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MULTIVARIATE INCREASING FAILURE RATE AVERAGE DISTRIBUTIONS[1]

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A class of multivariate distributions which is an extension of the class of univariate distributions with increasing failure rate averages is introduced. Properties of this class are studied and examples of distributions which are members of this class are given.

1. Introduction. In reliability theory, the class of (univariate) increasing failure rate average (IFRA) distributions plays a distinguished role. In particular, it is the smallest class of life distributions containing the exponentials which is closed under the formation of coherent systems and limits in distribution (see Barlow and Proschan (1975)).

Recently a new characterization of IFRA distributions was obtained by Block and Savits (1976) in terms of an integral inequality: a life distribution F is IFRA if and only if for every nonnegative nondecreasing function h,

$$\int h(x)\,dF(x) \leqslant \{\int h^{\alpha}(x/\alpha)\,dF(x)\}^{1/\alpha}, \qquad 0 < \alpha \leqslant 1. \tag{1.1}$$

This characterization proved very useful in establishing the fact that the class of IFRA distributions is closed under convolution.

In this paper we investigate the natural multivariate extension of (1.1). Properties of the class of distributions satisfying this extension are investigated in Section 2. In Section 3, we remove a technical assumption in the definition of Section 2 and give several characterizations of the class of multivariate IFRA distributions. Various examples of multivariate distributions which are in the class are given in Section 4 along with a method for constructing distributions which are in the class.

2. Definition and properties. Let $(T_1, \cdots, T_n)$ be a nonnegative random vector with distribution function F.

(2.0) DEFINITION. $(T_1, \cdots, T_n)$ is said to have a multivariate IFRA distribution if and only if

$$E[h(T_1, \cdots, T_n)] \leqslant E^{1/\alpha}[h^{\alpha}(T_1/\alpha, \cdots, T_n/\alpha)] \tag{2.1}$$

for all continuous nonnegative nondecreasing functions h and all $0 < \alpha \leqslant 1$.

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(2.2) REMARK. The continuity assumption on h is a technical simplification. More will be said about this in Section 3.

Any class of life distributions $\mathcal{C}$ which in some sense is to be designated as multivariate IFRA should possess at least some of the following properties:

(P1) $\mathcal{C}$ is closed under the formation of coherent systems.

(P2) $\mathcal{C}$ is closed under limits in distribution.

(P3) If $(T_1, \cdots, T_n) \in \mathcal{C}$, any joint marginals belong to $\mathcal{C}$.

(P4) If $(T_1, \cdots, T_n)$, $(S_1, \cdots, S_m) \in \mathcal{C}$ and are independent, then $(T_1, \cdots, T_n; S_1, \cdots, S_m) \in \mathcal{C}$.

(P5) $\mathcal{C}$ is closed under nonnegative scaling.

(P6) $\mathcal{C}$ is closed under convolution (whenever the operation makes sense).

(P7) If $(T_1, \cdots, T_n) \in \mathcal{C}$ and $\tau_1, \cdots, \tau_m$ are any coherent life functions of order n, then $(\tau_1(T_1, \cdots, T_n), \cdots, \tau_m(T_1, \cdots, T_n)) \in \mathcal{C}$.

See Esary and Marshall (1970) for definitions of coherent systems and life functions. Several other multivariate IFRA definitions (e.g., see Esary and Marshall (1979)) have been proposed but none of them satisfies all of the above properties. It will now be shown that distributions satisfying Definition (2.0) have all of the above properties.

(2.3) THEOREM. *The class* $\mathcal{S}$ *of multivariate* IFRA *distributions possesses all of the properties* (P1)–(P7).

Before we prove this theorem, we establish the following lemma.

