Mutual information and the F-theorem

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ABSTRACT: Mutual information is used as a purely geometrical regularization of entanglement entropy applicable to any QFT. A coefficient in the mutual information between concentric circular entangling surfaces gives a precise universal prescription for the monotonous quantity in the c-theorem for $d = 3$. This is in principle computable using any regularization for the entropy, and in particular is a definition suitable for lattice models. We rederive the proof of the c-theorem for $d = 3$ in terms of mutual information, and check our arguments with holographic entanglement entropy, a free scalar field, and an extensive mutual information model.

Contents

1 Introduction

A great part of the problem of understanding the structure of the space of quantum field theories (QFT) reduces to the one of classifying possible critical points. These critical points must be free of ultraviolet problems which greatly constrains the consistent QFT's. In this picture, the theories with mass scales are obtained by perturbing critical points and following the corresponding renormalization group (RG) trajectories. However, interestingly, not all critical points are joined to each other in such flows. For example, nontrivial constraints arise by anomaly matching conditions implying the ultraviolet (UV) and infrared (IR) fixed points have the same chiral anomalies [1].

A different class of constraints are provided by c-theorems. Such theorems state that a certain c-function must decrease along any RG trajectory and so they illustrate the irreversible character of RG flow. The possibility of such a structure first emerged over

twenty-five years ago with Zamolodchikov's proof of the c-theorem in two dimensions [2]. However, it is only recently that a long sought proof was found for the corresponding a-theorem in four dimensions [3, 4] — see also [5]. Both the $d = 2$ and $d = 4$ cases conform with a broader conjecture by Cardy [6] that for any even number of spacetime dimensions, there exists a c-theorem where at the fixed points, the c-function coincides with the coefficient of the Euler term in the trace anomaly, the so-called A-type anomaly term [7]. Unfortunately, progress towards proving such c-theorems in higher dimensions (e.g., $d = 6$ [8–10]) has been limited. Of course, Cardy's proposal cannot be extended to odd dimensions where the trace anomaly vanishes.

Recently, a new perspective on c-theorems was proposed in [11, 12], in particular, one which incorporates both *odd* and even dimensions. Motivated by holographic studies, they suggested that the c-function could be identified as the universal coefficient appearing in the entanglement entropy of a region enclosed by a sphere. In fact, in even dimensions, this universal coefficient yields precisely the central charge in the A-type trace anomaly at fixed points of the RG flows [11–13] — see also [14–16]. Hence this new proposal provides a new perspective on Cardy's conjecture for even dimensions. However, more importantly, entanglement entropy provides a framework to consider c-theorems for quantum field theories in an odd number of dimensions. Recently, entanglement entropy was also discussed as a framework to extend the two-dimensional g-theorem [17, 18] to higher dimensional boundary c-theorems [19, 20] — see also [21].

Much of the subsequent work focused on the case of $d = 3$. In particular, an independent formulation of the three-dimensional c-theorem was proposed in [22], where the c-function was identified at conformal fixed points with the constant term in the free energy evaluated on a three-sphere. A great deal of evidence for this F-theorem was gathered from a wide variety of field theory examples, e.g., [23–29].[1] However, it was also known that the F-theorem coincides with the previous proposal using entanglement entropy [13]. In particular, for any odd d, it was shown that the free energy of a d-sphere and the entanglement entropy of a $(d − 2)$-sphere can be mapped to each other in a CFT [13, 30]. Next, a proof was found to show that the constant term in the entanglement entropy for a circle decreases monotonically under RG flow in $d = 3$ QFT's [32]. This proof of the three-dimensional F-theorem relies only on unitarity and Lorentz invariance of the underlying QFT, as well as strong subadditivity of entanglement entropy [33]. Unfortunately, this proof does not (easily) extend to higher dimensions due to the appearance of more singular terms in the entanglement entropy.

In the context of QFT, an essential feature of entanglement entropy is that it cannot be defined without reference to a UV regulator because the result is dominated by short-distance correlations in the vicinity of the entangling surface. Considering a d-dimensional QFT, if we choose a spherical entangling surface of radius R in flat space,[2] the entanglement

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[1]See [31] for an interesting application of the F-theorem to derive nonperturbative results for strongly interacting quantum field theories.

[2]Implicitly, we also assume that the theory is in its ground-state throughout the following.

entropy generically has an expansion

$$S = c_{d-2}\left(\frac{R}{\delta}\right)^{d-2} + c_{d-3}\left(\frac{R}{\delta}\right)^{d-3} + c_{d-4}\left(\frac{R}{\delta}\right)^{d-4} + \cdots \qquad (1.1)$$
$$+ \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, c_0 \log(R/\delta) + \cdots & \text{even } d\,, \\ (-)^{\frac{d-1}{2}}\, 2\pi\, c_0\ +\ \cdots & \text{odd } d\,. \end{cases}$$

where δ is the short-distance UV cut-off. For a CFT, the coefficients c_k are pure numbers but for a generic QFT, they will depend on other mass scales μ_i through the dimensionless combination $\mu_i \delta$ [34–36]. Of course, the first term in the above expansion is the celebrated 'area law' term [37–39].

The coefficients of the power law divergent terms depend on the details of the UV regulator and only c_0 can be universal. In an even number of spacetime dimensions, this universality is guaranteed by the logarithmic character of the associated singluarity in eq. (1.1). However, for odd dimensions where c_0 appears in the finite contribution, there are a variety of potential ambiguities which may ruin the universality of this constant. For example, the regulator may limit the radius R from being determined with a resolution of better than the UV cut-off δ. However, any shift $R \to R' = R + \alpha\,\delta$ will result in mixing c_0 with all of the higher order coefficients in the expansion (1.1). We will expand on such ambiguities further in section 2.1.

Important progress in this regard was made in [34, 40][3] where a class of regulators was identified which yield an unambiguous definition of c_0. The key feature was that the regulator must be covariant or 'geometric,' and as a result, the UV divergent terms in the entanglement entropy all have a geometric character. That is, these contributions can all be expressed as integrals of local geometrical quantities over the entangling surface.[4] However, identifying various properties to define a 'nice regulator' leaves the discussion somewhat incomplete, in that one might be left to conclude that c_0 is only an artifact of the choice of a geometric regulator. If this constant is a physical quantity, we should be able to use any regularization scheme which defines the continuum theory of the underlying QFT.

Given these issues, we introduce a list of requirements needed to properly establish a c-theorem:

(a) The c-function must be a dimensionless, well-defined quantity, which is independent of the regularization scheme. This means that it must be a renormalized physical quantity, which can in principle be computed with any regulator.

(b) Evaluated at a given fixed point, the c-function yield a quantity which is intrinsic to the fixed-point theory of interest. In particular, for an RG flow running between two fixed points, the c-function must be independent of the details of the RG flow when evaluated at the infrared fixed point.

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[3]See also [41, 42] for discussion from the perspective of the free energy.

[4]Note that with such a covariant regulator, the expansion (1.1) will be restricted to contributions involving odd (even) powers of R/δ for odd (even) d. That is, all of the coefficients c_{d-3-2k} will vanish.

(c) Finally, for the c-theorem to hold, the c-function must decrease monotonically along any RG flow connecting a UV fixed point to an IR fixed point.

Our proposal is to use the concept of mutual information to define the c-function and we will demonstrate that the corresponding c-theorem in three dimensions satisfies all three of the above conditions. The mutual information is another measure of information between two systems[5] and for non-intersecting regions A and B, it may be defined as

$$I(A, B) = S(A) + S(B) - S(A \cup B). \qquad (1.2)$$

Unlike entanglement entropy, mutual information is finite and independent of the UV cut-off of the theory, i.e., the divergences in $S(A)$ and $S(B)$ are canceled by those in $S(A \cup B)$. We note that, in fact, mutual information can be defined without any reference to entanglement entropy. The latter definition uses the Araki formula for the relative entropy of two different states in an operator algebra [50] — see also [51]. Unfortunately, this approach seems unsuited for practical calculations and so our discussion will only consider the definition above in eq. (1.2).

