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Mutual Entrainment in Oscillator Lattices with Nonvariational Type Interaction

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A model system for limit cycle oscillators distributed on a d-dimensional cubic lattice is studied. This model has the form $\dot{\phi}_i=\omega_i-K\sum_{j\in N_i}\{\sin(\phi_i-\phi_j+\alpha)-\sin\alpha\}$, $i=1,2\cdots L^d$, where ϕ_i is the phase of the oscillator on the i-th site, the natural frequencies may change from site to site and N_i represents the set of the nearest neighbor sites of the i-th site. In a previous work we studied the case of vanishing α and discussed the possibility of phase transition. In the present work we deal with the case of nonvanishing α, and show by computer simulation that the parameter α has a strong effect on global entrainment. Two types of effects of α are discussed. One is that it suppresses large scale phase fluctuations, thus facilitating global entrainment. The other is associated with its effect on mutual entrainment in the presence of vortices.

§ 1. Introduction

Systems of coupled limit cycle oscillators have been the subject matter of extensive experimental and theoretical studies. Many of them have been motivated by the fact that self-entrainment of oscillators are functionally much relevant to living organism, such as the heart and intestine. A most remarkable feature of many-oscillator systems is that they exhibit phase transition-like phenomena when the natural frequencies are distributed and this problem was studied by using various model equation.[1)~18)] Inspired by Winfree's earlier idea of phase description,[1)] Kuramoto proposed a well-defined model and obtained analytical results in the case of mean field coupling.[3)] The model employed there is given by

$$\dot{\phi}_i=\omega_i-\frac{K}{N}\sum_{j=1}^{N}\sin(\phi_i-\phi_j)\,, \tag{1}$$

where ϕ_i represents the phase of the i-th oscillator, and N the total number of the oscillators. The natural frequencies ω_i generally change randomly; the normalized number density of the oscillators having natural frequency ω is denoted as $g(\omega)$. The above system shows a phase transition such that collective oscillations appear only above a critical coupling strength. For this special model the mean field approximation holds exactly and analytical expressions for various quantities can be obtained. The mean field coupling is not very realistic, however, and we considered the case of finite-range coupling in a previous paper.[18)] There the oscillators were regularly arranged on a d-dimensional cubic lattice, and they were assumed to interact only with their nearest neighbors. The model system is then given by a generalization of Eq. (1) in the form

$$\dot{\phi}_i=\omega_i-\sum_{j\in N_i}K\sin(\phi_i-\phi_j)\,,\quad (i=1,2\cdots N=L^d) \tag{2}$$

where N_i is a set of the nearest neighbor sites of the i-th site. As the coupling strength is increased, the oscillators first exhibit local entrainment, and the entrained clusters thus formed become larger and larger. It was revealed in the previous work[18] that there exists a critical coupling strength above which a macroscopic number of oscillators are mutually entrained into a unique frequency provided the system dimension equals three or higher.

The interaction assumed in systems described by Eq. (1) or Eq. (2) is very special, while the theory of the phase description states nothing more than that the general form of coupling should be a 2π-periodic function of the phase difference $\phi_i-\phi_j$ between the interacting pair.[4),13)] Moreover, the model in Eq. (2) cannot consistently explain the following fact. That is, in real living systems such as the mammalian intestine and heart, the cells forming a tissue oscillate with frequencies larger than when they are isolated from each other.[2),10)] These facts naturally lead us to the study of the second simplest model:

$$\dot{\phi}_i=\omega_i-K\sum_{j\in N_i}\{\sin(\phi_i-\phi_j+\alpha)-\sin\alpha\}, \quad -\frac{\pi}{2}<\alpha<\frac{\pi}{2}. \tag{3}$$

The interaction here is of attractive type and tends to increase the frequencies when α is positive. Note that the interaction cannot be derived from any kinetic potential,[5)] and we call this type of interaction nonvariational. The nonvariational type interaction seems to be characteristic to general nonequilibrium systems. We discuss in this paper some remarkable effects of α on local and global entrainments.

