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MUSIELAK-ORLICZ-SOBOLEV SPACES ON METRIC MEASURE SPACES

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Abstract. Our aim in this paper is to study Musielak-Orlicz-Sobolev spaces on metric measure spaces. We consider a Hajłasz-type condition and a Newtonian condition. We prove that Lipschitz continuous functions are dense, as well as other basic properties. We study the relationship between these spaces, and discuss the Lebesgue point theorem in these spaces. We also deal with the boundedness of the Hardy-Littlewood maximal operator on Musielak-Orlicz spaces. As an application of the boundedness of the Hardy-Littlewood maximal operator, we establish a generalization of Sobolev's inequality for Sobolev functions in Musielak-Orlicz-Hajłasz-Sobolev spaces.

Keywords: Sobolev space; metric measure space; Sobolev's inequality; Hajłasz-Sobolev space; Newton-Sobolev space; Musielak-Orlicz space; capacity; variable exponent

MSC 2010: 46E35, 31B15

1. Introduction

Sobolev spaces on metric measure spaces have been studied during the last two decades, see e.g. [6], [21], [23], [33], [51]. The theory was generalized to Orlicz-Sobolev spaces on metric measure spaces in [4], [5], [53]. We refer to [2], [3], [15], [54] for Sobolev spaces on $\mathbb{R}^N$, [9], [14] for variable exponent Sobolev spaces, [50] for Musielak-Orlicz spaces, [16] for the study of differential equations of divergence form in Musielak-Sobolev spaces and [17] for the study of uniform convexity of Musielak-Orlicz-Sobolev spaces and its applications to variational problems. In the last decade, variable exponent Sobolev spaces on metric measure spaces have been developed, see

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e.g. [19], [20], [31], [32], [49]. The purpose of this paper is to define Musielak-Orlicz-Sobolev spaces on metric measure spaces and prove the basic properties of such spaces.

There are two ways to define first order Sobolev spaces on metric measure spaces. Hajłasz [21] showed that a p-integrable function u, $1 < p < \infty$, belongs to $W^{1,p}(\mathbb{R}^N)$ if and only if there exists a nonnegative p-integrable function g such that

$$|u(x) - u(y)| \leqslant |x - y|(g(x) + g(y)) \tag{1.1}$$

for almost every $x, y \in \mathbb{R}^N$. If we replace $|x - y|$ by the distance of the points x and y, (1.1) can be stated in metric measure spaces. Spaces defined using (1.1) are called Hajłasz-Sobolev spaces. See also [23], [33]. The theory was generalized to Orlicz-Sobolev spaces by Aïssaoui (see [4], [5]). For the Sobolev capacity on Hajłasz-Sobolev spaces, see [38]. By the classical Lebesgue differentiation theorem, almost every point is a Lebesgue point for a locally integrable function. For the Lebesgue point theorem in Hajłasz-Sobolev spaces, we refer the reader to [36].

Another way is to use weak upper gradients. A nonnegative Borel measurable function h is said to be an upper gradient of u if

$$|u(x) - u(y)| \leqslant \int_\gamma h \,\mathrm{d}s \tag{1.2}$$

for every x, y and every curve γ connecting x to y. Upper gradients were introduced by Heinonen and Koskela [34] as a tool to study quasiconformal maps. If (1.2) holds for a function u on every curve not belonging to an exceptional family of p-modulus zero in metric measure spaces, we call h a weak upper gradient of u. We call these spaces Newtonian spaces or Newton-Sobolev spaces. The study of Newton-Sobolev spaces was initiated by Shanmugalingam [51]. See also [6]. The theory was generalized to Orlicz-Sobolev spaces by Tuominen [53].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [9], [14]). See also [24], [27]. Harjulehto, Hästö and Pere [31] studied basic properties of the variable exponent Hajłasz-Sobolev space and the variable exponent Newton-Sobolev space. For the Lebesgue point theorem in variable exponent spaces, see e.g. [25].

The Hardy-Littlewood maximal operator is a classical tool in harmonic analysis and the study of Sobolev functions and partial differential equations, and plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see e.g. [7], [35], [41], [52]). It is well known that the Hardy-Littlewood maximal operator is bounded on the Lebesgue space $L^p(\mathbb{R}^N)$ if $p > 1$ (see [52]).

See e.g. [8] for Orlicz spaces, [10], [12] for variable exponent Lebesgue spaces $L^{p(\cdot)}$, [42], [47] for the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. These spaces are special cases of so-called Musielak-Orlicz spaces [44], [50]. For general Musielak-Orlicz spaces, see [11]. In bounded doubling metric measure spaces, the boundedness of the Hardy-Littlewood maximal operator on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was studied in [20], [32]. See also [1].

One of the important applications of the boundedness of the Hardy-Littlewood maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha * f\|_{L^{p^*}(\mathbb{R}^N)} \leqslant C\|f\|_{L^p(\mathbb{R}^N)}$$

for $f \in L^p(\mathbb{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/N$ (see e.g. [2], Theorem 3.1.4). This result was extended to Orlicz spaces in [8], [48]. In Euclidean setting, variable exponent versions were discussed e.g. in [13], [39], [40], [44], [47]. For variable exponent versions on metric measure spaces, see e.g. [20], [28].

In this paper, we define Musielak-Orlicz-Newton-Sobolev spaces as well as Musielak-Orlicz-Hajłasz-Sobolev spaces on metric measure spaces and prove the basic properties of such spaces.

The paper is organized as follows. In Section 2, we define Musielak-Orlicz spaces on metric measure spaces.

In Section 3, we study basic properties of Musielak-Orlicz-Hajłasz-Sobolev spaces. We show that Lipschitz continuous functions are dense and study a related Sobolev-type capacity. We prove that every point except for a small set is a Lebesgue point for Sobolev functions in Musielak-Orlicz-Hajłasz-Sobolev spaces.

In Section 4, we study basic properties of Musielak-Orlicz-Newton-Sobolev spaces. We show that Lipschitz continuous functions are dense if the measure is doubling and study a related Sobolev-type capacity. We discuss the Lebesgue point theorem in Musielak-Orlicz-Newton-Sobolev spaces.

In Section 5, we study the relationship between Musielak-Orlicz-Hajłasz-Sobolev spaces and Musielak-Orlicz-Newton-Sobolev spaces in a metric measure space (see Theorem 5.4).

In Section 6, we show that the Hardy-Littlewood maximal operator is bounded on Musielak-Orlicz spaces in our setting (see Theorem 6.3).

In Section 7, as an application of the boundedness of the Hardy-Littlewood maximal operator, we give a general version of Sobolev's inequality for Sobolev functions in Musielak-Orlicz-Hajłasz-Sobolev spaces (see Theorem 7.7). In such a general setting, we can obtain new results (e.g., Corollaries 7.6 and 7.8).

In Section 8, we discuss Fuglede's theorem for Musielak-Orlicz-Sobolev spaces in Euclidean setting.

2. Musielak-Orlicz spaces

Throughout this paper, let C denote positive constant independent of the variables in question.

We denote by (X, d, μ) a metric measure space, where X is a set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r, and $d_\Omega = \sup\{d(x, y)\colon x, y \in \Omega\}$ for a set $\Omega \subset X$.

For a measurable function $Q(\cdot)$ satisfying

$$0 < Q^- := \inf_{x\in X} Q(x) \leqslant \sup_{x\in X} Q(x) =: Q^+ < \infty,$$

we say that a measure μ is lower Ahlfors $Q(x)$-regular if there exists a constant $c_0 > 0$ such that

$$\mu(B(x, r)) \geqslant c_0 r^{Q(x)}$$

for all $x \in X$ and $0 < r < d_X$. Further, μ is Ahlfors $Q(x)$-regular if there exists a constant $c_1 > 0$ such that

$$c_1^{-1} r^{Q(x)} \leqslant \mu(B(x, r)) \leqslant c_1 r^{Q(x)}$$

for all $x \in X$ and $0 < r < d_X$. We say that the measure μ is a doubling measure, if there exists a constant $c_2 > 0$ such that $\mu(B(x, 2r)) \leqslant c_2 \mu(B(x, r))$ for every $x \in X$ and $0 < r < d_X$. We say that X is a doubling space if μ is a doubling measure.

We consider a function

$$\Phi(x, t) = t\phi(x, t)\colon X \times [0, \infty) \to [0, \infty)$$

satisfying the following conditions (Φ1)–(Φ4):

(Φ1) $\phi(\cdot, t)$ is measurable on X for each $t \geqslant 0$ and $\phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in X$;

(Φ2) there exists a constant $A_1 \geqslant 1$ such that $A_1^{-1} \leqslant \phi(x, 1) \leqslant A_1$ for all $x \in X$;

(Φ3) $\phi(x, \cdot)$ is uniformly almost increasing, namely, there exists a constant $A_2 \geqslant 1$ such that $\phi(x, t) \leqslant A_2 \phi(x, s)$ for all $x \in X$ whenever $0 \leqslant t < s$;

(Φ4) there exists a constant $A_3 > 1$ such that $\phi(x, 2t) \leqslant A_3 \phi(x, t)$ for all $x \in X$ and $t > 0$.

Note that (Φ2), (Φ3) and (Φ4) imply $0 < \inf_{x\in X} \phi(x, t) \leqslant \sup_{x\in X} \phi(x, t) < \infty$ for each $t > 0$.

Let $\bar{\phi}(x,t) = \sup_{0\leqslant s\leqslant t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,\mathrm{d}r$$

for $x \in X$ and $t \geqslant 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leqslant \overline{\Phi}(x,t) \leqslant A_2\Phi(x,t) \tag{2.1}$$

for all $x \in X$ and $t \geqslant 0$.

By (Φ3), we see that

$$\Phi(x,at) \begin{cases} \leqslant A_2 a\Phi(x,t) & \text{if } 0 \leqslant a \leqslant 1, \\ \geqslant A_2^{-1} a\Phi(x,t) & \text{if } a \geqslant 1. \end{cases} \tag{2.2}$$

We shall also consider the following conditions:

(Φ5) for every $\gamma_1, \gamma_2 > 0$, there exists a constant $B_{\gamma_1,\gamma_2} \geqslant 1$ such that $\phi(x,t) \leqslant B_{\gamma_1,\gamma_2}\phi(y,t)$, whenever $d(x,y) \leqslant \gamma_1 t^{-1/\gamma_2}$ and $t \geqslant 1$;

(Φ6) there exist $x_0 \in X$, a function $g \in L^1(X)$ and a constant $B_\infty \geqslant 1$ such that $0 \leqslant g(x) < 1$ for all $x \in X$ and $B_\infty^{-1}\Phi(x,t) \leqslant \Phi(x',t) \leqslant B_\infty\Phi(x,t)$, whenever $d(x',x_0) \geqslant d(x,x_0)$ and $g(x) \leqslant t \leqslant 1$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\ldots,k$, be measurable functions on X such that

(P1) $1 < p^- := \inf_{x\in X} p(x) \leqslant \sup_{x\in X} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x\in X} q_j(x) \leqslant \sup_{x\in X} q_j(x) =: q_j^+ < \infty$

for all $j = 1,\ldots,k$.

Set $L_c(t) = \log(c+t)$ for $c \geqslant \mathrm{e}$ and $t \geqslant 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies (Φ1), (Φ2), (Φ3) and (Φ4). $\Phi(x,t)$ satisfies (Φ5) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leqslant \frac{C_p}{L_{\mathrm{e}}(1/d(x,y))}$$

with a constant $C_p \geqslant 0$ and

(Q2) $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \leqslant \frac{C_{q_j}}{L_{\mathrm{e}}^{(j+1)}(1/d(x,y))}$$

with constants $C_{q_j} \geqslant 0$, $j = 1, \ldots, k$.

Example 2.2. Let $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on X satisfying (P1) and (Q1).

Then,

$$\Phi(x,t) = (1+t)^{p_1(x)}(1+1/t)^{-p_2(x)} L_c(t)^{q_1(x)} L_c(1/t)^{-q_2(x)}$$

satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p_j^- > 1$, $j = 1,2$ or $q_j^- \geqslant 0$, $j = 1,2$. As a matter of fact, it satisfies (Φ3) if and only if $p_j(\cdot)$ and $q_j(\cdot)$ satisfy the following conditions:

(1) $q_j(x) \geqslant 0$ at points x where $p_j(x) = 1$, $j = 1,2$;

(2) $\sup\limits_{\{x\colon\, p_j(x)>1\}} \{\min(q_j(x), 0) \log(p_j(x) - 1)\} < \infty$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5) if $p_1(\cdot)$ is log-Hölder continuous and $q_1(\cdot)$ is 2-log-Hölder continuous.

Example 2.3. Let $\Phi(\cdot,\cdot)$ be defined as in Example 2.1 and fix $x_0 \in X$. Let κ and c be positive constants. If μ satisfies $\mu(B(x_0,r)) \leqslant cr^\kappa$ for all $r \geqslant 1$ and

(P3) $p(\cdot)$ is log-Hölder continuous at ∞, namely $|p(x) - p(x')| \leqslant C_{p,\infty}/L_e(d(x,x_0))$ for $d(x',x_0) \geqslant d(x,x_0)$ with a constant $C_{p,\infty} \geqslant 0$,

then $\Phi(\cdot,\cdot)$ satisfies (Φ6) with $g(x) = 1/(1+d(x,x_0))^{\kappa+1}$.

Example 2.4. Let $\Phi(\cdot,\cdot)$ be defined as in Example 2.2 and fix $x_0 \in X$. Let κ and c be positive constants. If μ satisfies $\mu(B(x_0,r)) \leqslant cr^\kappa$ for all $r \geqslant 1$, $p_2(\cdot)$ satisfies (P3) and

(Q3) $q_2(\cdot)$ is 2-log-Hölder continuous at ∞, namely $|q_2(x) - q_2(x')| \leqslant C_{q_2,\infty}/L_c^{(2)}(d(x,x_0))$ for $d(x',x_0) \geqslant d(x,x_0)$ with a constant $C_{q_2,\infty} \geqslant 0$,

then $\Phi(\cdot,\cdot)$ satisfies (Φ6) with $g(x) = 1/(1+d(x,x_0))^{\kappa+1}$.

We say that u is a locally integrable function on X if u is an integrable function on all balls B in X. From now on, we assume that $\Phi(x,t)$ satisfies (Φ1), (Φ2), (Φ3) and (Φ4). Then the associated Musielak-Orlicz space

$$L^\Phi(X) = \left\{ f \in L^1_{\mathrm{loc}}(X)\colon\, \int_X \Phi(y,|f(y)|)\,\mathrm{d}\mu(y) < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(X)} = \inf\left\{\lambda > 0\colon \int_X \overline{\Phi}(y, |f(y)|/\lambda)\,\mathrm{d}\mu(y) \leqslant 1\right\}$$

(cf. [50]).

