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Multivariable MRAC Using Nussbaum Gains for Aircraft with Abrupt Damages

Yu Liu and *Gang Tao*

Department of Electrical and Computer Engineering
University of Virginia
Charlottesville, VA 22904

Abstract— **This paper derives a stable multivariable model reference adaptive control (MRAC) scheme for systems with abrupt parameter variations (which may cause uncertain sign changes in the system's high frequency gain matrix), motivated by the application to in-flight aircraft systems with damages. Such sign changes are illustrated by an aircraft model with asymmetric abrupt damages, and their uncertainty is handled by a Nussbaum gain based adaptive control design to control the aircraft for both healthy and post-damage situations, by adapting controller parameters autonomously after the damages occur, for which the knowledge of time instants, structures and values of the damages is not required. A piecewise continuous Lyapunov function is utilized to prove the desired system stability and tracking properties in the presence of damages.**

I. Introduction

Aircraft safety under structural damages has been one of the major foci in the research of aircraft flight control. Aircraft structural damages may cause unknown changes to aircraft mass distribution and aerodynamic features. Under asymmetric damages, the assumption of mass symmetry about x-z plane in aircraft body frame in standard aircraft modeling is no longer valid, and new aircraft modeling and control techniques are needed. Reference [11] presents a study on the aircraft dynamics with partial losses of left wing, vertical, and horizontal stabilizers. A neural network based adaptive control algorithm is introduced for the control of aircraft in the presence of structure uncertainties of asymmetric damages. In [2], motion equations are introduced in detail for aircraft with asymmetric mass loss. Simulation results are presented for the comparison between the developed motion equations and standard equations. In [7], we introduce a nonlinear aircraft model with partial wing damage, and the linearization of such an aircraft model is illustrated. In [6], the real time identification of a damaged aircraft model is studied. A hybrid adaptive control method is given in [10], applied to aircraft with damages, using a neural network based estimation scheme.

In [9], we introduced a multivariable model reference adaptive control (MRAC) design based on the LDS decomposition of the high frequency gain matrix for the control of aircraft with multiple wing damages. One of the key design conditions is that, all the nominal and post-damage systems have a uniform known modified interactor matrix and the leading principal minors of their high frequency gain matrices should be nonzero with unchanging signs. Such a condition can be satisfied for some cases as studied in [9], for which a standard MRAC design is applicable.

In this paper, we study the aircraft damage cases when such a design condition may not be satisfied. Under certain damage conditions, the leading principal minors of the high frequency gain matrix may change signs after a damage and such changes are generally uncertain. To handle such cases, a multivariable MRAC scheme needs to be designed with relaxed requirement on the sign knowledge of the high frequency gain matrix. For multivariable MRAC, the LDS, LDU or SDU decomposition of the high frequency gain matrix can be used to relax the knowledge of such a matrix [4], [12]. When the sign knowledge cannot be incorporated in the design of adaptive laws, the Nussbaum gain method is usually employed. In [3], a Nussbaum gain based multivariable MRAC design is proposed based on LDU decomposition of the high frequency gain matrix. In this paper, a similar approach will be utilized to develop an LDS decomposition based MRAC scheme incorporated with Nussbaum gains, to relax the assumption on the sign information of the high frequency gain matrix. As our new contributions, we give a complete design and stability analysis for the Nussbaum gain based multivariable MRAC scheme, and apply it to adaptive flight control of aircraft with damages. It may be interesting to note that an LDS decomposition based MRAC design employs a simpler controller structure than that of an LDU decomposition based design, while the latter has less parameters to update. This MRAC scheme redesigned with Nussbaum gains will be applied to the linearized aircraft model with successive asymmetric wing damages.

The paper is organized as follows. In Section II, we briefly introduce the model of aircraft with damages and the research motivation. In Section III, we formulate the problem of MRAC of damaged aircraft. Key design conditions will be specified, and a numerical case study will be presented. In Section IV, we illustrate the design of the Nussbaum gain based multivariable MRAC scheme for control of aircraft with damages. The adaptive control design ensures the closed-loop stability and asymptotic output tracking under damage conditions, which is shown using a piecewise continuous Lyapunov function.

II. Motivation

For the effective control of damaged aircraft, a model that describes the aircraft dynamics under asymmetric damages is essential. In this section, we will first introduce such a model, then illustrate the damage effects on the system high frequency gain matrix, which motivate our research work.

A. Modeling of Aircraft with Damages

In [2], a detailed study on the motion of the damaged asymmetric aircraft is addressed. The nonlinear model of aircraft with asymmetric damages can be expressed as

$$\bar{M}\dot{x} = f(x, U), \tag{1}$$

where x is state vector, U is the control vector, and $f(x,U) = [f_1(x,U), f_2(x,U), \dots, f_9(x,U)]^T$ consists of the force and moment equations of the aircraft model and the kinematic equations [1], and

$$\bar{M} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}. \tag{2}$$

The matrix $\bar{M}$ would be decoupled block-wise when there is no damages, i.e., $\bar{M}_{12}$ and $\bar{M}_{21}$ are zero. When damages occur, the aircraft center of gravity will shift (with its coordinates in the body frame denoted as Δx, Δy, and Δz), and the inertia products I_{xy} and I_{yz} will change from zero in standard aircraft modeling to nonzero due to the asymmetric mass distribution after damages. After damages, some of the above nonzero terms will appear in $\bar{M}_{12}$ and $\bar{M}_{21}$ making it a coupled matrix, which shows that the longitudinal and lateral dynamics become coupled under such conditions. The above terms also appear in $\bar{M}_{11}$ and $\bar{M}_{22}$ when damages occur.

