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# Mutations of Absolute Valued Algebras

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**Abstract**

Let $A$ be an absolute valued algebra such that there exists a nonzero algebraic element $e \in A$ satisfying some of the following conditions:

1. $e(xy) = x(ey)$ for all $x, y \in A$.
2. $(ex)e = e(xe)$ for all $x \in A$.

We prove that the norm of $A$ comes from an inner product. This generalizes previously known results in [21] and [10, 11] for the cases that $e$ is a left unit and $e$ is a central idempotent, respectively.

**Keywords:** Absolute valued algebra, right division algebra, idempotent, generalized left unit, algebraic (central, flexible) element

## 1 Introduction

Absolute valued algebras, introduced by Ostrowski [18] and studied basically by Albert [1, 2], Wright [27] and Urbanik-Wright [26], know nowadays a significant development [22]. A special attention has been given to those absolute valued algebras which are provided with an involution [25, 16, 9, 10, 12, 14, 20, 23] or have a left unit element [4, 21, 6, 19]. In these particular cases, it is known that the norm comes from an inner product [25, 10, 21]. We note that there are separable complete absolute valued algebras whose norm does not come from an inner product [21, Remark 3, page 938].

In the present paper we extend the above results to more general situations. Indeed, we prove that, if $A$ is an absolute valued algebra, and if there exists a nonzero algebraic element $e \in A$ satisfying some of the following conditions:

1. $e(xy) = x(ey)$ for all $x, y \in A$,
2. $(ex)e = e(xe)$ for all $x \in A$,

then the norm $||.||$ of $A$ comes from an inner product $(./.)$. Moreover, in the first case there exists a norm-one element $b \in A$ with $e(eb) = e$ such that

$$x(xb) = 2(e/x)x(eb) - ||x||^2 e \text{ for all } x \in A,$$

whereas in the second one there exists a norm-one element $c \in A$ with $ec = ce = e$ such that

$$(ex)(xe) = 2(c/x)exe - ||x||^2 e^2 \text{ for all } x \in A.$$

Our arguments use a deformation of the initial product to obtain a new absolute valued algebra containing a nonzero central idempotent. Then, we apply that the norm of any absolute valued algebra containing a nonzero central idempotent comes from an inner product [10].

Among those finite-dimensional absolute valued algebras $A$, existence of a nonzero $e \in A$ satisfying $e(xy) = x(ey)$ for all $x, y \in A$ which is not scalar multiple of left unit, can happen only in dimension 4 (Theorems 3.5, 3.6). We give also a new result (Theorem 4.9) characterizing the finite-dimensionality of third power-associative absolute valued algebras.

## 2 Notations and preliminary results

Absolute valued algebras (AVA) are defined as those real or complex algebras $A$ satisfying $||xy|| = ||x||.||y||$ for a given norm $||.||$ on $A$, and $x, y \in A$.

Given an element $a$ in an algebra $B$, we denote by $L_a$ (respectively, $R_a$) the operator of left (respectively, right) multiplication by $a$ on $B$. The element $a$ is said to be algebraic if the subalgebra $B(a)$ of $B$ generated by $a$ is finite-dimensional. $B$ is said to be right division algebra if $R_x$ is bijective for all nonzero $x \in B$. For $b, c \in B$, we denote by $Lin\{b, c\}$ the linear hull spanned by $\{b, c\}$.

It is well known that the only possible dimensions for finite-dimensional AVA are $1, 2, 4, 8$ [1], and that such algebras contain always a nonzero idempotent [24]. The celebrated noncommutative Albert-Urbanik-Wright Theorem states that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (quaternions), $\mathbb{O}$ (octonions), are the unique AVA with a unit [26].

Throughout this paper, $(A, ||.||)$ will denote an AVA.

We take from [26], [10] the following.

**Lemma 2.1** [26] *If all the elements of a subset $B$ of $A$ commute with each other, then the linear hull spanned by $B$ is a pre-Hilbert space.*

**Lemma 2.2** [26] *Let $x, y$ be norm-one commuting elements of $A$ with $||x - y|| = 2$. Then $x + y = 0$.*

**Theorem 2.3** [10] *The norm of any AVA containing a nonzero central idempotent comes from an inner product.*

Our study use a specific deformation of the product according to the following definition.

