Multivariate Lorenz dominance – A review and new perspectives

by

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Principal properties of the celebrated Lorenz curve

1. The Lorenz curve is visual: it indicates the degree of scatter 'the more the curve is bent'.
2. Up to a scale parameter, the Lorenz curve fully describes the underlying distribution.
3. The pointwise ordering of Lorenz functions is the Lorenz order.

Measuring dispersion

The Lorenz curve has been employed in virtually every field of statistical application.

In the economic and social sciences,

- economic inequality and poverty (Kolm, 1969; Atkinson, 1970; Sen, 1973),
- industrial concentration (Hart, 1971),
- social segregation (Alker, 1965),
- the distribution of publishing activities (Goldie, 1977), and
- many others

are measured by the Lorenz curve.

Further applications

The Lorenz curve is connected, e.g., with

- a fundamental notion of risk (Rothschild and Stiglitz, 1970, 1971) and
- the evaluation of decisions under risk (**?**),
- the measurement of ecological diversity (Thompson, 1976) and
- notions of increasing efficiency and power in statistical inference.

For a comprehensive guide to the widely dispersed literature up to 1993 see Mosler and Scarsini (1993).

Multivariate dispersion

In many applications it is natural to consider several variables instead of a single one and to measure their multivariate dispersion. E.g., in

- economic and social inequality: income, wealth, education, health,
- poverty: the same,
- decision under risk: multiple attributes, multiple goals.

See the early literature on social choice (Tobin, 1970; Sen, 1973; Kolm, 1977) and its multidimensional measurement (Atkinson and Bourguignon, 1982, 1989; Kolm, 1977; Maasoumi, 1986; Mosler, 1994; Tsui, 1995).

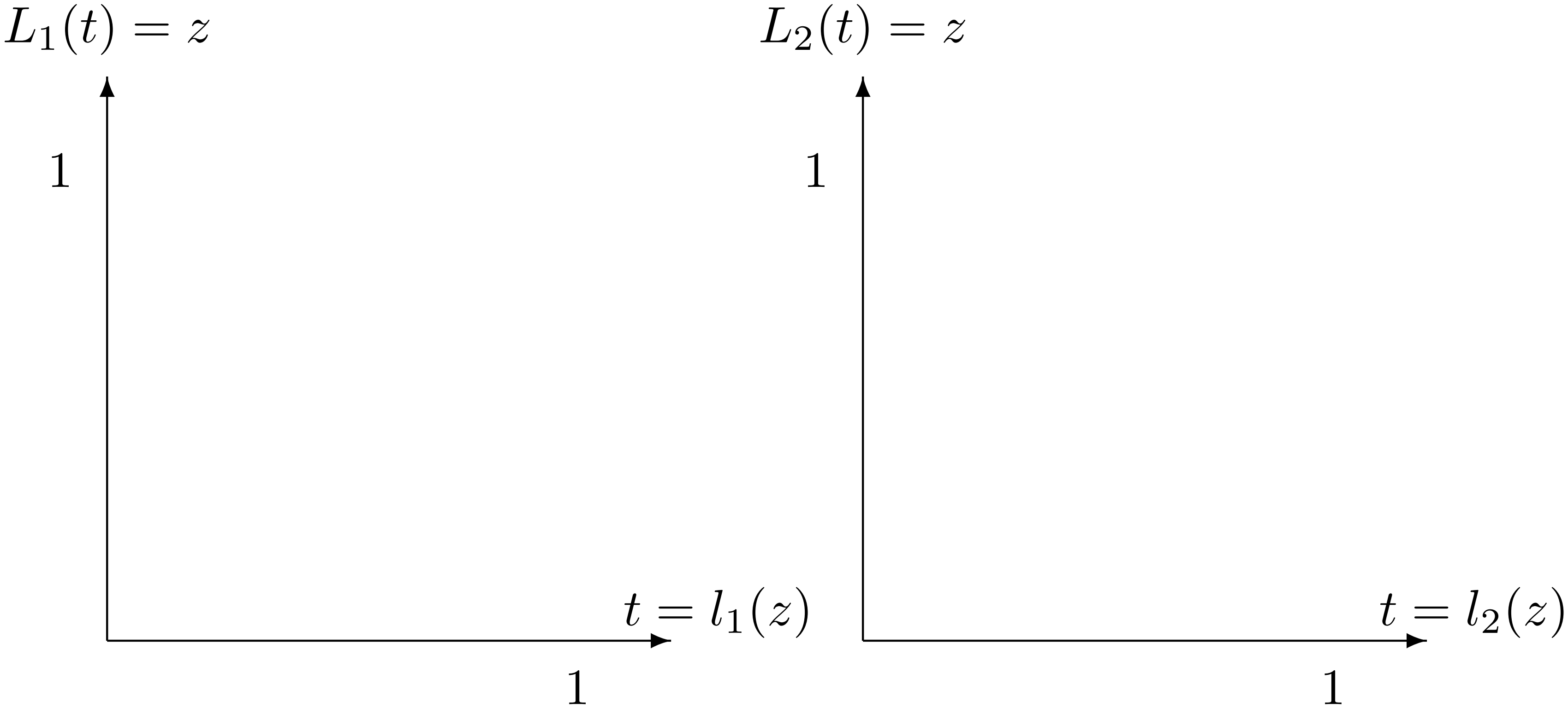
Overview

- Inverse Lorenz function
- Three multivariate Lorenz orderings
- Dominance of absolute endowments
- Other multivariate dominance relations
- Transfers
- Outlook

E.g., consider three households having (wealth, income) pairs

$$(1000, 0)\,, \qquad (0, 80)\,, \qquad (0, 20)\,.$$

The usual univariate Lorenz curve for income and wealth alone are:



Distribution matrix and matrix of shares

Let $\boldsymbol{A} = [a_{ik}]$ denote the *distribution matrix* of d commodities among n households, and let $\breve{\boldsymbol{A}} = [\breve{a}_{ik}]$ be the *matrix of shares,* $\breve{a}_{ik} = a_{ik}/\sum_i a_{ik}$.

