

MULTIVARIATE COUPLED FIXED POINT THEOREMS ON ORDERED PARTIAL METRIC SPACES

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ABSTRACT. A partial metric, also called a nonzero self-distance, is motivated by experience from computer science. Besides a lot of properties of partial metric analogous to those of metric, fixed point theorems in partial metric spaces have been studied recently. We establish several kinds of extended fixed point theorems in ordered partial metric spaces with higher dimension under generalized notions of mixed monotone mappings.

1. Introduction

The fixed point theory has been applied for the study of nonlinear analysis, and its results have been developed in various areas of mathematics and engineering. In 1994 S. Matthews [15] introduced the notion of partial metric space motivated by experience from computer science, and extended the Banach contraction principle to the setting of partial metric spaces. In several decades many scholars studied fixed point theorems in partial metric spaces (see [2, 17, 18, 20, 22, 27]).

Bhaskar and Lakshmikantham [6] introduced the notions of a mixed monotone mapping and a coupled fixed point, and proved some coupled fixed point theorems for mixed mappings in ordered metric spaces. Many different kinds of coupled fixed point theorems with applications have been studied (see [1, 7, 11, 13, 14, 24, 26]). A few years later Harjani, López, and Sadarangani [8] have developed fixed point theorems for mixed monotone operators and have applied to certain integral equations. In this article we generalize the concept of coupled fixed point and mixed monotone mapping, and show some fixed point theorems in ordered partial metric spaces.

First, we recall some definitions and properties of partial metric, also called a nonzero self-distance (see [15, 25]).

Definition 1.1. A *partial metric* on a nonempty set X is a function $p : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

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- (P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
 (P2) $p(x, x) \leq p(x, y)$,
 (P3) $p(x, y) = p(y, x)$,
 (P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A *partial metric space* is a pair (X, p) such that X is a nonempty set and p is a partial metric on X .

Note that the self-distance of any point need not be zero, and hence, the idea of generalizing metrics is that a metric on a nonempty set X is precisely a partial metric p on X such that $p(x, x) = 0$ for any $x \in X$. We give some examples of a partial metric space.

- (1) $([0, \infty), p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$.
- (2) (\mathbb{R}^n, p) , where $p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| + a$ for some $a \geq 0$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- (3) Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and define $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then, (X, p) is a partial metric space.

Furthermore, there is an elegant way to construct a metric space from a partial metric space (see [25, Section 1]).

Lemma 1.1. *Let (X, p) be a partial metric space, and let $d_p : X \times X \rightarrow [0, \infty)$ be defined by*

$$(1.1) \quad d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad \forall x, y \in X.$$

Then (X, d_p) is a metric space.

Definition 1.2. Let (X, p) be a partial metric space. For any $x \in X$ and $\epsilon > 0$, we define the *open* and *closed ball* for the partial metric p respectively by setting

$$B_\epsilon(x) = \{y \in X : p(x, y) < \epsilon\}, \quad \overline{B}_\epsilon(x) = \{y \in X : p(x, y) \leq \epsilon\}.$$

In a partial metric space (X, p) , the set of open balls is the basis of a T_0 topology on X , called *the partial metric topology* and denoted by τ_p .

Definition 1.3. Let (X, p) be a partial metric space. Then

- (i) A sequence $\{x_n\}$ in the space (X, p) converges to $x \in X$, written as $\lim_{n \rightarrow \infty} x_n = x$, if for any $\epsilon > 0$ such that $x \in B_\epsilon(x)$, there exists $N \geq 0$ so that for any $n \geq N$, $x_n \in B_\epsilon(x)$.
- (ii) A sequence $\{x_n\}$ in the space (X, p) is said to be a *Cauchy sequence* if $\lim_{n, m \rightarrow \infty} p(x_n, x_m) < \infty$.
- (iii) The partial metric space (X, p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.
- (iv) A mapping $F : X \rightarrow X$ is said to be *continuous at $x_0 \in X$* if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_\delta(x_0)) \subseteq B_\epsilon(F(x_0))$.

The following lemma on a partial metric space can be derived easily (see, e.g., [2, 15, 17, 18, 27]).

Lemma 1.2. *Let (X, p) be a partial metric space. Then the following are satisfied.*

- (i) *A sequence $\{x_n\}$ in the space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n) = 0$.*
- (ii) *A sequence $\{x_n\}$ in the space (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_p) .*
- (iii) *The partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover,*

$$p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = \lim_{n \rightarrow \infty} p(x, x_n) \Leftrightarrow \lim_{n \rightarrow \infty} d_p(x_n, x) = 0.$$

- (iv) *Suppose that $F : X \rightarrow X$ is continuous at $x_0 \in X$. Then, for each sequence $\{x_n\}$ in X , $x_n \rightarrow x_0$ in τ_p implies $F(x_n) \rightarrow F(x_0)$ in τ_p .*

2. Extended notions of mixed monotone mapping and fixed point

In this section we introduce certain type of mixed monotone property and fixed point of the mapping $F : X^n \rightarrow X$ with respect to $\Phi = (\varphi_1, \dots, \varphi_n)$, where I and J are nonempty disjoint subsets of $S = \{1, 2, \dots, n\}$ with $S = I \cup J$, and Φ is an n -tuple of self-maps on S with

$$\varphi_i(I) \subset I, \quad \varphi_i(J) \subset J, \quad \varphi_j(I) \subset J, \quad \varphi_j(J) \subset I$$

for $i \in I$ and for $j \in J$.

Definition 2.1. Let (X, \leq) be a partially ordered set, $n(\geq 2)$ a positive integer, I and J nonempty disjoint subsets of $S = \{1, 2, \dots, n\}$ with $S = I \cup J$, and $F : X^n \rightarrow X$ a given mapping. We say that F has the (I, J) -mixed monotone property if $F(x_1, \dots, x_n)$ is monotone nondecreasing in each component x_i for $i \in I$, and monotone nonincreasing in each component x_j for $j \in J$.

The above definition is a generalization of m -mixed monotone property for the mapping $F : X^n \rightarrow X$ introduced by M. Berzig and B. Samet [5], where $n \geq 2$ and $1 \leq m < n$.

Definition 2.2. Let X be a nonempty set, $n(\geq 2)$ a positive integer, I and J nonempty disjoint subsets of $S = \{1, 2, \dots, n\}$ with $S = I \cup J$, and $F : X^n \rightarrow X$ a given mapping. Let $\Phi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ be an n -tuple of self-maps on S with

$$\varphi_i(I) \subset I, \quad \varphi_i(J) \subset J, \quad \varphi_j(I) \subset J, \quad \varphi_j(J) \subset I$$

for $i \in I$ and $j \in J$. An element $\mathbf{x} = (x_1, x_2, \dots, x_n) \in X^n$ is called a (I, J) -fixed point of F with respect to Φ if

$$(2.1) \quad x_k = F(x_{\varphi_k(1)}, \dots, x_{\varphi_k(n)}) \quad \text{for all } k \in S.$$

Remark 2.1. We explain that the above definition is a generalization of several kinds of fixed points.

- (1) (See [6]). An element $(x, y) \in X^2$ is called a *coupled fixed point* of a mapping $F : X^2 \rightarrow X$ if

$$x = F(x, y) \text{ and } y = F(y, x).$$

In Definition 2.2 for $n = 2$, let $I = \{1\}$ and $J = \{2\}$. Consider $\Phi_2 = (\varphi_1, \varphi_2)$ defined by

$$\begin{aligned} \varphi_1(1) &= 1, \varphi_1(2) = 2, \\ \varphi_2(1) &= 2, \varphi_2(2) = 1. \end{aligned}$$

Thus, a coupled fixed point is the (I, J) -fixed point of F with respect to Φ_2 .