The linearization of the aircraft model under asymmetric damages is preformed following the same procedure of small perturbation linearization used in [8]. The equilibrium is chosen to be a rectilinear wing-level flight condition, which can include straight horizontal, ascending, or descending flight. The state and control vectors are chosen as

$$x = [\,u \quad w \quad q \quad \theta \quad v \quad r \quad p \quad \phi \quad \psi\,]^T \tag{3}$$

$$U = [\,\delta_e \quad \delta_{t_l} \quad \delta_{t_r} \quad \delta_a \quad \delta_r\,]^T \tag{4}$$

where the notation "δ" has been dropped from δx and δU for simplicity of presentation. Here u, v, and w represent the velocity perturbations along body axes, p, q and r are the angular velocity perturbations, θ, ϕ and ψ are the pitch, roll and yaw angle perturbations, and δ_e, δ_a, δ_r are the deflection perturbations of the elevator, aileron and rudder. δ_{t_l} and δ_{t_r} are the left and right throttle perturbations.

The linearized aircraft model can be obtained as

$$\begin{aligned} \bar{M}\dot{x} &= \bar{A}x + \bar{B}U \\ &= \begin{bmatrix} A^{(1)}_{4\times4} & A^{(2)}_{4\times5} \\ A^{(3)}_{5\times4} & A^{(4)}_{5\times5} \end{bmatrix} x + \begin{bmatrix} B^{(1)}_{4\times3} & B^{(2)}_{4\times2} \\ B^{(3)}_{5\times3} & B^{(4)}_{5\times2} \end{bmatrix} U, \end{aligned} \tag{5}$$

where $A^{(2)}$ and $B^{(2)}$ are zero, and

$$A^{(1)} = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial w} & \frac{\partial f_1}{\partial q} & -g\cos\theta_o \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial w} & \frac{\partial f_2}{\partial q} & -g\sin\theta_o \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial w} & \frac{\partial f_3}{\partial q} & \frac{\partial f_3}{\partial \theta} \\ 0 & 0 & 1 & 0 \end{bmatrix}, A^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial f_6}{\partial u} & \frac{\partial f_6}{\partial w} & \frac{\partial f_6}{\partial q} & \frac{\partial f_6}{\partial \theta} \\ 0 & 0 & \frac{\partial f_7}{\partial q} & \frac{\partial f_7}{\partial \theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{(4)} = \begin{bmatrix} \frac{\partial f_5}{\partial v} & \frac{\partial f_5}{\partial r} & \frac{\partial f_5}{\partial p} & g\cos\theta_o & 0 \\ \frac{\partial f_6}{\partial v} & \frac{\partial f_6}{\partial r} & \frac{\partial f_6}{\partial p} & \frac{\partial f_6}{\partial \phi} & 0 \\ \frac{\partial f_7}{\partial v} & \frac{\partial f_7}{\partial r} & \frac{\partial f_7}{\partial p} & \frac{\partial f_7}{\partial \phi} & 0 \\ 0 & \tan\theta_o & 1 & 0 & 0 \\ 0 & \frac{1}{\cos\theta_o} & 0 & 0 & 0 \end{bmatrix}, B^{(1)} = \begin{bmatrix} \frac{\partial f_1}{\partial \delta_e} & \frac{\partial f_1}{\partial \delta_{tl}} & \frac{\partial f_1}{\partial \delta_{tr}} \\ \frac{\partial f_2}{\partial \delta_e} & 0 & 0 \\ \frac{\partial f_3}{\partial \delta_e} & \frac{\partial f_3}{\partial \delta_{tl}} & \frac{\partial f_3}{\partial \delta_{tr}} \\ 0 & 0 & 0 \end{bmatrix},$$

$$B^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial f_6}{\partial \delta_{tl}} & \frac{\partial f_6}{\partial \delta_{tr}} \\ 0 & \frac{\partial f_7}{\partial \delta_{tl}} & \frac{\partial f_7}{\partial \delta_{tr}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^{(4)} = \begin{bmatrix} \frac{\partial f_5}{\partial \delta_a} & \frac{\partial f_5}{\partial \delta_r} \\ \frac{\partial f_6}{\partial \delta_a} & \frac{\partial f_6}{\partial \delta_r} \\ \frac{\partial f_7}{\partial \delta_a} & \frac{\partial f_7}{\partial \delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

The derivatives in $\bar{A}$ and $\bar{B}$ are subject to uncertain changes due to the unknown changes to aircraft mass, aerodynamic forces and moments due to wing damages.

So the linearized aircraft model can be re-expressed as

$$\dot{x} = \bar{M}^{-1}\bar{A}x + \bar{M}^{-1}\bar{B}U, \tag{7}$$

with $\bar{M}^{-1}\bar{A} \triangleq A$ and $\bar{M}^{-1}\bar{B} \triangleq B$. For the interest of conciseness, the explicit expressions of the derivatives in A and B are not shown in the paper.

B. Damage Effects

Since certain elements in $\bar{A}$, $\bar{B}$, and $\bar{M}$ have uncertain and abrupt changes when damages occur, it can be seen that system matrices A and B will have uncertain changes due to the variations of aircraft mass, mass distribution, and aerodynamic characteristics after damages. Different to standard decoupled aircraft model, the linearized aircraft model under damages is a coupled one, with coupling terms changing from zero to nonzero when damages occur.