§ 2. Mutual entrainment between two oscillators

We begin with an elementary consideration of a system of two oscillators:

$$\dot{\phi}_1=\omega_1-K\{\sin(\phi_1-\phi_2+\alpha)-\sin\alpha\},$$

$$\dot{\phi}_2=\omega_2-K\{\sin(\phi_2-\phi_1+\alpha)-\sin\alpha\}. \tag{4}$$

The mean phase $\psi=(\phi_1+\phi_2)/2$, and the phase difference $\phi_{12}=\phi_1-\phi_2$ obey the equations

$$\dot{\psi}=w+K(1-\cos\phi_{12})\sin\alpha,$$

$$\dot{\phi}_{12}=\omega_{12}-2K\cos\alpha\sin\phi_{12}, \tag{5}$$

where $w=(\omega_1+\omega_2)/2$ and $\omega_{12}=\omega_1-\omega_2$. If $|\omega_{12}/2K\cos\alpha|>1$, ϕ_{12} changes monotonically with t implying that the oscillators undergo quasiperiodic oscillations with independent frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$, where $\tilde{\omega}_i$ are defined and given by

$$\tilde{\omega}_i=\lim_{T\to\infty}\frac{\phi_i(T)-\phi_i(0)}{T}, \tag{6}$$

$$\tilde{\omega}_1=w+\frac{\omega_{12}}{2}\sqrt{1-\left(\frac{2K\cos\alpha}{\omega_{12}}\right)^2}+K\sin\alpha,$$

$$\tilde{\omega}_2=w-\frac{\omega_{12}}{2}\sqrt{1-\left(\frac{2K\cos\alpha}{\omega_{12}}\right)^2}+K\sin\alpha. \tag{7}$$

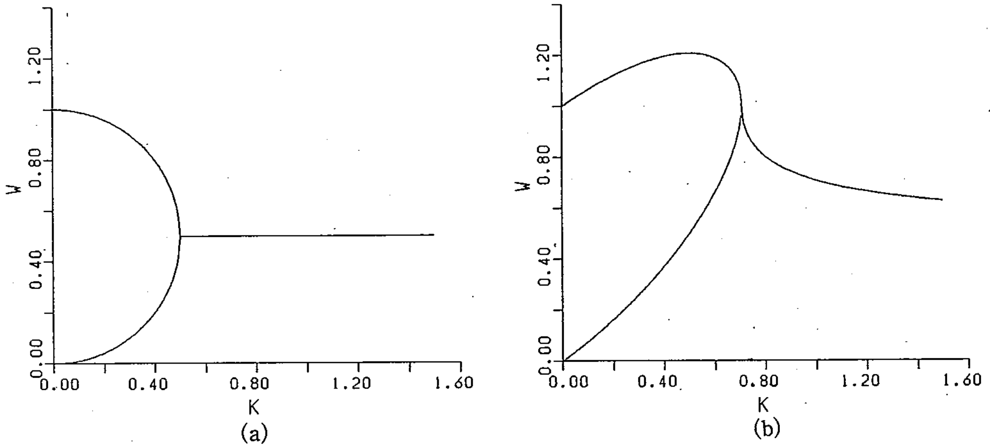


Fig. 1. Effective frequencies $\tilde{\omega}$ versus coupling strength K in a two-oscillator system. (a) $\alpha=0$, (b) $\alpha=\pi/4$.

If $|\omega_{12}/2K\cos\alpha|<1$, ϕ_{12} finds its stable equilibrium at

$$\phi_{12}=\sin^{-1}\left(\frac{\omega_{12}}{2K\cos\alpha}\right), \qquad K\cos\alpha\cos\phi_{12}>0\,. \tag{8}$$

The two frequencies are then identical and given by

$$\tilde{\omega}_1=\tilde{\omega}_2=w-\frac{|\omega_{12}|}{2}\tan\alpha\sqrt{\left(\frac{2K\cos\alpha}{\omega_{12}}\right)^2-1}+K\sin\alpha\,. \tag{9}$$

Thus we have a critical coupling strength for mutual entrainment. The effective frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ change with the coupling strength as shown in Fig. 1. When $0<\alpha<\pi/2$ and the mutual entrainment has been established, the effective frequencies become larger than the simple mean of the natural frequencies. This is because the phase difference ϕ_{12} increases the instantaneous frequencies through the term $(1-\cos\phi_{12})\sin\alpha$ in Eq. (5). The increase in frequency through the phase difference causes many interesting phenomena as we see in later sections.