In the example we have

$$\boldsymbol{A} = \begin{pmatrix} 1000 & 0 \\ 0 & 80 \\ 0 & 20 \end{pmatrix}, \qquad \breve{\boldsymbol{A}} = \begin{pmatrix} 1 & 0 \\ 0 & .8 \\ 0 & .2 \end{pmatrix}.$$

Basic idea: Inverse Lorenz function

$d = 1$: The usual Lorenz curve can also be seen as the graph of the *inverse Lorenz function*: Given a *portion of the total income*, the inverse Lorenz function indicates the *maximum percentage of the population* that receives this portion.

$d \geq 1$: Consider *shares of the total endowments* in each commodity and determine the *largest percentage of the population* by which these shares or less are held. This percentage, depending on the vector of shares, is the *d-variate inverse Lorenz function.*

Its graph is named the *Lorenz hypersurface*.

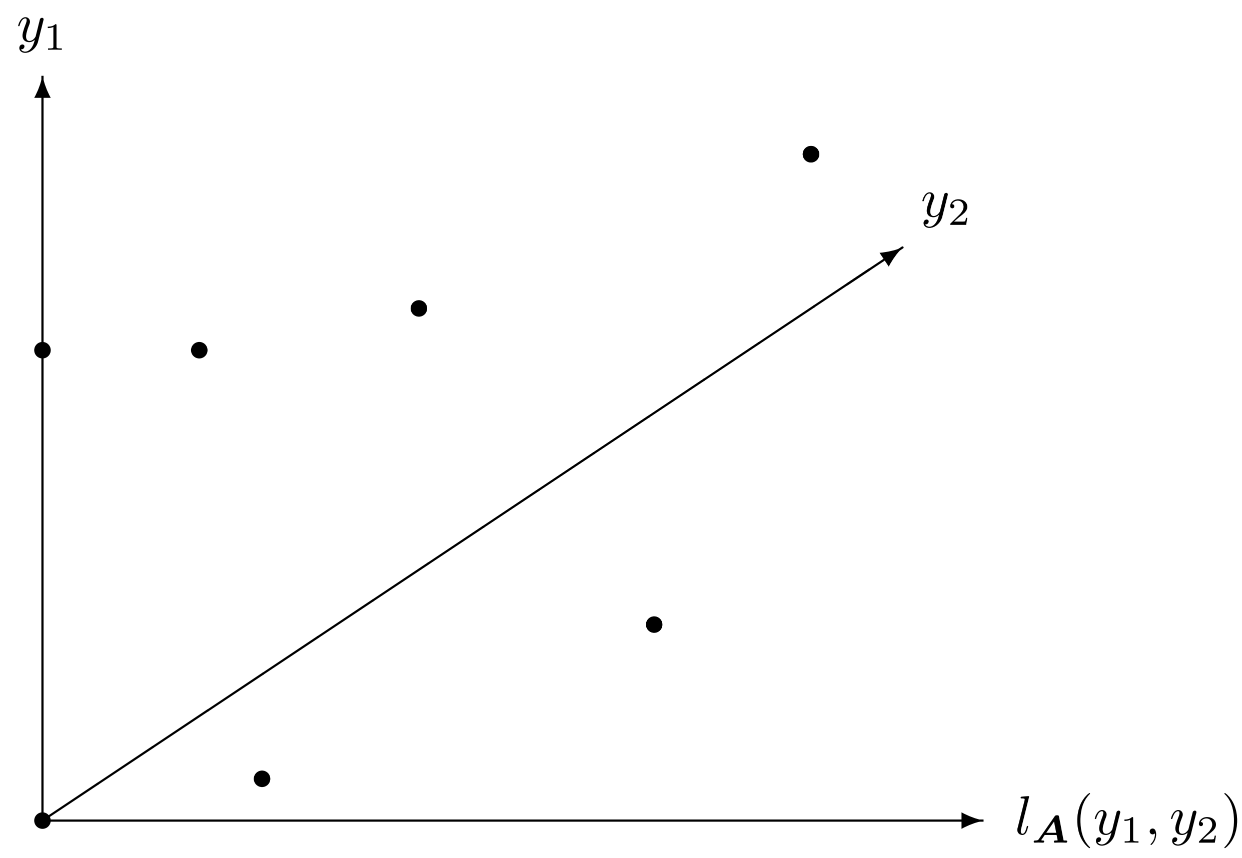


Figure 1: Lorenz hypersurface in the income-wealth example.

Inverse Lorenz function (ILF) of an empirical distribution

An $n \times d$ distribution matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]'$ corresponds to the n-point empirical distribution $F_{\boldsymbol{A}}$ in $\mathbb{R}_+^d$.