(2.4) LEMMA. *Let* $(T_1, \cdots, T_n) \in \mathcal{S}$ *and* $\psi_1, \cdots, \psi_m$ *be any functions of* n*-variables which are continuous, nondecreasing and satisfy the inequality* $\psi_i(x_1/\alpha, \cdots, x_n/\alpha) \leqslant (1/\alpha)\psi_i(x_1, \cdots, x_n)$ *for all* $(x_1, \cdots, x_n) \in \mathbb{R}_n$ *and* $0 < \alpha \leqslant 1$. *Then, setting* $S_i = \psi_i(T_1, \cdots, T_n)$ *for* $i = 1, \cdots, m$, *it follows that* $(S_1, \cdots, S_m) \in \mathcal{S}$.

PROOF. Let h be any continuous nonnegative nondecreasing function of m-variables. Then for $0 < \alpha \leqslant 1$,

$$\begin{aligned} E[h(S_1, \cdots, S_m)] &= E[h(\psi_1(T_1, \cdots, T_n), \cdots, \psi_m(T_1, \cdots, T_n))] \\ &\leqslant E^{1/\alpha}[h^\alpha(\psi_1(T_1/\alpha, \cdots, T_n/\alpha), \cdots, \psi_m(T_1/\alpha, \cdots, T_n/\alpha))] \\ &\leqslant E^{1/\alpha}[h^\alpha(1/\alpha\psi_1(T_1, \cdots, T_n), \cdots, 1/\alpha\psi_m(T_1, \cdots, T_n))] \\ &= E^{1/\alpha}[h^\alpha(S_1/\alpha, \cdots, S_m/\alpha)]. \end{aligned}$$

PROOF OF THEOREM (2.3).

(P1) and (P7). Since (P7) reduces to (P1) when $m = 1$, we only need prove (P7). So let $\tau_1, \cdots, \tau_m$ be coherent life functions of order n corresponding to the coherent structure functions $\phi_1, \cdots, \phi_m$ of order n respectively. Let $P_{il}, \cdots, P_{ip_i}$ be the minimum path sets for ϕ_i. Since $\tau_i(x_1, \cdots, x_n) = \max_{1 \leq k \leq p_i} \min_{j \in P_{ik}} x_j$, the result follows from Lemma 2.4.

(P2) Suppose that for every k, $(T_{1k}, \cdots, T_{nk}) \in \mathbb{S}$ and converges weakly to $(T_1, \cdots, T_n)$ as $k \to \infty$. Let h be any continuous nonnegative nondecreasing function, $0 < \alpha \leq 1$, and N any nonnegative real number. By abuse of notation, we also let N denote the constant function whose value is N. Then by definition of weak convergence we have that

$$E[h \wedge N(T_{1k}, \cdots, T_{nk})] \to E[h \wedge N(T_1, \cdots, T_n)] \text{ and}$$

$$E\left[(h \wedge N)^\alpha(T_{1k}/\alpha, \cdots, T_{nk}/\alpha)\right] \to E\left[(h \wedge N)^\alpha(T_1/\alpha, \cdots, T_n/\alpha)\right]$$

as $k \to \infty$, where $h \wedge N = \min(h, N)$. It then follows that

$$E[(h \wedge N)(T_1, \cdots, T_n)] \leq E^{1/\alpha}\left[(h \wedge N)^\alpha(T_1/\alpha, \cdots, T_n/\alpha)\right].$$

Now let $N \to \infty$.

(P3) This is a special case of (P7).

(P4) Let $(T_1, \cdots, T_n)$, $(S_1, \cdots, S_m)$ have distribution functions F and G respectively. Let $h(x_1, \cdots, x_n; y_1, \cdots, y_m)$ be continuous, bounded, nonnegative and nondecreasing. Then

