite memory
558 **Xiaxi Ding**, Superlinear Conservation Law with Viscosity
559 **J.L. Ericksen**, Liquid Crystals with Variable Degree of Orientation
560 **F. Robert Ore, Jr. and Xinfu Chen**, Electro-Optic Modulation in an Arbitrary Cross-Section Waveguide
561 **M.V. Tratnik**, Multivariable biorthogonal continuous-discrete Wilson and Racah polynomials
562 **Yisong Yang**, Existence of Solutions for a Generalized Yang-Mills Theory
563 **Peter Gritzmann, Laurent Habsieger and Victor Klee**, Good and Bad Radii of Convex Polygons
564 **Martin Golubitsky, Martin Krupa and Chjan. C. Lim**, Time-Reversibility and Particle Sedimentation
565 **G. Yin**, Recent Progress in Parallel Stochastic Approximations
566 **G. Yin**, On H-Valued SA: Finite Dimensional Approximations
567 **Chien-Cheng Chang**, Accurate Evaluation of the Effect of Diffusion and Conductivity in Certain Equations
568 **Chien-Cheng Chang and Ruey-Ling Chern**, The Effect of Viscous Diffusion in Discrete Vortex Dynamics for Slightly Viscous Flows
569 **Li Ta-Tsien (Li Da-qian) and Zhao Yan-Chun**, Global Existence of Classical Solutions to the Typical Free Boundary Problem for General Quasilinear Hyperbolic Systems and its Applications
570 **Thierry Cazenave and Fred B. Weissler**, The Structure of Solutions to the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation
571 **Marshall Slemrod and Athanasios E. Tzavaras**, A Limiting Viscosity Approach for the Riemann Problem in Isentropic Gas Dynamics
572 **Richard D. James and Scott J. Spector**, The Formation of Filamentary Voids in Solids
573 **P.J. Vassiliou**, On the Geometry of Semi-Linear Hyperbolic Partial Differential Equations in the Plane Integrable by the Method of Darboux
574 **Jerome V. Moloney and Alan C. Newell**, Nonlinear Optics
575 **Keti Tenenblat**, A Note on Solutions for the Intrinsic Generalized Wave and Sine-Gordon Equations
576 **P. Szmolyan**, Heteroclinic Orbits in Singularly Perturbed Differential Equations
577 **Wenxiong Liu**, A Parabolic System Arising In Film Development