§ 3. One-dimensional oscillator lattices

In equilibrium systems, the appearance of long range order becomes increasingly difficult as the spatial dimension is lowered. This is due to long wavelength fluctuations of the order parameter. Also in our oscillator lattices with vanishing α, there are no phase transitions. In fact, as we showed in the previous paper, the linear analysis of Eq. (2) gives expression for the mean square fluctuation of wavenumber k as

$$\langle\phi_k{}^2\rangle\sim\frac{(\Delta\omega)^2}{k^4}\,, \tag{10}$$

where $\Delta\omega$ is the root mean square of the frequency distribution $g(\omega)$. The "infra-red" divergence of phase fluctuations destroys the macroscopic order in lower dimensional systems. In contrast, nonvanishing α seems to have the effect of suppressing the long

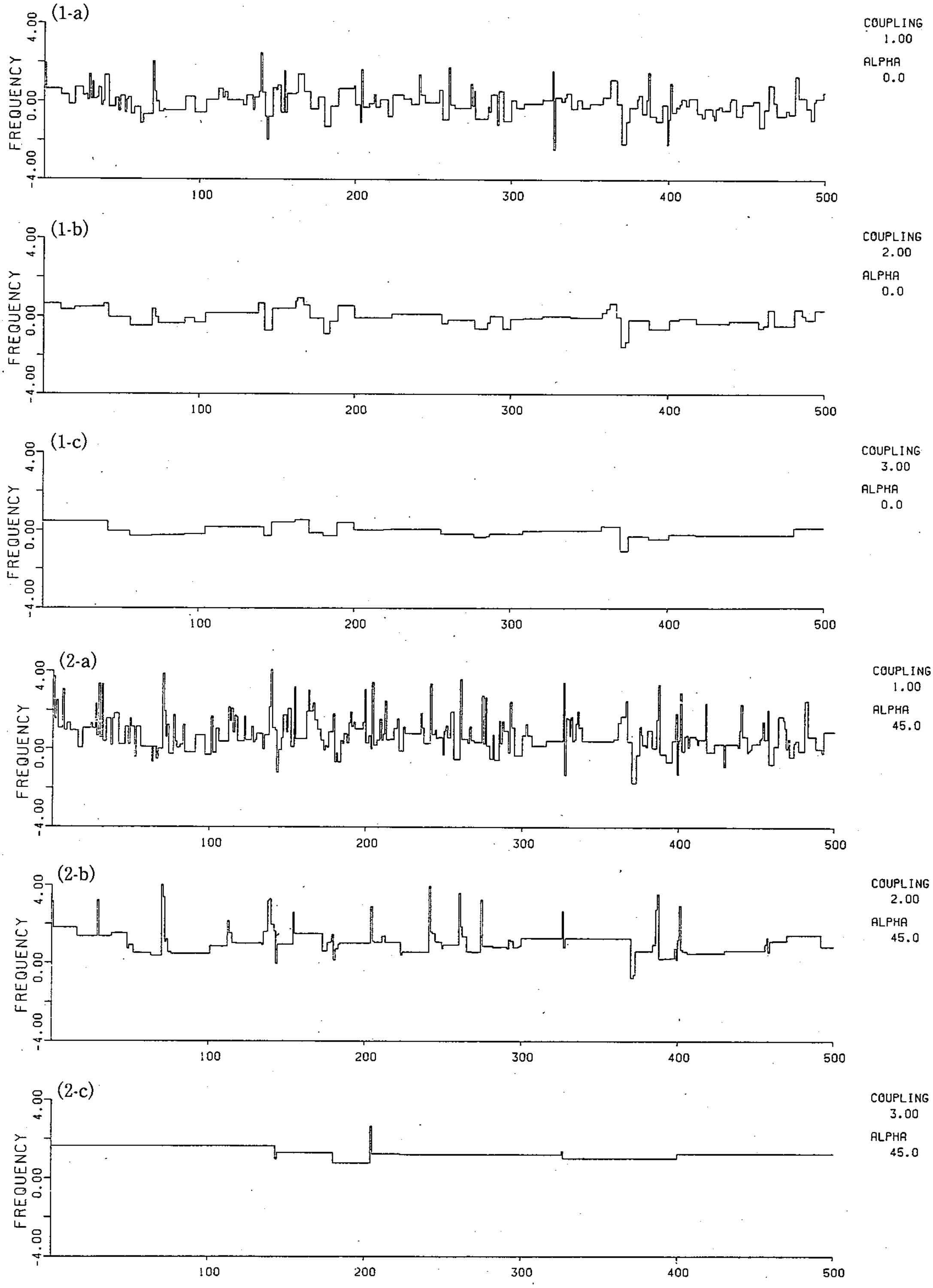


Fig. 2. Frequency patterns in a one-dimensional oscillator lattice.
(1-a) $\alpha=0$ $K=1.0$, (1-b) $\alpha=0$ $K=2.0$, (1-c) $\alpha=0$ $K=3.0$, (2-a) $\alpha=\pi/4$ $K=1.0$, (2-b) $\alpha=\pi/4$ $K=2.0$, (2-c) $\alpha=\pi/4$ $K=3.0$.

wavelength fluctuations through the nonlinear term, $\sin\alpha\{1-\cos(\phi_i-\phi_j)\}$ in Eq. (3). In order to see the situation in further detail, a computer simulation was carried out in a one-dimensional lattice of 10000 oscillators. The distribution $g(\omega)$ is assumed to be Gaussian with variance 1. As the method of numerical integration we used the Euler method where the elementary time increment was taken to be 0.1. This may look to be too crude an approximation to the ordinary differential equation. Actually, however, the errors produced then are unimportant for our purposes as we stated in the previous paper for similar situations.