For a measurable function f on X, we define the modular $\varrho_\Phi(f)$ by

$$\varrho_\Phi(f) = \int_X \overline{\Phi}(y, |f(y)|)\,\mathrm{d}\mu(y).$$

Lemma 2.5 ([45], Lemma 2.2, and [50], Theorem 8.14). *Let $\{f_i\}$ be a sequence in $L^\Phi(X)$. Then $\varrho_\Phi(f_i)$ converges to 0 if and only if $\|f_i\|_{L^\Phi(X)}$ converges to 0.*

3. Musielak-Orlicz-Hajłasz-Sobolev spaces $M^{1,\Phi}(X)$

3.1. Basic properties. We say that a function $u \in L^\Phi(X)$ belongs to Musielak-Orlicz-Hajłasz-Sobolev spaces $M^{1,\Phi}(X)$ if there exists a nonnegative function $g \in L^\Phi(X)$ such that

$$|u(x) - u(y)| \leqslant d(x,y)(g(x) + g(y)) \tag{3.1}$$

for μ-almost every $x, y \in X$. Here, we call the function g a Hajłasz gradient of u. We define the norm

$$\|u\|_{M^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)},$$

where the infimum is taken over all Hajłasz gradients of u. For the case when $\Phi(x,t) = t^p$, the spaces $M^{1,p}(X)$ were first introduced by P. Hajłasz [21] as a generalization of the classical Sobolev spaces $W^{1,p}(\mathbb{R}^N)$ to the general setting of quasi-metric measure spaces. For variable exponent spaces $M^{1,p(\cdot)}(X)$, see [31].

Since $L^\Phi(X)$ is a Banach space, standard arguments yield the following propositions (see [31]).

Proposition 3.1 (cf. [31], Proposition 3.1). *If $L^\Phi(X)$ is reflexive, then for every $u \in M^{1,\Phi}(X)$, there exist Hajłasz gradients of u which minimize the norm. Moreover, if $\|\cdot\|_{L^\Phi(X)}$ is a uniformly convex norm, then there exists a unique Hajłasz gradient of u which minimizes the norm.*

Remark 3.2. We say that $\Phi(x,t)$ is uniformly convex on X if for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$|a - b| \leqslant \varepsilon \max\{|a|, |b|\} \quad \text{or} \quad \overline{\Phi}\Big(x, \frac{|a+b|}{2}\Big) \leqslant (1-\delta)\frac{\overline{\Phi}(x,|a|) + \overline{\Phi}(x,|b|)}{2}$$

for all $a, b \in \mathbb{R}$ and $x \in X$. By [14], Section 2.4, if $\Phi(x,t)$ is uniformly convex on X, then the norm $\|\cdot\|_{L^{\Phi}(X)}$ is a uniformly convex norm.

Proposition 3.3 (cf. [31], Theorem 3.3). *$M^{1,\Phi}(X)$ is a Banach space.*

Proposition 3.4 (cf. [21], Theorem 5). *For every $u \in M^{1,\Phi}(X)$ and $\varepsilon > 0$, there exists a Lipschitz function $h \in M^{1,\Phi}(X)$ such that*

(1) $\mu(\{x \in X\colon\ u(x) \neq h(x)\}) \leqslant \varepsilon$;
(2) $\|u - h\|_{M^{1,\Phi}(X)} \leqslant \varepsilon$.

Proof. For $u \in M^{1,\Phi}(X)$, we take $g \in L^{\Phi}(X)$ which is a Hajłasz gradient of u. Set

$$E_\lambda = \{x \in X\colon\ |u(x)| \leqslant \lambda \text{ and } g(x) \leqslant \lambda\}.$$

Note that u is Lipschitz continuous with the constant 2λ on E_λ. By the McShane extension [46], we extend u to a Lipschitz function $\bar{u}$ on X, where

$$\bar{u}(x) = \inf_{y \in E_\lambda} \{u(y) + 2\lambda \operatorname{dist}(x,y)\}.$$

We modify this extension by truncating

$$u_\lambda = (\operatorname{sign} \bar{u}) \min\{|\bar{u}|, \lambda\}.$$

Then u_λ is Lipschitz continuous with the constant 2λ, $u = u_\lambda$ on E_λ and $|u_\lambda| \leqslant \lambda$. For every $\lambda > 1$, we see from (Φ2), (Φ3), (Φ4) and (2.2) that

$$\begin{aligned}
\mu(\{x \in X\colon\ u(x) \neq u_\lambda(x)\}) &\leqslant \mu(X \setminus E_\lambda) \\
&\leqslant A_1 A_2 \int_{X\setminus E_\lambda} \Phi\Big(x, \frac{|u(x)| + g(x)}{\lambda}\Big) \,\mathrm{d}\mu(x) \\
&\leqslant A_1 A_2^2 \Big\{ \int_{X\setminus E_\lambda} \Phi\Big(x, \frac{2|u(x)|}{\lambda}\Big) \,\mathrm{d}\mu(x) + \int_{X\setminus E_\lambda} \Phi\Big(x, \frac{2g(x)}{\lambda}\Big) \,\mathrm{d}\mu(x) \Big\} \\
&\leqslant \frac{A_1 A_2^3}{\lambda} \Big\{ \int_{X\setminus E_\lambda} \Phi(x, 2|u(x)|) \,\mathrm{d}\mu(x) + \int_{X\setminus E_\lambda} \Phi(x, 2g(x)) \,\mathrm{d}\mu(x) \Big\} \\
&\leqslant \frac{2A_1 A_2^3 A_3}{\lambda} \Big\{ \int_{X\setminus E_\lambda} \Phi(x, |u(x)|) \,\mathrm{d}\mu(x) + \int_{X\setminus E_\lambda} \Phi(x, g(x)) \,\mathrm{d}\mu(x) \Big\}.
\end{aligned}$$

Hence we have $\mu(\{x \in X\colon\ u(x) \neq u_\lambda(x)\}) \to 0$ as $\lambda \to \infty$. Since $u_\lambda \leqslant \lambda \leqslant |u| + g$ in $X \setminus E_\lambda$, we find by (Φ3) and (Φ4) that

$$\begin{aligned}
&\int_X \Phi(x, |u(x) - u_\lambda(x)|)\,\mathrm{d}\mu(x) \\
&\qquad = \int_{X\setminus E_\lambda} \Phi(x, |u(x) - u_\lambda(x)|)\,\mathrm{d}\mu(x) \\
&\qquad \leqslant A_2 \int_{X\setminus E_\lambda} \Phi(x, |u(x)| + |u_\lambda(x)|)\,\mathrm{d}\mu(x) \\
&\qquad \leqslant A_2^2 \int_{X\setminus E_\lambda} \{\Phi(x, 2|u(x)|) + \Phi(x, 2|u_\lambda(x)|)\}\,\mathrm{d}\mu(x) \\
&\qquad \leqslant 2A_2^2A_3 \int_{X\setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, |u_\lambda(x)|)\}\,\mathrm{d}\mu(x) \\
&\qquad \leqslant 2A_2^3A_3 \int_{X\setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, |u(x)| + g(x))\}\,\mathrm{d}\mu(x) \\
&\qquad \leqslant 4A_2^4A_3^2 \int_{X\setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, |u(x)|) + \Phi(x, g(x))\}\,\mathrm{d}\mu(x) \\
&\qquad \leqslant 8A_2^4A_3^2 \int_{X\setminus E_\lambda} \{\Phi(x, |u(x)|) + \Phi(x, g(x))\}\,\mathrm{d}\mu(x).
\end{aligned}$$

Since $u, g \in L^\Phi(X)$ and $\mu(X \setminus E_\lambda) \to 0$ as $\lambda \to \infty$, $\varrho_\Phi(u - u_\lambda)$ converges to 0 as $\lambda \to \infty$. Therefore, we see from Lemma 2.5 and (2.1) that $\|u - u_\lambda\|_{L^\Phi(X)}$ converges to 0 as $\lambda \to \infty$.

Next we consider the function $g_\lambda = (g + 3\lambda)\chi_{X\setminus E_\lambda}$, where χ_E denotes the characteristic function of E. Note that g_λ is a Hajłasz gradient of $u - u_\lambda$. We have by (Φ3) and (Φ4) that

$$\begin{aligned}
\int_X \Phi(x, g_\lambda(x))\,\mathrm{d}\mu(x) &= \int_{X\setminus E_\lambda} \Phi(x, g(x) + 3\lambda)\,\mathrm{d}\mu(x) \\
&\leqslant 8A_2A_3^3 \int_{X\setminus E_\lambda} \{\Phi(x, g(x)) + \Phi(x, \lambda)\}\,\mathrm{d}\mu(x) \\
&\leqslant 8A_2^2A_3^3 \int_{X\setminus E_\lambda} \{\Phi(x, g(x)) + \Phi(x, |u(x)| + g(x))\}\,\mathrm{d}\mu(x) \\
&\leqslant 32A_2^3A_3^4 \int_{X\setminus E_\lambda} \{\Phi(x, g(x)) + \Phi(x, |u(x)|)\}\,\mathrm{d}\mu(x)
\end{aligned}$$

and the above discussion implies that $\|g_\lambda\|_{L^\Phi(X)}$ converges to 0 as $\lambda \to \infty$. Thus the proposition is proved. □

For a locally integrable function u on X and a ball $B(x,r) \subset X$, we define the mean integral:

$$u_{B(x,r)} = \fint_{B(x,r)} u(y)\,\mathrm{d}\mu(y) = \frac{1}{\mu(B(x,r))}\int_{B(x,r)} u(y)\,\mathrm{d}\mu(y).$$

We introduce a fractional sharp maximal operator. For every locally integrable function u on X, we define

$$u^{\sharp}(x) = \sup_{r>0}\frac{1}{r}\fint_{B(x,r)} |u(x) - u_{B(x,r)}|\,\mathrm{d}\mu(x).$$

For a locally integrable function u on X, the Hardy-Littlewood maximal function Mu is defined by

$$Mu(x) = \sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u(y)|\,\mathrm{d}\mu(y).$$

The following is a generalization of [22], Theorem 3.4, [23], Theorem 3.1, and [31], Theorem 5.2, (see also [18]).

For $a, b \in \mathbb{R}$, we write $a \sim b$ if $C^{-1}a \leqslant b \leqslant Ca$ for a constant $C > 0$.

Theorem 3.5. *Let X be a doubling space. Suppose the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$. Then the following three statements are equivalent:*

(i) $u \in M^{1,\Phi}(X)$;

(ii) $u \in L^{\Phi}(X)$ *and there exists a nonnegative function* $g \in L^{\Phi}(X)$ *such that the Poincaré inequality*

$$\fint_{B(z,r)} |u(x) - u_{B(z,r)}|\,\mathrm{d}\mu(x) \leqslant Cr\fint_{B(z,r)} g(x)\,\mathrm{d}\mu(x)$$

holds for every $z \in X$ *and* $r > 0$;

(iii) $u \in L^{\Phi}(X)$ *and* $u^{\sharp} \in L^{\Phi}(X)$.

Moreover, we obtain $\|u\|_{M^{1,\Phi}(X)} \sim \|u\|_{L^{\Phi}(X)} + \|u^{\sharp}\|_{L^{\Phi}(X)}$ *for all* $u \in L^{\Phi}(X)$.

This theorem is proved in the same way as [22], Theorem 3.4.

3.2. Sobolev capacity on Musielak-Orlicz-Hajłasz-Sobolev spaces. For $u \in M^{1,\Phi}(X)$, we define

$$\widetilde{\varrho}_{\Phi}(u) = \varrho_{\Phi}(u) + \inf \varrho_{\Phi}(g),$$

where the infimum is taken over all Hajłasz gradients of u. For $E \subset X$, we write

$$S_\Phi(E) = \{u \in M^{1,\Phi}(X)\colon\ u \geqslant 1 \text{ in an open set containing } E\}.$$

The Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces is defined by

$$C_\Phi(E) = \inf_{u \in S_\Phi(E)} \widetilde{\varrho}_\Phi(u).$$

In the case $S_\Phi(E) = \emptyset$, we set $C_\Phi(E) = \infty$.

Remark 3.6. We can redefine the Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces by

$$C_\Phi(E) = \inf_{u \in S'_\Phi(E)} \widetilde{\varrho}_\Phi(u),$$

since $M^{1,\Phi}(X)$ is a lattice (see [38], Lemma 2.4), where

$$S'_\Phi(E) = \{u \in S_\Phi(X)\colon\ 0 \leqslant u \leqslant 1\}.$$

A standard argument yields the following results (see [31], Theorem 3.11, and [38], Theorem 3.2, Remark 3.3 and Lemma 3.4).

Proposition 3.7. *The set function $C_\Phi(\cdot)$ satisfies the following properties:*

(1) $C_\Phi(\cdot)$ *is an outer measure;*
(2) $C_\Phi(\emptyset) = 0$;
(3) $C_\Phi(E_1) \leqslant C_\Phi(E_2)$ *for* $E_1 \subset E_2 \subset X$;
(4) $C_\Phi(E) = \inf_{\{E \subset U, U\colon \text{open}\}} C_\Phi(U)$ *for* $E \subset X$ ($C_\Phi(\cdot)$ *is an outer capacity*);
(5) *if* $K_1 \supset K_2 \supset \ldots$ *are compact sets on* X, *then* $\lim_{i\to\infty} C_\Phi(K_i) = C_\Phi\Big(\bigcap_{i=1}^{\infty} K_i\Big)$.

Furthermore, as in the proof of [37], Theorem 4.1, we have the following consequence of [14], Theorem 2.2.8.

Proposition 3.8. *If $L^\Phi(X)$ is reflexive and $E_1 \subset E_2 \subset \ldots$ are subsets of X, then*

$$\lim_{i\to\infty} C_\Phi(E_i) = C_\Phi\Big(\bigcup_{i=1}^{\infty} E_i\Big).$$

We say that a property holds C_Φ-q.e. (quasi-everywhere) in X, if it holds everywhere except for a set $F \subset X$ with $C_\Phi(F) = 0$.