The share vector of the i-th household is

$$\breve{\boldsymbol{a}}_i = \frac{\boldsymbol{a}_i}{\sum_{j=1}^n \boldsymbol{a}_j} = \left(\frac{a_{i1}}{\sum_{j=1}^n a_{j1}}, \ldots, \frac{a_{id}}{\sum_{j=1}^n a_{jd}} \right) .$$

The inverse Lorenz function of $\boldsymbol{A}$ is defined by

$$l_{\boldsymbol{A}}(\boldsymbol{y}) = \max \left\{ \frac{1}{n} \sum_{i=1}^n \theta_i \ : \ \sum_{i=1}^n \theta_i \breve{\boldsymbol{a}}_i \le \boldsymbol{y}, 0 \le \theta_i \le 1 \right\} , \quad \boldsymbol{y} \in [0,1]^d . \tag{1}$$

Economic interpretation

The ILF is defined on the unit cube of $\mathbb{R}^d$. Its argument $\boldsymbol{y}$ represents a vector of relative endowments with respect to mean endowments in the d attributes.

$l_F(\boldsymbol{y})$ equals the maximum percentage of households who hold a vector of shares less than or equal to $\boldsymbol{y} = (y_1, \ldots, y_d)$. θ_i is a weight by which the endowment of the i-th household enters $\boldsymbol{y}$.

In case $d = 1$ the ILF is the inverse function of the Lorenz function. The Lorenz dominance between univariate distributions consists in the pointwise ordering of their Lorenz functions or, equivalently, of their ILFs.

ILF of a general probability distribution

Consider a probability distribution function F on $\mathbb{R}^d_+$ whose first moment exists and is strictly positive, in symbols $F \in \mathcal{P}_{\infty+}$.

The *inverse Lorenz function (ILF)* is given by

$$l_F(\boldsymbol{y}) = \max \int_{\mathbb{R}^d_+} g(\boldsymbol{x}) dF(\boldsymbol{x}) \,, \quad \boldsymbol{y} \in [0,1]^d \,, \tag{2}$$

where the maximum extends over all measurable functions $g : \mathbb{R}^d_+ \to [0,1]$ for which

$$\int_{\mathbb{R}^d_+} g(\boldsymbol{x}) \frac{\boldsymbol{x}}{\boldsymbol{\epsilon}(F)} dF(\boldsymbol{x}) \leq \boldsymbol{y} \,.$$

The graph $\{(z_0, \boldsymbol{y}) : z_0 = l_F(\boldsymbol{y}), \boldsymbol{y} \in [0,1]^d\}$ of the ILF is named the *Lorenz hypersurface.*

The function $g : \mathbb{R}^d_+ \to [0,1]$ can be considered as a *selection* of some part of the population Ω: Of all those households which have endowment vector $\boldsymbol{x}$ the percentage $g(\boldsymbol{x})$ is selected.

$\int_{\mathbb{R}^d_+} \frac{\boldsymbol{x}}{\boldsymbol{\epsilon}(F)} g(\boldsymbol{x}) dF(\boldsymbol{x})$ amounts to the *total portion vector* held by this subpopulation and $\int_{\mathbb{R}^d_+} g(\boldsymbol{x}) dF(\boldsymbol{x})$ is the size of the subpopulation selected by g.

Given a vector $\boldsymbol{y} \in [0,1]^d$, we determine the maximum of

$$\int_{\mathbb{R}^d_+} g(\boldsymbol{x}) dF(\boldsymbol{x})$$

over all measurable g such that

$$\int_{\mathbb{R}^d_+} \frac{\boldsymbol{x}}{\boldsymbol{\epsilon}(F)} g(\boldsymbol{x}) dF(\boldsymbol{x}) \le \boldsymbol{y}\,.$$

Proposition 1 (Properties of the ILF) .

(i) l_F *is monotone increasing, concave, and has values in* $[0,1]$,

(ii) $l_F(\mathbf{1}) = 1, \qquad l_F(\mathbf{0}) = F(\mathbf{0})$,

(iii) $l_F(\boldsymbol{y}_J, \mathbf{1}_{-J}) = l_{F_J}(\boldsymbol{y}_J)$.

Here,
F_J denotes the marginal distribution with respect to coordinates in $J \subset \{1, \ldots, d\}$, and
$(\boldsymbol{y}_J, \mathbf{1}_{-J})$ is the vector in $\mathbb{R}^d$ with components $y_j, j \in J$, and remaining components equal to 1.