[13),18)] The uniform initial distribution and the periodic boundary condition are chosen. The effective frequency of the i-th oscillator is calculated from the approximate formula

$$\tilde{\omega}_i=(\phi_i(T)-\phi_i(0))/T\ , \tag{11}$$

where T is the total time of the numerical integration subtracted by the equilibration time, and equals 600 in the present simulation.

How the spatial distribution of $\tilde{\omega}_i$ in the same part of an oscillator chain changes with the coupling strength and α is shown in Fig. 2. The flat regions in the frequency patterns imply that the oscillators are locally entrained. The mean size of these entrained clusters becomes larger as the coupling strength is increased. For $\alpha=0$ the size distribution has a peak about the mean size, while for $\alpha=\pi/4$ and relatively large K the frequency pattern comes to be composed of a small number of very large clusters and many small ones. In the calculation we adopted the following criterion for the mutual entrainment between the i-th and the neighboring j-th oscillators:

$$|\tilde{\omega}_i-\tilde{\omega}_j|<\pi/T\ .$$

Figure 3 shows the decrease of the total number of the entrained clusters as the coupling strength is increased. For $\alpha=0$ it decreases roughly as $1/K^2$, while for

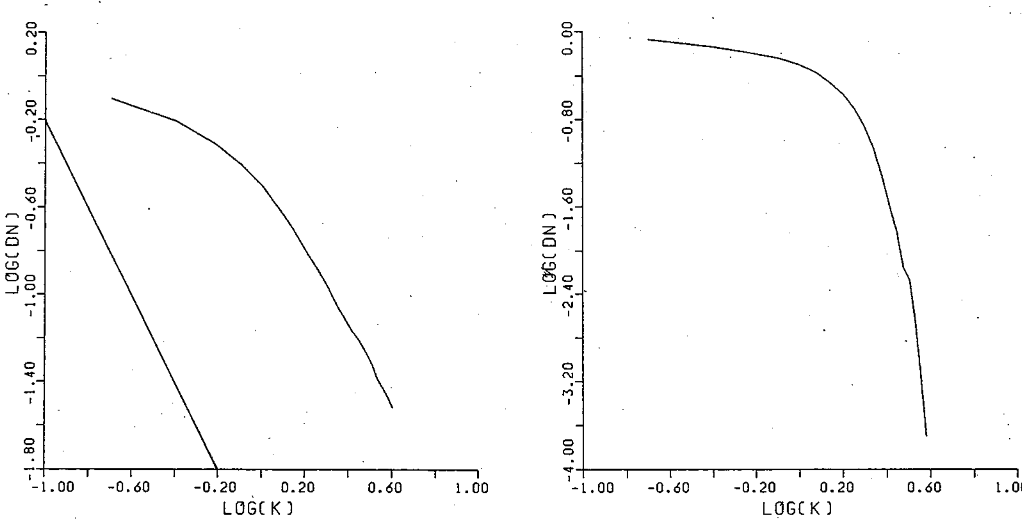


Fig. 3. Log-log plot of number density of entrained clusters versus coupling strength K in a one-dimensional oscillator lattice. (a) $\alpha=0$, (b) $\alpha=\pi/4$.