**Principal Mutations 2.4** *Let $a$ be a norm-one element of $A$. We define new products*

$$x_{la} \odot y = (ax)y, \;\; x_{ra} \odot y = (xa)y, \;\; x \odot_{la} y = x(ay), \;\; x \odot_{ra} y = x(ya),$$

$$x_a \odot y = a(xy), \;\; x \odot_a y = (xy)a$$

*on the normed space of $A$ to obtain new AVA $\cdot_{la}A$, $\cdot_{ra}A$, $A_{la}$, $A_{ra}$, $\cdot_a A$, $A_a$ , respectively, which will be called principal mutations of $A$.*

# 3 Generalized left units

**Definition 3.1** *A nonzero element $a$ of an algebra $B$ is said to be a generalized left unit if it satisfies $a(xy) = x(ay)$ for all $x, y \in B$. Clearly, a nonzero scalar multiple of a left unit (and particularly a left unit) is a generalized left unit.*

**Proposition 3.2** *Any right division AVA $A$ with a generalized left unit $a$ is finite-dimensional. If moreover $a$ is an idempotent, then $a$ is a left unit.*

**Proof:** Putting $y = a$ in the equality $a(xy) = x(ay)$, we get $L_a \circ R_a = R_{a^2}$, and hence $L_a = R_{a^2} \circ R_a^{-1}$ is an invertible operator on $A$. By [22, Theorem 2.2], $A$ is finite-dimensional. If moreover $a^2 = a$, then $L_a = I_A$ is the identity operator on $A$.□

We deduce the following characterization in finite-dimensional case

**Corollary 3.3** *Let $A$ be a finite-dimensional AVA and let $e$ be an element of $A$, then the following are equivalent*

1. *$e$ is both idempotent and generalized left unit,*

*2. e is a left unit.*□

Let $\mathbb{A}$ be one of (alternative) AVA $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ with standard involution $x \mapsto \overline{x}$. Any nonzero subalgebra of $\mathbb{A}$ contains 1 [24] and so is invariant under the standard involution of $\mathbb{A}$. Artin's theorem shows that for any $x, y \in \mathbb{A}$, the set $\{x, y, \overline{x}, \overline{y}\}$ is contained in an associative subalgebra of $\mathbb{A}$. This fact will be used in the sequel without further reference.

Given linear isometries $f, g : A \to A$, we denote by $A_{f.g}$ the AVA obtained from $A$ by replacing its product with the one $\odot$ defined by $x \odot y = f(x)g(y)$. It is known that finite-dimensional AVA are isometrically isomorphic to $\mathbb{A}_{f,g}$, where $\mathbb{A}$ stands for $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$, and $f, g$ are linear isometries of $\mathbb{A}$, fixing 1 [3].

According to above notations, we state the following preliminary result:

**Lemma 3.4** *Assume that algebra $\mathbb{A}_{f,g} := (\mathbb{A}, \odot)$ contains (norm-one) generalized left unit $e$. Then $g(x) = \overline{f(e)}xf(e)$ for all $x \in \mathbb{A}$ (equivalently $g = L_{\overline{f(e)}} \circ R_{f(e)}$).*

**Proof:** We have $g(f(e)) = 1 \odot (e \odot 1) = e \odot (1 \odot 1) = f(e)$. So

$$f(e)g(x) = e \odot (f^{-1}(x) \odot 1) = f^{-1}(x) \odot (e \odot 1) = xg(f(e)) = xf(e).$$

We get $g(x) = \overline{f(e)}xf(e)$.□

Among algebras $\mathbb{A}_{f,g}$, those containing a generalized left unit are characterized as follows.

**Theorem 3.5** *Let $\mathbb{A}$ be one of AVA $\mathbb{R}, \mathbb{C}$ or $\mathbb{O}$ and $f, g$ be linear isometries of $\mathbb{A}$ fixing* 1*. Then the following are equivalent*

1. *$\mathbb{A}_{f,g}$ contains generalized left units,*
2. *$\mathbb{A}_{f,g}$ contains a left unit,*
3. *$g = I_A$ is the identity operator of $\mathbb{A}$.*

*In such a cases* 1 *is left unit, and those generalized left units are (nonzero) scalar multiples of* 1.

**Proof:** The equivalence **2.** $\Leftrightarrow$ **3.** is proved in [19] and the implication **2.** $\Rightarrow$ **1.** is clear. Assume now that $\mathbb{A}_{f,g} := (\mathbb{A}, \odot)$ contains (norm-one) generalized left unit $e$. By Lemma **3.4** we have $g(x) = \overline{f(e)}xf(e)$ and we distinguish the following two cases:

1. If $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$, then $g(x) = \overline{f(e)}xf(e) = x$, that is, $g = I_A$.