When system matrices A and B have uncertain changes due to damages, the high frequency gain matrix of aircraft system may also have uncertain structural variations with the signs of its principal minors changed. This is verified by a numerical example presented in Section III.B. When such damage conditions occur, an adaptive control design which does not require the sign information of the high frequency gain matrix is needed, which will be developed in the next sections.

III. Problem Formulation

In this section, we shall formulate the problem of multivariable adaptive control of aircraft with damages. The design conditions for a multivariable MRAC design based on Nussbaum gain will be specified. A numerical case study will be presented to show that in certain damage situation, the relaxation of the assumption on the sign knowledge of the high frequency gain is necessary.

A. System Description

We consider a linear system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \; y(t) = Cx(t) \tag{8}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{m\times n}$ are unknown parameter matrices, $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^m$ are the state, input and output vectors. To represent an aircraft model, we let A and B be expressed as

$$A = A_0 + \Delta A, \qquad B = B_0 + \Delta B \tag{9}$$

where A_0 and B_0 are the nominal parameter matrices for the aircraft dynamics without damages, and ΔA and ΔB contain the unknown coupling terms and derivative changes caused by damages to the aircraft.

The objective is to design a control vector signal $u(t)$ such that the plant output $y(t)$ tracks a given reference output

$$y_m(t) = W_m(s)[r](t) \in R^m \tag{10}$$

for a stable $m \times m$ transfer matrix $W_m(s)$ and a bounded reference signal $r(t) \in R^m$, despite the uncertain damages.

For control design, we make the following assumptions:

(A0): the parameter matrices A, B and C are piecewise constant, with a finite number of unknown and constant jumps (A_i, B_i, C_i), $i = 1, 2, \ldots, \mathcal{N}$.

For each value (A_i, B_i, C_i) of (A, B, C), we define the transfer matrix $G_i(s) = C_i(sI - A_i)^{-1}B_i$ and assume:

(A1): All zeros of $G_i(s)$ are stable. (A2): An upper bound $\bar{\nu}$ on the observability index of $G_i(s)$ is known. (A3): $G_i(s)$ is strictly proper with full rank and has a known modified interactor matrix $\xi_m(s)$ such that $\lim_{s\to\infty} \xi_m(s)G_i(s) = K_{pi}$, the high frequency gain matrix of $G_i(s)$, is finite and non-singular. (A4): $W_m(s) = \xi_m^{-1}(s)$. (A5): All leading principal minors of the matrix K_{pi} are nonzero.

Assumption (A1)–(A4) are some basic assumptions for multivariable MRAC design. For MRAC, the plants need to be minimum phase systems. This condition could be satisfied for some aircraft systems. The need of the uniform $\xi_m(s)$ of $G_i(s)$ is to show how a multivariable MRAC scheme can be used to handle the system piecewise-constant parameter variations which can occur in aircraft systems with damages. Differing from that in [9], Assumption (A5) is largely relaxed and does not require that the signs of all leading principal minors of K_{pi} to be known and remain the same.

B. Discussion on Assumption (A5)

To illustrate the necessity of the relaxation of the assumption on the sign knowledge, we conduct a numerical study on a linearized model of a large transport aircraft. To simplify the analysis, we use a further simplified aircraft model introduced in [9]. Such a model is obtained by assuming that the center of gravity shift is small and the angular velocities are within a neighborhood of zero. The engine settings are assumed to be identical, so only one δ_t command is present. For the aircraft without damages, $A^{(2)}$, $A^{(3)}$, $B^{(2)}$, and $B^{(3)}$ are zero matrices, and the other system matrices are

$$A^{(1)} = \begin{bmatrix} -0.0058 & 0.0523 & -28.8870 & -32.1450 \\ -0.0003 & -0.6796 & 776.4800 & -1.1959 \\ 0.0021 & -0.0048 & -0.7694 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A^{(4)} = \begin{bmatrix} -0.1290 & -774.9200 & 28.3280 & 32.1450 & 0 \\ 0.0040 & -0.1757 & -0.0409 & -0.0001 & 0 \\ -0.0120 & 0.9409 & -1.4419 & -0.00001 & 0 \\ 0 & 0.0372 & 1 & 0 & 0 \\ 0 & 1.0007 & 0 & 0 & 0 \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} 3.6982 \times 10^{-4} \\ -7.1163 \times 10^{-8} \\ 6.2477 \times 10^{-6} \\ 0 \end{bmatrix}, \quad B^{(4)} = \begin{bmatrix} 0.4669 \\ -0.0382 \\ 0.01998 \\ 0 \\ 0 \end{bmatrix}. \tag{11}$$

The control inputs are engine throttle δ_t and rudder δ_r. The system outputs are pitch angle θ and yaw angle ψ.

This plant is a minimum phase system with an observability index of 5. The interactor matrix $\xi_m(s)$ can be chosen as $\xi_m(s) = \mathrm{diag}\{s^2 + s + 1, s^2 + s + 1\}$ yielding a nonsingular high frequency gain matrix $K_{p1} = \mathrm{diag}\{6.2477 \times 10^{-6}, -0.03822\}$, whose leading principal minors are 6.2477×10^{-6} and -2.3876×10^{-7} respectively. The seemingly small values are due to the physics of the aircraft model, and should not be considered as trivial.