$$\begin{aligned}
&E[h(T_1, \cdots, T_n; S_1, \cdots, S_m)] \\
&\quad = \int\int h(x_1, \cdots, x_n; y_1, \cdots, y_m)\, dF(x_1, \cdots, x_n)\, dG(y_1, \cdots, y_m) \\
&\quad \leq \int\left\{\int h^\alpha(x_1/\alpha, \cdots, x_n/\alpha; y_1, \cdots, y_m)\, dF(x_1, \cdots, x_n)\right\}^{1/\alpha} \\
&\qquad \times dG(y_1, \cdots, y_m) \\
&\quad \leq \left\{\int\int h^\alpha(x_1/\alpha, \cdots, x_n/\alpha; y_1/\alpha, \cdots, y_m/\alpha)\, dF(x_1, \cdots, x_n)\right. \\
&\qquad \left. \times dG(y_1, \cdots, y_m)\right\}^{1/\alpha} \\
&\quad = E^{1/\alpha}[h^\alpha(T_1/\alpha, \cdots, T_n/\alpha; S_1/\alpha, \cdots, S_m/\alpha)].
\end{aligned}$$

If h is not bounded, then consider $h \wedge N$ and let $N \to \infty$.

(P5) Let $a_1, \cdots, a_n \geq 0$ and set $\psi_i(x_1, \cdots, x_n) = a_i x_i (1 \leq i \leq n)$. Now apply Lemma 2.4.

(P6) If $(T_1, \cdots, T_n), (S_1, \cdots, S_n) \in \mathbb{S}$ and are independent, then the convolution corresponds to $(T_1 + S_1, \cdots, T_n + S_n)$. By (P4), $(T_1, \cdots, T_n; S_1, \cdots, S_n) \in \mathbb{S}$. Set $\psi_i(x_1, \cdots, x_n; y_1, \cdots, y_n) = x_i + y_i (1 \leq i \leq n)$ and apply Lemma 2.4.

(2.5) Remark. If $(T_1, \cdots, T_n) \in \mathbb{S}$ and $b_1, \cdots, b_n \geq 0$, then $(T_1 + b_1, \cdots, T_n + b_n) \in \mathbb{S}$ for let $\psi_i(x_1, \cdots, x_n) = x_i + b_i$. Since $\psi_i(x_1/\alpha, \cdots, x_n/\alpha) = (x_i/\alpha) + b_i \leq (1/\alpha)(x_i + b_i) = (1/\alpha)\psi_i(x_1, \cdots, x_n)$ for $0 < \alpha \leq 1$, we can apply Lemma (2.4) again.

(2.6) Note. Using Lemma 2.4, it is easy to show that a generalized version of (P6) holds, i.e., $(T_1, \cdots, T_n) \in \mathbb{S}$ and $S_1, \cdots, S_m$ nonempty subsets of $\{1, \cdots, n\}$ implies that $(\Sigma_{i \in S_1} T_i, \cdots, \Sigma_{i \in S_m} T_i) \in \mathbb{S}$.

3. Equivalent criteria. In this section we will remove the continuity assumption on h. Other derived results lead to alternative characterizations of multivariate IFRA, some of which are more amenable to verification than the definition.

A subset $D \subset \mathbb{R}_n$ is said to be an *upper set* if whenever $\mathbf{x} \in D$ and $\mathbf{y} \geqslant \mathbf{x}$, then $\mathbf{y} \in D$. When D is open, we speak of *upper domains*. Clearly $\{\mathbf{y} : \mathbf{y} \geqslant \mathbf{x}\}$ is an upper set and $\{\mathbf{y} : \mathbf{y} > \mathbf{x}\}$ is an upper domain for $\mathbf{x} \in \mathbb{R}_n$ given. The latter type is called an *upper quadrant domain*. A finite union of upper quadrant domains is called a *fundamental upper domain*. It should be noted that $f = I_D$, where D is an upper set, if and only if f is a binary increasing function. Furthermore, f is a left continuous binary increasing function if and only if $f = I_D$ where D is an upper domain. Other characterizations are possible for the other quantities defined above.

The following results, which closely follow those of Esary, Proschan and Walkup (1967), allow us to remove the continuity assumption on h.