- (2) (See [4]). An element $(x, y, z) \in X^3$ is called a *triple fixed point* of a mapping $F : X^3 \rightarrow X$ if

$$F(x, y, z) = x, F(y, x, y) = y \text{ and } F(z, y, x) = z.$$

In Definition 2.2 for $n = 3$, let $I = \{1, 3\}$ and $J = \{2\}$. Then a triple fixed point is the (I, J) -fixed point of F with respect to $\Phi_3 = (\phi_1, \phi_2, \phi_3)$ defined by

$$\begin{aligned} \phi_1(1) &= 1, \phi_1(2) = 2, \phi_1(3) = 3, \\ \phi_2(1) &= 2, \phi_2(2) = 1, \phi_2(3) = 2, \\ \phi_3(1) &= 3, \phi_3(2) = 2, \phi_3(3) = 1. \end{aligned}$$

- (3) (See [5]). An element $(x, y, z, w) \in X^4$ is called a *quadruple fixed point* of a mapping $F : X^4 \rightarrow X$ if

$$F(x, y, z, w) = x, F(x, y, w, z) = y, F(z, w, y, x) = z, \text{ and } F(z, w, x, y) = w.$$

In Definition 2.2 for $n = 4$, let $I = \{1, 2\}$ and $J = \{3, 4\}$. Then a quadruple fixed point is the (I, J) -fixed point of F with respect to $\Phi_4 = (\psi_1, \psi_2, \psi_3, \psi_4)$ defined by

$$\begin{aligned} \psi_1(1) &= 1, \psi_1(2) = 2, \psi_1(3) = 3, \psi_1(4) = 4, \\ \psi_2(1) &= 1, \psi_2(2) = 2, \psi_2(3) = 4, \psi_2(4) = 3, \\ \psi_3(1) &= 3, \psi_3(2) = 4, \psi_3(3) = 2, \psi_3(4) = 1, \\ \psi_4(1) &= 3, \psi_4(2) = 4, \psi_4(3) = 1, \psi_4(4) = 2. \end{aligned}$$

Definition 2.3. For a positive integer k , an element $x \in X$ is called a *fixed point of order n* of $F : X^n \rightarrow X$ if $F(x, x, \dots, x) = x$.

3. Main results

Let (X, \leq) be a partially ordered set, $n (\geq 2)$ a positive integer, and I, J nonempty disjoint subsets of $S = \{1, 2, \dots, n\}$ with $S = I \cup J$. We endow the product set X^n with the partial order \leq_I defined by

$$(x_1, \dots, x_n) \leq_I (y_1, \dots, y_n) \iff \begin{cases} x_i \leq y_i, & i \in I \\ x_j \geq y_j, & j \in J. \end{cases}$$

For $\mathbf{x} = (x_1, x_2, \dots, x_n) \in X^n$ and a self-map φ on S , we denote for the notational convenience

$$\mathbf{x}_\varphi := (x_{\varphi(1)}, x_{\varphi(2)}, \dots, x_{\varphi(n)}).$$

Our first result is following.

Theorem 3.1. *Let (X, \leq) be a partially ordered set and p a partial metric on X such that (X, p) be a complete partial metric space. Let $n(\geq 2)$ be an integer, I and J nonempty disjoint subsets of $S = \{1, 2, \dots, n\}$ with $S = I \cup J$, $F : X^n \rightarrow X$ a given mapping, and let $\Phi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ be an n -tuple of self-maps on S with*

$$(3.1) \quad \varphi_i(I) \subset I, \quad \varphi_i(J) \subset J, \quad \varphi_j(I) \subset J, \quad \varphi_j(J) \subset I$$

for $i \in I$ and $j \in J$.

Suppose that the following conditions hold:

- (i) F is continuous;
- (ii) F has the (I, J) -mixed monotone property;
- (iii) for each $k \in S$, there exists $(p, q) \in S \times S$ such that $k = \varphi_p(q)$;
- (iv) there exists $\mathbf{x}^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)}) \in X^n$ such that

$$(3.2) \quad \begin{aligned} x_i^{(0)} &\leq F(x_{\varphi_i(1)}^{(0)}, \dots, x_{\varphi_i(n)}^{(0)}), & i \in I, \\ x_j^{(0)} &\geq F(x_{\varphi_j(1)}^{(0)}, \dots, x_{\varphi_j(n)}^{(0)}), & j \in J; \end{aligned}$$

- (v) there exist altering distance functions ϕ and ψ such that

$$(3.3) \quad \phi(p(F(\mathbf{x}), F(\mathbf{y}))) \leq \phi\left(\max_{k \in S}\{p(x_k, y_k)\}\right) - \psi\left(\max_{k \in S}\{p(x_k, y_k)\}\right)$$

for all $\mathbf{x} = (x_1, \dots, x_n) \leq_I \mathbf{y} = (y_1, \dots, y_n)$.

Then F has an (I, J) -fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of F with respect to Φ .

Proof. We consider the sequences $\{\mathbf{x}^{(r)}\}_{r=0}^\infty$ defined by

$$\mathbf{x}^{(r)} = (x_1^{(r)}, \dots, x_n^{(r)}), \quad r \in \mathbb{N},$$

where $x_k^{(r)} = F(x_{\varphi_k}^{(r-1)}) = F(x_{\varphi_k(1)}^{(r-1)}, x_{\varphi_k(2)}^{(r-1)}, \dots, x_{\varphi_k(n)}^{(r-1)})$ for all $k = 1, 2, \dots, n$.

In order that the proof is more comprehensive, we will divide it in several steps.

Step 1. $\{\mathbf{x}^{(r)}\}_{r=0}^\infty$ is the nondecreasing sequence with respect to \leq_I , that is,

$$\begin{aligned} \{x_i^{(r)}\}_{r=0}^\infty, & \quad i \in I \text{ are nondecreasing sequences, and} \\ \{x_j^{(r)}\}_{r=0}^\infty, & \quad j \in J \text{ are nonincreasing sequences.} \end{aligned}$$

We will show by induction that

$$x_i^{(r)} \geq x_i^{(r-1)} \quad \text{and} \quad x_j^{(r)} \leq x_j^{(r-1)}$$

for all $i \in I$ and for all $j \in J$.

By (3.2), we have

$$x_i^{(1)} = F(\mathbf{x}_{\varphi_i}^{(0)}) \geq x_i^{(0)} \quad \text{and} \quad x_j^{(1)} = F(\mathbf{x}_{\varphi_j}^{(0)}) \leq x_j^{(0)}.$$

Again by the induction hypothesis and the (I, J) -mixed monotone property of F , we have

$$x_i^{(r+1)} = F(\mathbf{x}_{\varphi_i}^{(r)}) \geq F(\mathbf{x}_{\varphi_i}^{(r-1)}) = x_i^{(r)}$$

and

$$x_j^{(r+1)} = F(\mathbf{x}_{\varphi_j}^{(r)}) \leq F(\mathbf{x}_{\varphi_j}^{(r-1)}) = x_j^{(r)}.$$

This proves our claim.