Egalitarian distribution

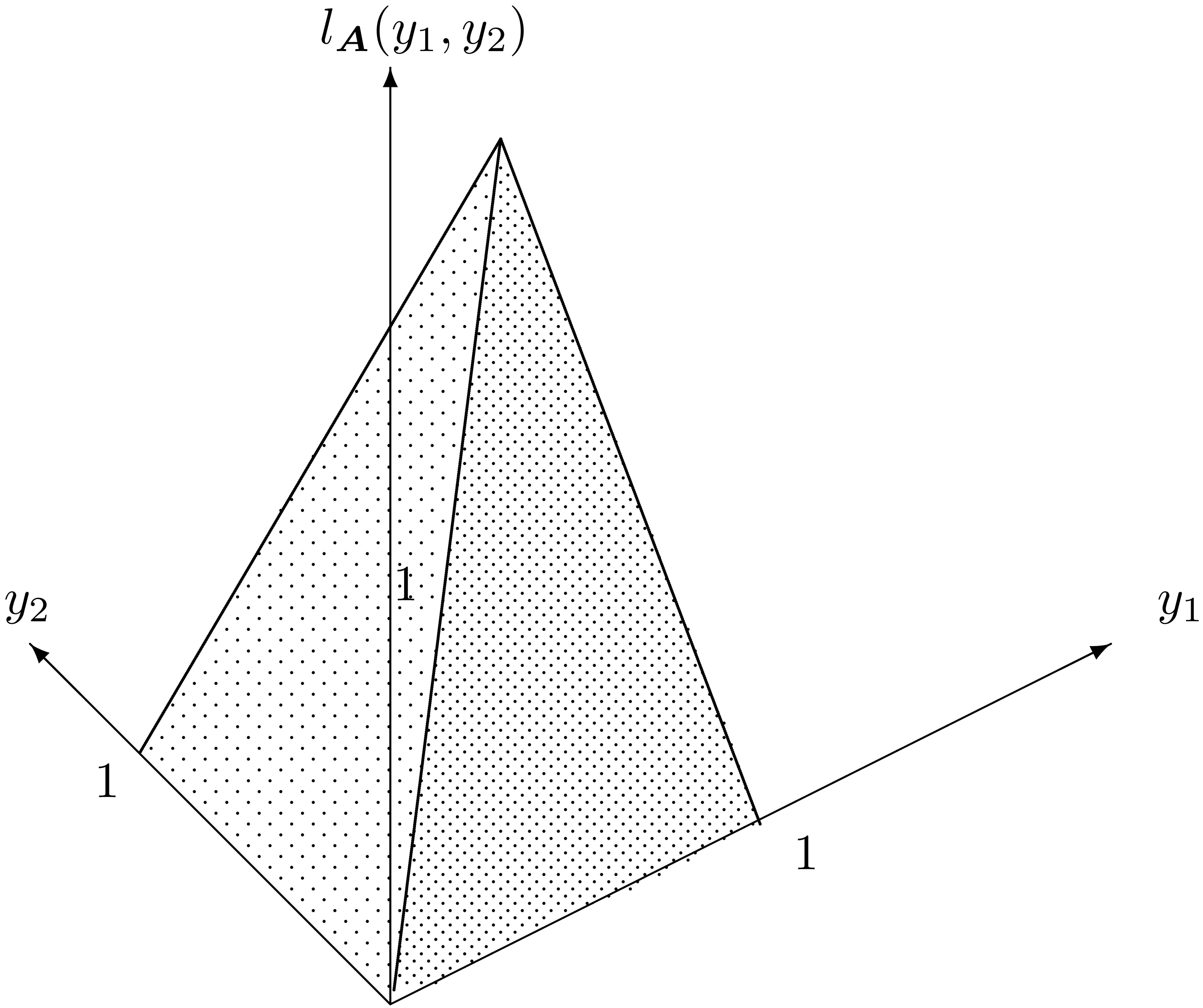
The *egalitarian distribution*, δ_ξ, puts unit mass to some point $\xi \in \mathbb{R}^d_+$.
It has ILF

$$l_{\delta_\xi}(\boldsymbol{y}) = \min_{i=1,\dots d} y_i \quad \text{for } \ \boldsymbol{y} \in [0,1]^d \,,$$

independently of ξ.

The next figure shows the graph of l_{δ_ξ} when $d = 2$. In this case,

$$\int_{\mathbb{R}^d_+} g(\boldsymbol{x}) dF(\boldsymbol{x}) = g(\xi) \quad \text{and} \quad \int_{\mathbb{R}^d_+} g(\boldsymbol{x}) \frac{\boldsymbol{x}}{\boldsymbol{\epsilon}(F)} dF(\boldsymbol{x}) = (g(\xi), g(\xi))\,.$$


$l_{\boldsymbol{A}}(y_1, y_2)$
y_2
y_1
1
1
1

ILF and Lorenz zonoid

Definition 1 (Lorenz zonoid) *The* Lorenz zonoid $LZ(F)$ *of a distribution* $F \in \mathcal{P}_{1+}$ *is the lift zonoid of the scaled distribution* $\widetilde{F}$*, where* $\widetilde{F}(\boldsymbol{x}) = F(\boldsymbol{\epsilon}(F) \cdot \boldsymbol{x})$*. In symbols,*

$$LZ(F) = \widehat{Z}(\widetilde{F})\,.$$

Proposition 2 (Lorenz zonoid boundary) *The graph of* l_F *is the intersection of the unit cube* $[0,1]^{d+1}$ *with the boundary of the set* $LZ(F) + (\mathbb{R}_- \times \mathbb{R}_+^d)$.

Equivalently,

$$[0,1]^{d+1} \cap \big(LZ(F) + (\mathbb{R}_- \times \mathbb{R}_+^d)\big)$$

is the hypograph of l_F.