Beginning with eq. (1.2), we provide a new definition of c_0 where the mutual information serves as a 'universal' geometric regulator of the circle entanglement entropy, which can be applied for any QFT with any UV regulator.[6] Since the mutual information is finite and independent of the UV regulator, the constant c_0 defined in this way automatically satisfies our first criterion above. With further care in formulating our prescription, we are able to show that the second criterion is also satisfied. Hence with the mutual information, we are able to identify c_0 as a renormalized physical quantity, which can be calculated using any regulator. Finally, we will also show the proof of the c-theorem can be formulated in terms of our mutual information prescription. The latter clarifies the validity of certain assumptions in [32], as well as establishing that our final requirement is satisfied within this new framework.

An overview of the paper is as follows: we begin the next section with a discussion of some ambiguities which arise in evaluating c_0 directly the entanglement entropy (1.1). In section 2.2, we provide a cleaner definition of a similar entanglement 'central charge' using the mutual information. Then in section 2.3, we establish that the new definition protects the coefficient $\tilde{c}_0$ from the high energy details of the theory. In section 3, we argue that in fact $\tilde{c}_0 = c_0$ with a suitable choice of regulator. In particular, we examine with which regulators, the charge defined by the mutual information can be obtained directly from that in the entanglement entropy of a circle. Further, with a detailed examination of the example of a free massless scalar, we highlight some ambiguities in defining c_0, even when using some geometrical regularizations, and we show that they can be eliminated using our mutual information prescription. In section 4, we prove the monotonic decrease of c_0 in terms of the mutual information. We conclude with a brief discussion of the results in section 5. In appendix A, we calculate the mutual information for a circle within

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[5]For example, this quantity can be used to bound the connected correlators of observables in two regions [43]. See [44–49] for various studies of general properties of mutual information in QFT.

[6]Similar ideas were discussed previously in [46, 47, 52].

a holographic framework. Appendix C presents calculations of mutual information in a particular model where the mutual information satisfies a certain extensivity property [47]. Finally, in appendix D, we calculate the entanglement entropy of a circle for a free massless scalar, first by mapping the problem to the hyperbolic cylinder $R \times H^2$ and then mapping to de Sitter space.

The discussion in sections 2 and 3, as well as the examples in the appendices, can be easily generalized to any number of spacetime dimensions. However, the proof of monotonicity is section 4 is only valid for $d = 3$. For conciseness, the material in sections 2 and 3, as well as in the appendices will be limited to considering $d = 3$.

2 Mutual information as a 'geometric regulator'

A central issue raised in the introduction was that entanglement entropy in QFT cannot be defined without explicit reference to a UV regulator. Further, without restricting the class of allowed regulators, this introduces various potential ambiguities in trying to define c_0 directly from eq. (1.1) for odd dimensions, in particular, for $d = 3$. In section 2.1 below, we discuss these ambiguities in more detail and, in particular, we explicitly show that evaluating c_0 with a lattice regulator is problematic. Next in section 2.2, we consider using mutual information as a geometric regulator for entanglement entropy of a circle and this allows us to define a related coefficient $\tilde{c}_0$. Since mutual information is a well defined (finite) quantity, $\tilde{c}_0$ can be regarded as a physical quantity, which is independent of the details of the regulator. However, our prescription still has a certain freedom which introduces ambiguities in this constant which are somewhat analogous to those found using the entanglement entropy. Hence we refine the definition of $\tilde{c}_0$ in section 2.3 to ensure that at RG fixed points, it is a quantity intrinsic to the underlying CFT, i.e., that it is independent of any higher UV scales. With this refined definition, we also explicitly show that $\tilde{c}_0$ can be computed using a lattice regulator. Hence with this refined approach, our candidate c-function constructed with mutual information satisfies both the first and second criteria which required for a legitimate c-theorem. Of course, it will still remain to show that this new c-function actually decreases monotonically along RG flows, our third criterion. We will return to this point in section 4.

Let us add at this point that throughout our discussion distance scales, e.g., the radius of an entangling surface, appear as a proxy for the usual energy scales that would appear in a discussion of c-theorems oriented towards more conventional particle physics techniques, e.g., [2–4]. Of course, this replacement is one of the basic differences that appears in the construction of c-theorems using entanglement entropy, e.g., [32, 44]. The idea comes from the usual intuition that short (long) distances or wavelengths correspond to high (low) energies. In particular, the density matrix for a finite region, e.g., a disk, will be insensitive to modes with wavelengths much larger than the size of the region. While this can be seen in explicit calculations with e.g., free fields, holographic entanglement entropy [53, 54] is nice framework where one can gain some intuition for this feature. In particular, the extremal surface only probes the bulk spacetime to a certain depth that is determined by the geometry of the entangling surface in the boundary theory, e.g., see

appendix A. Let us add, however, that while the a density matrix, and the corresponding entanglement entropy, are sensitive to wavelengths matching the size of the region, they are also sensitive to all shorter wavelengths. As we will see below, eliminating this sensitivity to short wavelengths is the essential challenge in establishing our second criterion above, as well as the first, with an entanglement construction of a c-theorem.

2.1 Ambiguities in c_0

The F-theorem in three dimensions was originally proposed in terms of the constant c_0 appearing in the entanglement entropy (1.1) of a circle [11, 12]. At a conformal fixed point, we have

$$S(R) = 2\pi R\left(\frac{a}{\delta} + b\right) - 2\pi\,c_0\,, \tag{2.1}$$

where R is the radius of the circle and δ is a short distance cut-off. As noted above, with a covariant regulator, it can be shown that c_0 always decreases along RG flows [32]. However, without further information on the regularization procedure, this dimensionless constant is ambiguous.

The latter ambiguity is readily demonstrated with a square lattice regulator [55]. The results of an explicit calculation for a free scalar on a square lattice are shown in figure 1. In this figure, c_0 is determined by evaluating the entanglement entropy of a circle as a function of radius and then subtracting the term proportional to R in a linear fit to the data. According to eq. (2.1), the result should correspond to $-2\pi c_0$, however, rather than converging to a specific value, the result fluctuates apparently forever as a function of the circle radius. It is straightforward to understand the size of the fluctuations as follows: a typical error in the radius of the circle will be of order of the lattice spacing δ. Given the form of eq. (2.1), this error will then feed into c_0 with errors of order a. In the example shown here, from the coefficient of the linear fit, one finds $2\pi a \sim 0.5$ which then matches the approximate size of the noise in figure 1. We emphasize that this situation will *not* be improved by simply carrying out the lattice calculations with greater resolution. Attempting to describe a circle on a square lattice simply fails at the scale of the lattice spacing δ. That is, we simply cannot be sure what the precise geometry of the entangling surface is with a resolution of better than δ and so the length of this surface cannot be determined with an accuracy of better than δ. Hence, as described above, in a direct calculation of the entanglement entropy, the result for c_0 will always 'polluted' by UV data because of this intrinsic error. Therefore one must expect the fluctuations in figure 1 will never diminish even if the calculations were extended out to arbitrarily large values of R. Hence we must be able to find a better definition of c_0 if this constant represents a true physical quantity.

The above issue actually arises more broadly. Even if the underlying theory is already taken in the continuum limit, it is generally a subtle problem how to define c_0 unambiguously through eq. (2.1) because evaluating the entanglement entropy explicitly requires a cut-off. In particular, it may be that the latter sets a limit δ on the resolution of distances for the entire theory and so there is a lack of precision in defining R in eq. (2.1). Now again if we shift $R \to R + \alpha\,\delta$, c_0 at the IR fixed point gets mixed with a and hence is
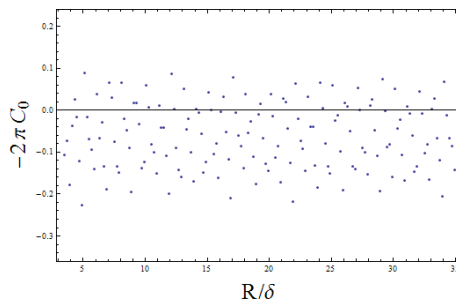
Figure 1. The entropy of the set of lattice points inside circles of radius R for a free scalar in a square lattice. The radius is taken from 3.5 to 35, with equal spacing of 0.2. The values of $-2\pi c_0$ shown are obtained by computing $S(R)$ and then subtracting the term proportional to R in a linear fit to the data.

polluted by contributions from much higher UV scales. We illustrate this issue further with calculations for a free scalar field in section 3.