Step 2. For any $k \in S$ and $r \geq 1$, $\mathbf{x}_{\varphi_k}^{(r)}$ is comparable to $\mathbf{x}_{\varphi_k}^{(r-1)}$ with respect to \leq_I .

Suppose $k \in I$. Then by (3.1),

$$\varphi_k(i) \in I \quad \text{for } i \in I \quad \text{and} \quad \varphi_k(j) \in J \quad \text{for } j \in J$$

and by the definition of $\mathbf{x}_{\varphi_k}^{(r)}$ and Step 1, we get

$$(3.4) \quad \begin{cases} \left(\mathbf{x}_{\varphi_k}^{(r)}\right)_i = x_{\varphi_k(i)}^{(r)} \geq x_{\varphi_k(i)}^{(r-1)} = \left(\mathbf{x}_{\varphi_k}^{(r-1)}\right)_i & \text{for } i \in I, \\ \left(\mathbf{x}_{\varphi_k}^{(r)}\right)_j = x_{\varphi_k(j)}^{(r)} \leq x_{\varphi_k(j)}^{(r-1)} = \left(\mathbf{x}_{\varphi_k}^{(r-1)}\right)_j & \text{for } j \in J. \end{cases}$$

Consequently, $\mathbf{x}_{\varphi_k}^{(r)} \geq_I \mathbf{x}_{\varphi_k}^{(r-1)}$. The case of $k \in J$ can be similarly proven.

Step 3. $\lim_{r \rightarrow \infty} p(x_k^{(r+1)}, x_k^{(r)}) = 0$ for each $k \in S$.

For $r \geq 1$ and $k \in S$ we denote

$$D_k^{(r)} := \max_{l \in S} \left\{ p\left(x_{\varphi_k(l)}^{(r)}, x_{\varphi_k(l)}^{(r-1)}\right) \right\} \quad \text{and} \quad E^{(r)} := \max_{k \in S} \left\{ p\left(x_k^{(r)}, x_k^{(r-1)}\right) \right\}.$$

Then, for each $k \in S$,

$$D_k^{(r)} \leq E^{(r)},$$

because $\varphi_k(S) \subset S$.

From the contractive condition (v) and Step 2, we obtain

$$(3.5) \quad \begin{aligned} \phi\left(p\left(x_k^{(r+1)}, x_k^{(r)}\right)\right) &= \phi\left(p\left(F\left(\mathbf{x}_{\varphi_k}^{(r)}\right), F\left(\mathbf{x}_{\varphi_k}^{(r-1)}\right)\right)\right) \\ &\leq \phi\left(D_k^{(r)}\right) - \psi\left(D_k^{(r)}\right) \\ &\leq \phi\left(D_k^{(r)}\right) \leq \phi\left(E^{(r)}\right). \end{aligned}$$

Using the fact that ϕ is nondecreasing, we have

$$p\left(x_k^{(r+1)}, x_k^{(r)}\right) \leq E^{(r)}.$$

Hence,

$$E^{(r+1)} = \max_{k \in S} \left\{ p\left(x_k^{(r+1)}, x_k^{(r)}\right) \right\} \leq E^{(r)},$$

and thus, the sequence $\{E^{(r)}\}$ is decreasing and nonnegative. This implies that there exists $\alpha \geq 0$ such that

$$\lim_{r \rightarrow \infty} E^{(r)} = \alpha.$$

It is easily seen that if $\phi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, $\phi(\max\{a_1, a_2, \dots, a_n\}) = \max\{\phi(a_1), \phi(a_2), \dots, \phi(a_n)\}$ for $a_1, a_2, \dots, a_n \in [0, \infty)$. Taking into account this and (3.5) we get

$$\phi(E^{(r+1)}) \leq \phi(D_k^{(r)}) - \psi(D_k^{(r)}) \leq \phi(D_k^{(r)}) \leq \phi(E^{(r)}).$$

Since ϕ is a continuous function, letting $r \rightarrow +\infty$ in the above inequalities yields

$$\phi(\alpha) \leq \lim_{r \rightarrow \infty} (\phi(D_k^{(r)}) - \psi(D_k^{(r)})) \leq \lim_{r \rightarrow \infty} \phi(D_k^{(r)}) \leq \phi(\alpha).$$

The squeeze theorem gives us

$$\lim_{r \rightarrow \infty} (\phi(D_k^{(r)}) - \psi(D_k^{(r)})) = \lim_{r \rightarrow \infty} \phi(D_k^{(r)}) = \phi(\alpha),$$

and this implies $\lim_{r \rightarrow \infty} \psi(D_k^{(r)}) = 0$. Since ψ is an altering distance function,

$$(3.6) \quad \lim_{r \rightarrow \infty} D_k^{(r)} = \lim_{r \rightarrow \infty} \max_{l \in S} \left\{ p(x_{\varphi_k(l)}^{(r)}, x_{\varphi_k(l)}^{(r-1)}) \right\} = 0, \quad k \in S.$$

From assumption (iii) we have that for each $k \in S$, there exists $(k_1, k_2) \in S \times S$ such that $\varphi_{k_1}(k_2) = k$. Thus, by (3.6), we obtain

$$\begin{aligned} 0 &\leq \lim_{r \rightarrow \infty} p(x_k^{(r+1)}, x_k^{(r)}) = \lim_{r \rightarrow \infty} p(x_{\varphi_{k_1}(k_2)}^{(r+1)}, x_{\varphi_{k_1}(k_2)}^{(r)}) \\ &\leq \lim_{r \rightarrow \infty} \max_{l \in S} \left\{ p(x_{\varphi_{k_1}(l)}^{(r)}, x_{\varphi_{k_1}(l)}^{(r-1)}) \right\} = \lim_{r \rightarrow \infty} D_{k_1}^{(r)} = 0, \end{aligned}$$

and this proves our claim.

Step 4. For each $k \in S$,

$$\lim_{\alpha, \beta \rightarrow \infty} p(x_k^{(\alpha)}, x_k^{(\beta)}) = 0.$$

Assume that $\lim_{\alpha, \beta \rightarrow \infty} p(x_k^{(\alpha)}, x_k^{(\beta)}) \not\rightarrow 0$ for some $k \in S$. This implies that

$$\lim_{\alpha, \beta \rightarrow \infty} \max_{k \in S} \left\{ p(x_k^{(\alpha)}, x_k^{(\beta)}) \right\} \not\rightarrow 0.$$

So there exists an $\varepsilon > 0$ for which we can find two subsequences $\{x_k^{(\alpha(r))}\}_{r=0}^\infty$ and $\{x_k^{(\beta(r))}\}_{r=0}^\infty$ of $\{x_k^{(r)}\}_{r=0}^\infty$ such that $\beta(r)$ is the smallest index satisfying

$$(3.7) \quad \beta(r) > \alpha(r) > r, \quad \max_{k \in S} \left\{ p(x_k^{(\alpha(r))}, x_k^{(\beta(r))}) \right\} \geq \varepsilon.$$

This means that

$$(3.8) \quad \max_{k \in S} \left\{ p(x_k^{(\alpha(r))}, x_k^{(\beta(r)-1)}) \right\} < \varepsilon.$$

For each k , (P4) and (3.8) give us

$$\begin{aligned}
 (3.9) \quad & p\left(x_k^{(\beta(r))}, x_k^{(\alpha(r))}\right) \\
 & \leq p\left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)}\right) + p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r))}\right) - p\left(x_k^{(\beta(r)-1)}, x_k^{(\beta(r)-1)}\right) \\
 & \leq p\left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)}\right) + p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r))}\right) \\
 & < p\left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)}\right) + \varepsilon.
 \end{aligned}$$