(3.1) LEMMA. *Let C be either an upper closed set or an upper domain in $\mathbb{R}_n$. For $\mathbf{T} = (T_1, \cdots, T_n)$ multivariate* IFRA, *it follows that* (2.1) *holds for $h = I_C$*, i.e.,

$$P(\mathbf{T} \in C) \leqslant P^{1/\alpha}(\mathbf{T} \in \alpha C), \qquad 0 < \alpha \leqslant 1.$$

PROOF. As in the proof of Lemma 3.2 of EPW (1967), for a closed upper set C a sequence of continuous functions $\langle h_k \rangle, 0 \leqslant h_k \leqslant 1$, can be constructed so that $h_k \downarrow I_C$. Similarly if C is an upper domain, there exist continuous nondecreasing $\langle h_k \rangle$ such that $h_k \uparrow I_C$. The result follows from the monotone convergence theorem.

(3.2) LEMMA. *Let D be any Borel measureable upper set in $\mathbb{R}_n$. Then if $\mathbf{T} = (T_1, \cdots, T_n)$ is multivariate* IFRA, (2.1) *holds for $h = I_D$*, i.e.,

$$P(\mathbf{T} \in D) \leqslant P^{1/\alpha}(\mathbf{T} \in \alpha D), \qquad 0 < \alpha \leqslant 1.$$

PROOF. As in the proof of Theorem 3.3 of EPW (1967), D can be approximated below by a closed upper set so that

$$P(\mathbf{T} \in D) - \varepsilon \leqslant P(\mathbf{T} \in C) \leqslant P^{1/\alpha}(\mathbf{T} \in \alpha C) \leqslant P^{1/\alpha}(\mathbf{T} \in \alpha D)$$

where the second inequality follows from Lemma 3.1 above. Let $\varepsilon \downarrow 0$ to obtain the result.

(3.3) REMARKS.

(1). An associate editor has suggested an alternate method of obtaining the result of Lemma 3.2. This result, due to E. Arjas and A. O. Pittenger, proceeds by writing $\int_A dF(x) = \mu(A)$. Then for $0 < \varepsilon < 1, \beta > 1$ and any Borel measureable upper set A with interior $A^\circ, \beta A \subset (1 - \varepsilon)\beta A^0 \subset (1 - \varepsilon)\beta A$, so that

$$\mu(\beta A) \leqslant \mu((1 - \varepsilon)\beta A^\circ) \leqslant [\mu(A^\circ)]^{(1-\varepsilon)\beta} \leqslant (\mu(A))^{(1-\varepsilon)\beta}$$

where the second inequality follows from Lemma 3.1. Let $\varepsilon \downarrow 0$ to obtain the result.

(2). It should be noticed that Lemma 3.1 gives that if (2.1) holds for continuous nonnegative nondecreasing functions then (2.1) holds for binary nondecreasing right continuous and nondecreasing binary left continuous functions. Similarly the proof of Lemma 3.2 shows that if (2.1) holds for nondecreasing binary right (or left) continuous, then it holds for binary nondecreasing Borel measureable functions. It only remains to show that if (2.1) holds for binary nondecreasing Borel measureable functions, then it holds for arbitrary nondecreasing Borel measureable functions. This is contained in our next result.

(3.4) THEOREM. *The random vector* $\mathbf{T} = (T_1, \cdots, T_n)$ *is multivariate* IFRA *if and only if* (2.1) *is valid for all Borel measureable nonnegative nondecreasing functions h.*

PROOF. Clearly we only need prove the necessity, i.e., the "only if" part of the theorem. Let h be a Borel measureable nonnegative nondecreasing function and let $D_{ik} = \{\mathbf{x} : h(\mathbf{x}) > i2^{-k}\}, i = 1, \cdots, k \cdot 2^k, k = 1, 2, \cdots$. Now set $h_k = 2^{-k}\sum_{i=1}^{k \cdot 2^k} I_{D_{ik}}$. If follows from Lemma 3.2 and the Minkowski inequality that (2.1) is valid for each h_k. Since $h_k \uparrow h$, the desired result follows from the monotone convergence theorem.