In addition, gauge fields introduce further ambiguities in the entanglement entropy that spoil the universality of the constant term appearing in eq. (2.1). In particular, for theories with gauge fields, the entropy depends on the exact prescription for assigning a gauge-invariant operator algebra to a given region [76, 77]. Prescriptions that eliminate superselection sectors for the local algebras (in order to remove the classical Shannon contribution to the entanglement entropy arising from the probability distribution assigned to these sectors) are related to certain gauge fixings. Hence one finds an alarming gauge dependence in entanglement entropy.

Returning to eq. (2.1), let us consider the two constants, a and b, appearing in the area law term. Of course, the dimensionless constant a appearing as the coefficient of the linear divergence depends on the details of the UV regulator. The appearance of the finite constant b is a slightly more subtle issue. Recall that eq. (2.1) describes the entanglement entropy at a conformal fixed point. However, this expression also clearly requires that b have the dimensions of mass and so naïvely, it would seem that a finite contribution of this form simply cannot appear in eq. (2.1). However, we are assuming here that the fixed point theory may arise in the deep infrared of an RG flow from some UV fixed point. In such a scenario, the b term will typically appear with mass scales from this RG flow, e.g., [36, 56–58].[7] Hence, we wish to emphasize that even in the far infrared where the theory flows to a CFT, the 'area law' term in eq. (2.1) contains data external to the CFT, e.g., masses which appear at higher scales. So, interestingly this contribution to the entanglement entropy in the infrared is not an intrinsically infrared quantity but rather it depends on the whole running of the theory.

With an appropriate regulator, the fixed point theory will be well-defined in the continuum and implicitly, the meaning of distances between spacetime points is defined with infinite precision. For example, generally, a CFT can be thought of as defined by the spectrum of its operators and their correlation functions at finite separation. In such a frame-

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[7]Further explicit examples of this b term appear in our calculations with the extensive model in the appendix C. Let us also note that similar terms will appear in our regularization using the mutual information.

work, the above ambiguities can be avoided following the proposal of [34] — see also [40]. This approach implicitly assumes that the regulator respects the property $S(A) = S(\bar{A})$ for global pure states, where $\bar{A}$ is the region complementary to A. Further as discussed above, the regulator must be covariant or 'geometric.' As a result, all of the divergent terms in eq. (1.1) can be expressed as integrals of local geometrical quantities over the entangling surface. For the circle in three dimensions, the latter only includes the area law term but in higher dimensions, a rich array of such terms can appear [34–36, 58, 59]. Further for 'scalable' geometries, these geometric contributions can be eliminated to isolate the c_0 coefficient in eq. (1.1) by combining various derivatives of the entanglement entropy [34], i.e., for the circle in three dimensions, $c_0 = \frac{1}{2\pi} [R \, \partial_R S(R) - S(R)]$. This construction has the virtue that it removes the entire area law contribution and so eliminates the finite b term, as well as the divergent term proportional to a/δ.

As noted in the introduction, we feel the preceding approach is somewhat unsatisfactory. In particular, we should be able to identify c_0 as a physical quantity of the underlying quantum field theory. That is, it should be defined with any regularization scheme which defines the continuum theory. Without such a definition, one cannot be sure that this constant is not an artifact of the 'nice choice' of a geometric regulator. We believe that the mutual information provides precisely such a physical definition.

2.2 Mutual information

As commented in the introduction, we propose to define the appropriate c-function with mutual information. The mutual information $I(A, B)$ between two non-intersecting regions A and B is given by

$$I(A, B) = S(A) + S(B) - S(A \cup B). \tag{2.2}$$

Let us reiterate that mutual information can be defined without any reference to entanglement entropy [50, 51]. However, we only consider the above definition since it is better suited for our calculations. The advantage of working with this quantity is that the mutual information is a well-defined finite quantity. In particular, it is independent of any regularization scheme used to define the underlying QFT. In eq. (2.2), the divergences in $S(A)$ and $S(B)$ are canceled by those in $S(A \cup B)$. Moreover, mutual information in the continuum theory does not suffer any ambiguities in gauge theories [76, 77]. Hence in defining the c-function with mutual information, we will have automatically satisfied the first criterion which we listed in the introduction to establish a legitimate c-theorem.

Now given a disk A of radius R, the mutual information can be used to construct a 'regulated' form of the entanglement entropy (2.1) as follows:[8] First we replace the original entangling surface by two new (concentric) circles — see figure 2. The first has a slightly smaller radius R_- enclosing a smaller disk A^- and the second with a slightly larger radius R_+ which is taken as the boundary of A^+, the exterior of this circle. The quantity of interest is then the mutual information between A^+ and A^-, i.e.,

$$I(A^+, A^-) = S(A^+) + S(A^-) - S(A^+ \cup A^-). \tag{2.3}$$

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[8]A similar strategy for regulating calculations of entanglement entropy was recently discussed in [52].
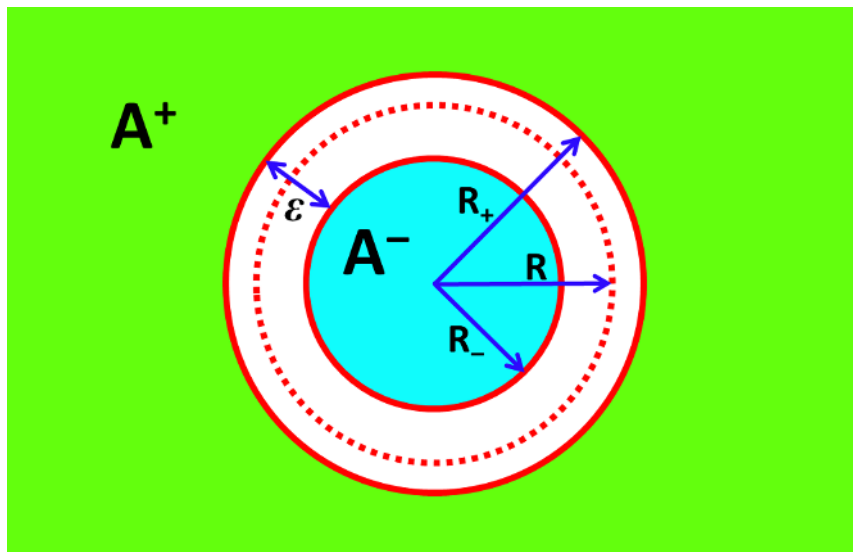
Figure 2. We compute the mutual information between the interior of the circle of radius R_- and the exterior of a concentric circle of radius R_+ with $R_+ > R_-$.

Let us denote the separation of the two radii as $\varepsilon \equiv R_+ - R_-$. We will always be thinking of the two radii and their separation as macroscopic scales much larger than any UV cut-off implicit in defining the underlying quantum field theory, i.e., $R_+, R_-, \varepsilon \gg \delta$.

In the following, we focus on evaluating this mutual information in the groundstate of the underlying quantum field theory. Hence given that we are working with a pure quantum state, we can also expect that the entanglement entropies of a region and its complement will be the same, i.e., $S(A) = S(\bar{A})$. Hence in eq. (2.3), $S(A^+)$ can replaced by $S(\bar{A}^+)$, the entanglement entropy of the disk enclosed by the second circle of radius R_+. Similarly, $S(A^+ \cup A^-)$ can be replaced by $S(\overline{A^+ \cup A^-})$, the entanglement entropy of the annular strip between the two circles. Then the mutual information (2.3) becomes

$$I(A^+, A^-) = S_{\mathrm{disk}}(A^-) + S_{\mathrm{disk}}(\bar{A}^+) - S_{\mathrm{strip}}(\overline{A^+ \cup A^-}). \tag{2.4}$$

However, we must emphasize that in general, care must be taken with these substitutions, as will be discussed in section 3.

Now we wish to consider the mutual information $I(A^+, A^-)$ when the separation of the circles is small, i.e., in the limit $\varepsilon/R \to 0$.[9] However, we must first designate the precise relation between the original radius R and the two new radii $R_\pm$:

$$R_- = R - \left(\frac{1}{2} - \alpha\right)\varepsilon, \qquad R_+ = R + \left(\frac{1}{2} + \alpha\right)\varepsilon. \tag{2.5}$$

Alternatively, we have

$$R = \frac{R_+ + R_-}{2} - \alpha\,\varepsilon. \tag{2.6}$$

We allow the dimensionless parameter α to take some value in $-1/2 \leq \alpha \leq 1/2$ so that the original radius lies in the range $R_- \leq R \leq R_+$. However, ultimately we will fix $\alpha = 0$ below.