Using (3.7) and (3.9), we get

$$\varepsilon \leq \max_{k \in S} \left\{ p\left(x_k^{(\alpha(r))}, x_k^{(\beta(r))}\right) \right\} < \max_{k \in S} \left\{ p\left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)}\right) \right\} + \varepsilon.$$

Letting $r \rightarrow \infty$ in the last inequality and using Step 3, we obtain that

$$(3.10) \quad \lim_{r \rightarrow \infty} \max_{k \in S} \left\{ p\left(x_k^{(\alpha(r))}, x_k^{(\beta(r))}\right) \right\} = \varepsilon.$$

Again, for each k , (P4) and (3.8) give us

$$\begin{aligned}
 & p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)}\right) \\
 & \leq p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r))}\right) + p\left(x_k^{(\alpha(r))}, x_k^{(\alpha(r)-1)}\right) - p\left(x_k^{(\alpha(r))}, x_k^{(\alpha(r))}\right) \\
 & \leq p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r))}\right) + p\left(x_k^{(\alpha(r))}, x_k^{(\alpha(r)-1)}\right) \\
 & < \varepsilon + p\left(x_k^{(\alpha(r))}, x_k^{(\alpha(r)-1)}\right),
 \end{aligned}$$

which implies to

$$(3.11) \quad \max_{k \in S} \left\{ p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)}\right) \right\} < \max_{k \in S} \left\{ p\left(x_k^{(\alpha(r))}, x_k^{(\alpha(r)-1)}\right) \right\} + \varepsilon.$$

From (P4) we have, for each k ,

$$\begin{aligned}
 & p\left(x_k^{(\beta(r))}, x_k^{(\alpha(r))}\right) \\
 & \leq p\left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)}\right) + p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)}\right) + p\left(x_k^{(\alpha(r)-1)}, x_k^{(\alpha(r))}\right) \\
 & \quad - p\left(x_k^{(\beta(r)-1)}, x_k^{(\beta(r)-1)}\right) - p\left(x_k^{(\alpha(r)-1)}, x_k^{(\alpha(r)-1)}\right) \\
 & \leq p\left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)}\right) + p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)}\right) + p\left(x_k^{(\alpha(r)-1)}, x_k^{(\alpha(r))}\right),
 \end{aligned}$$

and by (3.7) we get

$$\begin{aligned}
 (3.12) \quad & \varepsilon \leq \max_{k \in S} \left\{ p\left(x_k^{(\beta(r))}, x_k^{(\alpha(r))}\right) \right\} \\
 & \leq \max_{k \in S} \left\{ p\left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)}\right) \right\} + \max_{k \in S} \left\{ p\left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)}\right) \right\} \\
 & \quad + \max_{k \in S} \left\{ p\left(x_k^{(\alpha(r)-1)}, x_k^{(\alpha(r))}\right) \right\}.
 \end{aligned}$$

By (3.11) and (3.12) we have

$$\begin{aligned} & \varepsilon - \max_{k \in S} \left\{ p \left(x_k^{(\beta(r))}, x_k^{(\beta(r)-1)} \right) \right\} - \max_{k \in S} \left\{ p \left(x_k^{(\alpha(r)-1)}, x_k^{(\alpha(r))} \right) \right\} \\ & \leq \max_{k \in S} \left\{ p \left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)} \right) \right\} \\ & < \max_{k \in S} \left\{ p \left(x_k^{(\alpha(r))}, x_k^{(\alpha(r)-1)} \right) \right\} + \varepsilon. \end{aligned}$$

Letting $r \rightarrow \infty$ in the last inequality, and by Step 3, we obtain that

$$(3.13) \quad \max_{k \in S} \left\{ p \left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)} \right) \right\} = \varepsilon.$$

Since $\beta(r) > \alpha(r) > r$, by Step 1

$$(x_1^{(\alpha(r)-1)}, x_2^{(\alpha(r)-1)}, \dots, x_n^{(\alpha(r)-1)}) \leq_I (x_1^{(\beta(r)-1)}, x_2^{(\beta(r)-1)}, \dots, x_n^{(\beta(r)-1)}).$$

Using the contractive condition (v) we can obtain

$$\begin{aligned} & \phi \left(p \left(x_l^{(\beta(r))}, x_l^{(\alpha(r))} \right) \right) \\ & = \phi \left(p \left(F \left(\mathbf{x}_{\varphi_l}^{(\beta(r)-1)} \right), F \left(\mathbf{x}_{\varphi_l}^{(\alpha(r)-1)} \right) \right) \right) \\ & \leq \phi \left(\max_{k \in S} \left\{ p \left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)} \right) \right\} \right) - \psi \left(\max_{k \in S} \left\{ p \left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)} \right) \right\} \right) \end{aligned}$$

for any $l = 1, 2, \dots, n$. Thus,

$$(3.14) \quad \begin{aligned} & \phi \left(\max_{k \in S} \left\{ p \left(x_k^{(\beta(r))}, x_k^{(\alpha(r))} \right) \right\} \right) \\ & \leq \phi \left(\max_{k \in S} \left\{ p \left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)} \right) \right\} \right) - \psi \left(\max_{k \in S} \left\{ p \left(x_k^{(\beta(r)-1)}, x_k^{(\alpha(r)-1)} \right) \right\} \right). \end{aligned}$$

Finally, letting $r \rightarrow \infty$ in (3.14) and using (3.10), (3.13), and the continuity of ϕ and ψ , we get

$$\phi(\varepsilon) \leq \phi(\varepsilon) - \psi(\varepsilon)$$

and, consequently, $\psi(\varepsilon) = 0$. Since ψ is an altering distance function, $\varepsilon = 0$, and this is a contradiction. This proves our claim.

Step 5. The sequence $\mathbf{x}^{(r)}$ converges to $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in X^n$ and

$$\lim_{r \rightarrow \infty} p(\bar{x}_k, x_k^{(r)}) = p(\bar{x}_k, \bar{x}_k) = 0 \quad \text{for all } k \in S.$$

By (1.1), for $k \in S$, we have

$$(3.15) \quad d_p(x_k^{(r_1)}, x_k^{(r_2)}) \leq 2p(x_k^{(r_1)}, x_k^{(r_2)}).$$

Letting $r_1, r_2 \rightarrow \infty$ in (3.15) and using Step 4, we get that

$$\lim_{r_1, r_2 \rightarrow \infty} d_p(x_k^{(r_1)}, x_k^{(r_2)}) = 0.$$

It means that $\{x_k^{(r)}\}$, $k \in S$ is a Cauchy sequence in the metric space (X, d_p) . Since (X, p) is complete, by Lemma 1.2(iii), it is also the case for (X, d_p) . Then there exists $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ such that

$$(3.16) \quad \lim_{r \rightarrow \infty} d_p(x_k^{(r)}, \bar{x}_k) = 0, \quad k \in S.$$

On the other hand, we have

$$d_p(x_k^{(r)}, \bar{x}_k) = 2p(x_k^{(r)}, \bar{x}_k) - p(x_k^{(r)}, x_k^{(r)}) - p(\bar{x}_k, \bar{x}_k).$$

Letting $r \rightarrow \infty$ in the above equation, using (3.16) and Step 4, we get

$$(3.17) \quad \lim_{r \rightarrow \infty} p(x_k^{(r)}, \bar{x}_k) = p(\bar{x}_k, \bar{x}_k), \quad k \in S.$$

From (P2) and (P3) we have $p(\bar{x}_k, \bar{x}_k) \leq p(\bar{x}_k, x_k^{(r)}) = p(x_k^{(r)}, \bar{x}_k)$ for all $r \in \mathbb{N}$. Taking $r \rightarrow \infty$, we have that

$$(3.18) \quad p(\bar{x}_k, \bar{x}_k) \leq \lim_{r \rightarrow \infty} p(x_k^{(r)}, \bar{x}_k), \quad k \in S.$$

Combining (3.17) and (3.18), we get that

$$\lim_{r \rightarrow \infty} p(x_k^{(r)}, \bar{x}_k) = p(\bar{x}_k, \bar{x}_k) = 0, \quad k \in S.$$

This proves our claim.