Our next result, Theorem 3.5, gives a characterization of multivariate IFRA in terms of indicator functions of fundamental upper domains. The technical condition $\bar{F}(\mathbf{0}) = P(T_1 > 0, \cdots, T_n > 0) = 1$ is assumed here only because we restrict ourselves to fundamental upper domains of $\mathbb{R}_n^+ = \{\mathbf{x} : x_i > 0 \text{ all } i\}$. If we use fundamental upper domains of $\mathbb{R}_n$ instead, no such condition is necessary. The assumption $\bar{F}(\mathbf{0}) = 1$ implies that no one dimensional marginal distribution has any mass at zero. On the other hand, if $(T_1, \cdots, T_n)$ is multivariate IFRA and we assume that no one dimensional marginal is concentrated at zero, then $\bar{F}(\mathbf{0}) = 1$. To see this, suppose $\bar{F}(\mathbf{0}) \neq 1$. Then there exists an i such that $P(T_i = 0) = P(T_i = 0, T_j \geqslant 0 \text{ all } j \neq i) > 0$. But by property (P3), T_i is univariate IFRA and so $P(T_i = 0) = 1$.

(3.5) THEOREM. *Assume* $\bar{F}(\mathbf{0}) = 1$. *Then* $(T_1, \cdots, T_n)$ *is multivariate* IFRA *if and only if inequality* (2.1) *is valid for the indicator function of every fundamental upper domain in* $\mathbb{R}_n^+$.

PROOF. The necessity of the condition follows from Lemma 3.1. Thus we need only prove the sufficiency. Let $D \subset \mathbb{R}_n^+$ be any upper domain. For $k = 1, 2, \cdots$ and any positive integers $i_1, \cdots, i_{n-1}$, let

$$a_k(i_1, \cdots, i_{n-1}) = \inf\{t : (i_1 2^{-k}, \cdots, i_{n-1} 2^{-k}, t) \in D\}.$$

If $\{\ \} = \phi$, set $a_k(i_1, \cdots, i_{n-1}) = +\infty$. We set $D_k(i_1, \cdots, i_{n-1}) = \{\mathbf{x} : \mathbf{x} > (i_1 2^{-k}, \cdots, i_{n-1} 2^{-k}, a_k(i_1, \cdots, i_{n-1}))\}$ and put $D_k = \cup_{1 < i_1, \cdots, i_{n-1} \leqslant k \cdot 2^k} D_k(i_1, \cdots, i_{n-1})$. Clearly D_k is a fundamental upper domain in $\mathbb{R}_n^+$ and it is not hard to show that $I_{D_k} \uparrow I_D$. It thus follows from the assumptions and the monotone convergence theorem that $P^{1/\alpha}(\mathbf{T} \in \alpha D) \geqslant P(\mathbf{T} \in D)$ for $0 < \alpha \leqslant 1$. By the second remark of (3.3), the result follows for all Borel measureable (and so continuous) nonnegative nondecreasing functions restricted to $\mathbb{R}_n^+$. But since $\bar{F}(\mathbf{0}) = 1$, we may remove this condition, so that $\mathbf{T}$ is multivariate IFRA.

4. Examples of multivariate IFRA distributions.

A. *Generated from univariate independent* IFRA *distributions.* The following theorem gives functions of independent IFRA distributions which are multivariate IFRA.

(4.1) THEOREM. *Let* $X_1, \cdots, X_n$ *be independent* IFRA *random variables and let* $\phi \neq S_i \subset \{1, \cdots, n\}$ *for* $i = 1, \cdots, m$.

(i) $(X_1, \cdots, X_n)$ *is multivariate* IFRA.

(ii) *If* $T_i = \min_{j \in S_i} X_j, i = 1, \cdots, m$, *then* $(T_1, \cdots, T_m)$ *is multivariate* IFRA.

(iii) *If* $\tau_1, \cdots, \tau_m$ *are coherent life functions of order* n, *then* $(\tau_1(X_1, \cdots, X_n), \cdots, \tau_m(X_1, \cdots, X_n))$ *is multivariate* IFRA.