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[9]There is an implicit order of limits here which still maintains $\varepsilon \gg \delta$.

If we take the limit $\varepsilon \to 0$ at a conformal fixed point, the mutual information $I(A^+, A^-)$ will diverge in a form reminiscent of eq. (2.1) with ε playing the role of the cut-off δ. That is, we will have an expansion of the form

$$I(A^+, A^-) = 2\pi R \left(\frac{\tilde{a}}{\varepsilon} + \tilde{b} \right) - 4\pi \tilde{c}_0 + O(\varepsilon) \,, \qquad (2.7)$$

where $\tilde{a}$, $\tilde{b}$ and $\tilde{c}_0$ are three (new) constants. Heuristically, our construction is motivated by the intuition that in the $\varepsilon \to 0$ limit, this mutual information yields twice the entanglement entropy of a single circle, i.e., $I(A^+, A^-) \simeq 2S(A)$ [45–47]. This expectation suggests that we will recover the same constants as in eq. (2.1) in this limit, however, since $I(A^+, A^-)$ is a well defined quantity, the coefficients, $\tilde{a}$, $\tilde{b}$ and $\tilde{c}_0$, can all be regarded as physical quantities. That is, all of these coefficients are independent of the regularization scheme. In particular, this intuition guides us in defining the constant term in eq. (2.7) with an additional factor of 2 compared to eq. (2.1). However, we must add that generally this naïve intuition is spoiled by divergences in QFT. Nonetheless, we will argue in section 3 that $\tilde{c}_0 = c_0$ under the appropriate conditions.

Our proposal will be to construct a c-function, using the mutual information for the geometry above, which yields the constant $\tilde{c}_0$ at fixed points. In particular, in analogy to the discussion of the entropic c-function [32, 34], we may define

$$C(R) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \left(R \frac{\partial I(A^+, A^-)}{\partial R} - I(A^+, A^-) \right) \qquad (2.8)$$

so that $C(R) = \tilde{c}_0$ for fixed points of the RG flows. In fact, a slightly refined version (4.20) of this expression will appear in proving the F-theorem in section 4.

Our previous discussion of the entanglement entropy (2.1) provides a useful intuitive understanding of the nature of the parameters, $\tilde{a}$, $\tilde{b}$ and $\tilde{c}_0$, in eq. (2.7). Heuristically, we can say $\tilde{a}$ contains information about physics at high scales of order $1/\varepsilon$ while $\tilde{b}$ tells us about the RG flow down to the infrared scale $1/R$.[10] Finally, we can expect that the constant $\tilde{c}_0$ characterizes the fixed point theory.

However, one might observe that our definition of the constant $\tilde{c}_0$ remains fairly imprecise. In particular, while $\tilde{c}_0$ is a well-defined regulator-independent quantity, it depends on our choice for the intermediate radius R, or equivalently on α. If we change $\alpha \to \alpha'$, the expansion (2.7) becomes

$$I(A^+, A^-) = 2\pi R' \left(\frac{\tilde{a}}{\varepsilon} + \tilde{b} \right) - 4\pi \tilde{c}_0' + O(\varepsilon) \,, \qquad (2.9)$$

with

$$\tilde{c}_0' = \tilde{c}_0 + \frac{\alpha - \alpha'}{2} \tilde{a} \,. \qquad (2.10)$$

Hence depending on our choice of α, the value of $\tilde{c}_0$ is contaminated by the coefficient $\tilde{a}$, which depends on the details of the underlying QFT at high scales. That is, for generic

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[10]Let us note here that in principle if the RG flow is driven by operators with sufficiently large anomalous dimensions, one may find additional 'area law' terms with diverge in the limit of vanishing ε — see further discussion in appendix A.1.

values of α, $\tilde{c}_0$ does not correspond to an intrinsic quantity of the IR theory alone and so further attention is required in choosing α if a legitimate c-theorem is to be defined here. We will address this question in the next section. However, let us observe that if the constant term in the mutual information is decreasing from UV to IR for some choice of α, it is also decreasing for any other α since $\tilde{a}$ and hence the difference in eq. (2.10) remains constant. Finally, we note that the choice of α is creating an ambiguity here which is similar to that discussed for the entanglement entropy in the previous section, where the resolution in determining R was limited by the UV regulator.

Strategy: let us take a moment to outline our overall strategy in using the mutual information to examine RG flows. As noted above, the mutual information is a finite regulator-independent quantity and in our geometric construction above, R and ε are both macroscopic distance scales much larger that the UV cut-off. In other words, we are calculating the mutual information $I(A^+, A^-)$ for the renormalized quantum field theory in its continuum limit. Now we wish to examine an RG flow between two fixed points in this continuum theory and in such a situation, there must be some mass scales m_i which are introduced in perturbing the UV fixed point to generate the flow. We will choose $1/\varepsilon$ to be much larger than the largest of these scales, i.e., $\varepsilon \ll 1/m_i$. While ε is held fixed, the radius R is our probe scale and we will vary the radius from $R \ll 1/m_i$ (still with $\varepsilon \ll R$) where the mutual information is (only) probing the correlations in the UV CFT to $R \gg 1/m_i$ where the IR physics of the second fixed point is being probed. In particular then, we expect to find

$$\varepsilon \ll R \ll 1/m_i : \qquad I(A^+, A^-) = 2\pi R \frac{\tilde{a}}{\varepsilon} - 4\pi \, \tilde{c}_0^{UV} + O(\varepsilon) \,, \qquad (2.11)$$

$$\varepsilon \ll 1/m_i \ll R : \qquad I(A^+, A^-) = 2\pi R \left(\frac{\tilde{a}}{\varepsilon} + \tilde{b} \right) - 4\pi \, \tilde{c}_0^{IR} + O(\varepsilon) \,. \qquad (2.12)$$

As we will discuss below, the area term appears with the same coefficient $\tilde{a}$ in both expressions since this is determined by the UV physics at the scale $1/\varepsilon$. However, the RG flow will generally also induce a finite contribution proportional to the area, i.e., $2\pi R \tilde{b}$, which appears in the mutual information at IR scales. Of course, the central object in our analysis will be $\tilde{c}_0$, which is fixed-point value of the c-function (2.8) in our approach.

2.3 UV independence of $\tilde{c}_0$

As described above, the mutual information (2.7) provides a regulator-independent means of defining a candidate c-function (2.8) and so this approach automatically satisfies our first criterion for a proper c-theorem. However, at an RG fixed point, the c-function yields the parameter $\tilde{c}_0$, which fails to satisfy our second prerequisite for general α because it depends on physics at higher UV scales. Hence the question is if for some specific α, $\tilde{c}_0$ is a quantity which is intrinsic to the fixed point of interest.

Let us consider evaluating the mutual information $I(A^+, A^-)$ at some RG fixed point. As prescribed above, the geometry is constructed with $R \gg \varepsilon$. However, we also take the radius to be much larger than the distance scale set by the lowest mass m appearing in the definition of the fixed point, i.e., $R \gg L_m \sim 1/m$. As we are interested in the limit
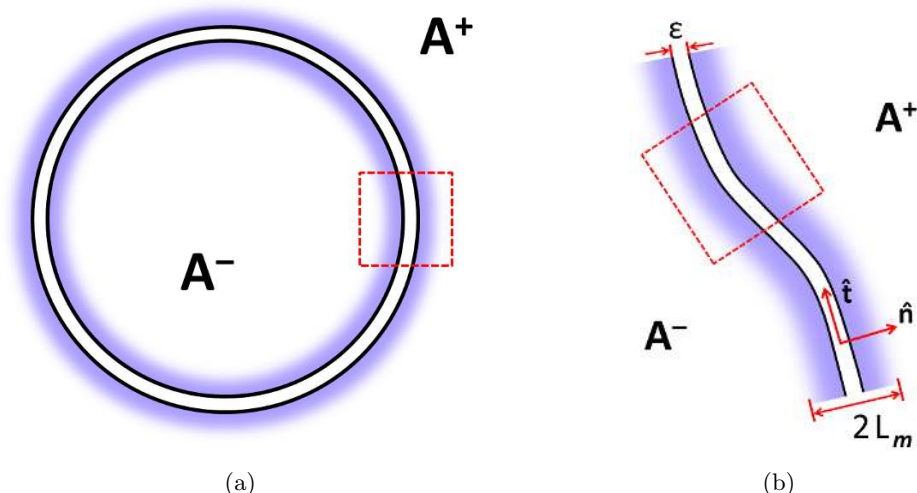
Figure 3. In both figures, the shaded areas represent regions on both sides of the boundaries where the correlations between A^+ and A^- contributing to the mutual information are sensitive to higher UV scales. As indicated in panel (b), this band has a width of roughly $2L_m$ in the regime where $\varepsilon \ll L_m$. These (nonconformal) correlations give local contributions calculable from a small region, such as that enclosed by the dashed (red) box. In panel (a), these local regions tend to become flat in the limit of large R and so the local factor can be written as a series in inverse powers of R, as in eq. (2.13). Panel (b) illustrates the more general case where the local factor can be expressed in terms of local geometric quantities in terms of the unit normal and tangent vectors, $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$, as in eq. (2.14).