Now, we show that $\bar{\mathbf{x}}$ is an (I, J) -fixed point of F with respect to Φ , that is,

$$(3.19) \quad \bar{x}_k = F(\bar{x}_{\varphi_k(1)}, \bar{x}_{\varphi_k(2)}, \dots, \bar{x}_{\varphi_k(n)}) = F(\bar{\mathbf{x}}_{\varphi_k}), \quad k \in S.$$

By (P1) and Step 5, to see this, it suffices to show that

$$p(F(\bar{\mathbf{x}}_{\varphi_k}), F(\bar{\mathbf{x}}_{\varphi_k})) = p(F(\bar{\mathbf{x}}_{\varphi_k}), \bar{x}_k) = 0, \quad k \in S.$$

Since $\bar{\mathbf{x}}_{\varphi_k} = (\bar{x}_{\varphi_k(1)}, \bar{x}_{\varphi_k(2)}, \dots, \bar{x}_{\varphi_k(n)})$ is greater than or equal to itself with respect to \leq_I ,

$$\phi(p(F(\bar{\mathbf{x}}_{\varphi_k}), F(\bar{\mathbf{x}}_{\varphi_k}))) \leq \phi(\max_{l \in S} \{p(\bar{x}_{\varphi_k(l)}, \bar{x}_{\varphi_k(l)})\}) - \psi(\max_{l \in S} \{p(\bar{x}_{\varphi_k(l)}, \bar{x}_{\varphi_k(l)})\}).$$

From Step 5, the right-hand side of the above inequality is zero, and so we obtain $\phi(p(F(\bar{\mathbf{x}}_{\varphi_k}), F(\bar{\mathbf{x}}_{\varphi_k}))) = 0$, and thus,

$$(3.20) \quad p(F(\bar{\mathbf{x}}_{\varphi_k}), F(\bar{\mathbf{x}}_{\varphi_k})) = 0.$$

Since $x_k^{(r)} \rightarrow \bar{x}_k$ for all $k \in S$ as $r \rightarrow \infty$ in (X, p) and (i) F is continuous, we get $F(\mathbf{x}_{\varphi_k}^{(r)}) \rightarrow F(\bar{\mathbf{x}}_{\varphi_k})$ as $r \rightarrow \infty$ in (X, p) . Then, by (3.20) we get

$$(3.21) \quad \lim_{r \rightarrow \infty} p(F(\bar{\mathbf{x}}_{\varphi_k}), x_k^{(r+1)}) = \lim_{r \rightarrow \infty} p(F(\bar{\mathbf{x}}_{\varphi_k}), F(\mathbf{x}_{\varphi_k}^{(r)})) = p(F(\bar{\mathbf{x}}_{\varphi_k}), F(\bar{\mathbf{x}}_{\varphi_k})) = 0.$$

On the other hand,

$$\begin{aligned} p(F(\bar{\mathbf{x}}_{\varphi_k}), \bar{x}_k) &\leq p(F(\bar{\mathbf{x}}_{\varphi_k}), x_k^{(r+1)}) + p(x_k^{(r+1)}, \bar{x}_k) - p(x_k^{(r+1)}, x_k^{(r+1)}) \\ &\leq p(F(\bar{\mathbf{x}}_{\varphi_k}), x_k^{(r+1)}) + p(x_k^{(r+1)}, \bar{x}_k). \end{aligned}$$

Letting $r \rightarrow \infty$ in the above inequality, and using Step 5 and (3.21), we get that

$$p(F(\bar{\mathbf{x}}_{\varphi_k}), \bar{x}_k) = 0.$$

This completes our proof. □

The previous result is still valid for F not necessarily continuous. Instead, we require that the underlying partial metric space X has an additional property. We discuss this in the following theorem.

Theorem 3.2. *If in Theorem 3.1 we replace the continuity of F by*

- (i) *if $\{x_n\}$ is a nondecreasing sequence and there is $x \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x) = 0$, then $x_n \leq x$ for all n ,*
- (ii) *if $\{x_n\}$ is a nonincreasing sequence and there is $x \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x) = 0$, then $x_n \geq x$ for all n ,*

then F has an (I, J) -fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of F with respect to Φ .

Proof. From Step 5 and (3.20) in the proof of Theorem 3.1, we have that

$$p(\bar{x}_k, \bar{x}_k) = p(F(\bar{\mathbf{x}}_{\varphi_k}), F(\bar{\mathbf{x}}_{\varphi_k})) = 0, \quad k \in S.$$

Assume that the sequence $\{\mathbf{x}^{(r)}\}$ in X^n satisfies the following: for $i \in I$ and $j \in J$,

$$\begin{aligned} \{x_i^{(r)}\}_{r=0}^{\infty} &\text{ is nondecreasing and } \lim_{r \rightarrow \infty} p(x_i^{(r)}, \bar{x}_i) = p(\bar{x}_i, \bar{x}_i) = 0; \\ \{x_j^{(r)}\}_{r=0}^{\infty} &\text{ is nonincreasing and } \lim_{r \rightarrow \infty} p(x_j^{(r)}, \bar{x}_j) = p(\bar{x}_j, \bar{x}_j) = 0. \end{aligned}$$

To see that $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ is an (I, J) -fixed point of F with respect to Φ , by (P1) it remains to show that $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ satisfies that

$$p(\bar{x}_k, F(\bar{\mathbf{x}}_{\varphi_k})) = 0, \quad k \in S.$$

By our assumption, $x_i^{(r)} \leq \bar{x}_i$ and $x_j^{(r)} \geq \bar{x}_j$ for every $r \in \mathbb{N}$, and thus, $\mathbf{x}^{(r)} \leq_I \bar{\mathbf{x}}$. Applying the contractive condition of altering distance functions ϕ and ψ , we have

$$\begin{aligned} \phi\left(p\left(F(\bar{\mathbf{x}}), F(\mathbf{x}^{(r)})\right)\right) &\leq \phi\left(\max_{k \in S} \left\{p\left(\bar{x}_k, x_k^{(r)}\right)\right\}\right) - \psi\left(\max_{k \in S} \left\{p\left(\bar{x}_k, x_k^{(r)}\right)\right\}\right) \\ &\leq \phi\left(\max_{k \in S} \left\{p\left(\bar{x}_k, x_k^{(r)}\right)\right\}\right). \end{aligned}$$

Since ϕ is nondecreasing, we obtain

$$(3.22) \quad p\left(F(\bar{\mathbf{x}}), F(\mathbf{x}^{(r)})\right) \leq \max_{k \in S} \left\{p\left(\bar{x}_k, x_k^{(r)}\right)\right\}.$$