$\alpha=\pi/4$ it exhibits a more rapid decrease. These results suggest that the local entrainment is facilitated by nonvanishing α.

In our simulation the distribution $g(\omega)$ is Gaussian as we stated. Then, due to the presence of oscillators with arbitrary high frequencies, complete entrainment is no longer possible as far as the system size is infinite. To learn the effect of α we apply a linear analysis to the case that $g(\omega)$ has high- and low-frequency cutoffs. Let the system size be L and the coupling strength be strong enough so that the condition $|K/(\omega_i-\omega_j)|\gg 1$ may be satisfied for any nearest neighbor pair (i, j). Because the phase differences between the nearest neighbor oscillators are expected to be so small, a power series expansion of Eq. (3) in the phase difference may be permitted. Up to the second order in the phase difference, we obtain

$$\dot{\phi}_i=\omega_i-K\sum_{j\in N_i}\cos\alpha(\phi_i-\phi_j)-K\sum_{j\in N_i}\frac{1}{2}\sin\alpha(\phi_i-\phi_j)^2\,. \tag{12}$$

When α is zero, the third term on the r.h.s. vanishes and the mean square of the phase difference can be easily calculated by using the Fourier transform. We have

$$\langle(\phi_{i+1}-\phi_i)^2\rangle\propto\left(\frac{\Delta\omega}{K}\right)^2 L\,. \tag{13}$$

Equation (13) implies that the phase difference increases indefinitely with the system size L, so that the complete entrainment is collapsed when L becomes as large as $O((K/\Delta\omega)^2)$. In a system of infinite size, typical phase difference between the nearest neighbor oscillators in a cluster of size L may also be expected to be $O((\Delta\omega/K)\sqrt{L})$, because no serious influences from the neighboring clusters are expected. By definition the phase difference between any nearest neighbor pair in an entrained cluster should be smaller than π, so that the above argument implies that a typical cluster size should be proportional to $(K/\Delta\omega)^2$. Figure 3 implies that the mean cluster size is proportional to K^2 for $\alpha=0$ and seems to support the argument above.

When α is nonzero, the third term of Eq. (12) is important. If we go over to the continuum approximation, Eq. (12) becomes

$$\begin{aligned}&\dot{\phi}(x)=\omega(x)+K\cos\alpha\,\nabla^2\phi+K\sin\alpha(\nabla\phi)^2\,,\\&\omega(x)=\omega_i\qquad\text{for } i\le x\le i+1\,,\end{aligned} \tag{14}$$

where the lattice constant is taken to be 1. By the Cole-Hopf transform[19] $\phi(x)=\cot\alpha\ln Q(x)$, Eq. (14) becomes

$$-\frac{1}{\cos\alpha}\frac{dQ}{dt}=\left(-\frac{\tan\alpha}{\cos\alpha}\omega(x)\right)Q-K\nabla^2 Q\,. \tag{15}$$

This is a Schrödinger equation with a random potential. For this type of systems the Anderson localization is known to occur for one-dimensional systems. The n-th eigenfunction $\hat{Q}_n(x)$ with eigenvalue $(-\lambda_n/\cos\alpha)$ will be localized about x_n with some localization length ξ_n:

$$\hat{Q}_n(x)\sim\exp\left\{-\frac{|x-x_n|}{\xi_n}\right\}. \tag{16}$$

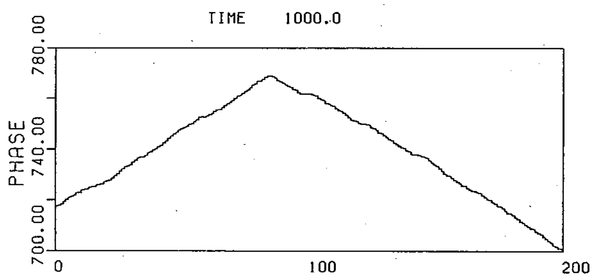


Fig. 4. Phase pattern in a one-dimensional oscillator lattice with $K=1.0$ and $\alpha=\pi/4$ at time 1000. This value of K is large enough for complete entrainment to be established.