$\varepsilon \to 0$, we also suppose that $\varepsilon \ll L_m$. Now in this regime, the region where the correlations contributing to the mutual information are sensitive to m is restricted to a narrow band near the annulus separating A^+ and A^-, as shown in figure 3. Beyond this region, the correlations between A^+ and A^- are the purely conformal ones of the fixed point theory. Hence in the regime where $\varepsilon \ll L_m \ll R$, the contribution to the mutual information, which is sensitive to m and higher scales, will be local and extensive on the annulus. That is, this high energy contribution will take the form

$$I(A^+, A^-)_{HE} = 2\pi R\left(\sigma_0 + \frac{\sigma_1}{R} + \frac{\sigma_2}{R^2} + \cdots\right) \tag{2.13}$$

where the factor in parentheses is some local contribution calculable from any small region, such as that enclosed by the dashed box in figure 3. If we increase the radius R while keeping the size of the box fixed, this region grows flatter and eventually becomes a straight strip of width ε in the limit $R \to \infty$. This is why the local factor can be arranged in the series in inverse powers of R shown in eq. (2.13). The first term σ_0 in this series corresponds to the local contribution for the straight strip. Now we see that if σ_1 is nonvanishing, the second term in the local factor produces a constant contribution in $I(A^+, A^-)_{HE}$ which is independent of R. This constant $2\pi\sigma_1$ would then be included as a part of $-4\pi\tilde{c}_0$ in eq. (2.7) and hence our candidate c-function (2.8) would be sensitive to higher UV scales. Therefore our objective is to find a choice of α in eq. (2.5) which ensures that σ_1 vanishes.

To proceed, let us think about more general situations where the two boundaries defining the 'interior' region A^- and the 'exterior' region A^+ are two parallel curves defining some curved strip of width ε. The only restriction[11] on the strip geometry will be that everywhere along the curves, the scale of their curvatures is much larger than both ε and L_m. Now as above in this regime, the high energy contribution to the mutual information will be local and extensive on this narrow ribbon. Hence, we will be able to write $I(A^+, A^-)_{HE}$ here in terms of local geometric quantities integrated along the strip. In particular, we can write

$$I(A^+, A^-)_{HE} = \int ds \left(\sigma_0 - \sigma_1 \, \hat{\mathbf{n}} \cdot \partial_s \hat{\mathbf{t}} - \sigma_2 \, \hat{\mathbf{t}} \cdot \partial_s^2 \hat{\mathbf{t}} + \cdots \right) , \qquad (2.14)$$

where $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ are the unit normal and unit tangent vectors, respectively, while s is the coordinate running along the length of the strip.[12] In the case of a circular strip of radius R, this expression reduces to eq. (2.13). Again, σ_0 corresponds to the local contribution for a straight strip of width ε, while σ_1 gives the problematic contribution which would make $\tilde{c}_0$ sensitive to higher UV scales in our construction above.

At this point, we turn to the choice of α and in particular, we choose $\alpha = 0$. The latter places the curve along which the integral is performed precisely half way between A^- and A^+. Hence this choice is distinguished because it establishes a symmetry between A^- and A^+ where $I_{HE}(A^+, A^-) = I_{HE}(A^-, A^+)$ and this symmetry will then be reflected in the expansion in eq. (2.14). Specifically, it requires that the local contribution must be unchanged when the direction of the normal vector $\hat{\mathbf{n}}$ is reversed, i.e., under the replacement $\hat{\mathbf{n}} \rightarrow -\hat{\mathbf{n}}$. Therefore, with the choice $\alpha = 0$, the term proportional to $\hat{\mathbf{n}} \cdot \partial_s \hat{\mathbf{t}}$ must be absent in eq. (2.14) and there is no constant contribution in $I(A^+, A^-)_{HE}$. Hence $\tilde{c}_0$ is protected from the details of the UV physics with $\alpha = 0$. Let us add that in the next section, we will show $\tilde{c}_0 = c_0$, i.e., the constant terms agree using the mutual information and the entanglement entropy (with a covariant regulator). Now we have verified these results with various explicit calculations, however, before describing those, it is useful to examine eq. (2.14) in more detail.

First, recall that σ_0 corresponds to the local contribution for a straight strip of width ε. Further, we observe that the term proportional to σ_0 in eq. (2.14) will make a contribution proportional to $2\pi R$ in the mutual information and in particular, this term will produce the contribution proportional to $\tilde{a}\,R/\varepsilon$ in eq. (2.7). Now considering eq. (1.2) for the mutual information, we see that the latter contribution must come from $S(A^+ \cup A^-)$ since the entanglement entropies for the individual circular boundaries will be independent of ε. Therefore combining these observations, we may identify the coefficient $\tilde{a}$ as that appearing in the universal contribution to the entanglement entropy of a straight strip. That is, in a three-dimensional quantum field theory, the latter takes the form, e.g., see [35, 55]:

$$S_{strip} = c_1 \frac{L}{\delta} - \tilde{a} \frac{L}{\varepsilon} + \cdots \qquad (2.15)$$

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[11]We are also assuming that the strip is not self-intersecting.

[12]A variety of geometric identities have been used to simplify eq. (2.14), e.g., one can show that $\hat{\mathbf{n}} \cdot \partial_s^2 \hat{\mathbf{t}}$ is a total derivative. The expression above contains all of the distinct geometric terms up to second order in s derivatives and discarding total derivatives.

for a straight strip of length L and width ε with $L \gg \varepsilon$ and where the ellipsis denotes terms that vanish in the limit that $\delta, \varepsilon \to 0$.[13] The universal coefficient $\tilde{a}$ is known for free scalar and fermion fields [55], and has been studied holographically [35]. In the present context, it is clear that this universal term is determined by the physics of high scales of order $1/\varepsilon$ and hence in RG flows, $\tilde{a}$ depends entirely on the UV fixed point.

To confirm the previous arguments, we have computed mutual information in a holographic framework in appendix A. As expected, we find that only the case $\alpha = 0$ gives place to a constant term which is independent of UV terms and in fact, the constant term appearing in the mutual information is precisely twice that for a circle. That is, comparing eqs. (2.1) and (2.7), we have $\tilde{c}_0 = c_0$ in the holographic model. For $\alpha \neq 0$, the constant term is contaminated with the linear coefficient $\tilde{a}$, which we noted above is equivalent to the coefficient in the entanglement entropy of a straight strip. We also computed the mutual information in a model with extensive mutual information in appendix C. In this example, we can follow explicitly the renormalization group flow of the constant term and check that it decreases as R increases and that in the infrared, it is independent of the details of the RG flow provided $\alpha = 0$.