More properties of the ILF

Proposition 3 (Uniqueness) *The ILF defines the underlying distribution uniquely, up to a vector of scale factors.*

Proposition 4 (Continuity) *Let a sequence* F^n *in* $\mathcal{P}_{1+}$ *weakly converge to* F. *Then* l_{F^n} *converges uniformly to* l_F *if* F^n *is uniformly integrable.*

In particular, F^n *is uniformly integrable if one of the following three restrictions holds:*
(i) *There exists a compact set that includes the supports of all* F^n,
(ii) *there exists a distribution* G *such that* $l_{F^n} \leq l_G$ *for all* n,
(iii) *there exists a distribution* G *such that* l_{F^n} *converges uniformly to* l_G.

Proposition 5 (Law of large numbers) *Let* $F \in \mathcal{P}_{1+}$ *and* F^n_{emp} *be the empirical distribution function of a random sample* $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ *from* F*. Then the ILF of* F^n_{emp} *converges uniformly to the ILF of* F *with probability one, that is,*

$$P\left(\lim_{n\to\infty} \sup_{\boldsymbol{y}} |l_{F^n_{emp}}(\boldsymbol{y}) - l_F(\boldsymbol{y})| = 0\right) = 1\,.$$

The ILF of an empirical distribution $F_{\boldsymbol{A}}$

The ILF of an empirical distribution is a piecewise linear function, and the Lorenz hypersurface is a piecewise linear hypersurface with extreme points as follows:

A point $(\frac{|I|}{n}, \sum_{i\in I} \breve{\boldsymbol{a}}_{i,J}, \mathbf{1}_{-J})$ is an extreme point of the Lorenz hypersurface if and only if the vector $\boldsymbol{\theta} = (\mathbf{1}_I, \mathbf{0}_{-I})$ provides a solution to the following maximization problem:

$$\max\Big\{\frac{1}{n}\sum_{i=1}^{n}\theta_i \,:\, \sum_{i=1}^{n}\theta_i\breve{\boldsymbol{a}}_{i,J} \le \sum_{i\in I}\breve{\boldsymbol{a}}_{i,J},\ \theta_i \in [0,1]\Big\}. \tag{3}$$

Obviously, in the case $d = 1$, such points have the form $(\frac{k}{n}, \sum_{i=1}^{k} \breve{a}_{(i)})$, $k = 0, \ldots, n$, where $\breve{a}_{(1)}, \ldots, \breve{a}_{(n)}$ are the shares ordered from below.

Ordering of ILFs

There are many possibilities to extend the classic Lorenz dominance to a multivariate setting. An economically meaningful notion is the following:

Definition 2 (ILF dominance) *F dominates G, $F \succeq_{PL} G$, if*

$$l_F(\boldsymbol{y}) \geq l_G(\boldsymbol{y}) \quad \textit{for all } \boldsymbol{y} \in [0,1]^d \,. \tag{4}$$

The ILF of an *empirical distribution* can be described by the extreme points of its graph.

ILF dominance

The pointwise ordering of ILFs yields a partial order among probability distributions having finite strictly positive first moments. It has been considered by many authors under different names,

- multivariate majorization by positive combinations ((Joe and Verducci, 1992)),
- price Lorenz order (Koshevoy and Mosler (1999)),
- convex-posilinear order (Müller and Stoyan (2002)),
- exchange rate Lorenz order (Arnold 2005).

When $d = 1$, the order is the usual Lorenz order.

With two univariate empirical distributions $F_{\boldsymbol{A}}$ on $a_1, \ldots, a_n$ and $F_{\boldsymbol{B}}$ on $b_1, \ldots, b_n$,
$F_{\boldsymbol{A}} \succeq_{PL} F_{\boldsymbol{B}}$ if and only if

$$\sum_{i=1}^{k} \breve{a}_{(i)} \leq \sum_{i=1}^{k} \breve{b}_{(i)} \,, \quad k = 1, \ldots, n-1 \,.$$

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be distributed as F and G, respectively. If $F \succeq_{PL} G$, we notate

$$\boldsymbol{X} \succeq_{PL} \boldsymbol{Y} \,.$$

Proposition 6 (Dominance by positive combinations) $\boldsymbol{X} \succeq_{PL} \boldsymbol{Y}$ *if and only if*

$$\boldsymbol{X}\boldsymbol{p}' \succeq_L \boldsymbol{Y}\boldsymbol{p}' \quad \textit{for all } \boldsymbol{p} \in \mathbb{R}^d_+ \,.$$

Proposition 7 (Convex scaled dominance, $d = 1$) $X \succeq_L Y$ *if and only if*

$$E\left[\phi\left(\frac{X}{E[X]}\right)\right] \geq E\left[\phi\left(\frac{X}{E[X]}\right)\right] \quad \textit{for all convex } \phi : \mathbb{R}_+ \to \mathbb{R},\,,$$

as far as the expectations exist.

Notate $\tilde{\boldsymbol{X}} = \left(\frac{X_1}{E[X_1]}, \dots, \frac{X_d}{E[X_d]}\right)$.

Proposition 8 (Convex-posilinear dominance) *$\boldsymbol{X} \succeq_{PL} \boldsymbol{Y}$ if and only if*

$$E[\psi(\tilde{\boldsymbol{X}}\boldsymbol{p}')] \geq E[\psi(\tilde{\boldsymbol{Y}}\boldsymbol{p}')] \quad \text{for all convex } \psi : \mathbb{R}_+ \to \mathbb{R} \quad \text{and all } \boldsymbol{p} \in \mathbb{R}^d_+ ,$$

as far as the expectations exist.