Then the solution of Eq. (13) is written as

$$Q(x,\,t)=\sum_n a_n(x)\exp\left\{-\frac{|x-x_n|}{\xi_n}\right\}\exp(\lambda_n t)\,, \tag{17}$$

where $a_n(x)$ is the amplitude of the n-th eigenfunction $\widehat{Q}_n(x)$ including its deviation from the exponential behavior as in Eq. (16). In the summation in Eq. (17), the term corresponding to the largest λ_n, which we denote as λ_m, becomes increasingly dominant with t. Thus the approximate solution of Eq. (14) is given by

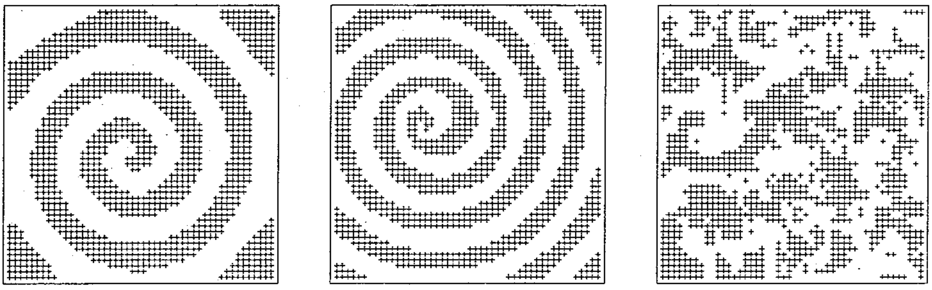
$$\begin{aligned}\phi(x,\,t)&=\cot\alpha\,\ln Q(x,\,t)\\&\simeq\cot\alpha\left\{\ln\,a_m(x)-\frac{|x-x_m|}{\xi_m}+\lambda_m t\right\}.\end{aligned} \tag{18}$$

Note that λ_m should be smaller than $\tan\alpha\cdot\max\omega_i$ so that it remains finite provided $g(\omega)$ has cutoffs. The solution in Eq. (18) shows that the phase pattern is centered at x_m and ϕ decreases linearly with the distance from x_m. A simulation result is shown in Fig. 4, where $g(\omega)$ is assumed to be constant over the interval $[0,1]$ and vanishing outside. The length scale ξ_m is related to the coupling strength K, and is roughly scaled as $\xi_m\sim\sqrt{K}\,\bar{\xi}_m$. The phase difference is roughly $\cot\alpha/\xi_m$ and is proportional to $1/\sqrt{K}$ so that complete entrainment is possible even if L goes to infinity. Thus, nonvanishing α facilitate large scale entrainment when $g(\omega)$ has frequency cutoffs. Without cutoffs, complete entrainment is impossible, still nonvanishing α facilitates the formation of large entrained clusters as was confirmed by our computer simulation.

§ 4. Two-dimensional oscillator lattices

Two-dimensional systems admit topological defects called vortices. In equilibrium X-Y spin systems, vortices play an important role in the phase transition as we know by the term Kosterlitz-Thouless transition. One may ask how about the role played by vortices in oscillator lattices.

In what follows (Figs. 5~8) we show some results of computer simulations on our two-dimensional system with two types of patterns. The first one is called the phase pattern which shows a snap shot of ϕ_i's, and the second is the frequency pattern which

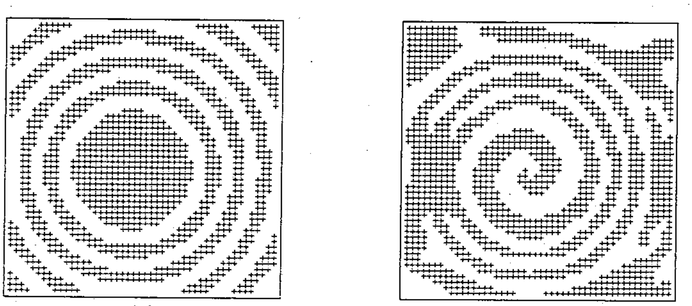


(a) (b) (c)

Fig. 5. Phase patterns in a two-dimensional system without frequency distribution. The initial condition is so chosen that a vortex is formed at the center.
(a) $\alpha=\pi/6$, (b) $\alpha=\pi/4$, (c) $\alpha=\pi/3$.

clearly shows how mutually entrained oscillators are distributed. The frequency pattern tells us whether the oscillators are mutually entrained and the phase pattern tells us how the oscillators are mutually entrained. Wherever oscillators are strongly syncronized, their phase changes smoothly in space, while if the oscillators are desynchronized, the phase pattern becomes nonsmooth in general.