Theorem 3.9. *For each Cauchy sequence of functions in $M^{1,\Phi}(X)\cap C(X)$, there is a subsequence which converges pointwise C_Φ-q.e. in X. Moreover, the convergence is uniform outside a set of arbitrary small Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces.*

Proof. Let $\{u_i\}$ be a Cauchy sequence of functions in $M^{1,\Phi}(X)\cap C(X)$. Since for all $0<\varepsilon<1$, $\|u\|_{M^{1,\Phi}(X)}<\varepsilon$ implies $\widetilde{\varrho}_\Phi(u)<\varepsilon$, we can take a subsequence of $\{u_i\}$, which we still denote by $\{u_i\}$, such that $\widetilde{\varrho}_\Phi(u_i-u_{i+1})\leqslant 2^{-i}A_2^{-1}(2A_3)^{-i-1}$ for each positive integer i. Consider the sets

$$E_i=\{x\in X\colon\ |u_i(x)-u_{i+1}(x)|>2^{-i}\}$$

and $F_j=\bigcup\limits_{i=j}^{\infty}E_i$. Here note that $2^i|u_i-u_{i+1}|\in S_\Phi(E_i)$ by the continuity of u_i. Since g_i is also a Hajłasz gradient of $|u_i-u_{i+1}|$ if g_i is a Hajłasz gradient of u_i-u_{i+1}, we have by (Φ4) and (2.1) that

$$C_\Phi(E_i)\leqslant\widetilde{\varrho}_\Phi(2^i|u_i-u_{i+1}|)\leqslant A_2(2A_3)^{i+1}\widetilde{\varrho}_\Phi(u_i-u_{i+1})\leqslant 2^{-i}.$$

Then it follows from Proposition 3.7 that

$$C_\Phi(F_j)\leqslant\sum_{i=j}^{\infty}C_\Phi(E_i)\leqslant 2^{-j+1}.$$

Hence, we obtain

$$C_\Phi\biggl(\bigcap_{j=1}^{\infty}F_j\biggr)\leqslant\lim_{j\to\infty}C_\Phi(F_j)=0$$

and $\{u_i\}$ converges in $X\setminus\bigcap\limits_{j=1}^{\infty}F_j$. Moreover, we find

$$|u_j(x)-u_k(x)|\leqslant\sum_{i=j}^{k-1}|u_i(x)-u_{i+1}(x)|\leqslant 2^{-j+1},$$

whenever $x\in X\setminus F_j$ for every $k>j$, which implies that $\{u_i\}$ converges uniformly in $X\setminus F_j$. □

We say that a function u is C_Φ-quasicontinuous on X if, for any $\varepsilon>0$, there is a set E such that $C_\Phi(E)<\varepsilon$ and u is continuous on $X\setminus E$. By Proposition 3.4 and Theorem 3.9, we have the following result.

Proposition 3.10. *For each $u \in M^{1,\Phi}(X)$, there is a C_Φ-quasicontinuous function $v \in M^{1,\Phi}(X)$ such that $u = v$ μ-a.e. in X.*

As in the proof of [38], Lemma 4.1, we have the following result.

Lemma 3.11. *$\mu(E) \leqslant CC_\Phi(E)$ for every $E \subset X$.*

In fact, note that for $u \in S_\Phi(E)$

$$\mu(E) \leqslant A_1A_2 \int_X \Phi(x, |u(x)|)\,\mathrm{d}\mu(x) \leqslant 2A_1A_2A_3\varrho_\Phi(u)$$

by (2.1), (Φ2) and (Φ3).

Theorem 3.12. *Suppose $\Phi(x,t)$ satisfies (Φ5). Then there exists a constant $C > 0$ such that $C_\Phi(B(x_0,r)) \leqslant C\Phi(x_0,r^{-1})\mu(B(x_0,2r))$ for all $x_0 \in X$ and $0 < r \leqslant 1$.*

P r o o f. Define

$$u(x) = \begin{cases} \dfrac{2r - d(x,x_0)}{r}, & x \in B(x_0,2r) \setminus B(x_0,r), \\ 1, & x \in B(x_0,r), \\ 0, & x \in X \setminus B(x_0,2r) \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{1}{r}, & x \in B(x_0,2r), \\ 0, & x \in X \setminus B(x_0,2r). \end{cases}$$

Then note from [38], Theorem 4.6, that g is a Hajłasz gradient of u and $u \in S_\Phi(B(x_0,r))$. Hence, we have by (Φ2), (Φ3), (Φ5) and (2.1)

$$\begin{aligned} C_\Phi(B(x_0,r)) &\leqslant \int_{B(x_0,2r)} \overline{\Phi}(x,u(x))\,\mathrm{d}\mu(x) + \int_{B(x_0,2r)} \overline{\Phi}(x,g(x))\,\mathrm{d}\mu(x) \\ &\leqslant A_2 \int_{B(x_0,2r)} \Phi(x,u(x))\,\mathrm{d}\mu(x) + A_2 \int_{B(x_0,2r)} \Phi(x,r^{-1})\,\mathrm{d}\mu(x) \\ &\leqslant A_1A_2^2\mu(B(x_0,2r)) + A_2B_{2,1}\Phi(x_0,r^{-1})\mu(B(x_0,2r)) \\ &\leqslant A_2(A_1^2A_2^2 + B_{2,1})\Phi(x_0,r^{-1})\mu(B(x_0,2r)), \end{aligned}$$

as required. □

3.3. Lebesgue points in Musielak-Orlicz-Hajłasz-Sobolev spaces. Let X be a doubling space. We recall from [36], Section 3, the definition of a discrete maximal function. Fix $r > 0$ and let $B(x_i, r)$, $i = 1, 2, \ldots$, be a family of balls covering X such that every point $x \in X$ belongs to at most θ balls $B(x_i, 6r)$. Here, θ can be chosen to depend only on the doubling constant c_2. Let $\{\varphi_i\}$ be a set of functions such that $0 \leqslant \varphi_i \leqslant 1$, $\varphi_i = 0$ in the complement of $B(x_i, 3r)$, $\varphi_i \geqslant c_3 > 0$ in $B(x_i, r)$, φ_i is Lipschitz with a constant c_3/r and $\sum_{i=1}^{\infty} \varphi_i = 1$ on X. We set

$$u_r(x) = \sum_{i=1}^{\infty} \frac{\varphi_i(x)}{\mu(B(x_i, 3r))} \int_{B(x_i,3r)} |u(y)| \,\mathrm{d}\mu(y).$$

Let $\{r_j\}$ be an enumeration of positive rationals. For every radius r_j, we choose a covering $\{B(x_i, r_j)\}$ as above. We define the discrete maximal function related to the covering $\{B(x_i, r_j)\}$ by

$$M^* u(x) = \sup_j u_{r_j}(x).$$

Note that the discrete maximal function related to the covering $\{B(x_i, r_j)\}$ depends on the chosen coverings. However, by [36], Lemma 3.1, the inequalities

$$c_M^{-1} Mu(x) \leqslant M^* u(x) \leqslant c_M Mu(x) \tag{3.2}$$

hold for every $x \in X$ and every $u \in L^1_{\mathrm{loc}}(X)$. Here the constant $c_M \geqslant 1$ depends only on the doubling constant.

Lemma 3.13. *Let X be a doubling space. Suppose the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$. Then there exists a constant $C > 0$ such that*

$$C_{\Phi}(\{x \in X \colon Mu(x) > \lambda\}) \leqslant C\lambda^{-\log_2(2A_3)} \|u\|_{M^{1,\Phi}(X)}$$

for all $0 < \lambda < 1$ and $u \in M^{1,\Phi}(X)$ with $\|u\|_{M^{1,\Phi}(X)} \leqslant 1$.

Proof. Let $u \in M^{1,\Phi}(X)$ with $\|u\|_{M^{1,\Phi}(X)} \leqslant 1$ and let g be a Hajłasz gradient of u. By our assumption, there exists a constant $B_M > 0$ such that $\|Mv\|_{L^{\Phi}(X)} \leqslant B_M \|v\|_{L^{\Phi}(X)}$ for all $v \in L^{\Phi}(X)$.

By (3.2), we have $\{x \in X \colon Mu(x) > \lambda\} \subset E_\lambda$, where set $E_\lambda = \{x \in X \colon c_M M^* u(x) > \lambda\}$ is open, since the supremum of continuous functions is lower semicontinuous.

Note, from the proof of [36], Theorem 3.6, that $c_M M^*u/\lambda \in S_\Phi(E_\lambda)$ and cMg is a Hajłasz gradient of M^*u for some constant $c \geqslant 1$. We have by ($\Phi3$), ($\Phi4$) and (2.2)

$$\begin{aligned}
&C_\Phi(E_\lambda)\\
&\quad\leqslant \int_X \overline{\Phi}(x, c_M M^*u(x)/\lambda)\,\mathrm{d}\mu(x) + \int_X \overline{\Phi}(x, cc_M Mg(x)/\lambda)\,\mathrm{d}\mu(x)\\
&\quad\leqslant A_2 \int_X \Phi(x, c_M M^*u(x)/\lambda)\,\mathrm{d}\mu(x) + A_2 \int_X \Phi(x, cc_M Mg(x)/\lambda)\,\mathrm{d}\mu(x)\\
&\quad\leqslant 2A_2^2A_3\Big(\frac{cc_M}{\lambda}\Big)^{\log_2(2A_3)}\Big\{\int_X \Phi(x, M^*u(x))\,\mathrm{d}\mu(x) + \int_X \Phi(x, Mg(x))\,\mathrm{d}\mu(x)\Big\}.
\end{aligned}$$

Since $\|Mu/B_M\|_{L^\Phi(X)} \leqslant \|u\|_{L^\Phi(X)} \leqslant 1$, we find by ($\Phi3$), ($\Phi4$), (2.2) and (3.2) that

$$\begin{aligned}
\int_X \Phi(x, M^*u(x))\,\mathrm{d}\mu(x) &\leqslant A_2 \int_X \Phi(x, c_M Mu(x))\,\mathrm{d}\mu(x)\\
&\leqslant 2A_2^2A_3(c_MB_M)^{\log_2(2A_3)} \int_X \Phi(x, Mu(x)/B_M)\,\mathrm{d}\mu(x)\\
&\leqslant 4A_2^2A_3^2(c_MB_M)^{\log_2(2A_3)} \int_X \overline{\Phi}(x, Mu(x)/B_M)\,\mathrm{d}\mu(x)\\
&\leqslant 4A_2^2A_3^2(c_MB_M)^{\log_2(2A_3)}\|Mu/B_M\|_{L^\Phi(X)}\\
&\leqslant 4A_2^2A_3^2(c_MB_M)^{\log_2(2A_3)}\|u\|_{L^\Phi(X)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_X \Phi(x, Mg(x))\,\mathrm{d}\mu(x) &\leqslant 2A_2A_3(B_M)^{\log_2(2A_3)} \int_X \Phi(x, Mg(x)/B_M)\,\mathrm{d}\mu(x)\\
&\leqslant 4A_2A_3^2(B_M)^{\log_2(2A_3)}\|g\|_{L^\Phi(X)}.
\end{aligned}$$

Thus we obtain the required result. □

As in the proof of [36], Theorem 4.5, we can show the following result by Lemma 3.13.

Theorem 3.14. *Let X be a doubling space and let $u \in M^{1,\Phi}(X)$. Suppose the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$. Then there exists a set $E \subset X$ of zero Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces such that*

$$\widetilde{u}(x) = \lim_{r\to 0} u_{B(x,r)}$$

for every $x \in X \setminus E$, where $\widetilde{u}$ is the C_Φ-quasicontinuous representative of u.

4.1. Basic properties. A curve γ in the set X is a nonconstant continuous map $\gamma\colon I \to X$, where $I = [a,b]$ is a closed interval in $\mathbb{R}$. The image of γ is denoted by $|\gamma|$. Let Γ be a family of rectifiable curves in X. We denote by $F(\Gamma)$ the set of all admissible functions, that is, all Borel measurable functions $h\colon X \to [0,\infty]$ such that

$$\int_\gamma h \,\mathrm{d}s \geqslant 1$$

for every $\gamma \in \Gamma$, where $\mathrm{d}s$ represents integration with respect to path length. We define the Φ-modulus of Γ by

$$M_\Phi(\Gamma) = \inf_{h\in F(\Gamma)} \varrho_\Phi(h).$$

If $F(\Gamma) = \emptyset$, then we set $M_\Phi(\Gamma) = \infty$.

Lemma 4.1 (cf. [30], Lemma 2.1)**.** *$M_\Phi(\cdot)$ is an outer measure.*

P r o o f. Since it is obvious that $M_\Phi(\emptyset) = 0$ and $\Gamma_1 \subset \Gamma_2$ implies $M_\Phi(\Gamma_1) \leqslant M_\Phi(\Gamma_2)$, we show that $M_\Phi(\cdot)$ is a countably subadditive capacity. For $\varepsilon > 0$, we take $h_i \in F(\Gamma_i)$ such that

$$\int_X \overline{\Phi}(x, h_i(x)) \,\mathrm{d}\mu(x) \leqslant M_\Phi(\Gamma_i) + \varepsilon 2^{-i}.$$

We set $h = \sup\limits_i h_i$. Noting that h satisfies $\int_\gamma h\,\mathrm{d}s \geqslant 1$ for every $\gamma \in \bigcup\limits_{i=1}^{\infty} \Gamma_i$, we have

$$M_\Phi\Big(\bigcup_{i=1}^{\infty}\Gamma_i\Big) \leqslant \varrho_\Phi(h) \leqslant \sum_{i=1}^{\infty}\int_X \overline{\Phi}(x, h_i(x))\,\mathrm{d}\mu(x) \leqslant \sum_{i=1}^{\infty} M_\Phi(\Gamma_i) + \varepsilon.$$

Letting $\varepsilon \to 0$, we have the required result. □

A family of curves Γ is said to be exceptional if $M_\Phi(\Gamma) = 0$. The following lemma is an extension of [31], Lemma 4.1. The proof is the same as the proof of [30], Lemma 2.2.

Lemma 4.2 (Fuglede's lemma). *Let $\{u_i\}$ be a sequence of nonnegative Borel functions in $L^{\Phi}(X)$ converging to zero in $L^{\Phi}(X)$. Then there exist a subsequence $\{u_{i_k}\}$ and an exceptional family Γ of rectifiable curves such that for every $\gamma \notin \Gamma$ we have*

$$\lim_{k\to\infty} \int_\gamma u_{i_k}\,\mathrm{d}s = 0.$$

Let u be a real-valued function on X. A nonnegative Borel measurable function h is said to be a Φ-weak upper gradient of u if there exists a family Γ of rectifiable curves with $M_\Phi(\Gamma) = 0$ and

$$|u(x) - u(y)| \leqslant \int_\gamma h\,\mathrm{d}s$$

for every rectifiable curve $\gamma \notin \Gamma$ with endpoints x and y. Here note that the basic properties of p-weak upper gradients can be extended to the basic properties of Φ-weak upper gradients as in [6], Chapter 1.