When unknown damages occurs, the derivatives in (5) have unknown changes and nonzero coupling terms also appear. To characterize such damage effects, we vary the derivatives in the above numerical model up to 40% of their nominal values. The first, second, and fourth rows of $A^{(2)}$ are all chosen as zero, and the third row is chosen as $[0.01, -0.02, 0.01, -0.02, 0]$. The first, fourth, and fifth rows of $A^{(3)}$ are zero, and its second and third rows are chosen as $[-0.02, -0.02, 0.01, -0.04]$ and $[-0.01, 0.01, 0.03, 0.04]$ respectively. We also choose $B^{(2)} = [0, 0, 0.001, 0]^T$ and $B^{(3)} = [0, -0.01, 0.001, 0, 0]^T$. For this system with derivative changes, we can verify that it is still a minimum phase system with an observability index of 5. The previous modified interactor matrix is valid for this system with a nonsingular high frequency gain matrix

$$K_{p2} = \begin{bmatrix} 8.7468 \times 10^{-6} & 0.001 \\ -0.010007 & -0.0535 \end{bmatrix} \tag{12}$$

whose leading principal minors are 8.7467×10^{-6} and 9.5390×10^{-6} respectively. It can be seen that the second leading principal minors of K_{p1} and K_{p2} (i.e., their determinants) have different signs.

Next we shall have a further discussion on the signs of leading principal minors for this numerical model. For this damaged aircraft system, it can be verified that all the transfer functions in $G(s)$ have a relative degree of 2. Based on linear system theory, we can determine the high frequency gain matrix of this system as $K_{p2} = CAB + \alpha_{n-1}CB$, where α_{n-1} is the coefficient of s^{n-1} in $\det(sI - A)$. Noting that $CB = 0$ for this system, we have

$$K_{p2} = CAB = \begin{bmatrix} 8.7468 \times 10^{-6} & B_3^{(2)} \\ 1.0007B_2^{(3)} & -0.03823 \end{bmatrix}, \tag{13}$$

where $B_3^{(2)}$ is the third element of $B^{(2)}$, and $B_2^{(3)}$ is the second element of $B^{(3)}$. More specifically, $B_3^{(2)}$ represents the influence of rudder to pitch rate and $B_2^{(3)}$ is the effect of engine thrust to yaw rate. Such coupling terms are due to the shift of the center of gravity under asymmetric damages. Next, as an example, we fix the coupling values in $A^{(2)}$, $A^{(3)}$, and $B_3^{(2)}$, and determine the condition on $B_2^{(3)}$ (i.e., effect of engine thrust to yaw rate) for the determinant of K_{p2} to change its sign.

Note that the determinant of K_{p1} is negative for the nominal system, and the determinant of K_{p2} would be positive if the following inequality is satisfied:

$$B_3^{(2)}B_2^{(3)} < -3.3413 \times 10^{-7}. \tag{14}$$

From (14), we can obtain that for the chosen $B_3^{(2)} = 0.001$, the determinant of K_{p2} would be positive as long as $B_2^{(3)} <$

-0.0003341. Taking into consideration that the system must have stable zeros for an MRAC design, we can see by numerical tests that the damaged system is minimum phase with a positive determinant of K_{p2} when $B_2^{(3)} < -0.001804$. The physical meaning is that for this chosen numerical example, the determinant of the high frequency gain matrix would change its sign when the effect of the engine thrust to the yaw rate is large enough to the negative direction of z body-axis, that is, the shift of the center of gravity to the negative direction of y body-axis is large enough after an asymmetric damage.

The above numerical example shows that the signs of the leading principal minors of the high frequency gain matrix may change under certain damage conditions.

IV. Multivariable MRAC Design and Analysis

In this section, we demonstrate the design of an LDS decomposition based multivariable model reference adaptive control scheme for aircraft with damages, with the incorporation of Nussbaum gains. With such a design, *a priori* knowledge of the high frequency gain matrix can be relaxed. The closed-loop stability and asymptotic tracking properties are demonstrated in the stability analysis using a discontinuous Lyapunov function.

A. Plant-Model Matching Controllers

For model reference adaptive control design, we first define a nominal model reference controller which achieves the desired control objective when the system parameters A, B and C are known. Parameters of this controller, which are unknown, will be used in constructing an error model needed for adaptation of an adaptive controller.

Since the system parameters (A, B, C) may take any of (A_i, B_i, C_i), $i = 1, 2, \ldots, \mathcal{N}$, there is a set of such nominal controllers, and each of them has the structure

$$u^*(t) = \Theta_1^{*T}\omega_1(t) + \Theta_2^{*T}\omega_2(t) + \Theta_{20}^* y(t) + \Theta_3^*(t) r(t) \quad (15)$$

where $\omega_1(t) = F(s)[u](t)$, $\omega_2 = F(s)[y](t)$, $F(s) = \frac{A_F(s)}{\Lambda(s)}$, $A_F(s) = [I, sI, \ldots, s^{\bar{\nu}-2}I]^T$, $\Lambda(s)$ is a monic stable polynomial of degree $\bar{\nu} - 1$, with the upper bound $\bar{\nu}$ on the observability indices of $G_i(s)$. The nominal parameters $\Theta_1^* = [\Theta_{11}^*, \ldots, \Theta_{1\bar{\nu}-1}^*]^T$, $\Theta_2^* = [\Theta_{21}^*, \ldots, \Theta_{2\bar{\nu}-1}^*]^T$, Θ_{20}^*, Θ_3^*, $\Theta_{ij}^* \in R^{m\times m}$, $i = 1, 2$, $j = 1, \ldots, \bar{\nu} - 1$, are for plant-model matching, and are derived next.