2. If $\mathbb{A} = \mathbb{O}$, then $e$ belongs in $\mathbb{R}$. Indeed, we have

$$g(e \odot y) = g(f(e)g(y)) = \overline{f(e)}\Big(f(e)g(y)\Big)f(e) = g(y)f(e).$$

So

$$\begin{aligned} 0 &= e \odot (x \odot y) - x \odot (e \odot y) \\ &= f(e)g(x \odot y) - f(x)g(e \odot y) \\ &= f(e).\overline{f(e)}(x \odot y)f(e) - f(x).g(y)f(e) \\ &= (x \odot y)f(e) - f(x).g(y)f(e) \\ &= f(x)g(y).f(e) - f(x).g(y)f(e) \\ &= (f(x), g(y), f(e)). \end{aligned}$$

where $(.,.,.)$ means associator in algebra $\mathbb{O}$.

As $f, g$ are bijective, we have

$$(x, y, f(e)) = 0 \text{ for all } x, y \in \mathbb{O}.$$

This implies that $f(e)$ belongs in $\mathbb{R}$ as well as $e$. So $g = I_O$.□

**Theorem 3.6** *Let $f, g$ be linear isometries of $\mathbb{H}$ fixing 1. Then the following are equivalent*

1. *$\mathbb{H}_{f,g}$ contains generalized left units,*

2. *$g$ is positive, that is, $g = L_a \circ R_{\overline{a}}$ for norm-one $a \in \mathbb{H}$.*

*In theses cases generalized left units of $\mathbb{H}_{f,g}$ are (nonzero) scalar multiples of $f^{-1}(\overline{a})$.*

**Proof: 1. ⇒ 2.** Assume that $\mathbb{H}_{f,g}$ contains a (norm-one) generalized left $e$, then $g$ equals $L_{\overline{f(e)}} \circ R_{f(e)}$ which is linear positive isometry of $\mathbb{H}$.

**2. ⇒ 1.** If $g = L_a \circ R_{\overline{a}}$; $a$ being norm-one in $\mathbb{H}$, then it is easy to see that $f^{-1}(\overline{a})$ is (norm-one) generalized left unit of $\mathbb{H}_{f,g}$.□

Theorems **3.5, 3.6** show that in finite-dimensional case existence of generalized left units non scalar multiples of left units can happen only in dimension 4. It raises naturally the problem to know if there exists $\infty$-dimensional AVA containing generalized left unit non scalar multiples of left units.□

We need now the following two Lemmas in order to state the main result in this section

**Lemma 3.7** *Let $a \in A$ be norm-one left generalized unit. Then for any norm-one $b \in A$, AVA $(A_{rb}, ||.||)$ has central element $a$.*

**Proof:** The product $x \odot y = x(yb)$ on AVA $(A_{rb}, ||.||)$ satisfies

$$a \odot x = a(xb) = x(ab) = x \odot a.\square$$

**Lemma 3.8** *Let $a \in A$ be norm-one algebraic left generalized unit. Then there exists norm-one $b \in A$, for which $(A_{rb}, ||.||)$ has central idempotent $a$.*

**Proof:** The (well defined) mapping $A(a) \to A(a)$ $x \mapsto L_a^2(x) = a(ax)$ is onto, so there exists (norm-one) $b \in A(a)$ such that $a(ab) = a$. Now $a$ is an idempotent for algebra $(A_{rb}, ||.||) : a \odot a = a(ab) = a.\square$

**Corollary 3.9** *The norm of any AVA with algebraic left generalized unit comes from an inner product.*$\square$

Rodriguez [21] showed that for any AVA $(B, ||.||)$ with left unit $e$ and inner product $(./.)$, the left multiplications in $B$ satisfy $L_x^2 - 2(e/x)L_x + ||x||^2 I_A \equiv 0$ for all $x \in B$ and we have $(ab/c) = -(b/ac)$ for all $a, b, c \in A$ with $(a/e) = 0$. In other hand it follows immediately from [22, Theorem 3.2 and Proposition 3.3] that any AVA $(B, ||.||)$ with a nonzero central idempotent $e$ and inner product $(./.)$ satisfies $x^2 - 2(e/x)ex + ||x||^2 e = 0$ for all $x \in B$ (i.e. $B$ is e-quadratic [5]). We deduce a similar result for algebraic generalized left unit.