We define the norm

$$\|u\|_{N^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \inf \|h\|_{L^\Phi(X)},$$

where the infimum is taken over all Φ-weak upper gradients of u. We say that the function $u \in L^\Phi(X)$ belongs to Musielak-Orlicz-Newton-Sobolev spaces $N^{1,\Phi}(X)$ if $\|u\|_{N^{1,\Phi}(X)} < \infty$.

Remark 4.3. Let u be a real-valued function on X and let h be a Φ-weak upper gradient of u. Suppose Γ is a family of rectifiable curves γ satisfying the condition that there exists a rectifiable subcurve γ' of γ, that is, $|\gamma'| \subset |\gamma|$, such that

$$|u(x') - u(y')| \nleqslant \int_{\gamma'} h\,\mathrm{d}s,$$

where x' and y' are endpoints of γ'. Then note that $M_\Phi(\Gamma) = 0$ (see [6], Lemma 1.40).

Lemma 4.4 (cf. [36], Lemma 2.6, and [29], Lemma 3). *Suppose that $\{u_i\}$ is a sequence of measurable functions. Let g_i be a Φ-weak upper gradient of u_i. If $u = \sup_i u_i$ is finite almost everywhere, then $g = \sup_i g_i$ is a Φ-weak upper gradient of u.*

For $u \in N^{1,\Phi}(X)$, we set

$$\widehat{\varrho}_\Phi(u) = \varrho_\Phi(u) + \inf \varrho_\Phi(h),$$

where the infimum is taken over all Φ-weak upper gradients of u. For $E \subset X$, we denote

$$s_\Phi(E) = \{u \in N^{1,\Phi}(X)\colon\ u \geqslant 1 \text{ on } E\}.$$

We define the capacity in Musielak-Orlicz-Newton-Sobolev spaces by

$$c_\Phi(E) = \inf_{u \in s_\Phi(E)} \widehat{\varrho}_\Phi(u).$$

In the case $s_\Phi(E) = \emptyset$, we set $c_\Phi(E) = \infty$. For the definition of Sobolev capacity, see [6], Section 6.2.

By Lemma 4.4, we have the following result.

Proposition 4.5. *The set function $c_\Phi(\cdot)$ is an outer measure.*

Proof. Since it is obvious that $c_\Phi(\emptyset) = 0$ and $E_1 \subset E_2$ implies $c_\Phi(E_1) \leqslant c_\Phi(E_2)$, we only show that $c_\Phi(\cdot)$ is a countably subadditive capacity. Let E_i be subsets in X. We may assume that $\sum_{i=1}^{\infty} c_\Phi(E_i) < \infty$. For $\varepsilon > 0$, we take $u_i \in s_\Phi(E_i)$ such that

$$\int_X \overline{\Phi}(x, |u_i(x)|)\,\mathrm{d}\mu(x) + \int_X \overline{\Phi}(x, h_i(x))\,\mathrm{d}\mu(x) \leqslant c_\Phi(E_i) + \varepsilon 2^{-i},$$

where h_i is a Φ-weak upper gradient of u_i. Set $u = \sup_i u_i$ and $h = \sup_i h_i$. Noting that $u \in L^\Phi(X)$ and $h \in L^\Phi(X)$, we find that h is a Φ-weak upper gradient of u by Lemma 4.4 and $u \in s_\Phi\Bigl(\bigcup_{i=1}^{\infty} E_i\Bigr)$. Hence, we have

$$\begin{aligned} c_\Phi\Bigl(\bigcup_{i=1}^{\infty} E_i\Bigr) &\leqslant \widehat{\varrho}_\Phi(u) \\ &\leqslant \sum_{i=1}^{\infty}\Bigl\{\int_X \overline{\Phi}(x, |u_i(x)|)\,\mathrm{d}\mu(x) + \int_X \overline{\Phi}(x, h_i(x))\,\mathrm{d}\mu(x)\Bigr\} \\ &\leqslant \sum_{i=1}^{\infty} c_\Phi(E_i) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we have the required result. □

We denote by Γ_E the family of all rectifiable curves whose image intersects the set E.

Lemma 4.6. *Let $E \subset X$. If $c_\Phi(E) = 0$, then $M_\Phi(\Gamma_E) = 0$.*

Proof. Let $E \subset X$ with $c_\Phi(E) = 0$. Then for all positive integers i, we choose functions $u_i \in N^{1,\Phi}(X)$ with Φ-weak upper gradients κ_i such that $u_i(x) \geqslant 1$ for every $x \in E$ and

$$\int_X \overline{\Phi}(x, |u_i(x)|)\,\mathrm{d}\mu(x) + \int_X \overline{\Phi}(x, \kappa_i(x))\,\mathrm{d}\mu(x) \leqslant A_2^{-1}(2A_3)^{-i-1}.$$

Set $v_k = \sum\limits_{i=1}^{k} |u_i|$. Then note that $h_k = \sum\limits_{i=1}^{k} \kappa_i$ is a Φ-weak upper gradient of v_k. Since

$$\begin{aligned}\int_X \overline{\Phi}\Big(x, \frac{|u_i(x)|}{2^{-i}}\Big)\,\mathrm{d}\mu(x) &\leqslant A_2(2A_3)^i \int_X \Phi(x, |u_i(x)|)\,\mathrm{d}\mu(x)\\ &\leqslant A_2(2A_3)^{i+1} \int_X \overline{\Phi}(x, |u_i(x)|)\,\mathrm{d}\mu(x) \leqslant 1\end{aligned}$$

and

$$\int_X \overline{\Phi}\Big(x, \frac{\kappa_i(x)}{2^{-i}}\Big)\,\mathrm{d}\mu(x) \leqslant 1$$

by (2.1) and (Φ4), we have

$$\|v_l - v_m\|_{L^\Phi(X)} \leqslant \sum_{i=m+1}^{l} \|u_i\|_{L^\Phi(X)} \leqslant 2^{-m}$$

and

$$\|h_l - h_m\|_{L^\Phi(X)} \leqslant \sum_{i=m+1}^{l} \|\kappa_i\|_{L^\Phi(X)} \leqslant 2^{-m}$$

for every $l > m$. Hence $\{v_k\}$ and $\{h_k\}$ are Cauchy sequences in $L^\Phi(X)$. Therefore, $\{h_k\}$ converges to a function h in $L^\Phi(X)$, which we may assume to be a Borel function. Setting $v(x) = \lim\limits_{k\to\infty} v_k(x)$ for every $x \in X$, we find $v \in L^\Phi(X)$. Since $v_k(x) \geqslant k$ for $x \in E$, we have

$$E \subset E_\infty = \{x \in X\colon\ v(x) = \infty\}.$$

Hence it suffices to show that $M_\Phi(\Gamma_{E_\infty}) = 0$.

It follows from Lemma 4.2 that there exists a subsequence $\{h_{k_j}\}$ of $\{h_k\}$ such that there exists an exceptional family Γ_1 and

$$\lim_{j\to\infty} \int_\gamma |h_{k_j} - h|\,\mathrm{d}s = 0$$

for all rectifiable curves $\gamma \notin \Gamma_1$. Set

$$\Gamma_2 = \Big\{\gamma\colon\ \gamma \text{ is a rectifiable curve satisfying } \textstyle\int_\gamma v\,\mathrm{d}s = \infty\Big\}$$

and

$$\Gamma_3 = \Big\{\gamma\colon\ \gamma \text{ is a rectifiable curve satisfying } \textstyle\int_\gamma h\,\mathrm{d}s = \infty\Big\}.$$

We see from the convexity of $\overline{\Phi}$ that

$$M_\Phi(\Gamma_2) \leqslant \int_X \overline{\Phi}\Big(x, \frac{v(x)}{i}\Big)\,\mathrm{d}\mu(x) \leqslant \frac{\|v\|_{L^\Phi(X)}}{i}$$

for all $i \geqslant \|v\|_{L^\Phi(X)}$. Hence $M_\Phi(\Gamma_2) = 0$. Similarly, $M_\Phi(\Gamma_3) = 0$. We denote by $\Gamma_{4,i}$ the exceptional family of rectifiable curves for u_i in Remark 4.3 and by Γ_4 the union of $\Gamma_{4,i}$. By Remark 4.3 and Lemma 4.1, we have $M_\Phi(\Gamma_4) = M_\Phi(\bigcup \Gamma_{4,i}) = 0$. Hence we find $M_\Phi(\Gamma_0) = 0$, where $\Gamma_0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

To complete the proof, we show that $\Gamma_{E_\infty} \subset \Gamma_0$. Suppose $\gamma \notin \Gamma_0$. Since $\gamma \notin \Gamma_2$, there is $y \in |\gamma|$ with $v(y) < \infty$. For any $x \in |\gamma|$, we find that

$$v_{k_j}(x) \leqslant v_{k_j}(y) + |v_{k_j}(x) - v_{k_j}(y)| \leqslant v_{k_j}(y) + \int_\gamma h_{k_j}\,\mathrm{d}s,$$

since $\gamma \notin \Gamma_4$. Letting $j \to \infty$, we have

$$v(x) = \lim_{j\to\infty} v_{k_j}(x) \leqslant v(y) + \int_\gamma h\,\mathrm{d}s,$$

since $\gamma \notin \Gamma_1$. Since $\gamma \notin \Gamma_3$ and $v(y) < \infty$, we have $v(x) < \infty$ for all $x \in |\gamma|$, which implies $\gamma \notin \Gamma_{E_\infty}$, as required. □

Standard arguments and Lemma 4.6 yield the following proposition (see [31]).

Proposition 4.7 (cf. [31], Theorem 4.4)**.** *$N^{1,\Phi}(X)$ is a Banach space.*

We say that X supports a $(1,1)$-Poincaré inequality if there exists a constant $C > 0$ such that for all open balls B in X,

$$\frac{1}{\mu(B)} \int_B |u(x) - u_B|\,\mathrm{d}\mu(x) \leqslant C d_B \frac{1}{\mu(B)} \int_B h(x)\,\mathrm{d}\mu(x)$$

holds, whenever h is a Φ-weak upper gradient of u on B and u is integrable on B.

Lemma 4.8. *Let X be a doubling space that supports a $(1,1)$-Poincaré inequality. Assume that the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$. Then Lipschitz continuous functions are dense in $N^{1,\Phi}(X)$.*

Proof. Let $u \in N^{1,\Phi}(X)$ and let h be a Φ-weak upper gradient of u. By truncation, we may assume that u is a bounded function on X, say $|u| \leqslant u_0$ for $u_0 > 1$ (see [51], Lemma 4.3). Set

$$E_\lambda = \{x \in X\colon \ Mh(x) > \lambda\}.$$

As in the proof of [31], Theorem 4.5, we can define

$$u_\lambda(x) = \lim_{r\to 0} u_{B(x,r)}$$

for all $x \in X \setminus E_\lambda$ and u_λ is $c\lambda$-Lipschitz in $X \setminus E_\lambda$ with some constant $c > 1$. We extend u_λ as a Lipschitz function to all of X by the McShane extension [46], by setting

$$u_\lambda(x) = \inf_{y\in X\setminus E_\lambda} \{u_\lambda(y) + c\lambda d(x,y)\}.$$

We may assume that u_λ is still bounded by u_0 by truncation. Then we have by $(\Phi2)$, $(\Phi3)$ and $(\Phi4)$ that

$$\begin{aligned}
\int_X & \Phi(x, |u(x) - u_\lambda(x)|)\,\mathrm{d}\mu(x) \\
&= \int_{E_\lambda} \Phi(x, |u(x) - u_\lambda(x)|)\,\mathrm{d}\mu(x) \\
&\leqslant 2A_2^2A_3\Big\{\int_{E_\lambda} \Phi(x, |u(x)|)\,\mathrm{d}\mu(x) + \int_{E_\lambda} \Phi(x, |u_\lambda(x)|)\,\mathrm{d}\mu(x)\Big\} \\
&\leqslant 4A_2^3A_3 \int_{E_\lambda} \Phi(x, u_0)\,\mathrm{d}\mu(x) \\
&\leqslant 8A_1A_2^4A_3^2u_0^{\log_2(2A_3)}\mu(E_\lambda).
\end{aligned}$$

Hence we see from the boundedness of the Hardy-Littlewood maximal operator on $L^{\Phi}(X)$, Lemma 2.5 and (2.1) that $u_\lambda \to u$ in $L^{\Phi}(X)$. Since E_λ is open and $u - u_\lambda$ is zero μ-a.e. in $X \setminus E_\lambda$, we may assume that the Φ-weak upper gradient of $u - u_\lambda$ is zero in $X \setminus E_\lambda$ (see [51], Lemma 4.3). Since

$$\int_X \Phi(x, \lambda\chi_{E_\lambda}(x))\,\mathrm{d}\mu(x) \leqslant A_2 \int_X \Phi(x, Mh(x))\,\mathrm{d}\mu(x) < \infty$$

by the boundedness of the Hardy-Littlewood maximal operator on $L^{\Phi}(X)$, we find that the function $(c\lambda+h)\chi_{E_\lambda}\in L^{\Phi}(X)$ is a Φ-weak upper gradient of $u-u_\lambda$. Hence $u-u_\lambda\in N^{1,\Phi}(X)$ and therefore so does u_λ. We have

$$\begin{aligned}\int_X \Phi(x,(c\lambda+h)\chi_{E_\lambda}(x))\,\mathrm{d}\mu(x)\\ &\leqslant 4A_2^3A_3^2c^{\log_2(2A_3)}\Big\{\int_{E_\lambda}\Phi(x,\lambda)\,\mathrm{d}\mu(x)+\int_{E_\lambda}\Phi(x,h(x))\,\mathrm{d}\mu(x)\Big\}\\ &\leqslant 4A_2^4A_3^2c^{\log_2(2A_3)}\Big\{\int_{E_\lambda}\Phi(x,Mh(x))\,\mathrm{d}\mu(x)+\int_{E_\lambda}\Phi(x,h(x))\,\mathrm{d}\mu(x)\Big\}.\end{aligned}$$

Then the right hand side converges to zero as $\lambda\to\infty$. Hence $\{u_\lambda\}$ converges to u in $N^{1,\Phi}(X)$ by Lemma 2.5 and (2.1). □

4.2. Lebesgue points in Musielak-Orlicz-Newton-Sobolev spaces.

Lemma 4.9. *Let X be a doubling space that supports a $(1,1)$-Poincaré inequality. If the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$, then there exists a constant $C>0$ such that*

$$c_\Phi(\{x\in X\colon\ Mu(x)>\lambda\})\leqslant C\lambda^{-\log_2(2A_3)}\|u\|_{N^{1,\Phi}(X)}$$

for all $0<\lambda<1$ and $u\in N^{1,\Phi}(X)$ with $\|u\|_{N^{1,\Phi}(X)}\leqslant 1$.