We first introduce the following notation:

$$G_i(s) = C_i(sI - A_i)^{-1}B_i = Z_i(s)P_i^{-1}(s) \quad (16)$$

for some $m \times m$ right coprime polynomial matrices $Z_i(s)$ and $P_i(s)$ with $P_i(s)$ being column proper, $i = 1, 2, \ldots, \mathcal{N}$.

With the specification of $\Lambda(s)$, $\xi_m(s)$, $P_i(s)$, $Z_i(s)$, there exist Θ_1^*, Θ_2^*, Θ_{20}^*, $\Theta_3^* = K_{pi}^{-1}$ such that

$$\begin{aligned} &\Theta_1^{*T} A_F(s) P_i(s) + (\Theta_2^{*T} A_F(s) + \Lambda(s)\Theta_{20}^*) Z_i(s) \\ = \;&\Lambda(s)(P_i(s) - \Theta_3^* \xi_m(s) Z_i(s)). \end{aligned} \quad (17)$$

Since $\Lambda(s)$ and $Z_i(s)$ are stable, we have the plant-model transfer matrix matching equation

$$\begin{aligned} &I - \Theta_1^{*T}F(s) - \Theta_2^{*T}F(s)G_i(s) - \Theta_{20}^* G_i(s) \\ = \;&\Theta_3^* W_m^{-1}(s) G_i(s) \end{aligned} \quad (18)$$

from which the plant-model matching parameters Θ_1^*, Θ_2^*, and Θ_{20}^* can be determined with $\Theta_3^* = K_{pi}^{-1}$. For each (A_i, B_i, C_i), $i = 1, 2, \ldots, \mathcal{N}$, we can determine a set of constant parameters Θ_j^*, $j = 1, 2, 20, 3$, so that the plant-model matching parameters Θ_j^* are piecewise constant.

B. Adaptive Control Scheme

To design the adaptive control scheme, we first introduce the LDS decomposition of the high frequency gain matrix. Then we develop a model reference adaptive control design based on the Nussbaum gain method, which further relaxes the requirement on the knowledge of the signs of the leading principal minors of the high frequency gain matrix. The desired stability and tracking properties will be demonstrated.

LDS decomposition of K_p [4], [12]. Let Δ_j, $j = 1, 2, \ldots, m$, denote the leading principal minors of the high frequency gain matrix $K_p \in R^{m\times m}$ and assume that $\Delta_j \neq 0$, $j = 1, 2, \ldots, m$. The gain matrix K_p then has a non-unique decomposition

$$K_p = L_s D_s S, \quad (19)$$

where $S \in R^{m\times m}$ is a symmetric and positive definite matrix, L_s is an $m \times m$ unit lower triangular matrix, and

$$\begin{aligned} D_s &= \mathrm{diag}\{d_1, d_2, \ldots, d_m\} \\ &= \mathrm{diag}\left\{\mathrm{sign}[\Delta_1], \mathrm{sign}\left[\frac{\Delta_2}{\Delta_1}\right], \ldots, \mathrm{sign}\left[\frac{\Delta_m}{\Delta_{m-1}}\right]\right\}. \end{aligned} \quad (20)$$

It is important to note that different K_{pi} can have different D_s with different signs for their elements.

Adaptive controller. When plant parameters are uncertain, the controller parameters Θ_1^*, Θ_2^*, Θ_{20}^*, Θ_3^* are also unknown. The adaptive version of (15) is

$$u(t) = \Theta_1^T(t)\omega_1(t) + \Theta_2^T(t)\omega_2(t) + \Theta_{20}(t)y(t) + \Theta_3(t)r(t) \quad (21)$$

where $\Theta_1(t)$, $\Theta_2(t)$, $\Theta_{20}(t)$, and $\Theta_3(t)$ are estimates of Θ_1^*, Θ_2^*, Θ_{20}^*, and Θ_3^*, and will be adaptively updated.

Error dynamics. From the plant-model transfer matrix matching equation (18), for any $u(t)$, we have

$$u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^* y(t) = \Theta_3^* W_m^{-1}(s)[y](t), \quad (22)$$

from which, together with the reference model (10) and Assumption (A4), we obtain

$$\begin{aligned} &K_p\left(u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^* y(t) - \Theta_3^* r(t)\right) \\ &= \xi_m(s)[y - y_m](t). \end{aligned} \quad (23)$$

With the LDS decomposition in (19), we express (23) as

$$\begin{aligned} &D_s S\left(u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^* y(t) - \Theta_3^* r(t)\right) \\ &= L_s^{-1}\xi_m(s)[y - y_m](t). \end{aligned} \quad (24)$$

From (24) and the adaptive controller (21), we can have

$$\xi_m(s)[y - y_m](t) + \Theta_0^* \xi_m(s)[y - y_m](t) = D_s S \tilde{\Theta}^T(t)\omega(t), \quad (25)$$

where D_s is in (20), $S = S^T > 0$ in (19), $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ with $\Theta(t)$ being the estimate of $\Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^*, \Theta_3^*]^T$, $\omega(t) = [\omega_1^T(t), \omega_2^T(t), y^T(t), r^T(t)]^T$, and $\Theta_0^* = L_s^{-1} - I$. Choose $f(s)$ as a stable and monic polynomial whose degree is equal to the maximum degree of the modified interactor matrix $\xi_m(s)$, introduce the filter $h(s) = \frac{1}{f(s)}$, and define the filtered tracking error

$$\bar{e}(t) = \xi_m(s)h(s)[e](t) = [\bar{e}_1(t), \ldots, \bar{e}_m(t)]^T \qquad (26)$$

with $e(t) = y(t) - y_m(t)$. Operating both sides of (25) by $h(s)I_m$ leads to

$$\bar{e}(t) = -\Theta_0^*\bar{e}(t) + \Psi^* h(s)[\tilde{\Theta}^T\omega](t), \qquad (27)$$

with $\Psi^* \stackrel{\triangle}{=} D_s S$, which is nonsingular and symmetric.