(iv) *If* $T_i = \Sigma_{j \in S_i} X_j, i = 1, \cdots, m$, *then* $(T_1, \cdots, T_m)$ *is multivariate* IFRA.

PROOF. All of these properties follow from Theorem 2.3 and Lemma 2.4.

(4.2) COROLLARY. *The multivariate exponential distribution of Marshall and Olkin (see Barlow and Proschan* (1975), *page* 139) *is multivariate* IFRA.

PROOF. Let $(T_1, \cdots, T_n)$ be MVE. Then it is of the form of Theorem 4.1, (ii), where the X_i are exponential, and so IFRA.

(4.3) COROLLARY. *Let* $X_1, \cdots, X_n$ *be independent identically distributed* IFRA *random variables and* $Y_1, \cdots, Y_n$ *be the corresponding order statistics. Then* $(Y_1, \cdots, Y_n)$ *is multivariate* IFRA.

PROOF. Let τ_k be the life function corresponding to a $(n - k + 1)$-out-of-n system. Then $Y_k = \tau_k(X_1, \cdots, X_n)$. Since $(X_1, \cdots, X_n)$ is multivariate IFRA, it follows from (4.1) (ii) that $(Y_1, \cdots, Y_n)$ is multivariate IFRA.

(4.4) NOTE. It is clear in the previous corollary that the hypothesis can be weakened to $(X_1, \cdots, X_n)$ MIFRA.

B. *Random variables having exponential scaled minimums.* Esary and Marshall (1974) have considered the following class of distributions. Let $(T_1, \cdots, T_n)$ be a nonnegative random vector such that for every choice of nonnegative a_i we have $\min(a_i T_i)$ is exponential. Then $(T_1, \cdots, T_n)$ is multivariate IFRA. This follows by Theorem 3.5 for let a_{ij} be any nonnegative constants and define $T_{ij} = a_{ij} T_j (1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n)$. Then if $\phi \neq S \subset \{(i,j) : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n\}$, letting $S_j = \{i : (i,j) \in S\}$, we can see that $\min_{(i,j) \in S} T_{ij} = \min_{1 \leqslant j \leqslant n}((\min_{i \in S_j} a_{ij}) T_j)$ is exponential by assumption. Since the collection of random variables $\{T_{ij} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n\}$ has exponential minimums, it follows from Esary and Marshall (1974) that any coherent structure function of these random variables has a (univariate) IFRA distribution.

C. *Multivariate Weibull distributions.* Multivariate Weibull distributions have been discussed by Marshall and Olkin (1967), Arnold (1967), Moeschberger (1974), David (1974) and Lee and Thompson (1974). These turn out to be of two types, both of which are multivariate IFRA. The first of these was introduced by Marshall and Olkin and is of the form

$$(T_1', \cdots, T_n') = \left(T_1^{1/\alpha_1}, \cdots, T_n^{1/\alpha_n}\right)$$

where $\alpha_i > 0$ for $i = 1, 2, \cdots, n$ and $(T_1, \cdots, T_n)$ has the MVE distribution given in Corollary 4.2. If follows easily from Corollary 4.2 and Lemma 2.4 that $(T_1', \cdots, T_n')$ is multivariate IFRA if $\alpha_i \geq 1$ for $i = 1, \cdots, n$. Distributions of this form were studied extensively by Moeschberger (1974).

The second type of multivariate Weibull distribution was introduced by David (1974) and by Lee and Thompson (1974) and has the form $(T_1, \cdots, T_n)$ where $T_i = \min(U_J : i \in J)$, $\phi \neq J \subset \{1, \cdots, n\}$, $P(U_J > x) = e^{-\lambda_J x^{\alpha_J}}$, $x \geq 0$, and the U_J are independent. By Theorem 4.1, for $\alpha_J \geq 1, (T_1, \cdots, T_n)$ is multivariate IFRA. This distribution need not have Weibull marginals if the α_J's are not all equal. The Weibull distributions of Arnold (1967) have a similar form, but the restriction that they belong to an additive family forces $\alpha_J = \alpha$ for all J. This implies that these are also of the type discussed in the preceding paragraph.