Definition 3 (Convex-linear scaled dominance) *$\boldsymbol{X} \succeq^s_{lz} \boldsymbol{Y}$ if*

$$E[\psi(\tilde{\boldsymbol{X}}\boldsymbol{p}')] \geq E[\psi(\tilde{\boldsymbol{Y}}\boldsymbol{p}')] \quad \text{for all convex } \psi : \mathbb{R}_+ \to \mathbb{R} \quad \text{and all } \boldsymbol{p} \in \mathbb{R}^d ,$$

as far as the expectations exist.

Definition 4 (Convex scaled dominance) *$\boldsymbol{X} \succeq^s_{cx} \boldsymbol{Y}$ if*

$$E[\phi(\tilde{\boldsymbol{X}}\boldsymbol{p}')] \geq E[\phi(\tilde{\boldsymbol{Y}}\boldsymbol{p}')] \quad \text{for all convex } \phi : \mathbb{R}^d_+ \to \mathbb{R} ,$$

as far as the expectations exist.

It obviously holds

$$\boldsymbol{X} \succeq^s_{cx} \boldsymbol{Y} \quad \Rightarrow \quad \boldsymbol{X} \succeq^s_{lz} \boldsymbol{Y} \quad \Rightarrow \quad \boldsymbol{X} \succeq_{PL} \boldsymbol{Y} \,.$$

Proposition 9 (Properties of the three multivariate Lorenz orders)
For each of the three orders holds:

(i) *There exists a set of smallest elements containing all one-point distributions.*

(ii) *The order is scale invariant.*

(iii) *If two distribution are ordered, all their marginals are ordered in the same way.*

(iv) *With stochastically independent attributes, the order is equivalent to the usual Lorenz order of all univariate marginals.*

(v) *The order is continuous in the following sense:*
If the sequence G^n *converges weakly to* G *and* $F \succeq G^n$ *holds for all* n, *then* $F \succeq G$.
If the sequence G^n *is uniformly integrable and converges weakly to* G *and* $G^n \succeq F$ *holds for all* n, *then* $G \succeq F$.

Dominance of absolute endowments

Distributions of absolute endowments can be ordered in a similar way.

For a probability distribution F on $\mathbb{R}_+$, the *generalized Lorenz function* is given by

$$GL_F(t) = \int_0^t F^{-1}(s)ds, \quad 0 \leq t \leq 1 \,.$$

Hence $GL_F(0) = 0$, $GL_F(1) = \boldsymbol{\epsilon}(F)$, and $GL_F(t) = \boldsymbol{\epsilon}(F)L_F(t)$ if $\boldsymbol{\epsilon}(F)$ is finite. The *dual generalized Lorenz function* of F is

$$DGL_F(t) = \boldsymbol{\epsilon}(F) - GL_F(1-t), \quad 0 \leq t \leq 1 \,.$$

Supermajorization and submajorization, $d = 1$

For two univariate distributions F and G, consider the pointwise ordering of these functions,

$$F \succeq^w G \quad \text{if} \quad GL_F(t) \le GL_G(t) \ \text{ for all } t\,,$$
$$F \succeq_w G \quad \text{if} \quad DGL_F(t) \ge DGL_G(t) \ \text{ for all } t\,.$$

$F \succeq^w G$ is mentioned as *supermajorization*, and $F \succeq_w G$ as *submajorization* of F over G. Obviously

$$F \succeq^w G \quad \Rightarrow \quad \boldsymbol{\epsilon}(F) \le \boldsymbol{\epsilon}(G)\,,$$

$$F \succeq_w G \quad \Rightarrow \quad \boldsymbol{\epsilon}(F) \le \boldsymbol{\epsilon}(G)\,,$$

$$F \succeq^w G \ \text{ and } \ F \succeq_w G \quad \Rightarrow \quad F \succeq_{PL} G \ \text{ and } \ \boldsymbol{\epsilon}(F) = \boldsymbol{\epsilon}(G)\,.$$

On the other hand, if $\boldsymbol{\epsilon}(F) = \boldsymbol{\epsilon}(G)$, submajorization as well as supermajorization coincide with the usual Lorenz order.

Supermajorization and submajorization, $d \geq 1$

We use the notation $F_{\boldsymbol{p}}(t) = \int_{\boldsymbol{x}\boldsymbol{p}' \leq t} dF(\boldsymbol{x})$.

Definition 5 (Price supermajorization, price submajorization) *For F and $G \in \mathcal{P}_{1+}$, define*

$$F \succeq_P^w G \quad \text{if } F_{\boldsymbol{p}} \succeq^w G_{\boldsymbol{p}} \text{ for all } \boldsymbol{p} \in \mathbb{R}_+^d,$$

$$F \succeq_{wP} G \quad \text{if } F_{\boldsymbol{p}} \succeq_w G_{\boldsymbol{p}} \text{ for all } \boldsymbol{p} \in \mathbb{R}_+^d.$$

$\succeq_P^w$ *is called* price supermajorization, $\succeq_{wP}$ price submajorization. *Both $\succeq_P^w$ and $\succeq_{wP}$ are called* weak price majorizations.

