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MULTIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS GENERATED BY MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

Abstract. We define a multivariate negative binomial distribution (MVNB) as a bivariate Poisson distribution function mixed with a multivariate exponential (MVE) distribution. We focus on the class of MVNB distributions generated by Marshall–Olkin MVE distributions. For simplicity of notation we analyze in detail the class of bivariate (BVNB) distributions. In applications the standard data from [2] and [7] and data concerning parasites of birds from [4] are used.

1. Introduction. It is known that a univariate geometrical probability distribution function is a mixed Poisson distribution with exponentially distributed parameter. A univariate negative binomial distribution is a mixed Poisson distribution where the mixing parameter has a gamma distribution. Also it is easy to see, considering convolution and mixture, that mutually corresponding are: the class of negative binomial distributions and the class of gamma distributions. These univariate properties suggest the definition of a multivariate negative binomial (MVNB) distribution on the basis of multivariate exponential (MVE) distributions and convolution. There exist a few variants of MVE distributions with exponential marginals; we focus on the class of Marshall–Olkin MVE distributions. For simplicity of notation we consider in detail bivariate (BVNB) distributions defined by the BVE class.

We present three applications of BVNB distributions using the standard data from [2] and [7] concerning accidents, and the data concerning the number of parasites of the pheasant [4]. A new variant of MVNB distri-

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butions for random variables characterized by a small correlation may be useful.

In [7] the existence of a negative correlated MVNB distribution is suggested. In an example a negative correlated BVNB distribution is shown. We also present an example of a bivariate distribution with negative binomial marginals which do not belong to any class of BVNB distributions.

2. The negative binomial distribution. Let us recall the elementary notations and definitions concerning univariate negative binomial distributions, suitable for the construction of a multivariate analogue. A random variable X is *geometrically distributed* with parameter p if $P(X = k) = qp^k$, $k \geq 0$, $0 < p < 1$, $q = 1 - p$. Its probability generating function (pgf) is $\phi_p(u) = q(1 - pu)^{-1}$, and also we have $\mathrm{E}(X) = pq^{-1}$, $\mathrm{Var}(X) = pq^{-2}$. The random variable X is *negative binomial* with parameters p and r if $X \overset{\mathrm{d}}{=} \mathrm{E}_{\Lambda}(\Pi(\Lambda))$, where $\Pi(\Lambda)$ under the condition $\Lambda = \lambda$ is a Poisson random variable with parameter λ, and Λ has a gamma distribution with shape parameter r and scale parameter p. Its pgf is $\phi_{p,r}(u) = \phi_p^r(u)$. The parameter r is also called *aggregation*. This random variable has

$$\mathrm{E}(X) = rpq^{-1}, \qquad \mathrm{Var}(X) = rpq^{-2}. \tag{1}$$

The random variables (Λ_1, Λ_2) have a *Marshall–Olkin BVE distribution* [6] if there exist random variables U, V, W, mutually independent and exponentially distributed with parameters $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, respectively, $\alpha + \beta + \gamma > 0$, such that $\Lambda_1 = \min(U, W)$, $\Lambda_2 = \min(V, W)$. The

$$P(\Lambda_1 > x, \Lambda_2 > y) = \exp(-\alpha x - \beta y - \gamma \max(x, y)), \qquad x \geq 0,\ y \geq 0. \tag{2}$$

We say that (X_1, X_2) are *generated* by (Λ_1, Λ_2) if

$$(X_1, X_2) \overset{\mathrm{d}}{=} \mathrm{E}_{(\Lambda_1, \Lambda_2)}(\Pi_1(\Lambda_1), \Pi_2(\Lambda_2)) \overset{\mathrm{df}}{=} T(\Lambda_1, \Lambda_2), \tag{3}$$

where $\Pi_1(\Lambda_1), \Pi_2(\Lambda_2)$ under the condition $\Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2$ are independent Poisson random variables with parameters λ_1, λ_2, respectively. The relation (3) defines a transformation of the distributions of random variables which we also call a transformation of random variables.