On the other hand, for any $k \in S$, by (P4) and (3.22) we get

$$\begin{aligned} p(\bar{x}_k, F(\bar{\mathbf{x}}_{\varphi_k})) &\leq p\left(\bar{x}_k, x_k^{(r+1)}\right) + p\left(x_k^{(r+1)}, F(\bar{\mathbf{x}}_{\varphi_k})\right) - p\left(x_k^{(r+1)}, x_k^{(r+1)}\right) \\ &\leq p\left(\bar{x}_k, x_k^{(r+1)}\right) + p\left(x_k^{(r+1)}, F(\bar{\mathbf{x}}_{\varphi_k})\right) \end{aligned}$$

$$\begin{aligned}
 &= p\left(\bar{x}_k, x_k^{(r+1)}\right) + p\left(F\left(\mathbf{x}_{\varphi_k}^{(r)}\right), F\left(\bar{\mathbf{x}}_{\varphi_k}\right)\right) \\
 &\leq p\left(\bar{x}_k, x_k^{(r+1)}\right) + \max_{l \in S} \left\{ p\left(x_{\varphi_k(l)}^{(r)}, \bar{x}_{\varphi_k(l)}\right) \right\}.
 \end{aligned}$$

Since $\lim_{r \rightarrow \infty} p(x_k^{(r)}, \bar{x}_k) = p(\bar{x}_k, \bar{x}_k) = 0$ for all $k \in S$, taking $r \rightarrow \infty$ in the last inequality implies

$$p(\bar{x}_k, F(\bar{\mathbf{x}}_{\varphi_k})) = 0.$$

This completes the proof. □

4. Uniqueness of (I, J) -fixed point with respect to Φ

In this section, we consider some additional conditions to ensure the uniqueness of the (I, J) -fixed point and appropriate conditions to ensure that for the (I, J) -fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ we have $\bar{x}_1 = \bar{x}_2 = \dots = \bar{x}_n$.

Theorem 4.1. *Adding the following hypothesis to the hypotheses of Theorem 3.1 (resp. Theorem 3.2) :*

- (H) *For all $\mathbf{x}, \mathbf{y} \in X^n$, there exists $\mathbf{z} \in X^n$ that is comparable to \mathbf{x} and \mathbf{y} with respect to the partial order \leq_I ,*

then we obtain the uniqueness of the (I, J) -fixed point of F with respect to Φ .

Proof. By Theorem 3.1 (resp. Theorem 3.2), the set of (I, J) -fixed points of F is nonempty. Suppose that $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ and $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)$ in X^n are (I, J) -fixed points of F with respect $\Phi = (\varphi_1, \varphi_2, \dots, \varphi_n)$, that is, for $k \in S$

$$\bar{x}_k = F(\bar{\mathbf{x}}_{\varphi_k}) \quad \text{and} \quad \bar{y}_k = F(\bar{\mathbf{y}}_{\varphi_k}),$$

where $\bar{\mathbf{x}}_{\varphi_k} = (\bar{x}_{\varphi_k(1)}, \bar{x}_{\varphi_k(2)}, \dots, \bar{x}_{\varphi_k(n)})$ and $\bar{\mathbf{y}}_{\varphi_k} = (\bar{y}_{\varphi_k(1)}, \bar{y}_{\varphi_k(2)}, \dots, \bar{y}_{\varphi_k(n)})$, respectively. Again, by Theorem 3.1 (resp. Theorem 3.2), we get

$$(4.1) \quad p(\bar{x}_k, \bar{x}_k) = p(\bar{y}_k, \bar{y}_k) = 0, \quad k \in S.$$

By assumption, there exists $\mathbf{z} = (z_1, z_2, \dots, z_n)$ in X^n that is comparable to $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ with respect to \leq_I . We define the sequence $\{\mathbf{z}^{(r)}\}$ as follows

$$\mathbf{z}^{(0)} = \mathbf{z}, \quad \mathbf{z}^{(r)} = \left(z_1^{(r)}, z_2^{(r)}, \dots, z_n^{(r)} \right),$$

where $z_k^{(r+1)} = F\left(\mathbf{z}_{\varphi_k}^{(r)}\right)$ and $\mathbf{z}_{\varphi_k}^{(r)} = \left(z_{\varphi_k(1)}^{(r)}, z_{\varphi_k(2)}^{(r)}, \dots, z_{\varphi_k(n)}^{(r)} \right)$ for $k \in S$.

Since \mathbf{z} is comparable with $\bar{\mathbf{x}}$ with respect to \leq_I , we may assume that $\mathbf{z} \leq_I \bar{\mathbf{x}}$. Using the mathematical induction, it is easy to show that $\mathbf{z}^{(r)} \leq_I \bar{\mathbf{x}}$ for all $r \in \mathbb{N}$, and $\mathbf{z}_{\varphi_k}^{(r)}$ and $\bar{\mathbf{x}}_{\varphi_k}$ are comparable for all $k \in S$. Applying the contractive condition of altering distance functions ϕ and ψ we have

$$\begin{aligned}
 &\phi\left(p\left(\bar{x}_k, z_k^{(r+1)}\right)\right) \\
 &= \phi\left(p\left(F\left(\mathbf{x}_{\varphi_k}\right), F\left(\mathbf{z}_{\varphi_k}^{(r)}\right)\right)\right)
 \end{aligned}$$

$$\begin{aligned} &\leq \phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) - \psi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) \\ &\leq \phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right). \end{aligned}$$

By the last equation and using the fact that ϕ is nondecreasing, we obtain

$$\begin{aligned} &\phi \left(\max_{k \in S} \left\{ p \left(\bar{x}_k, z_k^{(r+1)} \right) \right\} \right) = \max_{k \in S} \left\{ \phi \left(p \left(\bar{x}_k, z_k^{(r+1)} \right) \right) \right\} \\ (4.2) \quad &\leq \phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) - \psi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) \\ &\leq \phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right). \end{aligned}$$

In fact, since $\varphi_k(S) \subset S$, (4.2) implies

$$(4.3) \quad \max_{k \in S} \left\{ p \left(\bar{x}_k, z_k^{(r+1)} \right) \right\} \leq \max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \leq \max_{k \in S} \left\{ p \left(\bar{x}_k, z_k^{(r)} \right) \right\}.$$

Consequently, the sequence $\left(\max_{k \in S} \left\{ p \left(\bar{x}_k, z_k^{(r)} \right) \right\} \right)$ is decreasing and nonnegative, and so, for certain $\alpha \geq 0$

$$(4.4) \quad \lim_{r \rightarrow \infty} \max_{k \in S} \left\{ p \left(\bar{x}_k, z_k^{(r)} \right) \right\} = \alpha.$$

Letting $r \rightarrow \infty$ in (4.2) with continuity of ϕ and using (4.3) and (4.4) we have

$$\begin{aligned} \phi(\alpha) &\leq \lim_{r \rightarrow \infty} \left[\phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) - \psi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) \right] \\ &\leq \lim_{r \rightarrow \infty} \phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) \\ &\leq \phi(\alpha). \end{aligned}$$

The squeeze theorem gives us

$$\begin{aligned} \phi(\alpha) &= \lim_{r \rightarrow \infty} \phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) \\ &= \lim_{r \rightarrow \infty} \phi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right) - \lim_{r \rightarrow \infty} \psi \left(\max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} \right), \end{aligned}$$

and this implies

$$\lim_{r \rightarrow \infty} \max_{l \in S} \left\{ p \left(\bar{x}_{\varphi_k(l)}, z_{\varphi_k(l)}^{(r)} \right) \right\} = 0.$$