Proof. Let $u\in N^{1,\Phi}(X)$ with $\|u\|_{N^{1,\Phi}(X)}\leqslant 1$ and $h\in L^{\Phi}(X)$ be a Φ-weak upper gradient of u. By (3.2), we have

$$\{x\in X\colon\ Mu(x)>\lambda\}\subset E_\lambda,$$

where $E_\lambda=\{x\in X\colon\ c_MM^*u(x)>\lambda\}$. Here, note from the boundedness of the Hardy-Littlewood maximal operator on $L^{\Phi}(X)$, Lemma 4.4 and [29], Lemma 5, that $M^*u\in L^{\Phi}(X)$ and $cMh\in L^{\Phi}(X)$ is a Φ-weak upper gradient of M^*u for some constant $c\geqslant 1$. Since $c_MM^*u/\lambda\in s_\Phi(E_\lambda)$, we have by ($\Phi3$), ($\Phi4$) and (2.2) that

$$\begin{aligned}c_\Phi(E_\lambda)&\leqslant\int_X\overline{\Phi}(x,c_MM^*u(x)/\lambda)\,\mathrm{d}\mu(x)+\int_X\overline{\Phi}(x,cc_MMh(x)/\lambda)\,\mathrm{d}\mu(x)\\ &\leqslant 2A_2^2A_3\Big(\frac{cc_M}{\lambda}\Big)^{\log_2(2A_3)}\Big\{\int_X\Phi(x,M^*u(x))\,\mathrm{d}\mu(x)+\int_X\Phi(x,Mh(x))\,\mathrm{d}\mu(x)\Big\}.\end{aligned}$$

Thus, as in the proof of Lemma 3.13, we obtain the required result. □

As in the proof of [29], Theorem 1, we can show the following consequence of Lemma 4.9.

Theorem 4.10. *Let X be a doubling space that supports a $(1,1)$-Poincaré inequality. If the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$ and $u \in N^{1,\Phi}(X)$, then there exists a set $E \subset X$ of zero Sobolev capacity in Musielak-Orlicz-Newton-Sobolev space such that*

$$u(x) = \lim_{r\to 0} u_{B(x,r)}$$

and

$$\lim_{r\to +0} \fint_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}\mu(y) = 0$$

for every $x \in X \setminus E$.

5. Equivalence of function spaces

Let $\mathbb{R}^N$ be the N-dimensional Euclidean space. In the case $X = \mathbb{R}^N$, let μ be the Lebesgue measure on $\mathbb{R}^N$ and let d be the Euclidean metric. We define the Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^N)$ by

$$W^{1,\Phi}(\mathbb{R}^N) = \{u \in L^{\Phi}(\mathbb{R}^N)\colon\ |\nabla u| \in L^{\Phi}(\mathbb{R}^N)\}.$$

The norm

$$\|u\|_{W^{1,\Phi}(\mathbb{R}^N)} = \|u\|_{L^{\Phi}(\mathbb{R}^N)} + \||\nabla u|\|_{L^{\Phi}(\mathbb{R}^N)}$$

makes $W^{1,\Phi}(\mathbb{R}^N)$ a Banach space.

We prove relations between the Musielak-Orlicz-Hajłasz-Sobolev space and the Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^N)$.

Proposition 5.1. $M^{1,\Phi}(\mathbb{R}^N) \subset W^{1,\Phi}(\mathbb{R}^N)$. *Moreover, if the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(\mathbb{R}^N)$, then $M^{1,\Phi}(\mathbb{R}^N) = W^{1,\Phi}(\mathbb{R}^N)$.*

Proof. First we show $M^{1,\Phi}(\mathbb{R}^N) \subset W^{1,\Phi}(\mathbb{R}^N)$. Let $u \in M^{1,\Phi}(\mathbb{R}^N)$ and let $g \in L^{\Phi}(\mathbb{R}^N)$ be a Hajłasz gradient of u. Since $t \leqslant A_1A_2\Phi(x,t)$ for $t \geqslant 1$ by (Φ2) and (2.2), we have $g \in L^1(B)$ for every ball B and hence ∇u exists and satisfies $|\nabla u(x)| \leqslant Cg(x)$ for a.e. $x \in \mathbb{R}^N$ by [33], Remark 5.13. Thus we have $M^{1,\Phi}(\mathbb{R}^N) \subset W^{1,\Phi}(\mathbb{R}^N)$.

Next we prove the second claim. Let $u \in W^{1,\Phi}(\mathbb{R}^N)$. Then we have by [21], Section 2,

$$|u(x) - u(y)| \leqslant |x-y|(M|\nabla u|(x) + M|\nabla u|(y))$$

for a.e. $x, y \in \mathbb{R}^N$. By the boundedness of the Hardy-Littlewood maximal operator on $L^{\Phi}(\mathbb{R}^N)$, we find that $M|\nabla u| \in L^{\Phi}(\mathbb{R}^N)$ is a Hajłasz gradient of u. Hence we obtain the required result. $\square$

Theorem 5.2. *$N^{1,\Phi}(\mathbb{R}^N) \subset W^{1,\Phi}(\mathbb{R}^N)$. Moreover, if $W^{1,\Phi}(\mathbb{R}^N)$ is reflexive and C^1-functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$, then $N^{1,\Phi}(\mathbb{R}^N) = W^{1,\Phi}(\mathbb{R}^N)$.*

Proof. The proof of the first claim is exactly the same as the proof of [31], Theorem 5.3. Hence we only show the second claim. Let $u \in W^{1,\Phi}(\mathbb{R}^N)$. Then we can take $\{u_i\} \subset W^{1,\Phi}(X) \cap C^1(X)$ such that u_i converges to u in $W^{1,\Phi}(X)$. By the proof of [30], Theorem 4.2, we see that the sum of absolute value of the distributional gradient of u_i is a Φ-weak upper gradient of u in $\mathbb{R}^N$. Hence we obtain the required result. □

Remark 5.3. By [43], Theorem 3.5, we know that C^1-functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$ if $\Phi(x,t)$ satisfies $(\Phi5)$ and $(\Phi6)$.

Theorem 5.4. *For $u \in M^{1,\Phi}(X)$, there exists a representative $\widetilde{u}$ of u such that*

$$\|\widetilde{u}\|_{N^{1,\Phi}(X)} \leqslant 4\|u\|_{M^{1,\Phi}(X)}.$$

Furthermore, if X is a doubling space that supports a $(1,1)$-Poincaré inequality and the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$, then $M^{1,\Phi}(X) \supset N^{1,\Phi}(X)$.

Proof. Let $u \in M^{1,\Phi}(X)$ and let $g \in L^\Phi(X)$ be a Hajłasz gradient of u. If u is continuous on X, we find that $4g$ is a Φ-weak upper gradient of u as in [51], Lemma 4.7. Since continuous functions are dense in $M^{1,\Phi}(X)$ by Proposition 3.4, we can take $\{u_i\} \subset M^{1,\Phi}(X)$ such that u_i is continuous on X, u_i converges to u in $M^{1,\Phi}(X)$ and

$$\|u_n - u_m\|_{N^{1,\Phi}(X)} \leqslant 4\|u_n - u_m\|_{M^{1,\Phi}(X)}$$

for all positive integers n, m. Therefore, $\{u_i\} \subset N^{1,\Phi}(X)$ is a Cauchy sequence. Hence there exists a $\widetilde{u} \in N^{1,\Phi}(X)$ such that

$$\|\widetilde{u}\|_{N^{1,\Phi}(X)} \leqslant 4\|u\|_{M^{1,\Phi}(X)},$$

since $N^{1,\Phi}(X)$ is a Banach space by Proposition 4.7. Noting that $u(x) = \widetilde{u}(x)$ for a.e. $x \in X$, we find that $\widetilde{u}$ is an equivalence class of u in $M^{1,\Phi}(X)$.

By our assumption and Theorem 3.5, we obtain that $M^{1,\Phi}(X) \supset N^{1,\Phi}(X)$. □

In this section, we show the boundedness of maximal operators on $L^{\Phi}(X)$. This proof with only a minor change appears in [44], but for reader's convenience, we give the proof.

For a nonnegative $f \in L^1_{\mathrm{loc}}(X)$, let

$$I(f, x, r) = \frac{1}{\mu(B(x,r))} \int_{X \cap B(x,r)} f(y) \, \mathrm{d}\mu(y)$$

and

$$J(f, x, r) = \frac{1}{\mu(B(x,r))} \int_{X \cap B(x,r)} \Phi(y, f(y)) \, \mathrm{d}\mu(y).$$

Lemma 6.1 (cf. [44], Lemma 3.1)**.** *Assume that μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies ($\Phi5$). Then there exists a constant $C > 0$ such that*

$$\Phi(x, I(f; x, r)) \leqslant CJ(f; x, r)$$

for all $x \in X$, $r > 0$ and for all nonnegative $f \in L^1_{\mathrm{loc}}(X)$ such that $f(y) \geqslant 1$ or $f(y) = 0$ for each $y \in X$ and $\|f\|_{L^{\Phi}(X)} \leqslant 1$.

P r o o f. Given f as in the statement of the lemma, $x \in X$ and $r > 0$, set $I = I(f; x, r)$ and $J = J(f; x, r)$. Note that $\|f\|_{L^{\Phi}(X)} \leqslant 1$ implies

$$J \leqslant 2A_3 \mu(B(x,r))^{-1} \leqslant 2A_3 c_0^{-1} r^{-Q(x)}$$

for $0 < r < d_X$ by (2.1) and lower Ahlfors $Q(x)$-regularity of μ.

By ($\Phi2$) and (2.2), $\Phi(y, f(y)) \geqslant (A_1 A_2)^{-1} f(y)$, since $f(y) \geqslant 1$ or $f(y) = 0$. Hence $I \leqslant A_1 A_2 J$. Thus, if $J \leqslant 1$, then

$$\Phi(x, I) \leqslant (A_1 A_2 J) A_2 \phi(x, A_1 A_2) \leqslant CJ.$$

Next, suppose $J > 1$. Since $\Phi(x,t) \to \infty$ as $t \to \infty$, there exists $K \geqslant 1$ such that

$$\Phi(x, K) = \Phi(x, 1) J.$$

Then $K \leqslant A_2 J$ by (2.2). With this K, we have

$$\int_{X \cap B(x,r)} f(y) \, \mathrm{d}\mu(y) \leqslant K\mu(B(x,r)) + A_2 \int_{X \cap B(x,r)} f(y) \frac{\phi(y, f(y))}{\phi(y, K)} \, \mathrm{d}\mu(y).$$

Since

$$1 \leqslant K \leqslant A_2 J \leqslant 2A_2A_3c_0^{-1}r^{-Q(x)} \leqslant Cr^{-Q^+},$$

by (Φ5) there is $\beta > 0$, independent of f, x, r, such that

$$\phi(x, K) \leqslant \beta\phi(y, K) \quad \text{for all } y \in B(x, r).$$

Thus, we have by (Φ2)

$$\begin{aligned}\int_{X\cap B(x,r)} f(y)\,\mathrm{d}\mu(y) &\leqslant K\mu(B(x,r)) + \frac{A_2\beta}{\phi(x,K)}\int_{X\cap B(x,r)} f(y)\phi(y, f(y))\,\mathrm{d}\mu(y)\\ &= K\mu(B(x,r)) + A_2\beta\mu(B(x,r))\frac{J}{\phi(x,K)}\\ &= K\mu(B(x,r))\Big(1 + \frac{A_2\beta}{\phi(x,1)}\Big) \leqslant K\mu(B(x,r))(1 + A_1A_2\beta).\end{aligned}$$

Therefore

$$I \leqslant (1 + A_1A_2\beta)K.$$

By (Φ2), (Φ3) and (Φ4), we obtain

$$\Phi(x, I) \leqslant C\Phi(x, K) \leqslant CJ$$

with $C > 0$ independent of f, x, r, as required. □

Lemma 6.2 (cf. [44], Lemma 3.2). *Suppose that $\Phi(x,t)$ satisfies (Φ6). Then there exists a constant $C > 0$ such that*

$$\Phi(x, I(f; x, r)) \leqslant C\{J(f; x, r) + \Phi(x, g(x))\}$$

for all $x \in X$, $r > 0$ and for all nonnegative $f \in L^1_{\mathrm{loc}}(X)$ such that $g(y) \leqslant f(y) \leqslant 1$ or $f(y) = 0$ for each $y \in X$, where g is the function appearing in (Φ6).

Proof. Given f as in the statement of the lemma, $x \in X$ and $r > 0$, let $I = I(f; x, r)$ and $J = J(f; x, r)$.

By Jensen's inequality, we have

$$\overline{\Phi}(x, I) \leqslant \frac{1}{\mu(B(x,r))}\int_{X\cap B(x,r)} \overline{\Phi}(x, f(y))\,\mathrm{d}\mu(y).$$

In view of (2.1),

$$\Phi(x, I) \leqslant 2A_2A_3\frac{1}{\mu(B(x,r))}\int_{X\cap B(x,r)} \Phi(x, f(y))\,\mathrm{d}\mu(y).$$

If $d(x, x_0) \geqslant d(y, x_0)$, then $\Phi(x, f(y)) \leqslant B_\infty \Phi(y, f(y))$ by (Φ6), where x_0 is the point appearing in (Φ6).

Let $d(x, x_0) < d(y, x_0)$. If $g(x) < f(y)$, then $\Phi(x, f(y)) \leqslant B_\infty \Phi(y, f(y))$ by (Φ6) again. If $g(x) \geqslant f(y)$, then $\Phi(x, f(y)) \leqslant A_2 \Phi(x, g(x))$ by (Φ3). Hence,

$$\Phi(x, f(y)) \leqslant C\{\Phi(y, f(y)) + \Phi(x, g(x))\}$$

in any case. Therefore, we obtain the required inequality. □

Theorem 6.3 (cf. [44], Theorem 4.1)**.** *Assume that X is a doubling space and μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x, t)$ satisfies (Φ5), (Φ6) and further assume:*

($\Phi 3^*$) *$t \mapsto t^{-\varepsilon_0}\phi(x, t)$ is uniformly almost increasing on $(0, \infty)$ for some $\varepsilon_0 > 0$.*

Then the Hardy-Littlewood maximal operator M is bounded from $L^\Phi(X)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^\Phi(X)} \leqslant C\|f\|_{L^\Phi(X)}$$

for all $f \in L^\Phi(X)$.