We also define

$$\xi(t) = \Theta^T(t)\zeta(t) - h(s)[\Theta^T\omega](t), \qquad (28)$$

$$\zeta(t) = h(s)[\omega](t), \qquad (29)$$

from which we have

$$\begin{aligned} h(s)[\tilde{\Theta}^T\omega](t) &= h(s)[(\Theta - \Theta^*)^T\omega](t) \\ &= h(s)[\Theta^T\omega](t) - \Theta^{*T}h(s)[\omega](t) \\ &= \Theta^T(t)\zeta(t) - \xi(t) - \Theta^{*T}\zeta(t). \end{aligned} \qquad (30)$$

Thus (27) can be re-written as

$$\begin{aligned} \bar{e}(t) &= -\Theta_0^*\bar{e}(t) + \Psi^*(\Theta^T(t)\zeta(t) - \xi(t) - \Theta^{*T}\zeta(t)) \\ &= \Psi^*(-\Theta_m^*\bar{e}(t) - \Theta^{*T}\zeta(t) + \Theta^T(t)\zeta(t) - \xi(t)), \end{aligned} \qquad (31)$$

with $\Theta_m^* \stackrel{\triangle}{=} \Psi^{*-1}\Theta_0^*$. Note that the last column of Θ_0^* is always zero, so is that of Θ_m^*. Thus we define $\Theta_s^* \in R^{m\times(m-1)}$ that contains the first $m-1$ columns of Θ_m^*, and

$$\bar{e}_s(t) = [\bar{e}_1(t), \ldots, \bar{e}_{m-1}(t)]^T \in R^{m-1}, \qquad (32)$$

from which we have

$$\bar{e}(t) = \Psi^*(-\Theta_s^*\bar{e}_s(t) - \Theta^{*T}\zeta(t) + \Theta^T(t)\zeta(t) - \xi(t)). \qquad (33)$$

Based on (33), we define

$$\hat{e}(t) = N\Psi(t)(-\Theta_s\bar{e}_s(t) - \Theta^T\zeta(t) + \Theta^T(t)\zeta(t) - \xi(t)) \qquad (34)$$

where $N = \text{diag}\{N_1(z_1), N_2(z_2), \ldots, N_m(z_m)\}$ with the functions $N_i(z_i), i = 1, 2, \ldots, m$, being the Nussbaum gains accounting for the unknown signs of the leading principal minors of K_p. The definitions of $N_i(z_i)$ and z_i, $i = 1, 2, \ldots, m$, will be introduced next. $\Theta_s \in R^{m\times(m-1)}$ is the estimate of Θ_s^*. Note that Ψ is not the estimate of $\Psi^* = D_s S$, but the estimate of S.

Based on (33) and (34), we define the estimation error $\epsilon(t) = [\epsilon_1(t), \epsilon_2(t), \ldots, \epsilon_m(t)]^T$ as

$$\epsilon(t) = \frac{1}{m^2(t)}(\bar{e}(t) - \hat{e}(t)). \qquad (35)$$

Thus we have

$$\begin{aligned} \epsilon m^2 &= \Psi^*(-\Theta_s^*\bar{e}_s(t) - \Theta^{*T}\zeta(t) + \Theta^T(t)\zeta(t) - \xi(t)) \\ &\quad - N\Psi(-\Theta_s\bar{e}_s(t) - \Theta^T\zeta(t) + \Theta^T(t)\zeta(t) - \xi(t)) \\ &= \Psi^*(\tilde{\Theta}_s\bar{e}_s(t) + \tilde{\Theta}^T\zeta(t)) - N\Psi\xi_s(t) + \Psi^*\xi_s(t) \end{aligned} \qquad (36)$$

where

$$\tilde{\Theta}_s(t) = \Theta_s(t) - \Theta_s^*, \qquad (37)$$

$$\xi_s(t) = -\Theta_s\bar{e}_s(t) - \xi(t), \qquad (38)$$

$$m^2(t) = 1 + \bar{e}_s^T(t)\bar{e}_s(t) + \zeta^T(t)\zeta(t) + \xi_s^T(t)\xi_s(t). \qquad (39)$$

Adaptive laws. We choose the adaptive laws

$$\dot{\Theta}_s(t) = N\,\epsilon(t)\bar{e}_s^T(t), \qquad (40)$$

$$\dot{\Theta}^T(t) = N\,\epsilon(t)\zeta^T(t), \qquad (41)$$

$$\dot{\Psi}(t) = N\,\Gamma_\psi\epsilon(t)\xi_s^T(t), \qquad (42)$$

where $\Gamma_\psi = \text{diag}\{\gamma_{\psi 1}, \ldots, \gamma_{\psi m}\}$, with $\gamma_{\psi i}$ positive, $i = 1, \ldots, m$.

Nussbaum gains. Note that the estimation error equation in (36) is similar to the parametric model of the SISO Nussbaum design in [5], and the Nussbaum gains $N_i(z_i)$, $i = 1, \ldots$ in N can be generated as below:

$$N_i(z_i) = z_i^2\cos z_i \qquad (43)$$

where

$$z_i(t) = w_i(t) + \frac{\Psi_i\Psi_i^T}{2\gamma_{\psi i}}, \qquad (44)$$

$$\dot{w}_i(t) = \epsilon_i^2 m^2, \qquad w_i(0) = 0, \qquad (45)$$

with Ψ_i being the ith row of Ψ.