From the condition (iii) in Theorem 3.1, we have

$$(4.5) \quad \lim_{r \rightarrow \infty} p \left(\bar{x}_k, z_k^{(r)} \right) = 0 \quad \text{for all } k \in S.$$

Using a similar argument for $\bar{\mathbf{y}} = (\bar{y}_1, \dots, \bar{y}_n)$, we can obtain

$$(4.6) \quad \lim_{r \rightarrow \infty} p\left(\bar{y}_k, z_k^{(r)}\right) = 0 \quad \text{for all } k \in S.$$

From (P4)

$$\begin{aligned} p(\bar{x}_k, \bar{y}_k) &\leq p\left(\bar{x}_k, z_k^{(r)}\right) + p\left(z_k^{(r)}, \bar{y}_k\right) - p\left(z_k^{(r)}, z_k^{(r)}\right) \\ &\leq p\left(\bar{x}_k, z_k^{(r)}\right) + p\left(z_k^{(r)}, \bar{y}_k\right). \end{aligned}$$

Letting $r \rightarrow \infty$, (4.5) and (4.6) yield $p(\bar{x}_k, \bar{y}_k) = 0$. Combining (4.1) and (P1), we have $\bar{x}_k = \bar{y}_k$ for all $k \in S$, and hence,

$$\bar{\mathbf{x}} = \bar{\mathbf{y}}. \quad \square$$

Theorem 4.2. *In addition to the hypotheses of Theorem 3.1 (resp. Theorem 3.2), suppose that for any $i \in I$ and $j \in J$*

$$x_i^{(0)} \leq x_j^{(0)}.$$

Then there exists a fixed point $x \in X$ of order n of F , that is, $F(x, \dots, x) = x$.

Proof. Theorem 3.1 (resp. Theorem 3.2) ensures that there exists an (I, J) -fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of F with respect to Φ , that is,

$$\bar{x}_k = F(\bar{x}_{\varphi_k(1)}, \dots, \bar{x}_{\varphi_k(n)}) \quad \text{for all } k \in S.$$

It is enough to show that $\bar{x}_1 = \bar{x}_2 = \dots = \bar{x}_n$. We will prove it by several steps.

Step 1. $\bar{x}_i \leq \bar{x}_j$ for any $i \in I$ and $j \in J$.

We will show that $x_i^{(r)} \leq x_j^{(r)}$ for all $i \in I, j \in J$ and $r \in \mathbb{N}$. Obviously, the inequality is satisfied for $r = 0$.

Suppose $x_i^{(r)} \leq x_j^{(r)}$ for any $i \in I$ and $j \in J$. Then by (3.1), we have

$$(4.7) \quad x_{\varphi_i(k)}^{(r)} \leq x_{\varphi_j(k)}^{(r)}, \quad k \in I \quad \text{and} \quad x_{\varphi_i(k)}^{(r)} \geq x_{\varphi_j(k)}^{(r)}, \quad k \in J.$$

By (4.7) and the (I, J) -mixed monotone property of F , we obtain

$$x_i^{(r+1)} = F\left(x_{\varphi_i(1)}^{(r)}, \dots, x_{\varphi_i(n)}^{(r)}\right) \leq F\left(x_{\varphi_j(1)}^{(r)}, \dots, x_{\varphi_j(n)}^{(r)}\right) = x_j^{(r+1)}.$$

In the proof of Theorem 3.1 (resp. Theorem 3.2), we have shown that for each k , $\{x_k^{(r)}\}$ is a Cauchy sequence in a complete metric space (X, d_p) and converges to \bar{x}_k . Hence, we obtain

$$(4.8) \quad \bar{x}_i \leq \bar{x}_j \quad \text{for any } i \in I, j \in J.$$

This proves our claim.

Step 2. $\bar{x}_s = \bar{x}_t$ for any $s, t \in S$.

In the proof of Theorem 3.1, we have shown that $p(\bar{x}_k, \bar{x}_k) = 0$ for all $k \in S$. By (P1), to see this claim, it suffices to show that $p(\bar{x}_s, \bar{x}_t) = 0$ for any $s, t \in S$.

Assume that $s \in I$ and $t \in J$. Suppose that for some $s_1 \in I$ and $t_1 \in J$

$$p(\bar{x}_{s_1}, \bar{x}_{t_1}) > 0.$$

Note from Step 1 that $\bar{x}_{\varphi_{s_1}(k)} \leq \bar{x}_{\varphi_{t_1}(k)}$ for $k \in I$ and $\bar{x}_{\varphi_{s_1}(k)} \geq \bar{x}_{\varphi_{t_1}(k)}$ for $k \in J$. By the contractive condition (v), we get

$$\begin{aligned} 0 &< \phi(p(\bar{x}_{s_1}, \bar{x}_{t_1})) \\ &= \phi\left(p\left(F\left(\bar{x}_{\varphi_{s_1}(1)}, \dots, \bar{x}_{\varphi_{s_1}(n)}\right), F\left(\bar{x}_{\varphi_{t_1}(1)}, \dots, \bar{x}_{\varphi_{t_1}(n)}\right)\right)\right) \\ &\leq \phi\left(\max_{k \in S} \left\{p\left(\bar{x}_{\varphi_{s_1}(k)}, \bar{x}_{\varphi_{t_1}(k)}\right)\right\}\right) - \psi\left(\max_{k \in S} \left\{p\left(\bar{x}_{\varphi_{s_1}(k)}, \bar{x}_{\varphi_{t_1}(k)}\right)\right\}\right) \\ &\leq \phi\left(\max_{k \in S} \left\{p\left(\bar{x}_{\varphi_{s_1}(k)}, \bar{x}_{\varphi_{t_1}(k)}\right)\right\}\right). \end{aligned}$$

Since ϕ is the altering distance function, we have

$$\max_{k \in S} \left\{p\left(\bar{x}_{\varphi_{s_1}(k)}, \bar{x}_{\varphi_{t_1}(k)}\right)\right\} > 0.$$

We take $s_2 \in I$ and $t_2 \in J$ such that $d(\bar{x}_{s_2}, \bar{x}_{t_2}) = \max_{k \in S} \left\{p\left(\bar{x}_{\varphi_{s_1}(k)}, \bar{x}_{\varphi_{t_1}(k)}\right)\right\} > 0$. Continuing in this way, we construct a sequence of pairs $\{(s_m, t_m)\}_m$ in $I \times J$ such that for any $m \in \mathbb{N}$

$$p(\bar{x}_{s_m}, \bar{x}_{t_m}) > 0 \quad \text{and} \quad p(\bar{x}_{s_{m+1}}, \bar{x}_{t_{m+1}}) = \max_{k \in S} \left\{p(\bar{x}_{\varphi_{s_m}(k)}, \bar{x}_{\varphi_{t_m}(k)})\right\}.$$

Then we have for all $m \in \mathbb{N}$

$$(4.9) \quad \phi(p(\bar{x}_{s_m}, \bar{x}_{t_m})) > \sum_{r=2}^m \psi(p(\bar{x}_{s_r}, \bar{x}_{t_r})).$$