We use the following result, which is a special case of the theorem for $\Phi(x, t) = t^{p_0}$ ($p_0 > 1$) (see [33], Theorem 2.2).

Lemma 6.4. *Let $p_0 > 1$. Suppose that X is a doubling space. Then there exists a constant $\tilde{c} > 0$ depending only on p_0 and c_2 for which the following holds: If f is a measurable function such that*

$$\int_X |f(y)|^{p_0}\, \mathrm{d}\mu(y) \leqslant 1,$$

then

$$\int_X [Mf(x)]^{p_0}\, \mathrm{d}\mu(x) \leqslant \tilde{c}.$$

P r o o f of Theorem 6.3. Set $p_0 = 1+\varepsilon_0$ for $\varepsilon_0 > 0$ in condition ($\Phi 3^*$) and consider the function

$$\Phi_0(x, t) = \Phi(x, t)^{1/p_0}.$$

Then $\Phi_0(x, t)$ also satisfies all the conditions (Φj), $j = 1, 2, \ldots, 6$. In fact, it trivially satisfies (Φj) for $j = 1, 2, 4, 5, 6$ with the same g as in (Φ6). Since

$$\Phi_0(x, t) = t\phi_0(x, t) \quad \text{with} \quad \phi_0(x, t) = [t^{-\varepsilon_0}\phi(x, t)]^{1/p_0},$$

condition ($\Phi 3^*$) implies that $\Phi_0(x, t)$ satisfies (Φ3).

Let $f \geqslant 0$ and $\|f\|_{L^\Phi(X)} \leqslant 1$. Let $f_1 = f\chi_{\{x:\, f(x) \geqslant 1\}}$, $f_2 = f\chi_{\{x:\, g(x) \leqslant f(x) < 1\}}$ with g from (Φ6) and $f_3 = f - f_1 - f_2$.

Since $\Phi(x,t) \geqslant 1/(A_1 A_2)$ for $t \geqslant 1$ by (Φ2) and (2.2),

$$\Phi_0(x,t) \leqslant (A_1 A_2)^{1-1/p_0} \Phi(x,t)$$

if $t \geqslant 1$. Hence there is a constant $\lambda > 0$ such that $\|f_1\|_{L^{\Phi_0}(X)} \leqslant \lambda$, whenever $\|f\|_{L^\Phi(X)} \leqslant 1$. Applying Lemma 6.1 to Φ_0 and f_1/λ, we have

$$\Phi_0(x, Mf_1(x)) \leqslant CM\Phi_0(\cdot, f_1(\cdot))(x).$$

Hence

$$\Phi(x, Mf_1(x)) \leqslant C[M\Phi_0(\cdot, f(\cdot))(x)]^{p_0} \tag{6.1}$$

for all $x \in X$ with a constant $C > 0$ independent of f.

Next, applying Lemma 6.2 to Φ_0 and f_2, we have

$$\Phi_0(x, Mf_2(x)) \leqslant C[M\Phi_0(\cdot, f_2(\cdot))(x) + \Phi_0(x, g(x))].$$

Noting that $\Phi_0(x, g(x)) \leqslant Cg(x)$ by (2.2) and (Φ2), we have

$$\Phi(x, Mf_2(x)) \leqslant C\{[M\Phi_0(\cdot, f(\cdot))(x)]^{p_0} + g(x)^{p_0}\} \tag{6.2}$$

for all $x \in X$ with a constant $C > 0$ independent of f.

Since $0 \leqslant f_3 \leqslant g \leqslant 1$, we have $0 \leqslant Mf_3 \leqslant Mg \leqslant 1$. Hence

$$\Phi(x, Mf_3(x)) \leqslant A_2 \Phi_0(x, Mg(x))^{p_0} \leqslant C[Mg(x)]^{p_0} \tag{6.3}$$

for all $x \in X$ with a constant $C > 0$ independent of f.

Combining (6.1), (6.2) and (6.3), and noting that $g(x) \leqslant Mg(x)$ for a.e. $x \in X$, we obtain

$$\Phi(x, Mf(x)) \leqslant C\{[M\Phi_0(\cdot, f(\cdot))(x)]^{p_0} + [Mg(x)]^{p_0}\} \tag{6.4}$$

for a.e. $x \in X$ with a constant $C > 0$ independent of f.

In view of (2.1),

$$\int_X \Phi_0(y, f(y))^{p_0} \, \mathrm{d}\mu(y) = \int_X \Phi(y, f(y)) \, \mathrm{d}\mu(y) \leqslant 2A_3$$

for all $x \in X$. Hence, applying Lemma 6.4 to $(2A_3)^{-1/p_0}\Phi_0(y, f(y))$, we have

$$\int_X [M\Phi_0(\cdot, f(\cdot))(y)]^{p_0}\, d\mu(y) \leqslant C$$

with a constant $C > 0$ independent of f.

By Lemma 6.4, we obtain

$$\int_X [Mg(y)]^{p_0}\, d\mu(y) \leqslant C$$

as $g \in L^{p_0}(X)$.

Thus, by (6.4), we finally obtain

$$\int_X \Phi(y, Mf(y))\, d\mu(y) \leqslant C.$$

This completes the proof. □

Corollary 6.5. *Suppose μ is Ahlfors $Q(x)$-regular. Let $\Phi(x,t)$ be defined as in Examples 2.1 and 2.4. Then the Hardy-Littlewood maximal operator M is bounded from $L^{\Phi}(X)$ into itself.*

In fact, $\Phi(x,t)$ satisfies ($\Phi3^*$) with $\varepsilon_0 = (p^- - 1)/2$.

Similarly to Theorem 6.3, we can show the following lemma.

Lemma 6.6. *Assume that X is a bounded doubling space. Suppose that $\Phi(x,t)$ satisfies ($\Phi3^*$) and ($\Phi5$). Then the Hardy-Littlewood maximal operator M is bounded from $L^{\Phi}(X)$ into itself.*

Corollary 6.7. *Assume that X is a bounded doubling space. Let $\Phi(x,t)$ be defined as in Example 2.1. Then the Hardy-Littlewood maximal operator M is bounded from $L^{\Phi}(X)$ into itself.*

By Proposition 5.1 and Theorem 6.3, we have the following result.

Proposition 6.8. *Suppose that $\Phi(x,t)$ satisfies ($\Phi3^*$), ($\Phi5$) and ($\Phi6$). Then $M^{1,\Phi}(\mathbb{R}^N) = W^{1,\Phi}(\mathbb{R}^N)$.*

7. Sobolev's inequality

In this section, we show a Sobolev-type inequality on Musielak-Orlicz-Hajłasz-Sobolev spaces. For this purpose, we first prove Sobolev's inequality for a Riesz-type operator in Musielak-Orlicz spaces.

Lemma 7.1 (cf. [44], Lemma 5.1). *Let $H(x,t)$ be a positive function on $X\times(0,\infty)$ satisfying the following conditions:*

(H1) *$H(x,\cdot)$ is continuous on $(0,\infty)$ for each $x\in X$;*

(H2) *there exists a constant $K_1\geqslant 1$ such that $K_1^{-1}\leqslant H(x,1)\leqslant K_1$ for all $x\in X$;*

(H3) *$t\mapsto t^{-\varepsilon'}H(x,t)$ is uniformly almost increasing for $\varepsilon'>0$; namely, there exists a constant $K_2\geqslant 1$ such that $t^{-\varepsilon'}H(x,t)\leqslant K_2 s^{-\varepsilon'}H(x,s)$ for all $x\in X$ whenever $0<t<s$.*

Set $H^{-1}(x,s)=\sup\{t>0\colon\ H(x,t)<s\}$ for $x\in X$ and $s>0$. Then:

(1) *$H^{-1}(x,\cdot)$ is nondecreasing.*

(2) *$H^{-1}(x,\lambda s)\leqslant (K_2\lambda)^{1/\varepsilon'}H^{-1}(x,s)$ for all $x\in X$, $s>0$ and $\lambda\geqslant 1$.*

(3) *$H(x,H^{-1}(x,t))=t$ for all $x\in X$ and $t>0$.*

(4) *$K_2^{-1/\varepsilon'}t\leqslant H^{-1}(x,H(x,t))\leqslant K_2^{2/\varepsilon'}t$ for all $x\in X$ and $t>0$.*

(5) *$\min\{1,(s/K_1K_2)^{1/\varepsilon'}\}\leqslant H^{-1}(x,s)\leqslant\max\{1,(K_1K_2s)^{1/\varepsilon'}\}$ for all $x\in X$ and $s>0$.*

Remark 7.2. $H(x,t)=\Phi(x,t)$ satisfies (H1), (H2) and (H3) with $K_1=A_1$, $K_2=A_2$ and $\varepsilon'=1$.

Lemma 7.3. *Assume that X is a bounded space. Suppose that μ is lower Ahlfors $Q(x)$-regular and $\Phi(x,t)$ satisfies (Φ5). Then there exists a constant $C>0$ such that*

$$\frac{1}{\mu(B(x,r))}\int_{X\cap B(x,r)}f(y)\,\mathrm{d}\mu(y)\leqslant C\Phi^{-1}(x,r^{-Q(x)})$$

for all $x\in X$, $0<r<d_X$ and $f\geqslant 0$ satisfying $\|f\|_{L^\Phi(X)}\leqslant 1$.

P r o o f. Let f be a nonnegative function on X such that $\|f\|_{L^\Phi(X)}\leqslant 1$. Then we have $\int_X\Phi(y,f(y))\,\mathrm{d}\mu(y)\leqslant 2A_3$ by (2.1). By Lemma 6.1, (Φ2), (Φ3) and (Φ4), we obtain

$$\begin{aligned}\Phi\biggl(x,\frac{1}{\mu(B(x,r))}\int_{X\cap B(x,r)}f(y)\,\mathrm{d}\mu(y)\biggr)&\leqslant C(1+\mu(B(x,r))^{-1})\\&\leqslant C(1+r^{-Q(x)})\leqslant C_1r^{-Q(x)}\end{aligned}$$

for some constant $C_1>1$ and for all $x\in X$ and $0<r<d_X$. Hence, we find by Lemma 7.1 with $H=\Phi$

$$\frac{1}{\mu(B(x,r))}\int_{X\cap B(x,r)}f(y)\,\mathrm{d}\mu(y)\leqslant A_2\Phi^{-1}(x,C_1r^{-Q(x)})\leqslant C_1A_2^2\Phi^{-1}(x,r^{-Q(x)}),$$

as required. □

For an open set $\Omega \subset X$, $f \in L^1_{\mathrm{loc}}(X)$ and $\alpha > 0$, we define the Riesz-type operator $J^\Omega_\alpha f$ of order α by

$$J^\Omega_\alpha f(x) = \sum_{2^i \leqslant 2d_\Omega} \frac{2^{i\alpha}}{\mu(B(x,2^i))} \int_{\Omega \cap B(x,2^i)} |f(y)| \, \mathrm{d}\mu(y).$$

If μ is a doubling measure, then $I^\Omega_\alpha f(x) \leqslant C J^\Omega_\alpha f(x)$ for a.e. $x \in X$, where

$$I^\Omega_\alpha f(x) = \int_\Omega \frac{d(x,y)^\alpha |f(y)|}{\mu(B(x,r))} \, \mathrm{d}\mu(y)$$

is the usual Riesz potential of order α (see e.g. [23]).

Lemma 7.4. *Suppose that X is a bounded space and μ is lower Ahlfors $Q(x)$-regular. Assume that $\Phi(x,t)$ satisfies (Φ5) and*

($\Phi\mu$) *there exist constants $\gamma > 0$ and $A_4 \geqslant 1$ such that $s^{\gamma+\alpha}\Phi^{-1}(x, s^{-Q(x)}) \leqslant A_4 t^{\gamma+\alpha}\Phi^{-1}(x, t^{-Q(x)})$ for all $x \in X$, whenever $0 \leqslant t < s$.*

Then there exists a constant $C > 0$ such that

$$\sum_{\delta < 2^i \leqslant 2d_X} \frac{2^{i\alpha}}{\mu(B(x,2^i))} \int_{X \cap B(x,2^i)} f(y) \, \mathrm{d}\mu(y) \leqslant C\delta^\alpha \Phi^{-1}(x, \delta^{-Q(x)})$$

for all $x \in X$, $0 < \delta < d_X$ and $f \geqslant 0$ satisfying $\|f\|_{L^\Phi(X)} \leqslant 1$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^\Phi(X)} \leqslant 1$. By Lemmas 7.1 and 7.3 and ($\Phi\mu$), we have

$$\begin{aligned}
&\sum_{\delta < 2^i \leqslant 2d_X} \frac{2^{i\alpha}}{\mu(B(x,2^i))} \int_{X \cap B(x,2^i)} f(y) \, \mathrm{d}\mu(y) \\
&\qquad \leqslant C \sum_{\delta < 2^i \leqslant 2d_X} 2^{i\alpha}\Phi^{-1}(x, 2^{-iQ(x)}) \leqslant C \int_\delta^\infty t^\alpha \Phi^{-1}(x, t^{-Q(x)}) \frac{\mathrm{d}t}{t} \\
&\qquad \leqslant C\delta^\alpha \Phi^{-1}(x, \delta^{-Q(x)}),
\end{aligned}$$

as required. □

Note that ($\Phi\mu$) implies

$$\lim_{t\to\infty} t\Phi(x,t)^{-\alpha/Q(x)} = \infty \quad \text{uniformly in } x \in X. \tag{7.1}$$

We consider a function $\Psi_\alpha(x,t)\colon X \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

($\Psi1$) $\Psi_\alpha(\cdot,t)$ is measurable on X for each $t\geqslant 0$ and $\Psi_\alpha(x,\cdot)$ is continuous on $[0,\infty)$ for each $x\in X$;

($\Psi2$) there is a constant $A_5\geqslant 1$ such that $\Psi_\alpha(x,at)\leqslant A_5 a\Psi_\alpha(x,t)$ for all $x\in X$, $t>0$ and $0\leqslant a\leqslant 1$;

($\Psi\Phi\mu$) there exists a constant $A_6\geqslant 1$ such that $\Psi_\alpha(x,t\Phi(x,t)^{-\alpha/Q(x)})\leqslant A_6\Phi(x,t)$ for all $x\in X$ and $t>0$.