Stability analysis. To demonstrate the stability of the closed-loop system, we choose a piece-wise continuous Lyapunov function. Based on Assumption (A0), there are $\mathcal{N} - 1$ finite jumps due to the damages, and totally $\mathcal{N}$ choices of (A_i, B_i, C_i). Assuming that the asymmetric damage occurs at time instant t_j, $j = 1, 2, \ldots, \mathcal{N} - 1$, we choose the following Lyapunov-like function

$$V = \frac{1}{2}\text{tr}[\tilde{\Theta}_s^T S\tilde{\Theta}_s] + \frac{1}{2}\text{tr}[\tilde{\Theta}S\tilde{\Theta}^T] \qquad (46)$$

for time intervals (t_{j-1}, t_j), $j = 1, \ldots, \mathcal{N}$, with $t_0 = 0$ and $t_{\mathcal{N}} = \infty$. Due to the changes of system parameters after the damages (which are finite), and the finite jumps of nominal parameters, there would be a finite jump of V for each jump of system parameters (A_i, B_i, C_i), i.e.,

$$V(t_j^+) - V(t_j^-) < \infty, j = 1, 2, \ldots, \mathcal{N} - 1. \qquad (47)$$

From (36), the adaptive laws (40)–(42), we can obtain

$$\begin{aligned} \dot{V} &= \sum_{i=1}^m d_i^{-1}N_i(z_i)(\epsilon_i^2 m^2 + \gamma_{\psi i}^{-1}\Psi_i\dot{\Psi}_i^T) - \sum_{i=1}^m \gamma_{\psi i}^{-1}\sum_{j=1}^m \dot{\psi}_{ij}s_{ij} \\ &= \sum_{i=1}^m d_i^{-1}N_i(z_i)\dot{z}_i - \sum_{i=1}^m \gamma_{\psi i}^{-1}\sum_{j=1}^m \dot{\psi}_{ij}s_{ij} \end{aligned} \qquad (48)$$

for $t \in (t_{j-1}, t_j)$, where ψ_{ij} and s_{ij} are the jth element of the vectors Ψ_i and S_i (the ith column of S).

Integrating both sides of (48), we have

$$\begin{aligned} V(t) &= V(t_{j-1}^+) + \sum_{i=1}^m d_i^{-1}\pi_i^{[j-1]}(t) \\ &\quad - \sum_{i=1}^m \gamma_{\psi i}^{-1}\sum_{j=1}^m s_{ij}(\psi_{ij}(t) - \psi_{ij}(t_{j-1})), \end{aligned} \qquad (49)$$

where, for any $t \in (t_{j-1}, t_j)$,

$$\pi_i^{[j-1]}(t) = \int_{z_i(t_{j-1})}^{z_i(t)} N_i(z_i) dz_i$$
$$= 2z_i(t)\cos z_i(t) + (z_i^2(t) - 2)\sin z_i(t)$$
$$-2z_i(t_{j-1})\cos z_i(t_{j-1}) - z_i^2(t_{j-1})\sin z_i(t_{j-1})$$
$$+2\sin z_i(t_{j-1}), \quad (50)$$

From the definition of z_i and w_i in (44) and (45), we know that $z_i(t) \geq 0$. By observing (50) and the right hand side of (49), we can see that when $z_i(t)$ gets large along with t, the oscillating terms $z_i^2(t)\sin z_i(t),\ i = 1, \ldots, m$, would dominate. Since $V(t) \geq 0$, $z_i(t)$ cannot go to infinity. Otherwise the condition $V(t) \geq 0$ may be violated. Thus $z_i(t)$ is bounded for $t \in (t_{j-1}, t_j)$.

From the definition of $z_i(t)$ in (44), we can see that bounded $z_i(t)$ implies bounded $\Psi(t)$ and $w_i(t)$. Thus, from (49) we conclude that $V(t)$ is bounded for $t \in (t_{j-1}, t_j)$. Since $V(t)$ is not continuous at instant t_j, $j = 1, 2, \ldots, \mathcal{N}-1$ and has only finite jumps at those instants, we can conclude that $V(t)$ is bounded for $t \in [0, \infty)$. So we can conclude that $\Theta_s(t) \in L^\infty$ and $\Theta(t) \in L^\infty$. The bounded and nonnegative $V(t)$ for $t \geq 0$ also implies that $z_i(t)$ is bounded for $t \geq 0$. Thus we have $w_i(t) \in L^\infty$ and $\Psi(t) \in L^\infty$.

For any $t \in (t_{j-1}, t_j)$, we have

$$w_i(t) = \int_{t_{j-1}}^{t} \epsilon_i^2(\nu) m^2(\nu) d\nu. \quad (51)$$

Since $\epsilon_i(t)$ and $m(t)$ are continuous, for $[0, t)$ with $t > t_{\mathcal{N}-1}$ (i.e., all damages have occurred), we also have $w_i(t) = \int_0^t \epsilon_i^2(\nu) m^2(\nu) d\nu$. Given $w_i(t) \geq 0$ and bounded, we have

$$\lim_{t\to\infty} w_i(t) = \int_0^\infty \epsilon_i^2(\nu) m^2(\nu) d\nu < \infty, \quad (52)$$

which implies that $\epsilon_i(t) m(t) \in L^2$. Since $m^2(t) = 1 + \bar{e}_s^T \bar{e}_s + \zeta^T \zeta + \xi_s^T \xi_s$, from (52) we obtain

$$\int_0^\infty \epsilon_i^2(\nu) d\nu + \int_0^\infty \epsilon_i^2(\nu)(\bar{e}_s^T \bar{e}_s + \zeta^T \zeta + \xi_s^T \xi_s) d\nu < \infty. \quad (53)$$

Because the two terms on the left side of (53) are both nonnegative, we have $\int_0^\infty \epsilon_i^2(\nu) d\nu < \infty$, leading to $\epsilon_i(t) \in L^2$.