Definition 1. The random variables (X_1, X_2) have a *bivariate geometrical distribution* if they are generated by (Λ_1, Λ_2) which have a BVE distribution:

$$P(X_1 = i,\ X_2 = j) = \mathrm{E}(p(i, \Lambda_1) p(j, \Lambda_2)), \qquad i, j \geq 0, \tag{4}$$

where $p(i, \lambda) = (\lambda^i / i!) e^{-\lambda}$, $i \geq 0$, $\lambda > 0$.

Proposition 1. *If (Λ_1, Λ_2) has a BVE distribution,*

$$\psi(s, t) = \mathrm{E}(\exp(-s\Lambda_1 - t\Lambda_2))$$

is the Laplace–Stieltjes transform and (X_1, X_2) *is generated by* (Λ_1, Λ_2), *then its pgf is*

$$\phi(u,v) = \mathrm{E}(u^{X_1} v^{X_2}) = \psi(1-u, 1-v). \tag{5}$$

Moreover,

$$\mathrm{E}(X_i) = \mathrm{E}(\Lambda_i), \qquad \mathrm{Var}(X_i) = \mathrm{Var}(\Lambda_i) + \mathrm{E}(\Lambda_i), \qquad i = 1, 2,$$
$$\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(\Lambda_1, \Lambda_2).$$

The proof is omitted.

DEFINITION 2. The random variable (X_1, X_2) has a *BVNB distribution* with aggregation r if its pgf is

$$\phi_r(u,v) = \mathrm{E}(u^{X_1} v^{X_2}) = \phi^r(u,v), \tag{6}$$

where $\phi(u,v)$ is the pgf of a bivariate geometrical distribution.

The pgf of a bivariate geometrical distribution function may be presented in terms of the pgf of the univariate geometrical distribution.

PROPOSITION 2. *If* (Λ_1, Λ_2) *is the BVE distribution* (2), *then the distribution function* (5) *is geometrical and it has the pgf*

$$\begin{aligned}\phi(u,v) = {}& \phi_\alpha(u)\phi_\beta(v) + \frac{\gamma}{\alpha+\gamma}\phi_\alpha(u)\phi_{\alpha+\gamma}(u)\overline{\phi}_\beta(v) \\ &+ \frac{\gamma}{\beta+\gamma}\overline{\phi}_\alpha(u)\phi_\beta(v)\phi_{\beta+\gamma}(v) \\ &+ \frac{\gamma}{\alpha+\beta+\gamma}\overline{\phi}_\alpha(u)\overline{\phi}_\beta(v)\phi_{1+\alpha+\beta+\gamma}(u+v),\end{aligned} \tag{7}$$

where $\overline{\phi} = 1 - \phi$.

Proof. The Laplace–Stieltjes transform of the BVE distribution (2) (see [6]) is

$$\begin{aligned}\psi(s,t) &= \mathrm{E}(\exp(-s\Lambda_1 - t\Lambda_2)) \\ &= \frac{(\alpha+\beta+\gamma+s+t)(\alpha+\gamma)(\beta+\gamma) + st\gamma}{(\alpha+\beta+\gamma+s+t)(\alpha+\gamma+s)(\beta+\gamma+t)}.\end{aligned}$$

Thus, by (5), the result can be restated as

$$\begin{aligned}\phi(u,v) = \frac{1}{(1-u+\alpha)(1-v+\beta)}\bigg[&\alpha\beta + \gamma\beta\frac{1-u}{1-u+\alpha+\gamma} \\ &+ \gamma\alpha\frac{1-v}{1-v+\beta+\gamma} + \gamma\frac{(1-u)(1-v)}{2-u-v+\alpha+\beta+\gamma}\bigg]. \ \blacksquare\end{aligned}$$

PROPOSITION 3. *If* (X_1, X_2) *has a BVNB distribution generated by* (Λ_1, Λ_2) *with BVE distribution with aggregation* r, *then*

$$
\begin{gathered}
\mathrm{E}(X_i) = r\mathrm{E}(\Lambda_i), \qquad \mathrm{Var}(X_i) = r\,\mathrm{Var}(\Lambda_i) + r\mathrm{E}(\Lambda_i), \qquad i = 1,2, \\
\mathrm{Cov}(X_1, X_2) = r\,\mathrm{Cov}(\Lambda_1, \Lambda_2), \\
P(X_1 = 0,\ X_2 = 0) = \left(\frac{(2+\alpha+\beta+\gamma)(\alpha+\gamma)(\beta+\gamma)+\gamma}{(2+\alpha+\beta+\gamma)(1+\alpha+\gamma)(1+\beta+\gamma)} \right)^r .
\end{gathered} \tag{8}
$$