Note: ($\Psi2$) implies that $\Psi_\alpha(x,\cdot)$ is uniformly almost increasing on $[0,\infty)$; ($\Psi2$), (7.1) and ($\Psi\Phi\mu$) imply that $\Psi_\alpha(\cdot,t)$ is bounded on X for each $t>0$.

Theorem 7.5. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies ($\Phi3^*$), ($\Phi5$) and ($\Phi\mu$), and that $\Psi_\alpha(x,t)$ satisfies ($\Psi1$), ($\Psi2$) and ($\Psi\Phi\mu$). Then there exist constants $C_1,C_2>0$, such that*

$$\int_X \Psi_\alpha(x,J_\alpha^X f(x)/C_1)\,\mathrm{d}\mu(x)\leqslant C_2$$

for all $f\geqslant 0$ satisfying $\|f\|_{L^\Phi(X)}\leqslant 1$.

Proof. Let f be a nonnegative measurable function on X satisfying $\|f\|_{L^\Phi(X)}\leqslant 1$. Write

$$\begin{aligned}J_\alpha^X f(x)&=\sum_{2^i\leqslant\delta}\frac{2^{i\alpha}}{\mu(B(x,2^i))}\int_{X\cap B(x,2^i)}f(y)\,\mathrm{d}\mu(y)\\&\quad+\sum_{\delta<2^i<2d_X}\frac{2^{i\alpha}}{\mu(B(x,2^i))}\int_{X\cap B(x,2^i)}f(y)\,\mathrm{d}\mu(y)=:J_1+J_2.\end{aligned}$$

We have by Lemma 7.4

$$J_2\leqslant C\delta^\alpha\Phi^{-1}(x,\delta^{-Q(x)}).$$

Since $J_1\leqslant C\delta^\alpha Mf(x)$, we find that

$$J_\alpha^X f(x)\leqslant C\{\delta^\alpha Mf(x)+\delta^\alpha\Phi^{-1}(x,\delta^{-Q(x)})\}.$$

Here, let $\delta=\min\{d_X,\Phi(x,Mf(x))^{-1/Q(x)}\}$.

If $d_X\leqslant\Phi(x,Mf(x))^{-1/Q(x)}$, then note from Lemma 7.1 that

$$Mf(x)\leqslant A_2\Phi^{-1}(x,d_X^{-Q(x)})\leqslant A_2\max\{1,A_1A_2d_X^{-Q(x)}\}\leqslant C.$$

Therefore $J_\alpha^X f(x)\leqslant C$.

Next, if $d_X>\Phi(x,Mf(x))^{-1/Q(x)}$, then we have

$$\Phi^{-1}(x,\delta^{-Q(x)})=\Phi^{-1}(x,\Phi(x,Mf(x)))\leqslant A_2^2Mf(x)$$

in view of Lemma 7.1. Hence we see that

$$J_\alpha^X f(x) \leqslant C_1 \max\{Mf(x)\Phi(x, Mf(x))^{-\alpha/Q(x)}, 1\}$$

for some constant $C_1 > 0$. By (Ψ2) and ($\Psi\Phi\mu$), we find

$$\begin{aligned}\Psi_\alpha(x, J_\alpha^X f(x)/C_1) &\leqslant A_5\{\Psi_\alpha(x, Mf(x)\Phi(x, Mf(x))^{-\alpha/Q(x)}) + \Psi_\alpha(x,1)\}\\ &\leqslant C\{\Phi(x, Mf(x)) + 1\}.\end{aligned}$$

Hence, by Lemma 6.6

$$\int_X \Psi_\alpha(x, J_\alpha^X f(x)/C_1)\,\mathrm{d}\mu(x) \leqslant C\bigg\{\int_X \Phi(x, Mf(x))\,\mathrm{d}\mu(x) + \mu(X)\bigg\} \leqslant C_2$$

for some constant $C_2 > 0$, as required. □

Corollary 7.6. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Let $\Phi(x,t)$ be defined as in Example* 2.1 *and set*

$$\Psi_\alpha(x,t) = \bigg(t\prod_{j=1}^{k}(L_c^{(j)}(t))^{q_j(x)/p(x)}\bigg)^{p^\sharp(x)}$$

for all $x \in X$ and $t > 0$, where $1/p^\sharp(x) = 1/p(x) - \alpha/Q(x)$. Suppose

$$\operatorname*{ess\,sup}_{x\in X}(\alpha p(x) - Q(x)) < 0. \tag{7.2}$$

Then there exists a constant $C > 0$ such that

$$\int_X \Psi_\alpha(x, J_\alpha^X f(x))\,\mathrm{d}\mu(x) \leqslant C$$

for all $f \geqslant 0$ satisfying $\|f\|_{L^\Phi(X)} \leqslant 1$.

Proof. First note that

$$\Phi^{-1}(x,t) \sim t^{1/p(x)}\prod_{j=1}^{k}(L_c^{(j)}(t))^{-q_j(x)/p(x)}$$

for all $x \in X$ and $t > 0$. Therefore, by (7.2), there exists a constant $\gamma > 0$ such that

$$t^{\gamma+\alpha}\Phi^{-1}(x, t^{-Q(x)}) \sim t^{\gamma+\alpha-Q(x)/p(x)}\prod_{j=1}^{k}(L_c^{(j)}(t^{-1}))^{-q_j(x)/p(x)}$$

is uniformly almost decreasing on t. Hence $\Phi(x,t)$ satisfies ($\Phi\mu$). Similarly, since $t^{-1}\Psi_\alpha(x,t)$ is uniformly almost increasing on t, we see that $\Psi_\alpha(x,t)$ satisfies ($\Psi2$).

Finally, since

$$\Psi_\alpha(x, t\Phi(x,t)^{-\alpha/Q(x)}) = \Psi_\alpha\biggl(x, t^{p(x)/p^\sharp(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{-\alpha q_j(x)/Q(x)}\biggr) \leqslant C\Phi(x,t)$$

for all $x \in X$ and $t > 0$, we see that $\Psi_\alpha(x,t)$ satisfies ($\Psi\Phi\mu$). Hence we obtain the required result by Theorem 7.5. $\square$

Theorem 7.7. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies ($\Phi3^*$), ($\Phi5$) and ($\Phi\mu$), and that $\Psi_1(x,t)$ satisfies ($\Psi1$), ($\Psi2$) and ($\Psi\Phi\mu$). Then for each ball $B \subset X$, there exist constants $C_1, C_2 > 0$ such that*

$$\int_B \Psi_1(x, |u(x) - u_B|/C_1)\,\mathrm{d}\mu(x) \leqslant C_2$$

for all u satisfying $\|u\|_{M^{1,\Phi}(X)} \leqslant 1$.

P r o o f. Let $u \in M^{1,\Phi}(X)$ and let $g \in L^\Phi(X)$ be a Hajłasz gradient of u. Integrating both sides in (3.1) over y and x, we obtain the Poincaré inequality

$$\int_B |u(x) - u_B|\,\mathrm{d}\mu(x) \leqslant C d_B \int_B g(x)\,\mathrm{d}\mu(x)$$

for every ball $B \subset X$. Here, if μ is a doubling measure, then we have by [23], Theorem 5.2,

$$|u(x) - u_B| \leqslant C J_1^X g(x)$$

for μ-a.e. $x \in B$. Hence we obtain the Sobolev-type inequality on Musielak-Orlicz-Hajłasz-Sobolev spaces by Theorem 7.5. $\square$

Corollary 7.8. *Assume that X is a bounded doubling space and μ is lower Ahlfors $Q(x)$-regular. Let $\Phi(x,t)$ and $\Psi_1(x,t)$ be defined as in Corollary 7.6. Suppose*

$$\operatorname*{ess\,sup}_{x\in X}(p(x) - Q(x)) < 0.$$

Then for each ball $B \subset X$, there exists a constant $C > 0$ such that

$$\int_B \Psi_1(x, |u(x) - u_B|)\,\mathrm{d}\mu(x) \leqslant C$$

for all u satisfying $\|u\|_{M^{1,\Phi}(X)} \leqslant 1$.

8. Appendix

8.1. Musielak-Orlicz-Sobolev capacity in $\mathbb{R}^N$. For $u \in W^{1,\Phi}(\mathbb{R}^N)$, we define

$$\breve{\varrho}_\Phi(u) = \varrho_\Phi(u) + \varrho_\Phi(\nabla u).$$

For $E \subset \mathbb{R}^N$, we denote

$$T_\Phi(E) = \{u \in W^{1,\Phi}(\mathbb{R}^N) \colon u \geqslant 1 \text{ in an open set containing } E\}.$$

The Musielak-Orlicz-Sobolev Cap_Φ-capacity is defined by $\mathrm{Cap}_\Phi(E) = \inf\limits_{u \in T_\Phi(E)} \breve{\varrho}_\Phi(u)$. In the case $T_\Phi(E) = \emptyset$, we set $\mathrm{Cap}_\Phi(E) = \infty$.

Remark 8.1. Let $u, v \in W^{1,\Phi}(\mathbb{R}^N)$. Since

$$\begin{aligned}\int_{B(x,1)} &|u(x)| \, \mathrm{d}x + \int_{B(x,1)} |\nabla u(x)| \, \mathrm{d}x \\ &\leqslant 2|B(x,1)| + A_1A_2\biggl\{\int_{B(x,1)} \Phi(x, |u(x)|) \, \mathrm{d}x + \int_{B(x,1)} \Phi(x, |\nabla u(x)|) \, \mathrm{d}x\biggr\} \\ &\leqslant 2|B(x,1)| + 2A_1A_2A_3\breve{\varrho}_\Phi(u)\end{aligned}$$

for all $x \in \mathbb{R}^N$ by (2.1), (Φ2) and (Φ3), we find $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. The symbol $|E|$ denotes the Lebesgue measure for a set $E \subset \mathbb{R}^N$. As in the proof of [26], Theorem 2.2, we have $\min\{u,v\}, \max\{u,v\} \in W^{1,\Phi}(\mathbb{R}^N)$,

$$\nabla \min\{u,v\}(x) = \begin{cases} \nabla u(x) & \text{for a.e. } x \in \{u \leqslant v\}, \\ \nabla v(x) & \text{for a.e. } x \in \{u \geqslant v\} \end{cases}$$

and

$$\nabla \max\{u,v\}(x) = \begin{cases} \nabla u(x) & \text{for a.e. } x \in \{u \geqslant v\}, \\ \nabla v(x) & \text{for a.e. } x \in \{u \leqslant v\}. \end{cases}$$

Lemma 8.2. *Let $\{u_j\}$ and $\{v_j\}$ be sequences in $W^{1,\Phi}(\mathbb{R}^N)$. Assume that $\{\breve{\varrho}_\Phi(u_j)\}$ is bounded. If $\{\breve{\varrho}_\Phi(u_j - v_j)\}$ converges to zero, then $\{\breve{\varrho}_\Phi(u_j) - \breve{\varrho}_\Phi(v_j)\}$ converges to zero.*

P r o o f. We have by (Φ3) and (Φ4) that

$$\begin{aligned}\Phi(x, |v_j(x)|) &\leqslant A_2\Phi(x, |u_j(x) - v_j(x)| + |u_j(x)|) \\ &\leqslant 2A_2^2A_3\{\Phi(x, |u_j(x) - v_j(x)|) + \Phi(x, |u_j(x)|)\}\end{aligned}$$

for all $x \in \mathbb{R}^N$. Hence $\{\breve{\varrho}_\Phi(v_j)\}$ is also bounded. For any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$|\overline{\Phi}(x,t_1) - \overline{\Phi}(x,t_2)| \leqslant \varepsilon\{\overline{\Phi}(x,t_1) + \overline{\Phi}(x,t_2)\} + C(\varepsilon)\overline{\Phi}(x,|t_1 - t_2|)$$

for all $x \in \mathbb{R}^N$ and $t_1, t_2 \geqslant 0$. Therefore we have

$$\begin{aligned} |\breve{\varrho}_\Phi(u_j) - \breve{\varrho}_\Phi(v_j)| &\leqslant \varepsilon\{\breve{\varrho}_\Phi(u_j) + \breve{\varrho}_\Phi(v_j)\} + C(\varepsilon)\breve{\varrho}_\Phi(u_j - v_j) \\ &\leqslant 2M\varepsilon + C(\varepsilon)\breve{\varrho}_\Phi(u_j - v_j), \end{aligned}$$

since $\breve{\varrho}_\Phi(u_j) \leqslant M$ and $\breve{\varrho}_\Phi(v_j) \leqslant M$ for some constant $M > 0$. Hence we find

$$\lim_{j\to\infty} |\breve{\varrho}_\Phi(u_j) - \breve{\varrho}_\Phi(v_j)| \leqslant 2M\varepsilon,$$

as required. □

Standard arguments and Lemma 8.2 yield the following results (see [26], Theorems 3.1 and 3.2).

Proposition 8.3. *The set function* $\mathrm{Cap}_\Phi(\cdot)$ *satisfies the following conditions:*

(1) $\mathrm{Cap}_\Phi(\emptyset) = 0$;
(2) *if* $E_1 \subset E_2 \subset \mathbb{R}^N$, *then* $\mathrm{Cap}_\Phi(E_1) \leqslant \mathrm{Cap}_\Phi(E_2)$;
(3) $\mathrm{Cap}_\Phi(\cdot)$ *is an outer capacity;*
(4) *for* $E_1, E_2 \subset \mathbb{R}^N$, $\mathrm{Cap}_\Phi(E_1 \cup E_2) + \mathrm{Cap}_\Phi(E_1 \cap E_2) \leqslant \mathrm{Cap}_\Phi(E_1) + \mathrm{Cap}_\Phi(E_2)$;
(5) *if* $K_1 \supset K_2 \supset \ldots$ *are compact sets of* $\mathbb{R}^N$, *then*

$$\lim_{i\to\infty} \mathrm{Cap}_\Phi(K_i) = \mathrm{Cap}_\Phi\left(\bigcap_{i=1}^\infty K_i\right);$$

(6) *if* $W^{1,\Phi}(\mathbb{R}^N)$ *is reflexive and* $E_1 \subset E_2 \subset \ldots$ *are subsets of* $\mathbb{R}^N$, *then*

$$\lim_{i\to\infty} \mathrm{Cap}_\Phi(E_i) = \mathrm{Cap}_\Phi\left(\bigcup_{i=1}^\infty E_i\right);$$

(7) *if* $W^{1,\Phi}(\mathbb{R}^N)$ *is reflexive and* $E_i \subset \mathbb{R}^N$ *for* $i = 1, 2, \ldots$, *then*

$$\mathrm{Cap}_\Phi\left(\bigcup_{i=1}^\infty E_i\right) \leqslant \sum_{i=1}^\infty \mathrm{Cap}_\Phi(E_i).$$

We say that a property holds Cap_Φ-q.e. in $\mathbb{R}^N$, if it holds everywhere except for a set $F \subset \mathbb{R}^N$ with $\mathrm{Cap}_\Phi(F) = 0$. Analogously to Theorem 3.9, we have the following result.