D. *Multivariate gamma distributions.* The multivariate distribution given on pages 216–219 of Johnson and Kotz (1977) is a special case of the form of (iv) of Theorem 4.1 where the X_j have gamma distributions with densities

$$f_j(x) = \left[\Gamma(\theta_j)\right]^{-1} x^{\theta_j - 1} e^{-x}, x > 0.$$

Consequently, by Theorem 4.1, the distribution is multivariate IFRA if $\theta_j \geq 1$ for all j. Similarly the bivariate exponential distribution with moment generating function given by (40), page 260, of the same reference is multivariate IFRA.

E. *Construction of multivariate IFRA distributions.* Suppose that $(X_1, \cdots, X_n)$ has a multivariate IFRA distribution and let Y be any nonnegative random variable on the same probability space. In this section we investigate conditions under which $(X_1, \cdots, X_n; Y)$ is also multivariate IFRA and use these conditions to construct a multivariate IFRA distribution.

Two lemmas are first needed. Let $\bar{G}(y \mid x_1, \cdots, x_n) = P\{Y > y \mid X_1 = x_1, \cdots, X_n = x_n\}$ for $x_1, \cdots, x_n \geq 0, y \geq 0$. The random variable Y is said to be *stochastically increasing* in $(X_1, \cdots, X_n)$ if $\bar{G}(y \mid x_1, \cdots, x_n)$ is nondecreasing in $x_1, \cdots, x_n$ (see Barlow and Proschan (1975), page 146). If $\bar{G}(y \mid x_1, \cdots, x_n)$ is continuous in $x_1, \cdots, x_n$ we will say Y is *stochastically continuous* in $X_1, \cdots, X_n$.

(4.5) LEMMA. *Assume Y is stochastically increasing and continuous in $X_1, \cdots, X_n$. Then $E(\phi(Y) \mid X_1 = x_1, \cdots, X_n = x_n)$ is continuous, nonnegative and nondecreasing in $x_1, \cdots, x_n$ for every continuous, nonnegative, nondecreasing and bounded function ϕ.*

PROOF. Nonnegativity is clear. Continuity follows from weak convergence considerations. Suppose that $\phi(y) = I_{(t,\infty)}(y)$. Then by assumption $E(\phi(Y)|X_1 = x_1, \cdots, X_n = x_n) = \bar{G}(t|x_1, \cdots, x_n)$ is nondecreasing in $(x_1, \cdots, x_n)$. Now take nonnegative linear combinations of such ϕ and pass to the limit.

(4.6). LEMMA. *If* $\bar{G}(y|x_1/\alpha, \cdots, x_n/\alpha) \leqslant \bar{G}^{1/\alpha}(\alpha y|x_1, \cdots, x_n)(0 < \alpha \leqslant 1)$, *then for every nonnegative and nondecreasing* ϕ,

$$E(\phi(Y)|X_1 = x_1/\alpha, \cdots, X_n = x_n/\alpha) \leqslant E^{1/\alpha}(\phi^\alpha(Y/\alpha)|X_1 = x_1, \cdots, X_n = x_n).$$

PROOF. If $\phi(y) = I_{(t,\infty)}(y), t \geqslant 0$, then by assumption,

$$\begin{aligned} E(\phi(y)|X_1 = x_1/\alpha, \cdots, X_n = x_n/\alpha) &= \bar{G}(t|x_1/\alpha, \cdots, x_n/\alpha) \\ &\leqslant \bar{G}^{1/\alpha}(\alpha t|x_1, \cdots, x_n) = E^{1/\alpha}(\phi^\alpha(Y/\alpha)|X_1 \\ &= x_1, \cdots, X_n = x_n). \end{aligned}$$

Similarly, the above is true of $\phi(y) = I_{[t,\infty)}(y), t \geqslant 0$. If follows from the Minkowski inequality that the inequality remains valid for nonnegative linear combinations of such ϕ. Now pass to the limit using the monotone convergence theorem.