It is well known [6] that (Λ_1, Λ_2) distributed as (2) has

$$
\begin{gathered}
\mathrm{E}(\Lambda_1) = \frac{1}{\alpha+\gamma}, \qquad \mathrm{E}(\Lambda_2) = \frac{1}{\beta+\gamma}, \\
\mathrm{Cov}(\Lambda_1, \Lambda_2) = \frac{\gamma}{(\alpha+\gamma)(\beta+\gamma)(\alpha+\beta+\gamma)}.
\end{gathered} \tag{9}
$$

Example 1. Let $(\Lambda_1, \Lambda_2) = (\alpha_1 U, \alpha_2 U)$, where $\alpha_1 > 0$, $\alpha_2 > 0$, and U is exponentially distributed with parameter $\gamma > 0$. Then the density function of (U,U) is singular, $f(u,u) = \gamma e^{-\gamma u}$, $u > 0$. From (4) for $i \ge 0$, $j \ge 0$ we have

$$
\begin{aligned}
P(X_1 = i,\ X_2 = j) &= \int_0^\infty \frac{\gamma \alpha_1^i \alpha_2^j u^{i+j}}{i!j!} e^{-(\alpha_1+\alpha_2+\gamma)u}\, du \\
&= \binom{i+j}{i} p_1^i p_2^j (1 - p_1 - p_2), \\
p_1 = \frac{\alpha_1}{\alpha_1+\alpha_2+\gamma}, &\qquad p_2 = \frac{\alpha_2}{\alpha_1+\alpha_2+\gamma}.
\end{aligned}
$$

The pgf of the above distribution is

$$
\phi(u,v) = \frac{1 - p_1 - p_2}{1 - p_1 u - p_2 v},
$$

and the marginals are geometrically distributed with parameters $(\alpha_1+\gamma)^{-1}$ and $(\alpha_2+\gamma)^{-1}$, respectively.

3. Convolutions. It is known that the class of negative binomial distributions with common scale parameter is closed with respect to convolution. Accordingly the multivariate analogue is defined as the class of multivariate geometrical distributions and their convolutions.

Recall two useful formulas. The convolution of bivariate distributions $\{p_{m,n}\}$ and $\{a_{m,n}\}$ is

$$
\{b_{m,n}\} = \{p_{m,n}\} * \{a_{m,n}\},
$$

where

$$
b_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{i,j} a_{m-i,n-j}, \qquad m \ge 0,\ n \ge 0.
$$

The bivariate square root of a distribution $\{p_{m,n}\}$ is

$$
\{c_{m,n}\} = \{p_{m,n}\}^{1/2},
$$

recursively defined, where

$$\begin{aligned}
c_{0,0} &= \sqrt{p_{0,0}}, \\
c_{0,n} &= \frac{1}{2c_{0,0}}\Big[b_{0,n} - \sum_{j=1}^{n} c_{0,j}c_{0,n-j}\Big], && n \ge 1, \\
c_{m,0} &= \frac{1}{2c_{0,0}}\Big[b_{m,0} - \sum_{i=1}^{m} c_{i,0}c_{m-i,0}\Big], && m \ge 1, \\
c_{m,n} &= \frac{1}{2c_{0,0}}\Big[b_{m,n} - \sum_{\substack{i=0 \\ (i,j)\ne(0,0),(m,n)}}^{m}\sum_{j=0}^{n} c_{i,j}c_{m-i,n-j}\Big], && m \ge 1, n \ge 1.
\end{aligned}$$

THEOREM 1. *Let f_1, f_2 denote two MVNB distributions and $f_3 = f_1 * f_2$ denote their convolution. If f_i is generated by g_i, $i = 1,2,3$, then $g_1 * g_2 = g_3$.*