Finally, this new approach should allow us to compute $\tilde{c}_0$ using a lattice regulator. Figure 4 shows the results of a preliminary calculation for a free massless scalar using a (square) lattice regulator. In particular, figure 4(a) shows the mutual information $I(A, B)$ as a function of the radius in lattice units R/δ. Here we have set $\alpha = 0$ and so $R = (R_+ + R_-)/2$. We approach the continuum limit by increasing R in terms of the lattice spacing, while holding the ratio $\varepsilon/R = 2/11$ fixed (or equivalently $R_+/R_- = 6/5$ held fixed). As shown in the figure, the data does indeed seem to converge quite well. As in the entanglement entropy calculation shown in figure 1, the noise arises because the circumference of the entangling surfaces is uncertain at a resolution of the order of the lattice spacing. That is, there is an uncertainty of order the lattice spacing δ in the radius R in the formula (2.7) for the mutual information. Hence we can expect fluctuations of the order of $2\pi\tilde{a}\,\delta/\varepsilon$ but these will vanish in the continuum limit where $\delta/\varepsilon \to 0$. Recall from eq. (2.15) that $\tilde{a}$ appears as the coefficient of the universal term in the entanglement entropy for a strip. Using this connection, the value of this coefficient can be computed analytically for a massless scalar [55]:[14] $\tilde{a} \simeq 0.0397$. Combining this value with the ratio $\varepsilon/R = 2/11$, our estimate for the expected fluctuations becomes $1.37\,\delta/R$, which seems to match quite well with the order of magnitude of the noise in the figure 4.

In figure 4(b), we plot $I_0(A) = I(A^+, A^-) - 2\pi R\,\tilde{a}/\varepsilon$ as a function of R/δ. Comparing with eq. (2.7), we see that this quantity should converge to the universal constant term $-4\pi\tilde{c}_0$ in the limit $\varepsilon/R \to 0$.[15] Averaging over the values in the figure for $R/\delta \in (25, 33)$, our lattice calculations yield an approximate value of $4\pi\tilde{c}_0 \simeq 0.110$. Comparing this result with the exact value [23, 60],

$$4\pi c_0^{scalar} \equiv \frac{1}{4}\left(\log 2 - \frac{3\zeta(3)}{2\pi^2}\right) \simeq 0.12761410956\,, \tag{2.16}$$

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[13]Additional 'area law' terms, which diverge in this limit, may appear in eq. (2.15) if the RG flow is driven by nearly marginal operators — see appendix A.1.

[14]It can also be obtained efficiently by directly evaluating the strip entropy on the square lattice.

[15]Note that $\tilde{b} = 0$ in the present case since we are not considering an RG flow.
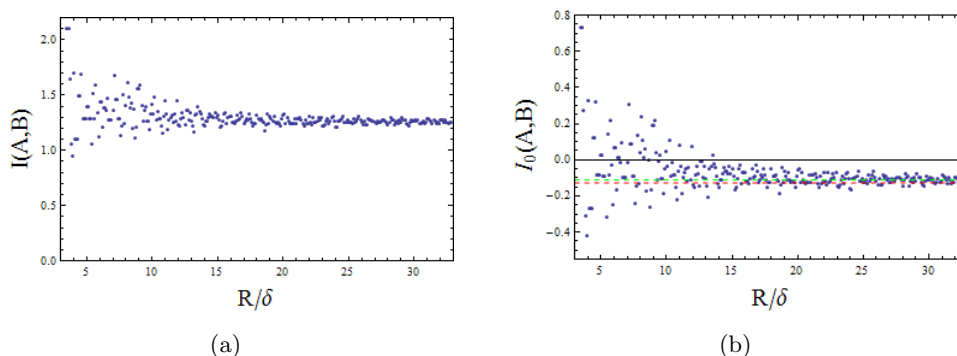
Figure 4. Panel (a) shows the mutual information $I(A^+, A^-)$ calculated on a square lattice for a free scalar. The mutual information is plotted as a function of R/δ while holding the ratio ε/R fixed with $\varepsilon/R = 2/11$. Panel (b) shows the same data as in panel (a) but now on the vertical axis, we have plotted the quantity $I_0(A) = I(A^+, A^-) - 2\pi R\,\tilde{a}/\varepsilon$. For the limit of small ε/R, this quantity should converge to the universal value of the constant term $-4\pi\tilde{c}_0$. The green dashed line shows the approximate value of $-4\pi\tilde{c}_0$ obtained by averaging over the values for $R \in (25, 33)$. The exact value shown with the red dashed line is slightly smaller.

we find that our preliminary lattice calculation has already produced a good approximation. In fact, the difference of roughly 15% between the exact and lattice values is as good as we could expect since the lattice calculation is limited to $\varepsilon/R = 2/11 \simeq 0.18$, as well as $R/\delta \approx 30$. Of course, the lattice result can be improved by making ε/R smaller which in turn, requires considering circles of larger radii.

Recall that the choice of $\alpha = 0$ was essential to protect the constant term in the mutual information from the details of the UV physics. That is, with a fixed large R and $\alpha = 0$, changing the UV behavior in an RG flow to a given IR conformal fixed point will result in a renormalization of the 'area' term in eq. (2.7) but the constant term remains unchanged. Notice with eq. (2.14), we have framed our argument in a very general context and so this property also holds for the mutual information across any smooth shape[16] as we scale the size of the region to infinity. The key point is that we must adopt the prescription of evaluating the area along the curve which is precisely half way between the boundaries of A^+ and A^-. However, for general shapes other than a circle, it is not expected that the constant term should decrease monotonically as the size of the region increases.

Finally, we note that our general argument for the UV independence of $\tilde{c}_0$ is very similar to those made for extracting a universal number for the constant term of the circle entropy [34]. However, those arguments require some specific properties for the regularization used in evaluating the entanglement entropy. In particular, as noted above, the regulator must be covariant which ensures that the UV contributions to the entropy have a geometric expansion analogous to that shown in eq. (2.14). Further, the regulator must respect the symmetry $S(A) = S(\bar{A})$, where $\bar{A}$ is the complement of A. The latter again requires that the entanglement entropy will be invariant under the replacement $\hat{\mathbf{n}} \to -\hat{\mathbf{n}}$. This invariance then ensures that there can not be a term linear in the curvature of the

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[16]Similarly, it also extends to smooth shapes in any odd number of dimensions.

entangling surface, i.e., the second term in the expansion (2.14) must again vanish, and hence the constant term is protected from UV contributions.

At this point, we have established that the mutual information for the geometry illustrated in figure 2 can be used to construct a candidate 'central charge' $\tilde{c}_0$ for three-dimensional conformal fixed points. In particular, this new charge satisfies the first two requirements which we set out in the introduction as necessary to properly establish a c-theorem. Of course, it remains to be shown that this quantity can be extended to a c-function $C(R)$ satisfying $C(R) = \tilde{c}_0$ at conformal fixed points and also satisfying the third criterion, i.e., that the c-function actually decreases monotonically along RG flows. In the next section, we will argue that generally we expect $\tilde{c}_0 = c_0$, i.e., our new universal coefficient will coincide with that extracted from the entanglement entropy (using a covariant regulator). The latter is already very suggestive with regards to the monotonic flow of $C(R)$ but we will return to this point in section 4.

3 Matching mutual information and entanglement entropy

In the previous section, we focused on the expansion (2.7) of the mutual information at a conformal fixed point. In particular, we showed that with $\alpha = 0$, i.e., with $R = (R_+ + R_-)/2$, the constant term is free of contamination from high energy scales. In this section, we discuss under which conditions the constant $\tilde{c}_0$ appearing there will equal c_0 in eq. (2.1), the constant appearing in the analogous expansion of the entanglement entropy of a circle.

Recall that the purity of the vacuum state yields the identity $S(A) = S(\bar{A})$ for the entanglement entropies of a region A and its complement $\bar{A}$.[17] This identity then allows us to rewrite the original expression (2.3) for the mutual information $I(A^+, A^-)$ in terms of the entanglement entropies of two disks and of a thin annular strip, as in eq. (2.4). Now the assertion above is that under suitable conditions, the constant term $4\pi\tilde{c}_0$ appearing in eq. (2.7) is equivalent to twice $2\pi c_0$ appearing in eq. (2.1) for a single disk. Given eq. (2.4), this match can only hold if the constant term in the entropy of the annular strip vanishes. Hence next we turn to understanding when the latter constant will vanish, e.g., what conditions are required for the regularization.

However, first let us remark that much of the following analysis still applies without applying $S(A) = S(\bar{A})$. Hence while the discussion is facilitated by considering the entanglement entropy of the annular strip, the same analysis establishes that the constant term vanishes in the entropy of the complementary region, $A^+ \cup A^-$. The robust result is then that if the constant term vanishes in the entropy of $A^+ \cup A^-$, then the constant term in the mutual information is given by the sum of those in $S_-(R)$ and $S_+(R)$, the entropies of the interior and exterior of a circle of radius R, respectively. Since our discussion is implicitly considering a conformal fixed point, the radius R of the circle in either of these entropies can be set freely.