(4.7) PROPOSITION. *Let* $(X_1, \cdots, X_n)$ *be multivariate* IFRA *and let* Y *be stochastically increasing and continuous in* $X_1, \cdots, X_n$ *and satisfy the inequality in Lemma* 4.6. *The* $(X_1, \cdots, X_n; Y)$ *is multivariate* IFRA.

PROOF. Let $h(x_1, \cdots, x_n; y)$ be any continuous, nonnegative, nondecreasing and bounded function. Then

$$\begin{aligned} &E[h(X, \cdots, X_n; Y)] \\ &\quad = \int\int H(x_1, \cdots, x_n; y)\, dG(y|x_1, \cdots, x_n)\, dF(x_1, \cdots, x_n). \end{aligned}$$

Since $h(x_1, \cdots, x_n; y)$ is nonnegative and nondecreasing in y, it follows from Lemma 4.6 that

$$\begin{aligned} &\int h(x_1, \cdots, x_n; y)\, dG(y|x_1, \cdots, x_n) \\ &\qquad \leqslant \{\int h^\alpha(x_1, \cdots, x_n; y/\alpha)\, dG(y|\alpha x_1, \cdots, \alpha x_n)\}^{1/\alpha} \end{aligned}$$

for all $0 < \alpha \leqslant 1$. But a slight extension of Lemma 4.5 yields that the right-hand side is continuous, nonnegative and nondecreasing in $x_1, \cdots, x_n$. Consequently, since $(X_1, \cdots, X_n)$ is multivariate IFRA,

$$\begin{aligned} E[h(X_1, \cdots, X_n; Y)] &\leqslant \{\int\int h^\alpha(x_1/\alpha, \cdots, x_n/\alpha; y/\alpha) \\ &\qquad \times dG(y|x_1, \cdots, x_n)\, dF(x_1, \cdots, x_n)\}^{1/\alpha} \\ &= E[h^\alpha(X_1/\alpha, \cdots, X_n/\alpha; Y/\alpha)]. \end{aligned}$$

If h is not bounded, consider $h \wedge N$ and let $N \to \infty$.

Applying Proposition 4.7 successively, we can prove the following corollary.

(4.8) COROLLARY. *Let* $(T_1, \cdots, T_n)$ *be a nonnegative random vector such that*

(i) T_1 *is (univariate)* IFRA *and*

(ii) *for* $k = 1, \cdots, n-1$, T_k *is stochastically increasing and continuous in* $T_1, \cdots, T_{k-1}$ *and satisfies the inequality of Lemma* 4.6. *Then* $(T_1, \cdots, T_n)$ *is multivariate* IFRA.

Corollary 4.8 can be used to construct distributions which are multivariate IFRA. For example, let (X, Y) be a distribution constructed as follows. Take X to be exponential with parameter $\lambda_1 \geqslant 0$ and set

$$\begin{aligned}\bar{G}(y|x) &= \exp(-\lambda_2 y), \qquad y < x \\ &= \exp(-(\lambda_2 + \lambda_{12})y + \lambda_{12}x), \qquad y \geqslant x\end{aligned}$$

where $\lambda_{12}, \lambda_2 \geqslant 0$. Then (X, Y) is multivariate IFRA with joint distribution

$$\begin{aligned}\bar{F}_{X,Y}(x,y) &= \exp(-\lambda_1 x - \lambda_2 y), \qquad y < x \\ &= \frac{\lambda_{12}}{\lambda_{12} - \lambda_1}\exp(-(\lambda_2 + \lambda_1)y) \\ &\quad - \frac{\lambda_1}{\lambda_{12} - \lambda_1}\exp(-(\lambda_1 - \lambda_{12})x - (\lambda_2 + \lambda_{12})y), \qquad y \geqslant x.\end{aligned}$$

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