Even without distribution in natural frequency, vortices may cause nonuniformity in local frequency and as a result the feature of mutual entrainment may drastically be changed. In the vicinity of a vortex the phase difference between nearest neighbor oscillators is expected to be as large as $\pi/2$. Because phase difference induces the increase in local frequency if α is positive, the vortex acts as a pacemaker entraining the surrounding medium. This is why a spiral pattern arises round the vortex. A spiral pattern with a phase singularity at the center has a phase gradient which is nearly constant along the radial direction. Figure 5 shows phase patterns for $\alpha=\pi/6$, $\pi/4$, $\pi/3$ and $\omega_i=0$ for all i. We demonstrate the phase patterns by assigning the mark + on lattice point i if the condition $\sin\phi_i(T)>0$ is satisfied. For $\alpha=\pi/6$, the spiral pattern rotates steadily. As α is increased, the spiral winds up more and more tightly, so that the phase gradient in the radial direction is steepened. The global entrainment collapses above a certain critical value α_c and then the center of the spiral oscillates at a frequency different from the surrounding medium. Thus the



(a) (b)

Fig. 6. Phase patterns in a two-dimensional system with $\alpha=\pi/6$ and $K=1.0$, where the central oscillators have natural frequencies higher than the surrounding medium by 0.8. Vortex is absent in (a) and present in (b).

vortex breaks the mutual entrainment even in homogeneous systems. The computer simulation for $\alpha=\pi/4$ shows that the center of the spiral moves around as a result of the desynchronization. For even larger α many vortex-antivortex pairs are created and the phase patterns become very chaotic.[21] When the initial phase distribution is relatively uniform, a perfectly uniform state is finally established as far as $|\alpha|<\pi/2$, as the uniform state is linearly stable then. It is important to realize that the final pattern of entrainment depends on the initial condition in a two-dimensional oscillator lattice.

When the system is not homogeneous in natural frequency, the local entrainment extends with the coupling strength. However, the pattern of entrainment is more complicated than in one-dimensional lattices because of the vortices involved. We show some simulation results which help understand the effect of the vortices on mutual entrainment. Figure 6 shows phase patterns with and without vortex. The natural frequencies assumed are $\omega_i=0.8$ in a circular domain of radius 12 and $\omega_i=0$ outside it, and $K=1.0$ and $\alpha=\pi/6$. When there is no vortex, all oscillators are entrained to the central region and a target-like phase pattern emerges then. When a vortex is introduced, it increases the frequencies of the central oscillators, and the entrainment over the entire system is no longer possible. A spiral pattern is then formed near the center, although it cannot extend outward. As a result of desynchronization between the inner and outer regions additional disturbances are produced. This gives a typical case that a vortex combined with the distribution of natural frequencies breaks the global entrainment.

Figure 7 shows the situation in which the complete entrainment can be established even in the presence of vortex and frequency distribution for $\alpha=\pi/4$. In contrast there is a situation that a vortex breaks global entrainment in the system without frequency distribution for the same value of α as is shown in Fig. 5. The origin of the desynchronization is that the central part oscillates so fast that the outer region is unable to follow it. It is therefore expected that the global entrainment is recovered if the natural frequencies in the central part are made suitably lower than the surrounding. In Fig. 7 the central four oscillators have natural frequencies lower by $(3/4)\sqrt{2}$ than the surrounding. Then a steadily rotating spiral pattern is clearly seen implying global entrainment.

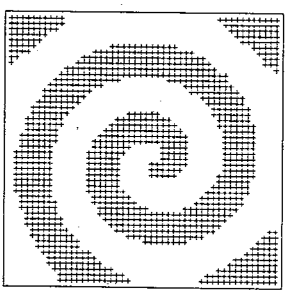


Fig. 7. Phase patterns in a two-dimensional system with $\alpha=\pi/4$ and $K=1.0$, where the central oscillators have frequencies lower than the surrounding medium by $(3/4)\sqrt{2}$.