Theorem 8.4 (cf. [26], Lemma 5.1). *Suppose that $W^{1,\Phi}(\mathbb{R}^N)$ is reflexive. Then, for each Cauchy sequence of functions in $W^{1,\Phi}(\mathbb{R}^N)\cap C(\mathbb{R}^N)$, there is a subsequence which converges pointwise* Cap_Φ*-q.e. in $\mathbb{R}^N$. Moreover, the convergence is uniform outside a set of arbitrary small Musielak-Orlicz-Sobolev* Cap_Φ*-capacity.*

We say that a function $u\colon \mathbb{R}^N \to \mathbb{R}$ is Cap_Φ-quasicontinuous, if for every $\varepsilon > 0$, there exists a open set E with $\mathrm{Cap}_\Phi(E) < \varepsilon$ such that u restricted to $\mathbb{R}^N \setminus E$ is continuous.

Corollary 8.5 (cf. [26], Theorem 5.2). *Suppose that $W^{1,\Phi}(\mathbb{R}^N)$ is reflexive and C^1-functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$. Then $u \in W^{1,\Phi}(\mathbb{R}^N)$ has a* Cap_Φ*-quasicontinuous representative of u.*

8.2. Fuglede's theorem in $\mathbb{R}^N$.

Lemma 8.6 (cf. [30], Lemma 3.1). *Suppose that C^1-functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$. Let $E \subset \mathbb{R}^N$. If $\mathrm{Cap}_\Phi(E) = 0$, then $M_\Phi(\Gamma_E) = 0$.*

Proof. Let $E \subset X$ with $\mathrm{Cap}_\Phi(E) = 0$. Then, for every positive integer i, we choose a function $u_i \in W^{1,\Phi}(\mathbb{R}^N)\cap C^1(\mathbb{R}^N)$ such that $u_i(x) \geqslant 1$ for every $x \in E$ and $\breve{\varrho}_\Phi(u_i) \leqslant A_2^{-1}(2A_3)^{-i-1}$. Set $v_k = \sum\limits_{i=1}^{k} |u_i|$. Since

$$\breve{\varrho}_\Phi\Big(\frac{u_i}{2^{-i}}\Big) \leqslant A_2(2A_3)^{i+1}\breve{\varrho}_\Phi(u_i) \leqslant 1$$

by (2.1) and (Φ4), we have $\|u_i\|_{W^{1,\Phi}(\mathbb{R}^N)} \leqslant 2^{-i}$. Therefore

$$\|v_l - v_m\|_{W^{1,\Phi}(\mathbb{R}^N)} \leqslant \sum_{i=m+1}^{l} \|u_i\|_{W^{1,\Phi}(\mathbb{R}^N)} \leqslant 2^{-m}$$

for every $l > m$. Hence $\{v_k\}$ is a Cauchy sequence in $W^{1,\Phi}(\mathbb{R}^N)$. Setting $v(x) = \lim\limits_{k\to\infty} v_k(x)$ for every $x \in X$, we see that $v \in W^{1,\Phi}(\mathbb{R}^N)$ is a Borel function. Thus, as in the proof of Lemma 4.6, we have the required result. □

We say that $u\colon \mathbb{R}^N \to \mathbb{R}$ is absolutely continuous on lines, $u \in \mathrm{ACL}(\mathbb{R}^N)$, if u is absolutely continuous on almost every line segment in $\mathbb{R}^N$ parallel to the coordinate axes. Note that an ACL function has classical derivatives almost everywhere. An ACL function is said to belong to $\mathrm{ACL}^\Phi(\mathbb{R}^N)$ if $|\nabla u| \in L^\Phi(\mathbb{R}^N)$. Since $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow W^{1,1}(\mathbb{R}^N)$ locally, we obtain the following result.

Lemma 8.7. $\mathrm{ACL}^{\Phi}(\mathbb{R}^N) \cap L^{\Phi}(\mathbb{R}^N) = W^{1,\Phi}(\mathbb{R}^N)$.

Let $u\colon \mathbb{R}^N \to \mathbb{R}$ and Γ be the family of rectifiable curves $\gamma\colon [0, l(\gamma)] \to \mathbb{R}^N$ such that $u \circ \gamma$ is not absolutely continuous on $[0, l(\gamma)]$. We say that u is absolutely continuous on curves, $u \in \mathrm{ACC}_{\Phi}(\mathbb{R}^N)$, if $M_{\Phi}(\Gamma) = 0$. It is clear that $\mathrm{ACC}_{\Phi}(\mathbb{R}^N) \subset \mathrm{ACL}(\mathbb{R}^N)$. An ACC_{Φ} function is said to belong to $\mathrm{ACC}^{\Phi}(\mathbb{R}^N)$ if $|\nabla u| \in L^{\Phi}(\mathbb{R}^N)$.

The proof of the following theorem is the same as the proof of [30], Theorem 4.2.

Theorem 8.8 (cf. [30], Theorem 4.2)**.** *Suppose that $W^{1,\Phi}(\mathbb{R}^N)$ is reflexive and C^1-functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$. Then $\mathrm{ACC}^{\Phi}(\mathbb{R}^N) \cap L^{\Phi}(\mathbb{R}^N) = W^{1,\Phi}(\mathbb{R}^N)$.*

References

[1] *T. Adamowicz, P. Harjulehto, P. Hästö*: Maximal operator in variable exponent Lebesgue spaces on unbounded quasimetric measure spaces. Math. Scand. *116* (2015), 5–22.

[2] *D. R. Adams, L. I. Hedberg*: Function Spaces and Potential Theory. Fundamental Principles of Mathematical Sciences 314, Springer, Berlin, 1996.

[3] *R. A. Adams*: Sobolev Spaces. Pure and Applied Mathematics 65, Academic Press, New York, 1975.

[4] *N. Aïssaoui*: Another extension of Orlicz-Sobolev spaces to metric spaces. Abstr. Appl. Anal. *2004* (2004), 1–26.

[5] *N. Aïssaoui*: Strongly nonlinear potential theory of metric spaces. Abstr. Appl. Anal. *7* (2002), 357–374.

[6] *A. Björn, J. Björn*: Nonlinear Potential Theory on Metric Spaces. EMS Tracts in Mathematics 17, European Mathematical Society, Zürich, 2011.

[7] *B. Bojarski, P. Hajłasz*: Pointwise inequalities for Sobolev functions and some applications. Stud. Math. *106* (1993), 77–92.

[8] *A. Cianchi*: Strong and weak type inequalities for some classical operators in Orlicz spaces. J. Lond. Math. Soc., II. Ser. *60* (1999), 187–202.

[9] *D. V. Cruz-Uribe, A. Fiorenza*: Variable Lebesgue Spaces. Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2013.

[10] *D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer*: Corrections to: "The maximal function on variable L^p spaces". Ann. Acad. Sci. Fenn., Math. *29* (2004), 247–249; Ann. Acad. Sci. Fenn., Math. *28* (2003), 223–238.

[11] *L. Diening*: Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. Bull. Sci. Math. *129* (2005), 657–700.

[12] *L. Diening*: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. Math. Inequal. Appl. *7* (2004), 245–253.

[13] *L. Diening*: Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. Math. Nachr. *268* (2004), 31–43.

[14] *L. Diening, P. Harjulehto, P. Hästö, M. Růžička*: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Berlin, 2011.

[15] *L. C. Evans, R. F. Gariepy*: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[16] *X. Fan*: Differential equations of divergence form in Musielak-Sobolev spaces and a sub-supersolution method. J. Math. Anal. Appl. *386* (2012), 593–604.

[17] *X. Fan, C.-X. Guan*: Uniform convexity of Musielak-Orlicz-Sobolev spaces and applications. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *73* (2010), 163–175.
[18] *B. Franchi, G. Lu, R. L. Wheeden*: A relationship between Poincaré-type inequalities and representation formulas in spaces of homogeneous type. Int. Math. Res. Not. *1996* (1996), 1–14.
[19] *T. Futamura, P. Harjulehto, P. Hästö, Y. Mizuta, T. Shimomura*: Variable exponent spaces on metric measure spaces. More Progresses in Analysis. Proceedings of the 5th International ISAAC Congress, Catania, Italy, 2005 (H. G. W. Begehr et al., eds.). World Scientific, Hackensack, 2009, pp. 107–121.
[20] *T. Futamura, Y. Mizuta, T. Shimomura*: Sobolev embeddings for variable exponent Riesz potentials on metric spaces. Ann. Acad. Sci. Fenn., Math. *31* (2006), 495–522.
[21] *P. Hajłasz*: Sobolev spaces on an arbitrary metric space. Potential Anal. *5* (1996), 403–415.
[22] *P. Hajłasz, J. Kinnunen*: Hölder quasicontinuity of Sobolev functions on metric spaces. Rev. Mat. Iberoam. *14* (1998), 601–622.
[23] *P. Hajłasz, P. Koskela*: Sobolev met Poincaré. Mem. Am. Math. Soc. *145* (2000), no. 688, 101 pages.
[24] *P. Harjulehto, P. Hästö*: A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces. Rev. Mat. Complut. *17* (2004), 129–146.
[25] *P. Harjulehto, P. Hästö*: Lebesgue points in variable exponent spaces. Ann. Acad. Sci. Fenn., Math. *29* (2004), 295–306.
[26] *P. Harjulehto, P. Hästö, M. Koskenoja, S. Varonen*: Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$. J. Funct. Spaces Appl. *1* (2003), 17–33.
[27] *P. Harjulehto, P. Hästö, M. Koskenoja*: Properties of capacities in variable exponent Sobolev spaces. J. Anal. Appl. *5* (2007), 71–92.
[28] *P. Harjulehto, P. Hästö, V. Latvala*: Sobolev embeddings in metric measure spaces with variable dimension. Math. Z. *254* (2006), 591–609.
[29] *P. Harjulehto, P. Hästö, V. Latvala*: Lebesgue points in variable exponent Sobolev spaces on metric measure spaces. Zb. Pr. Inst. Mat. NAN Ukr. *1* (2004), 87–99.
[30] *P. Harjulehto, P. Hästö, O. Martio*: Fuglede's theorem in variable exponent Sobolev space. Collect. Math. *55* (2004), 315–324.
[31] *P. Harjulehto, P. Hästö, M. Pere*: Variable exponent Sobolev spaces on metric measure spaces. Funct. Approximatio, Comment. Math. *36* (2006), 79–94.
[32] *P. Harjulehto, P. Hästö, M. Pere*: Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator. Real Anal. Exch. *30* (2005), 87–104.
[33] *J. Heinonen*: Lectures on Analysis on Metric Spaces. Universitext, Springer, New York, 2001.
[34] *J. Heinonen, P. Koskela*: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. *181* (1998), 1–61.
[35] *J. Kinnunen*: The Hardy-Littlewood maximal function of a Sobolev function. Isr. J. Math. *100* (1997), 117–124.
[36] *J. Kinnunen, V. Latvala*: Lebesgue points for Sobolev functions on metric spaces. Rev. Mat. Iberoam. *18* (2002), 685–700.
[37] *J. Kinnunen, O. Martio*: Choquet property for the Sobolev capacity in metric spaces. Proceedings on Analysis and Geometry (S. K. Vodop'yanov, ed.). Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000, pp. 285–290. (In Russian.)
[38] *J. Kinnunen, O. Martio*: The Sobolev capacity on metric spaces. Ann. Acad. Sci. Fenn., Math. *21* (1996), 367–382.

[39] *V. Kokilashvili, S. Samko*: Maximal and fractional operators in weighted $L^{p(x)}$ spaces. Rev. Mat. Iberoam. *20* (2004), 493–515.
[40] *V. Kokilashvili, S. Samko*: On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent. Z. Anal. Anwend. *22* (2003), 889–910.
[41] *J. L. Lewis*: On very weak solutions of certain elliptic systems. Commun. Partial Differ. Equations *18* (1993), 1515–1537.
[42] *F.-Y. Maeda, Y. Mizuta, T. Ohno*: Approximate identities and Young type inequalities in variable Lebesgue-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. Ann. Acad. Sci. Fenn., Math. *35* (2010), 405–420.
[43] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Approximate identities and Young type inequalities in Musielak-Orlicz spaces. Czech. Math. J. *63* (2013), 933–948.
[44] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces. Bull. Sci. Math. *137* (2013), 76–96.
[45] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Mean continuity for potentials of functions in Musielak-Orlicz spaces. Potential Theory and Its Related Fields (K. Hirata, ed.). Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 2013, pp. 81–100.
[46] *E. J. McShane*: Extension of range of functions. Bull. Am. Math. Soc. *40* (1934), 837–842.
[47] *Y. Mizuta, T. Ohno, T. Shimomura*: Sobolev's inequalities and vanishing integrability for Riesz potentials of functions in the generalized Lebesgue space $L^{p(\cdot)}(\log L)^{q(\cdot)}$. J. Math. Anal. Appl. *345* (2008), 70–85.
[48] *Y. Mizuta, T. Shimomura*: Continuity properties of Riesz potentials of Orlicz functions. Tohoku Math. J. (2) *61* (2009), 225–240.
[49] *Y. Mizuta, T. Shimomura*: Continuity of Sobolev functions of variable exponent on metric spaces. Proc. Japan Acad., Ser. A *80* (2004), 96–99.
[50] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics 1034, Springer, Berlin, 1983.
[51] *N. Shanmugalingam*: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoam. *16* (2000), 243–279.
[52] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series 30, Princeton University Press, Princeton, 1970.
[53] *H. Tuominen*: Orlicz-Sobolev spaces on metric measure spaces. Ann. Acad. Sci. Fenn. Math. Diss. *135* (2004), 86 pages.
[54] *W. P. Ziemer*: Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation, Graduate Texts in Mathematics 120, Springer, Berlin, 1989.

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