The entanglement entropy of the narrow annular strip is a purely UV contribution, i.e., it probes correlations at the (length) scale ε. This is because contributions of the

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[17]However, as we show below, the regularization can spoil this equality for subleading terms.

correlation functions across distances much larger than the width ε cancel due to the opposite contributions coming from the two sides of the strip. This cancellation becomes more transparent when thinking in terms of the Rényi entropies. For the Rényi entropies, there are two twist operators (e.g., see [61]) running along each of the boundaries of the strip but with opposite orientations. That is, opposite 'twists' will be induced in a local field operator if it is carried in the same direction around these line operators of either edge of the strip. Because of their opposite orientations, the connected correlators of the twist operators with a distant operator should go to zero with ε. In other words, we may think that the first term in the effective OPE (e.g., see [62]) of the parallel twist operators will be proportional to the identity operator.

The example of a free scalar field is useful to illustrate this cancellation. For a free scalar, the Renyi entropies for a region V can be expressed as [55],

$$S_n(V) = \frac{1}{1-n} \sum_{k=1}^{n-1} \log Z\left[e^{2\pi i \frac{k}{n}} \right] \tag{3.1}$$

where $Z[e^{ia}]$ is the partition function where a phase e^{ia} is imposed as a boundary condition on the field in approaching V from above and below. This boundary condition can be enforced by coupling a (complex) scalar field to an external gauge field A_μ, which is pure gauge everywhere except at the boundary of V. To be concrete consider an infinite straight strip running along the direction $\hat{\mathbf{t}}$ and with the edges separated by $\vec{\varepsilon}$. One choice of the gauge field is

$$A_\mu(x) = a\, \varepsilon_{\mu\nu\sigma}\, \hat{t}^\nu \left(\frac{x^\sigma}{|x|^2} - \frac{(x+\varepsilon)^\sigma}{|x+\varepsilon|^2} \right) . \tag{3.2}$$

Hence the magnitude of this gauge field vanishes as $|A| \sim \varepsilon/|x|^2$ for small ε and fixed x, and the effect of the two boundaries on distant regions vanish at least linearly in ε.

The UV character of the entanglement entropy of the thin annular strip is also evident in holographic models. As discussed in appendix A, the minimal surface determining the holographic entanglement entropy in this case is an annular region in the bulk connecting the boundaries of the strip and it resides entirely near the AdS boundary — see figure 10. A similar cancellation of large distance contributions from the two edges of the thin strip is also seen in the extensive mutual information model considered in appendix C. The contributions of the two boundaries have different signs due to the factor $\hat{\mathbf{t}}_x \cdot \hat{\mathbf{t}}_y$ in eq. (C.24), where $\hat{\mathbf{t}}_x$ and $\hat{\mathbf{t}}_y$ are the tangent vectors of the two boundaries.

Because of the UV character of the entanglement entropy of a narrow strip, it is natural to think that it should be local and extensive, i.e., it can be written as an integral of local quantities along the length of the strip. One can argue that deviations from extensivity will go to zero with some positive power of the width ε as follows: consider the mutual information between small portions of the strip with a linear size ε and which are separated by some distance $|x|$ along the strip. By the definition (1.2), the mutual information measures precisely the degree of nonextensivity of the entropy between two regions. This mutual information should go to zero with a positive power of $\varepsilon/|x|$ due to clustering of correlation functions and conformal invariance. This is known to hold for

free fields [55] and was recently proved for distant spherical regions in any conformal field theory (in any number of dimensions) [49]. In these cases, mutual information between distant portions on the strip would decays as $(\varepsilon/|x|)^\sigma$ where the power satisfies $4\Delta \leq \sigma \leq 6$ where Δ is the lowest dimension in the spectrum of CFT (i.e., $\Delta = 1/2$ for the conformal scalar in three dimensions) or $\sigma = 6$ if $\Delta > 3/2$.

As noted above, the local and extensive character of the entropy of the annular strip means it is given by an integral of local objects along the length of the strip. Therefore the same argument given in section 2.3 for the vanishing of the constant term in high energy contribution to the mutual information serves here to show the vanishing of the constant term in the entanglement entropy of the strip. The only complication is that rather than being purely geometric, the local quantities appearing in the integral are introduced by the UV regulator.[18] According to the discussion in section 2.3, a minimal requirement to ensure that the regulator does not spoil our argument is that the regularization respect the symmetry interchanging A^+ and A^-, which inverts the curvature of the strip. This is achieved, for example, if the divergences of the two edges of the strip are regulated with the same geometric prescription because the curvatures of the two boundaries have opposite signs (taking the normal to the boundary pointing outwards from the strip). In contrast, this symmetry is not respected with a square lattice since the details of the regularization depend strongly on the precise position of the boundaries. With the symmetry requirement, the annular strip's entropy is given by an expansion as in eq. (2.14) and the parity argument eliminates the term proportional to the curvature of the strip, i.e., $\hat{\mathbf{n}} \cdot \partial_s \hat{\mathbf{t}}$. As a consequence, there is no constant contribution independent of the radius R and from eq. (2.4), the constant term in the mutual information coincides with twice that in the entanglement entropy of a disk — or the sum of the constant terms in $S_+(R)$ and $S_-(R)$.

The vanishing of the constant term in the entanglement entropy of the annular strip was verified for holographic entanglement entropy in appendix A. Below we confirm our arguments for a free scalar field with heat kernel and lattice regularizations.

However, before proceeding, we must say that there is one situation which seems to evade these arguments, namely topologically ordered phases. Here we are referring to phases which despite being gapped still have some kind of long range entanglement, e.g., see [63]. In this situation, the edges of the strip decouple but the entanglement entropy of the annulus will still have a topological contribution which is constant, i.e., independent of R. This contribution is clearly nonlocal and depends on the organization of entanglement all along the closed edges of the annular strip. However, we expect that when ε is smaller than the correlation length of the topological sector, the annulus contribution becomes local and as a result, the corresponding constant term vanishes. We return to this point in section 5.

3.1 Scalar field

In a conformal field theory, the constant term in the entanglement entropy (2.1) across a circle is equal to the constant term in the free energy of a three-dimensional Euclidean

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[18]This complication does not arise for the mutual information since the latter is independent of the regulator.

sphere [13],[19] i.e., $F = -\log Z[S^3]$. This provides a relatively simple approach to evaluate c_0 for free fields. In this case, one of the best methods to calculate the free energy is using zeta-function regularization, which eliminates all of the divergent terms and leaves simply the constant term. We already quoted the result for a conformally coupled free scalar [23, 60] in eq. (2.16) when comparing to the results from our lattice calculations using the mutual information approach. Let us add that this number coincides with that found in [30] using a slightly different calculation. Using a radial lattice, the same value for c_0^{scalar} was also obtained numerically from the circle entropy in [34] — see also [66]. However, we note that in this numerical calculation, the authors had to choose a particular definition $R = (n + 1/2)\, \delta$ of the circle radius in terms of the lattice spacing δ to reproduce the 'correct' value of c_0^{scalar}. The prescription $R = (n+1/2)\, \delta$ also gives the correct constant for a free fermion [67]. We discuss the reason for the success of this choice in the radial lattice below.

We now show that the constant in the mutual information (2.7) coincides with $4\pi c_0^{scalar}$ for a free scalar, as expected. According to previous discussion, this agreement will follow provided we impose the same cut-off for both, $S_+(R)$ and $S_-(R)$. Of course, the same cut-off, which is imposed to regulate the divergent contributions of the two circles, must also be used for the inner and outer boundaries of the strip.