Proof. In the proof we use the transform T of random variables defined in (3). We have

$$\begin{aligned}
&T(\Lambda_1^{(1)},\Lambda_2^{(1)}) + T(\Lambda_1^{(2)},\Lambda_2^{(2)}) \\
&\quad = \mathrm{E}_{\Lambda_1^{(1)},\Lambda_2^{(1)}}(\Pi(\Lambda_1^{(1)}),\Pi(\Lambda_2^{(1)})) + \mathrm{E}_{\Lambda_1^{(2)},\Lambda_2^{(2)}}(\Pi(\Lambda_1^{(2)}),\Pi(\Lambda_2^{(2)})) \\
&\quad = \mathrm{E}_{\Lambda_1^{(1)},\Lambda_2^{(1)},\Lambda_1^{(2)},\Lambda_2^{(2)}}((\Pi(\Lambda_1^{(1)}),\Pi(\Lambda_2^{(1)})) + (\Pi(\Lambda_1^{(2)}),\Pi(\Lambda_2^{(2)}))) \\
&\quad = \mathrm{E}_{\Lambda_1^{(1)},\Lambda_1^{(2)},\Lambda_2^{(1)},\Lambda_2^{(2)}}(\Pi(\Lambda_1^{(1)}) + \Pi(\Lambda_1^{(2)}),\Pi(\Lambda_2^{(1)}) + \Pi(\Lambda_2^{(2)})) \\
&\quad = \mathrm{E}_{\Lambda_1^{(1)},\Lambda_1^{(2)},\Lambda_2^{(1)},\Lambda_2^{(2)}}(\Pi(\Lambda_1^{(1)} + \Lambda_1^{(2)}),\Pi(\Lambda_2^{(1)} + \Lambda_2^{(2)})) \\
&\quad = T(\Lambda_1^{(1)} + \Lambda_1^{(2)},\Lambda_2^{(1)} + \Lambda_2^{(2)}). \ \blacksquare
\end{aligned}$$

4. The Edwards–Gurland BVNB distribution. Edwards and Gurland [2] define a variant of BVNB distribution as a compound correlated bivariate Poisson distribution using the Campbell correlated bivariate Poisson distribution. It has been used by several authors (see [2]) to the description of the number of accidents in successive intervals of time. We recall this definition in the form which enables the comparison with Definition 2.

DEFINITION 3. Let X, Y, Z and Λ denote random variables such that X, Y, Z under the condition $\Lambda = \lambda$ are mutually independent with Poisson distribution with parameters $\alpha_1\lambda$, $\alpha_2\lambda$, $\alpha_{1,2}\lambda$, respectively, where $\lambda > 0$, $0 \le \alpha_{1,2} \le \min(\alpha_1,\alpha_2)$, $\alpha_1 + \alpha_2 + \alpha_{1,2} > 0$, and Λ has a gamma distribution with shape parameter r and scale parameter mr^{-1}.

Then $(X_1,X_2) = \mathrm{E}_\Lambda(X+Z, Y+Z)$ has the *Edwards–Gurland BVNB distribution* with parameters λ, α_1, α_2, $\alpha_{1,2}$.

The pgf of the Edwards–Gurland BVNB distribution is

$$f(u,v)=\left(1-\frac{m}{r}(\alpha_1(u-1)+\alpha_2(v-1)+\alpha_{1,2}(uv-1))\right)^{-r}, \tag{10}$$

and from this its moments can be obtained:

$$\begin{aligned}\mathrm{E}(X_i)&=(\alpha_i+\alpha_{1,2})\mathrm{E}(\Lambda),\\ \mathrm{Var}(X_i)&=(\alpha_i+\alpha_{1,2})^2\,\mathrm{Var}(\Lambda)+(\alpha_i+\alpha_{1,2})\mathrm{E}(\Lambda),\qquad i=1,2,\\ \mathrm{Cov}(X_1,X_2)&=(\alpha_1+\alpha_{1,2})(\alpha_2+\alpha_{1,2})\,\mathrm{Var}(\Lambda)+\alpha_{1,2}\mathrm{E}(\Lambda).\end{aligned}$$

Remark. If $\alpha_{1,2}=0$, then a Edwards–Gurland BVNB distribution is a Bates–Neyman distribution and also a BVNB distribution defined by (6) given in Example 1.