From (36), and the facts that $\frac{\bar{e}_s}{m} \in L^\infty$, $\frac{\zeta}{m} \in L^\infty$ and $\frac{\xi_s}{m} \in L^\infty$, we have $\epsilon_i(t) m(t) \in L^\infty \cap L^2$.

From (40) to (42), $\epsilon_i(t) m(t) \in L^\infty \cap L^2$, $\frac{\bar{e}_s}{m} \in L^\infty$, $\frac{\zeta}{m} \in L^\infty$ and $\frac{\xi_s}{m} \in L^\infty$, we can obtain $\dot{\Theta}_s(t) \in L^2 \cap L^\infty$, $\dot{\Theta}(t) \in L^2 \cap L^\infty$ and $\dot{\Psi}(t) \in L^2 \cap L^\infty$.

Based on these desired properties, we have

Theorem 1: The MRAC scheme consisting of (21), (40), (41) and (42), ensures closed-loop signal boundedness and asymptotic output tracking $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$, for the system (8) satisfying Assumptions (A0)–(A5).

The proof of this theorem can be obtained similarly to that in [12], based on the well-defined feedback structure for the closed-loop system with a small loop gain. Such a feedback structure is developed from the feedback controller with bounded parameters and the controlled plant with stable zeros. The smallness of its loop gain is ensured by the L^2 properties of the adaptive laws. The asymptotic tracking property follows from the complete parametrization of the error equation (36), the L^2 properties, and the signal boundedness of the closed-loop system.

V. Conclusions

In this paper we demonstrated the design of a multivariable model reference adaptive controller for aircraft with damages, without the knowledge of the signs of the leading principal minors of the high frequency gain matrix. We introduced the modeling of aircraft dynamics in the presence of damages, which captures the key characteristics of the aircraft dynamics under asymmetric damages with the loss of mass symmetry. An LDS decomposition based multivariable MRAC scheme is developed with the incorporation of Nussbaum gains. For the piecewise linear systems under MRAC, a discontinuous Lyapunov function is utilized to show that desired stability and tracking properties are ensured, despite the jumping system parameter variations. This work shows that multivariable MRAC designs are potentially useful for control of aircraft systems with larger classes of damages and uncertainties. A detailed simulation study of MRAC of aircraft systems with damages is to be conducted.

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References

[1] A. E. Bryson, Jr., *Control of Spacecraft and Aircraft*, Princeton University Press, NJ, 1994.
[2] B. J. Bacon and I. M. Gregory, "General equations of motion for a damaged asymmetric aircraft," *Proc. of AIAA Guidance, Navigation, and Control Conference*, AIAA-2007-6306, 2007.
[3] A. K. Imai, R. R. Costa and L. Hsu, "Multivariable MRAC using Nussbaum gain," *Proceedings of 2001 International Symposium on Adaptive and Intelligent Systems and Control,* Charlottesville, VA, June, 2001.
[4] A. K. Imai, R. R. Costa, L. Hsu, G. Tao and P. V. Kokotović, "Multivariable MRAC using high frequency gain matrix factorization," *IEEE Trans. on Auto. Contr.*, vol. 49, no. 7, pp. 1152–1157, 2004.
[5] P. Ioannou and J. Sun, *Robust Adaptive Control*, Prentice Hall, 1996.
[6] T. J. J. Lombaerts, Q. P. Chu, J. A. Mulder, and D. A. Joosten, "Real time damaged aircraft model identification for reconfiguring flight control," *Proc. of AIAA Atmospheric Flight Mechanics Conference,* AIAA-2007-6717, 2007.
[7] Y. Liu and G. Tao, "Model-based direct adaptive actuator failure compensation techniques with applications to aircraft flight control systems," *Proc. of AIAA Guidance, Navigation, and Control Conference*, AIAA-2006-6554, 2006.
[8] Y. Liu, X. Tang, G. Tao, and S. M. Joshi, "Adaptive compensation of aircraft actuation failures using an engine differential model," *IEEE Trans. on Control Systems Technology*, vol. 16, no. 5, pp. 971–982, September, 2008.
[9] Y. Liu and G. Tao, "Multivariable MRAC for aircraft with abrupt damages," *Proceedings of 2008 American Control Conference*, pp. 2981–2986, June, 2008.
[10] N. T. Nguyen, M. Bakhtiari-Nejad, and Y. Huang, "Hybrid adaptive flight control with bounded linear stability analysis," *Proc. of AIAA Guidance, Navigation and Control Conference*, AIAA-2007-6422, 2007.
[11] N. Nguyen, K. Krishnakumar, J. Kaneshige, and P. Nespeca, "Dynamics and adaptive control for stability recovery of damaged asymmetric aircraft," *Proc. of AIAA Guidance, Navigation, and Control Conference*, AIAA-2006-6049, 2006.
[12] G. Tao, *Adaptive Control Design and Analysis*, John Wiley & Sons, 2003.