**Proposition 3.10** *Let $(A, ||.||)$ an AVA with an algebraic generalized left unit $a$ and inner product $(./.)$. Then there exists norm-one $b \in A$, with $a(ab) = a$ for which the equality $x(xb) - 2(a/x)x(ab) + ||x||^2 a = 0$ holds for all $x \in A$.*

**Proof:** There exists (norm-one) $b \in A$ such that $a(ab) = a$ and algebra $(A_{rb}, ||.||) := (A, \odot)$ is $a$-quadratic, so

$$x(xb) = x \odot x = 2(a/x)a \odot x - ||x||^2 a = 2(a/x)a(xb) - ||x||^2 a = 2(a/x)x(ab) - ||x||^2 a.\square$$

# 4 Algebraic flexible elements

A natural generalization including nonzero central idempotent and left unit is the one of nonzero idempotent $e$ for which left and right multiplication operators $L_e, R_e$ commute, that is $(e, x, e) = 0$ for all $x$. Such an element will be called flexible idempotent.

We have the following characterization

**Proposition 4.1** *For any $e \in A$, the following are equivalent*

1. *$e$ is both flexible idempotent and generalized left unit,*
2. *$e$ is a left unit.*

**Proof: 1. $\Rightarrow$ 2.** Let $x$ be arbitrary element in $A$, we have $(ex)e = e(xe) = xe^2 = xe$. The result is then concluded by a right simplification by $e$.$\square$

**Corollary 4.2** *Let $A$ be an AVA containing a generalized left unit which is a central idempotent. Then $A$ contains unit element and isomorphic to $\mathbb{R}$,$\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*$\square$

We need for the sequel the following result

**Lemma 4.3** *Let $a \in A$ be norm-one central algebraic element. There exists (norm-one) $b \in A$ with $ba^2 = a$ for which AVA $(._bA, ||.||)$ has central idempotent $a$.*

**Proof:** As $A(a)$ is finite-dimensional, the mapping $A(a) \to A(a)$ $x \mapsto R_{a^2}(x)$ is onto, and there exists (norm-one) $b$ in $A(a)$ such that $a = R_{a^2}(b) = ba^2$. Now, AVA $(._bA, ||.||)$, with product $x \odot y = b(xy)$, contains a central idempotent $a$.$\square$

A concrete example of algebraic element from [11]:

**Proposition 4.4** *Let $a \in A$ be norm-one such that $(a, a, a) = (a, a, a^2) = (a, a^2, a) = 0$. Then $a$ is algebraic and $A(a)$ is a commutative subalgebra of $A$.*

**Proof:** Since $(a, a, a) = (a, a, a^2) = 0$, we have

$$\|a - aa^2\| = \|a\|\|a - aa^2\| = \|a^2 - (a^2)^2\| = \|a - a^2\|\|a + a^2\|.$$

By Lemma **2.1** $Lin\{a, a^2\} := H$ is an Hilbert space. We distinguish the following cases:

1. If $a$ is orthogonal to $a^2$ then $\|a-a^2\|\|a+a^2\|=2$, that is, $\|a-a^2a\|=2$. As $(a,a,a)=0$ and $(a,a^2,a)=0$, we see that $a^2a=aa^2$ commutes with $a$ and by Lemma **2.2** we get $a^2a=-a\in H$. Also $(a^2)^2=(a^2a)a=-a^2\in H$ and $H$ is a two-dimensional commutative subalgebra of $A$.

2. If $a$ is not orthogonal to $a^2$ then $H=Lin\{a,b\}$, where $b\in H$ is orthonormal to $a$. So $b=\alpha a+\beta a^2$ with $\alpha\beta\neq 0$ and we have

   $$2=\|a-b\|\|a+b\|=\|a^2-b^2\|=\|a^2-(\alpha^2a^2+\beta^2(a^2)^2+2\alpha\beta aa^2)\|.$$

   This gives
   $$\|a-(\alpha^2a+\beta^2aa^2+2\alpha\beta a^2)\|=2.$$

   Using Lemma **2.2**, we deduce that $a+(\alpha^2a+\beta^2aa^2+2\alpha\beta a^2)=0$. We conclude from $\beta\neq 0$, that $aa^2\in H$. Consequently, $(a^2)^2=a(aa^2)\in H$ and therefore $H$ is 2-dimensional commutative subalgebra of $A$.□

**Corollary 4.5** *The norm of any AVA $(A,||.||)$ having norm-one central algebraic element $a$ comes from an inner product $(./.)$. Moreover $x^2=2(a/x)ax-||x||^2a^2$ holds for all $x\in A$.*