Definition 6 (Price majorization) *If both $F \succeq_P^w G$ and $F \succeq_{wP} G$ hold, we say that F* price majorizes *G, in symbols, $F \succeq_P G$.*

Corollary 10 (Equivalent definitions)
(i)

$$F \succeq_P G \quad \Leftrightarrow \quad F \succeq_{PL} G \ \ \textit{and} \ \ \boldsymbol{\epsilon}(F) = \boldsymbol{\epsilon}(G)\,.$$

(ii) *If* $\boldsymbol{\epsilon}(F) = \boldsymbol{\epsilon}(G)$,

$$F \succeq_P G \quad \Leftrightarrow \quad F \succeq_{wP} G \quad \Leftrightarrow \quad F \succeq_P^w G\,.$$

For the weak majorizations we have another characterization which involves a generalized version of the ILF and its dual:

Definition 7 (Inverse generalized Lorenz function)

For $\boldsymbol{y} \in \mathbb{R}^d_+$ *consider*

$$gl_F(\boldsymbol{y}) = \max \int_{\mathbb{R}^d_+} g(\boldsymbol{x})\, dF(\boldsymbol{x}), \tag{5}$$

where the maximum is taken over all measurable $g : \mathbb{R}^d_+ \to [0,1]$ *with*

$$\int_{\mathbb{R}^d_+} g(\boldsymbol{x})\, \boldsymbol{x}\, dF(\boldsymbol{x}) \leq \boldsymbol{y}\,.$$

The function $gl_F : \mathbb{R}^d_+ \to [0,1]$ *is the* inverse generalized Lorenz function (IGLF) *of* F*.*

Definition 8 (Dual inverse generalized Lorenz function) *The function dgl_F is called the* dual inverse generalized Lorenz function (DIGLF) *of F,*

$$dgl_F(\boldsymbol{y}) = \min_g \int_{\mathbb{R}_+^d} g(\boldsymbol{x})\, dF(\boldsymbol{x}), \quad \boldsymbol{y} \in \mathbb{R}_+^d, \tag{6}$$

where the minimum is taken over all measurable functions $g : \mathbb{R}_+^d \to [0, 1]$ for which

$$\int_{\mathbb{R}_+^d} g(\boldsymbol{x})\, \boldsymbol{x}\, dF(\boldsymbol{x}) \geq \boldsymbol{y}\,.$$

price super-majorization	$F \succ_P^w G$: $\forall p \geq 0\ F_p \succ^w G_p$	$\Leftrightarrow$	ordering of IGLFs
	$\Uparrow$		$\Uparrow$
price majorization	$F \succ_P G$: $\epsilon(F) = \epsilon(G)$ and $\forall p \geq 0\ \widetilde{F}_p \succ \widetilde{G}_p$	$\Leftrightarrow$	$F \succ_P^w G$ and $F \succ_{wP} G$
	$\Downarrow$		$\Downarrow$
price sub-majorization	$F \succ_{wP} G$: $\forall p \geq 0\ F_p \succ_w G_p$	$\Leftrightarrow$	ordering of DIGLFs

Table 1: Price majorization and weak price majorizations and their characterization by inverse generalized Lorenz functions (IGLFs) and their duals (DIGLFs).

Atkinson-Bourguignon dominance

Atkinson and Bourguignon (1982) suggested two other orderings to rank distributions of commodity vectors:

$F \succeq_{lo} G$ if $F(\boldsymbol{x}) \leq G(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^d$,

$F \succeq_{locc} G$ if $\int_{\boldsymbol{y} \leq \boldsymbol{x}} F(\boldsymbol{y}) d\boldsymbol{y} \leq \int_{\boldsymbol{y} \leq \boldsymbol{x}} G(\boldsymbol{y}) d\boldsymbol{y}$ for every $\boldsymbol{x} \in \mathbb{R}^d$.

The orderings are known in the literature as the lower orthant ordering and the lower orthant concave ordering, respectively; see, e.g., Dyckerhoff and Mosler (1997).

In general, the first Atkinson-Bourguignon (A-B) dominance implies the second one, and both can be characterized by inequalities on expected utilities.

If $d = 1$, the two A-B orderings amount to usual first and second degree stochastic dominance, respectively.

In dimension $d = 2$, the first A-B dominance implies the order of IGLFs, that is, the price supermajorization:

A class of utility functions generating the price supermajorization:

$\mathcal{U}_{wP} = \left\{ u : u(\boldsymbol{x}) = v(\boldsymbol{p}\boldsymbol{x}'), \boldsymbol{x} \in \mathbb{R}^d, \text{ with } v : \mathbb{R} \to \mathbb{R} \text{ concave}, \ \boldsymbol{p} \in \mathbb{R}^d_+ \right\}$.

When $d = 2$, obtain

$$\mathcal{U}_{lo} = \left\{ u : \mathbb{R}^2 \to \mathbb{R} \ : \ \frac{\partial u}{\partial x_1} \geq 0, \frac{\partial u}{\partial x_2} \geq 0, \frac{\partial^2 u}{\partial x_1 \partial x_2} \leq 0 \right\},$$

and the inclusion $\mathcal{U}_{wP} \subset \mathcal{U}_{lo}$ holds. Then $F \succeq_{lo} G$ implies $F \succeq_{wP} G$.

For $d > 2$ the first A-B dominance does not imply the weak price majorization.

The second A-B dominance, in dimension $d = 1$, equals price supermajorization. But for $d \geq 2$ it neither implies nor is implied by price supermajorization.