5. The negative correlated BVNB distribution. BVNB distributions generated by Marshall–Olkin BVE distributions (and also Edwards–Gurland BVNB distributions) are positive correlated. Now we present two examples of negative correlated distributions. In Example 2 we define a distribution function whose marginals are negative binomial, but which cannot be generated by any BVE distribution function. In Example 3 we define a BVNB distribution which is generated by BVE distributions and has negative correlation.

Example 2. Consider the singular density function

$$f(x,y)=\begin{cases}\left(\frac{1}{2}\right)^i\lambda e^{-\lambda x} & \text{for } 0<x<a,\ y=ia+x,\ i\geq 1,\\ \left(\frac{1}{2}\right)^j\lambda e^{-\lambda y} & \text{for } x=ja+y,\ 0<y<a,\ j\geq 1,\end{cases}$$

where $a=\frac{1}{\lambda}\log 2$, $\lambda>0$.

To construct (X_1,X_2) with this density define three mutually independent random variables: U, geometrically distributed with parameter $p=1/2$; Z, exponentially distributed with parameter λ truncated to the interval $[0,a]$; and a binary random variable δ with probability $1/2$ for 0 and 1. Then

$$X_1\overset{\mathrm{d}}{=}(U+1)a\delta+Z,\qquad X_2\overset{\mathrm{d}}{=}(U+1)a(1-\delta)+Z.$$

The Laplace transform of f is

$$\psi(s,t)=\frac{\lambda}{s+t+\lambda}(1-e^{-(s+t+\lambda)a})\left(\frac{e^{-(s+\lambda)a}}{1-e^{-(s+\lambda)a}}+\frac{e^{-(t+\lambda)a}}{1-e^{-(t+\lambda)a}}\right),$$

the marginals are exponential with parameter λ, and the covariance and correlation are

$$\mathrm{Cov}(X_1,X_2)=2a\lambda^{-1}-\tfrac{3}{2}a^2,\qquad \varrho(X_1,X_2)=\log 2-\tfrac{3}{2}\log^2 2=-0.028.$$

EXAMPLE 3. Consider the bivariate distribution function

$$P(X_1 = i,\ X_2 = j) = \begin{cases} \left(\frac{1}{2}\right)^{i+1} & \text{for } j = 0,\ i \geq 1, \\ (\frac{1}{2})^{j+1} & \text{for } i = 0,\ j \geq 1. \end{cases} \tag{11}$$

The pgf is

$$\phi(u,v) = \sum_{i=1}^{\infty} u^i \left(\frac{1}{2}\right)^{i+1} + \sum_{j=1}^{\infty} v^j \left(\frac{1}{2}\right)^{j+1} = \frac{1}{2}\left(\frac{u}{2-u} + \frac{v}{2-v}\right);$$

the marginals are geometrical with $p = 1/2$ and we have $\mathrm{Cov}(X_1, X_2) = -1$, $\varrho(X_1, X_2) = -1/2$.

From Proposition 1 the Laplace transform of the BVE distribution generated by (11), if it exists, is

$$\psi(s,t) = \phi(1-s, 1-t) = \frac{1}{2}\left(\frac{1-s}{1+s} + \frac{1-t}{1+t}\right).$$

Unfortunately the BVE measure which corresponds to this transform is not probabilistic. If μ is such that

$$\psi(s,t) = \int_0^\infty \int_0^\infty e^{-sx-ty}\, \mu(dx, dy),$$

then

$$\mu\{(0,0)\} = -1, \quad \mu\{0 \times [y, \infty)\} = e^{-y}, \quad \mu\{[x,\infty) \times 0\} = e^{-x}.$$

6. Applications. In population models two random mechanisms may be described by BVNB distributions. The first is the variation of the population (in the examples: technical state of an omnibus, condition of a bird) which is represented here by random variables (Λ_1, Λ_2). The second mechanism is in creation of random variables for an individual (the number of accidents of an omnibus, the number of parasites of a bird, and so on). In the definition of a BVNB distribution both mechanisms are conditionally independent.

The key problem in the applications of MVNB distributions is the estimation of parameters of the distribution. Edwards and Gurland [2] and Subrahmaniam and Subrahmaniam [7] analyze this problem for some standard data and examine a few techniques. Here this problem is solved in two stages. In the first stage we consider the estimation of the parameters p, r of the marginal distributions by the maximum likelihood method; for computations of the maximum of the likelihood function we use directly its definition. In the second stage we estimate the parameters α, β and γ of the joint distribution defined according to the suggestion in [7] in such a way as to preserve the probability of (0,0) cells (Zero-Zero Cell Frequency Method), or alternatively, as to preserve the correlation coefficient (Moment Method).