We have seen that the mutual entrainment can sensitively be dependent on the distribution of natural frequencies and also on vortex distribution. We show in Fig. 8 another simulation result with frequency distribution showing that mutual entrainment can strongly be dependent on initial condition. The distribution $g(\omega)$ is taken to be Gaussian with variance 1 and the other parameters are fixed as $K=2.6$ and $\alpha=\pi/4$, and periodic boundary condition

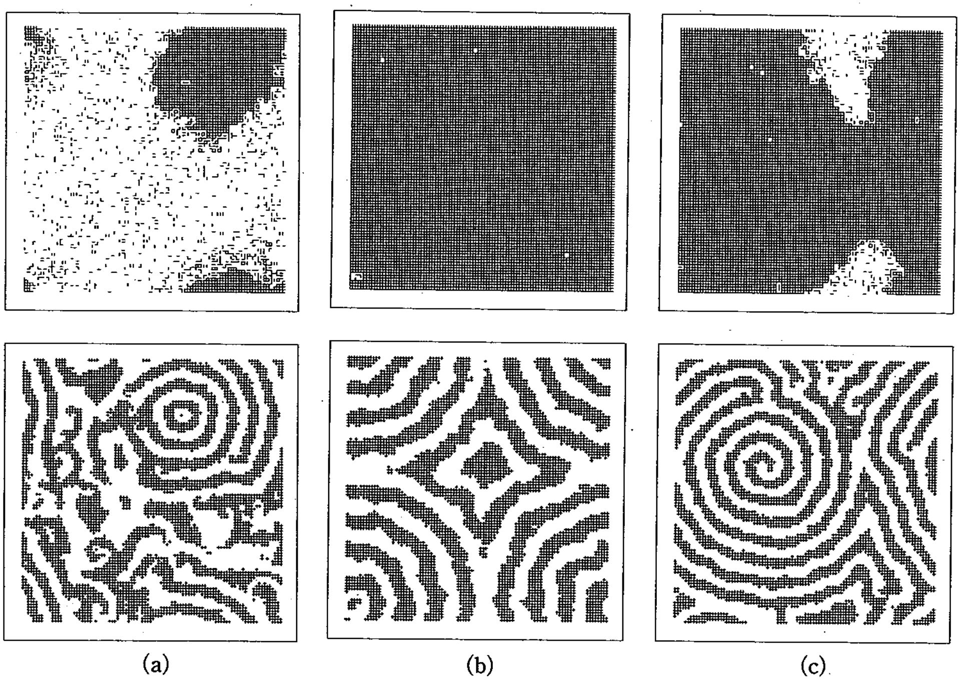


Fig. 8. Frequency patterns and phase patterns in a two-dimensional random system. The distribution $g(\omega)$ is taken to be a Gaussian with variance 1 and $\alpha=\pi/4$ and $K=2.6$. Patterns in (a), (b) and (c) correspond to different initial conditions.

is chosen.

Three different initial conditions are chosen, and the resulting frequency patterns and phase patterns are shown in Fig. 8. In the frequency pattern, a bond is assigned between the nearest neighbor sites if their frequencies are regarded identical. An entrained cluster linked mutually by bonds represents itself as a giant oscillator.[18] Quite different cluster distributions are obtained from different initial conditions for the same system. A spiral or target pattern, though imperfect, can be seen where the oscillators form a large entrained cluster. In contrast, where the oscillators are not mutually entrained, the phase pattern is chaotic. These two characteristic regions are clearly separated from each other.

We have shown some results of our computer simulations in two-dimensional oscillator lattice. We found that the nonvariational type coupling makes some topological defects involved act as pacemaker. This effect introduces an additional origin of complexity into pattern formation associated with mutual entrainment.

§ 5. Summary

We have shown some remarkable system behaviors characteristic to the nonvariational type interaction in oscillator lattices. This type of interaction increases the effective frequency through the phase gradient produced by the distribution of natural frequencies or by vortices. The system has a possible ability of adjusting local phase gradient so that the difference in natural frequency is compensated. Thus the mutual entrainment on a large scale can be facilitated. Conversely, the effective frequency increase by the phase difference can be the cause of desynchronization in the presence of vortices. This is because the effective frequency would be too much increased in the vicinity of phase singularity. Thus it may be said the nonvariational type interaction has ambivalent effects on mutual entrainment. Finally, we have to say that nothing is clear yet about the effect of α on phase transitions; this is completely open to future investigation.

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