Indeed,

$$\begin{aligned} 0 &< \phi(p(\bar{x}_{s_1}, \bar{x}_{t_1})) \\ &\leq \phi(p(\bar{x}_{s_2}, \bar{x}_{t_2})) - \psi(p(\bar{x}_{s_2}, \bar{x}_{t_2})) \\ &\leq \phi(p(\bar{x}_{s_3}, \bar{x}_{t_3})) - \psi(p(\bar{x}_{s_3}, \bar{x}_{t_3})) - \psi(p(\bar{x}_{s_2}, \bar{x}_{t_2})) \\ &\dots \\ &\leq \phi(p(\bar{x}_{s_m}, \bar{x}_{t_m})) - \sum_{r=2}^m \psi(p(\bar{x}_{s_r}, \bar{x}_{t_r})). \end{aligned}$$

Let $D = \{(s, t) \in I \times J : p(\bar{x}_s, \bar{x}_t) > 0\}$. Then D is nonempty, because it contains the sequence $\{(s_m, t_m)\}_m$ defined above. Since $D \subset I \times J$ has finite elements, we can take the minimum of the images of $\psi \circ p$ over D such as

$$\mu := \min\{\psi(p(\bar{x}_s, \bar{x}_t)) : (s, t) \in D\} > 0.$$

We choose $N \in \mathbb{N}$ such that

$$N \geq \frac{\max_{s \in I, t \in J} \{\phi(p(\bar{x}_s, \bar{x}_t))\}}{\mu} + 1.$$

Then we have

$$\phi(p(\bar{x}_{s_N}, \bar{x}_{t_N})) \leq \max_{s \in I, t \in J} \phi(p(\bar{x}_s, \bar{x}_t)) \leq (N - 1)\mu \leq \sum_{r=2}^N \psi(p(\bar{x}_{s_r}, \bar{x}_{t_r})).$$

This is a contradiction to (4.9). So for any $s \in I$ and $t \in J$

$$(4.10) \quad \bar{x}_s = \bar{x}_t.$$

Now, suppose that $s_1, s_2 \in I$ (resp., $s_1, s_2 \in J$). Then by (4.10), for some $t \in J$ (resp., $t \in I$) we have

$$\bar{x}_{s_1} = \bar{x}_t \quad \text{and} \quad \bar{x}_{s_2} = \bar{x}_t.$$

Thus, $\bar{x}_{s_1} = \bar{x}_{s_2}$ for $s_1, s_2 \in I$ (resp., $s_1, s_2 \in J$).

This completes the proof. □

Example 4.3. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Define $\varphi_i : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$, $i = 1, 2, 3$ by

$$\begin{aligned} \varphi_1(1) &= 1, & \varphi_1(2) &= 2, & \varphi_1(3) &= 3, \\ \varphi_2(1) &= 2, & \varphi_2(2) &= 1, & \varphi_2(3) &= 3, \\ \varphi_3(1) &= 3, & \varphi_3(2) &= 3, & \varphi_3(3) &= 1, \end{aligned}$$

and defined $F : X \times X \times X \rightarrow X$ by

$$F(x_1, x_2, x_3) = \begin{cases} \frac{x_1 + x_2 - x_3}{3}, & \text{if } x_1 + x_2 \geq x_3 \\ 0, & \text{if } x_1 + x_2 < x_3. \end{cases}$$

Then (X, p) is a partial metric space and the following conditions hold:

- (i) F is continuous;
- (ii) F has the (I, J) -mixed monotone property where $I = \{1, 2\}$ and $J = \{3\}$;
- (iii) for each $k \in S := \{1, 2, 3\}$, there exists $(p, q) \in S \times S$ such that $k = \varphi_p(q)$;
- (iv) there exists $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \in X^3$ such that

$$\begin{aligned} x_i^{(0)} &\leq F(x_{\varphi_i(1)}^{(0)}, x_{\varphi_i(2)}^{(0)}, x_{\varphi_i(3)}^{(0)}), & i \in I = \{1, 2\}, \\ x_j^{(0)} &\geq F(x_{\varphi_j(1)}^{(0)}, x_{\varphi_j(2)}^{(0)}, x_{\varphi_j(3)}^{(0)}), & j \in J = \{3\}; \end{aligned}$$

- (v) there exist two altering distance functions ϕ and ψ such that

$$\phi(p(F(\mathbf{x}), F(\mathbf{y}))) \leq \phi\left(\max_{k \in S} \{p(x_k, y_k)\}\right) - \psi\left(\max_{k \in S} \{p(x_k, y_k)\}\right)$$

for all $\mathbf{x} = (x_1, x_2, x_3) \leq_I \mathbf{y} = (y_1, y_2, y_3)$.

Thus by Theorem 3.1, F has an (I, J) -coupled fixed point with respect to $\Phi = (\varphi_1, \varphi_2, \varphi_3)$. Moreover, $(0, 0, 0)$ is the unique (I, J) -coupled fixed point of F .

Proof. The proofs of (i)-(iii) are clear.

The conditions (iv) is satisfied for $\mathbf{x}^{(0)} = (0, 0, 1)$. Indeed,

$$\begin{aligned} x_1^{(0)} &= 0 = F(0, 0, 1) = F\left(x_{\varphi_1(1)}^{(0)}, x_{\varphi_1(2)}^{(0)}, x_{\varphi_1(3)}^{(0)}\right), \\ x_2^{(0)} &= 0 = F(0, 0, 1) = F\left(x_{\varphi_2(1)}^{(0)}, x_{\varphi_2(2)}^{(0)}, x_{\varphi_2(3)}^{(0)}\right), \\ x_3^{(0)} &= 1 \geq \frac{1}{3} = F(1, 1, 0) = F\left(x_{\varphi_3(1)}^{(0)}, x_{\varphi_3(2)}^{(0)}, x_{\varphi_3(3)}^{(0)}\right). \end{aligned}$$

From the definition of the function F , we have

$$\begin{aligned} F(x_1, x_2, x_3) &\leq \max\left\{\frac{x_1 + x_2 - x_3}{3}, 0\right\}, \\ F(y_1, y_2, y_3) &\leq \max\left\{\frac{y_1 + y_2 - y_3}{3}, 0\right\} \end{aligned}$$

and

$$\begin{aligned} &p(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) \\ &\leq \max\left\{\max\left\{\frac{x_1 + x_2 - x_3}{3}, 0\right\}, \max\left\{\frac{y_1 + y_2 - y_3}{3}, 0\right\}\right\} \\ &= \max\left\{\frac{x_1 + x_2 - x_3}{3}, \frac{y_1 + y_2 - y_3}{3}, 0\right\} \\ &\leq \max\left\{\frac{x_1 + x_2}{3}, \frac{y_1 + y_2}{3}\right\} \\ &\leq \max\left\{\frac{2}{3} \max\{x_1, x_2, x_3\}, \frac{2}{3} \max\{y_1, y_2, y_3\}\right\} \\ &= \frac{2}{3} \max\{x_1, x_2, x_3, y_1, y_2, y_3\} \\ &= \frac{2}{3} \max\{\max\{x_1, y_1\}, \max\{x_2, y_2\}, \max\{x_3, y_3\}\} \\ &= \frac{2}{3} \max\{p(x_1, y_1), p(x_2, y_2), p(x_3, y_3)\}. \end{aligned}$$

Therefore, the condition (v) is satisfied for the altering distance functions $\phi = I$ and $\psi = \frac{1}{3}I$ (where I is an identity mapping). Since $X = [0, \infty)$ is a totally ordered set, by Theorem 4.1, $(0, 0, 0)$ is the unique coupled fixed point of F . □

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