In the consideration of the number of parasites of birds (see [3]–[5]) it is supposed that outliers in the data exist, e.g. with a small probability items with a very large number of parasites occur. Hence the moment or maximum likelihood estimators may be biased. Therefore in Application 3 a right side truncated distribution is used (in place of the kth extreme order statistic; here $k = 3$). For information on goodness of fit the chi-square statistic was computed. In the presentation of results the marginal classes are grouped in such a way that the frequencies exceed a fixed number (here 6). The negative value of the estimator in the case of positive parameter has been corrected to 0.

Application 1. In [2] (Table 1, see also [7], Table 5) Edwards and Gurland analyze some standard data concerning the number of accidents. Using this in our computations, under the assumption of negative binomial distribution the estimation of the parameters of the marginal distributions of 251 observations gives $p_1 = 0.79$, $r_1 = 0.91$, $p_2 = 0.77$, $r_2 = 1.01$. In the class of BVNB we have $r_1 = r_2$. For simplicity of further calculations let $r_1 = r_2 = r = 1$; then $p_1 = 0.7733$, $p_2 = 0.7724$. Using the marginal parameters and (0,0) cell frequency, we obtain $\alpha = 0.0005$, $\beta = 0.0019$, $\gamma = 0.2927$. The observed and expected frequencies are given in Table 1. Using the parameters of marginals and the correlation coefficient, after correction of negative values of the estimators, we obtain $\alpha = 0$, $\beta = 0$, $\gamma = 0.3792$. The observed and expected frequencies computed using (7) are presented in Table 2.

TABLE 1. The observed and expected frequencies for the Edwards–Gurland data ([2], Table 5) (using the Zero-Zero Cell Frequency Method) $r = 1$, $\alpha = 0$, $\beta = 0$, $\gamma = 0.3792$, $\chi^2 = 91.4$, $df = 45$

	0		1		2		3		4		5-7		≥ 8	
0	40	40.00	20	16.81	3	7.07	0	2.97	2	1.25	0	0.84	0	0.07
1	10	16.81	22	14.13	7	8.91	1	4.99	1	2.62	0	2.28	0	0.27
2	4	7.07	9	8.91	10	7.49	5	5.25	2	3.31	2	3.63	0	0.63
3	1	2.97	5	4.99	1	5.25	3	4.41	1	3.24	2	4.38	0	1.05
4	0	1.25	2	2.62	2	3.31	5	3.24	6	2.73	7	4.46	3	1.46
5-7	0	0.84	0	2.28	2	3.63	5	4.38	5	4.46	15	9.95	11	5.58
≥ 8	0	0.07	0	0.27	1	0.63	0	1.05	3	1.46	5	5.58	28	10.11

TABLE 2. The observed and expected frequencies for the Edwards–Gurland data ([2], Table 5) (using the Moment Method) $r = 1$, $\alpha = 0.0005$, $\beta = 0.0019$, $\gamma = 0.2927$, $\chi^2 = 60.3$, $df = 45$

	0		1		2		3		4		5-7		≥ 8	
0	40	32.07	20	13.98	3	6.10	0	2.66	2	1.16	0	0.84	0	0.09
1	10	14.01	22	12.20	7	7.97	1	4.63	1	2.53	0	2.34	0	0.33
2	4	6.13	9	7.99	10	6.95	5	5.05	2	3.30	2	3.84	0	0.76
3	1	2.69	5	4.65	1	5.06	3	4.40	1	3.36	2	4.82	0	1.32
4	0	1.19	2	2.55	2	3.31	5	3.36	6	2.93	7	5.10	3	1.90
5-7	0	0.89	0	2.38	2	3.88	5	4.85	5	5.12	15	12.23	11	7.97
≥ 8	0	0.13	0	0.37	1	0.80	0	1.36	3	1.94	5	8.04	28	19.39