Other multivariate majorizations

- The multivariate dilation order (Kolm, 1977; Russell and Seo, 1978): is generated by expected utility inequalities for all convex utility functions.
- The dilation order implies the lift zonoid order and equal expectation vectors, hence the price Lorenz order.

Transfers

In dimension one, the Lorenz order is closely connected with *Pigou-Dalton transfers*.

Since price Lorenz dominance is equivalent to univariate Lorenz order of expenditures, whatever the prices, it is tantamount saying that for any prices there exists a series of usual Pigou-Dalton transfers of *expenditures*.

A *multivariate Pigou-Dalton transfer* is a transfer of *commodity vectors* among two agents such that the resulting two endowments are in the convex hull of the previous. It leaves the total endowments unchanged.

A distribution matrix $\boldsymbol{A}$ is *chain-majorized* by another distribution matrix $\boldsymbol{B}$ if $\boldsymbol{A}$ is obtained by a finite number of multivariate Pigou-Dalton transfers from $\boldsymbol{B}$.

Chain-majorization implies matrix majorization and equal total endowments (hence convex scaled dominance), but for $n \geq 3$ and $d \geq 2$ the reverse is not true (Marshall and Olkin, 1979, p. 431).

Thus, the ordering induced by multivariate Pigou-Dalton transfers is stronger than dilation, convex scaled dominance, convex-linear dominance, and convex-posilinear (price Lorenz) dominance.

We started with three principal properties of the classical Lorenz curve:

- The pointwise ordering by Lorenz functions is the Lorenz order.
- The Lorenz curve is visual: it indicates the degree of disparity as the bow is bent.
- Up to a scale parameter, the Lorenz curve fully describes the underlying distribution. No information is lost when we look at the Lorenz curve instead of the distribution function or the density.

All these properties are shared by the graph of the multivariate inverse Lorenz function:

- The pointwise ordering of inverse Lorenz functions is equivalent to the price Lorenz order.
- The Lorenz hypersurface is visual in the same way as the Lorenz curve is: increasing deviation from the egalitarian hypersurface indicates more disparity.
- The Lorenz hypersurface determines the underlying distribution uniquely, up to a vector of scaling constants.

Outlook

References

ALKER, H. (1965). *Mathematics and Politics*. MacMillan, New York. Application of Lorenz curve.

ATKINSON, A. (1970). On the measurement of inequality. *Journal of Economic Theory* **2**, 244–263.

ATKINSON, A. and BOURGUIGNON, F. (1982). The comparison of multidimensioned distributions of economic status. *Review of Economic Studies* **49**, 183–201.

ATKINSON, A. and BOURGUIGNON, F. (1989). The design of direct taxation and family benefits. *Journal of Public Economics* **41**, 3–29.

DYCKERHOFF, R. and MOSLER, K. (1997). Orthant orderings of discrete random vectors. *Journal of Statistical Planning and Inference* **62**, 193–205.

GOLDIE, C. (1977). Convergence theorems for empirical Lorenz curves and their inverses. *Advances in Applied Probability* **9**, 765–791. Application of Lorenz curve.

Hart, P. (1971). Entropy and other measures of concentration. *Journal of the Royal Statistical Society, Series A* **134**, 73–89. Application of Lorenz curve.

Joe, H. and Verducci, J. (1992). Multivariate majorization by positive combinations. In M. Shaked and Y. Tong, eds., *Stochastic Inequalities*, 159–181. Hayward, California.

Kolm, S. C. (1969). The optimal production of social justice. In J. Marjolis and H. Guitton, eds., *Public Economics*, 145–200. MacMillan, New York.

Kolm, S. C. (1977). Multidimensional egalitarianisms. *Quarterly Journal of Economics* **91**, 1–13.

Koshevoy, G. and Mosler, K. (1999). Price majorization and the inverse Lorenz function. Discussion Papers in Statistics and Econometrics 3, Universität zu Köln.

Maasoumi, E. (1986). The measurement and decomposition of multidimensional inequality. *Econometrica* **54**, 991–997.

Marshall, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.

Mosler, K. (1994). Multidimensional welfarisms. In W. Eichhorn, ed., *Models and Measurement of Welfare and Inequality*, 808–820. Springer, Berlin.

Mosler, K. and Scarsini, M., eds. (1993). *Stochastic Orders and Applications. A Classified Bibliography*. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin.

Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*. J. Wiley, New York.

Rothschild, M. and Stiglitz, J. (1970). Increasing risk: I. A definition. *Journal of Economic Theory* **2**, 225–243.

Rothschild, M. and Stiglitz, J. (1971). Increasing risk: II. Its economic consequences. *Journal of Economic Theory* **3**, 66–84.

Russell, W. R. and Seo, T. K. (1978). Ordering uncertain prospects: The multivariate utility functions case. *Review of Economic Studies* **45**, 605–610.

Sen, A. K. (1973). *On Economic Inequality*. Oxford University Press, Oxford.

Thompson, W. (1976). Fisherman's luck. *Biometrics* **32**, 265–271. Application of Lorenz curve.

Tobin, J. (1970). On limiting the domain of inequality. *Journal of Law and Economics* **13**, 263–277.

Tsui, K. (1995). Multidimensional generalizations of the relative and absolute inequality indices: The Atkinson–Kolm–Sen approach. *Journal of Economic Theory* **67**, 251–265.