**Proof:** The first assertion is an immediate consequence of Lemma **4.3** and Theorem **2.3**. Now, according to notations in Lemma **4.3** and using the fact that algebra $._bA$ is $a$-quadratic, we have

$$bx^2=x\odot x=2(a/x)a\odot x-||x||^2a=2(a/x)b(ax)-||x||^2a=2(a/x)b(ax)-||x||^2ba^2.$$

The result is then concluded by a simplification by $b$.□

**Corollary 4.6** [11] *The norm of any AVA having norm-one central element $a$ such that $(a,a,a)=(a,a,a^2)=(a,a^2,a)=0$ comes from an inner product.*□

We state now the following important result:

**Theorem 4.7** *The norm $||.||$ of any AVA $A$ containing a nonzero flexible idempotent $e$ comes from an inner product $(./.)$. Moreover, $(ex)(xe)=2(e/x)exe-||x||^2e$ holds for all $x\in A$.*

**Proof:** The product $\odot$ in AVA $({}_{le}A)_{re}$ is given by $x\odot y=(ex)(ye)$ and we have
$$e\odot e=e \quad \text{and } e\odot x=e(xe)=(ex)e=x\odot e.$$

So $e$ is a central idempotent in AVA $(({}_{le}A)_{re}, ||.||)$ and first assertion is concluded by Theorem **2.3**. If $u \in A$ is orthogonal to $e$, we have $(ex)(xe) = x \odot x = -||x||^2 e$. Let now $x = (e/x)e + u$ be an orthogonal decomposition of an arbitrary $x \in A$. We have

$$\begin{aligned}(ex)(xe) &= x \odot x \\ &= ((e/x)e + u) \odot ((e/x)e + u)) \\ &= (e/x)^2 e \odot e + (e/x)(e \odot u + u \odot e) + u \odot u \\ &= (e/x)^2 e + 2(e/x)eue - ||u||^2 e \\ &= -(e/x)^2 e - ||u||^2 e + 2(e/x)e((e/x)e + u)e \\ &= 2(e/x)exe - ||x||^2 e. \square\end{aligned}$$

Theorem **4.7** can be refined as follows:

**Theorem 4.8** *The norm $||.||$ of any AVA $A$ containing norm-one algebraic flexible element $e$ comes from an inner product $(./.)$. Moreover, there exists (norm-one) $a \in A$ with $ea = ae = e$ for which the equality $(ex)(xe) = 2(a/x)exe - ||x||^2 e^2$ holds for all $x \in A$.*

**Proof:** The mapping $A(e) \to A(e)$ $x \mapsto L_e(x) = ex$ is onto, so there exists (norm-one) $a \in A(e)$ such that such that $ea = e$, and we have $e(ae) = (ea)e = e^2$. So $ae = e = ea$. Now, using the product $\odot$ in algebra $({}_{le}A)_{re}$, we have

$$a \odot x = (ea)(xe) = e(xe) = (ex)e = (ex)(ae) = x \odot a$$

for all $x \in A$. As $a \in A(e)$, the subalgebra $A^{\odot}(a)$ of $({}_{le}A)_{re}$ generated by $a$ is contained in $A(e, a) = A(e)$. So $a$ is an algebraic element of $({}_{le}A)_{re}$. Using Corollary **4.5**, we get the first assertion and we have

$$(ex)(xe) = x \odot x = 2(a/x)a \odot x - ||x||^2 a \odot a = 2(a/x)exe - ||x||^2 e^2. \square$$

We denote $\mathbb{P}$ the pseudo-octonions real algebra [17] and $\overset{*}{\mathbb{A}}$ the absolute valued algebra having underlying space $\mathbb{A}$ and product $x \odot y = \overline{x}\,\overline{y}$. We can now state the result

**Theorem 4.9** *For any AVA $(A, ||.||)$ satisfying $x^2 x = x x^2$, the following are equivalent*

1. *The norm $||.||$ comes from an inner product,*

2. *A is flexible,*

3. *A contains a norm-one algebraic flexible element,*

4. *A is finite-dimensional,*

5. *A is isomorphic to* $\mathbb{R}$,$\mathbb{C}$,$\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$,$\mathbb{O}$,$\overset{*}{\mathbb{O}}$ *or* $\mathbb{P}$.

**Proof:** The last assertion carry away all four above ones. In other hand, the implications **2.** $\Rightarrow$ **4.**, **1.** $\Rightarrow$ **4.**, **4.** $\Rightarrow$ **5.** are proved chronologically in [15], [7], [8] respectively. Moreover implication **3.** $\Rightarrow$ **1.** is consequence of Theorem **4.8**. This completes the proof.□

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