APPLICATION 2. In [2] (Table 1, see also [7], Table 2), to describe the number of accidents of omnibuses in London in two consecutive years, a negative binomial distribution is used. The estimation of the parameters of the marginals of 166 pairs of observations is a negative binomial distribution for the first variable with $p_1 = 0.2052$, $r_1 = 6.2533$ and for the second variable with $p_2 = 0.2710$, $r_2 = 4.1810$. Assuming $r_1 = r_2 = r$, we obtain $r = 5.045$. For simplicity of further calculations let (see [7]) $r = 5$; then $p_1 = 0.244$, $p_2 = 0.237$. Using the marginal parameters and (0,0) cell frequency, after the correction of negative values of parameters, we obtain $\alpha = 0$, $\beta = 0$, $\gamma = 3.2$. Table 3 shows the observed and expected frequencies computed using (7) and convolutions. Here $\chi^2 = 17.86$; using correlation, one obtains a similar goodness of fit ($\chi^2 = 17.66$).

TABLE 3. The observed and expected frequencies for the Edwards–Gurland data ([2], Table 1) (using Zero-Zero Cell Frequency Method) $r = 5$, $\alpha = 0$, $\beta = 0$, $\gamma = 3.24$, $\chi^2 = 17.9$, $df = 20$

	0		1		2		3		4-9	
0	15	15.00	15	14.32	4	8.20	2	3.65	1	2.09
1	17	14.32	18	16.39	9	10.95	3	5.57	5	3.77
2	4	8.20	16	10.95	12	8.36	6	4.79	5	3.79
3	2	3.65	6	5.57	5	4.79	2	3.04	4	2.80
4-9	2	2.09	4	3.77	5	3.79	0	2.80	4	3.29

APPLICATION 3. In [3]–[5] for some species of birds and some species of parasites it was established that the number of parasites of a bird is a negative binomial or a mixture of negative binomials. We consider the parasites of the species *Goniocotes chrysocephalus* and *Zlotorzyckiella colchici* of the pheasant (*Phasianus colchicus* L.). For the data on the number of parasites of these species for 50 pheasants see [4].

According to the supposition that outliers exist, the truncated random variables $\min(X_1, 95)$ in place of X_1 and $\min(X_2, 92)$ in place of X_2 are used. Assuming marginals with common r, by the maximum likelihood method we estimate $v_1 = 0.1691$, $p_1 = 0.9923$ for X_1 and $v_2 = 0.1487$, $p_2 = 0.9936$ for X_2. If it is assumed that $v_1 = v_2 = v$, then $r = 0.1520$, $p_1 = 0.9909$, $p_2 = 0.9914$. For simplicity of further calculations of the expected frequencies let $r = 0.15625 = \frac{1}{8} + \frac{1}{32}$ (then the technique of square roots and convolutions may be used). Then $p_1 = 0.9906$, $p_2 = 0.9911$.

We estimate the parameter γ by the Zero-Zero Cell Frequency Method. Using (1) and (9) we obtain $\alpha = 0.0087$, $\beta = 0.0082$, $\gamma = 0.0008$. Table 4 gives the observed and expected frequencies.

TABLE 4. The observed and expected frequencies for the number of parasites of the pheasant (using Zero-Zero Cell Frequency Method): $r = 0.18625$, $\alpha = 0.0087$, $\beta = 0.0082$, $\gamma = 0.0008$, $\chi^2 = 25.1$

		X_1													
		0		1		2-4		5-12		13-40		≥ 40		Total	
	0	15	15.00	0	1.36	3	1.10	1	1.33	3	2.44	1	2.89	23	24.11
	1	2	1.36	1	0.68	2	0.68	0	0.24	0	0.33	0	0.44	5	3.73
	2-4	3	1.09	1	0.68	0	1.43	0	0.76	1	0.32	0	0.56	5	4.85
X_2	5-12	3	1.31	0	0.24	0	0.75	1	1.87	1	0.50	1	0.54	6	5.21
	13-40	2	2.41	1	0.32	1	0.32	1	0.50	0	2.13	2	0.48	7	6.16
	≥ 41	0	2.73	1	0.42	0	0.53	1	0.51	0	0.45	2	1.30	4	5.94
	Total	25	23.91	4	3.70	6	4.82	4	5.18	5	6.18	6	6.21	50	50.00

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