In appendix D, we review the calculation of the entanglement entropy for the circle by conformally mapping the circle causal domain to the space $H^2 \times R$, where H^2 is the two-dimensional hyperbolic space [13]. The calculation reduces to the integration of the local entropy density at a fixed temperature over the hyperbolic space. One must then regulate the volume of H^2 to get a finite entropy. This can be done, for example, by integrating the entropy density in H^2 over a domain which corresponds to the region $|x| < R - \delta$ on the disk enclosed by the original circle, where δ is the short-distance cut-off in the CFT. This calculation yields

$$c_0 = \frac{3}{2}\, c_0^{scalar} \tag{3.3}$$

and hence the result seems to be off by a factor of $3/2$. We might generalize this scheme by choosing the domain in H^2 to correspond to the disk extending out to $r_{\max} = R - \hat{\delta}$ where

$$\hat{\delta} = \delta\, f(\delta/R) \quad \text{with} \ \ f(\delta/R) = f(0) + f'(0)\frac{\delta}{R} + f''(0)\frac{\delta^2}{R^2} + \cdots . \tag{3.4}$$

We are assuming that $f(\delta/R)$ has a well behaved Taylor expansion around $\delta/R = 0$. In this case, if we express the resulting entanglement entropy with an expansion in terms of δ/R, the constant becomes

$$c_0 = \left(\frac{3}{2} + \frac{f'(0)}{f(0)} \right) c_0^{scalar} . \tag{3.5}$$

This result is a manifestation of the regulator ambiguities in the constant term in the entropy.

Above we noted that a robust result would be that that if the constant term vanishes in the entropy of the strip, then the constant term in the mutual information is given by

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[19]Similar regularization ambiguities arise in identifying the universal coefficient using the free energy of a d-sphere in odd dimensions.

the sum of those in $S_-(R)$ and $S_+(R)$, the entropies of the interior and exterior of a circle of radius R. Further, we are guaranteed that the constant contribution vanishes for the annular strip if we use the same geometric regulator at both boundaries. Now as above, the calculation of the entanglement entropy of this exterior region can again be related to a calculation of thermal entropy in the same space $H^2 \times R$ [62, 68]. In addition, the calculation of $S_-(R)$ was regulated by choosing a region $|x| = R - \delta$ and so we must impose the same short-distance cut-off δ in calculating $S_+(R)$. That is, we should again only consider contributions up to a distance δ from the boundary and so we consider the region $|x| \geq R + \delta$ in calculating $S_+(R)$. While the latter region can be mapped directly to a domain on the hyperbolic plane, it is more informative to first map this region to the interior of the circle using the inversion

$$\hat{x}^\mu = \frac{R^2}{|x|^2}\, x^\mu \,. \tag{3.6}$$

With this conformal transformation the cut-off δ for the exterior region is mapped to the cut-off

$$\hat{\delta} = R - \frac{R^2}{R+\delta} = \delta \left(1 - \frac{\delta}{R} + \frac{\delta^2}{R^2} + \cdots \right) \tag{3.7}$$

for the inner disk. Comparing the above with eq. (3.4), we find with eq. (3.5) that the constant term for $S_+(R)$ is

$$c_0 = \frac{1}{2}\, c_0^{scalar} \,. \tag{3.8}$$

Now the constant appearing in the mutual information (2.7) is just the sum of those appearing in eqs. (3.3) and (3.8) and so we find

$$\tilde{c}_0 = \frac{c_0^+ + c_0^-}{2} = c_0^{scalar} \,. \tag{3.9}$$

Hence despite the ambiguities in defining the individual entanglement entropies, the mutual information has reproduced precisely coefficient expected from the calculation of the free energy.

It is instructive to test the robustness of this calculation using the general scheme outlined in eq. (3.4). Naïvely, averaging the contributions of the inner and outer regions with the same $\hat{\delta}$ only gives the desired constant as in eq. (3.9) if $f'(0) = 0$.[20] This apparent contradiction is easily understood and corrected by realizing that our specification was that the regularization scheme must have a geometric character, e.g., we should be able to apply it for general boundaries. In making the short-distance cut-off $\hat{\delta}$ explicitly depend on R in eq. (3.4), we are implicitly constructing a regularization procedure specific to the circle and the particular calculation of interest. However, we can restore the geometric nature of the cut-off by interpreting the R-dependence in eq. (3.4) as dependence on the curvature of the boundary (in which case, it could be applied with a generic boundary geometry).

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[20]Analogous calculations in higher odd dimensions would require that all of the odd derivatives of $f(\delta/R)$ vanish.

Then, since the curvature, i.e., $\hat{\mathbf{n}} \cdot \partial_s \hat{\mathbf{t}}$, of the circle is reversed when it is the boundary of the exterior region from when it is the boundary of the inner disk, we must choose

$$\delta_- = \delta \, f(\delta/R) \quad \text{and} \quad \delta_+ = \delta \, f(-\delta/R) \, . \tag{3.10}$$

With this additional sign reversal, it is straightforward to verify that this geometric cut-off again reproduces the correct result as in eq. (3.9).

Applying an analogous geometric cut-off in higher dimensions yields a result for $S_+(R) + S_-(R)$ with specific parity because of the symmetry of the conformal transformation (3.6). That is, this sum only contains even powers of δ/R for even dimensions and odd powers for odd dimensions, except for the universal terms, i.e., the constant term in odd dimensions and the $\log(\delta)$ term in even dimensions. Hence these universal contributions are independent of any of details in the choice of the function defining the cut-off in eq. (3.10). Similarly, the entanglement entropy of the annular strip cannot generate any contributions with the wrong parity and hence the universal coefficient in the mutual information is robust against the freedom in defining the (geometric) cut-off.[21]

Exactly the same approach solves the mismatch found for the calculation of c_0 with a mapping to de Sitter space, as described in appendix D. In the mapping to de Sitter space, the constant term is given directly by the corresponding constant in the free energy. This also explains why the results coincide with those obtained from the zeta-function regularization since this latter is clearly symmetric under inversions.

Lattice calculations. With these new insights, we can also understand why the numerical results using a radial lattice yield the correct constant provided that the circle radius is defined as $R = (n + 1/2)\, \delta$ with n being the number of lattice points inside the circle, and δ, the lattice spacing [34, 66, 67]. The entropies of the disk including the first n points and the complementary outer region, extending from the point $n + 1$ to infinity, are equal on the lattice due to the purity of the vacuum state. The prescription $R = (n + 1/2)\, \delta$ simply ensures that these contributions give the same constant term once one subtracts the contribution linear in the variable R from the entropy. The lattice representation of the annular strip includes the radial points from $n + 1$ to m, in the limit $n \to \infty$ with $m - n \ll n$. With the same cut-off prescription as above for the circle entropy, this is interpreted as an annulus extending from $r = R - \varepsilon/2 = (n+1/2)\, \delta$ to $r = R + \varepsilon/2 = (m+1/2)\, \delta$. The general argument indicates that the corresponding entanglement entropy has a vanishing constant contribution in the small ε expansion. Thus, the constant term in the mutual information must coincide with twice the numerical result for the circle with this prescription.

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[21]We might note that the cut-off used in [60] in a similar calculation of the constant term of the entanglement entropy — see the expansion in eq. (D.21) — gives the correct result without summing the contributions of the inner and outer regions. This calculation succeeds because this particular cut-off happens to be invariant under inversion (3.6). That is, in this case

$$\hat{\delta}[\delta, R] = R - \frac{R^2}{R + \hat{\delta}[\delta, -R]} \, .$$

Therefore in this case, the contributions of $S_+(R)$ and $S_-(R)$ are equal and it is sufficient to compute only $S_-(R)$. This choice of cut-off was motivated by comparing with the cut-off appearing the corresponding holographic calculations, which respects the conformal inversion invariance.

In order to verify this reasoning, we calculated the mutual information for the concentric circles numerically on a radial lattice. We computed mutual information for concentric circles with a radius R/δ from 30 to 100, and with ε/δ from 4 to 10. We used an infrared cut-off for the total lattice size from 500 to 1000 lattice points and fitted the entropies with the infrared cut-off first — the details of this procedure can be found in [115]. Then we fitted the mutual information for fixed ε as $x_1(R/\delta) + x_2 + x_3(R/\delta)^{-1}$. The constant term x_2 was subsequently fitted for different ε as $y_1 + y_2(\delta/\varepsilon)$. In this way, we found that the constant term $y_1 \simeq 0.1320$, which is compatible with the analytical result for the constant term on the mutual information of a scalar $4\pi c_0^{scalar} \simeq 0.1276$, as given in eq. (2.16), within numerical errors, i.e., roughly 3% for our simulations.

Let us now turn to regulating the scalar theory with a square lattice. As illustrated in figure 4, the expected value for c_0 for a scalar can be extracted from an explicit numerical computation of the mutual information of concentric circles on a square lattice. However, the constant term cannot be obtained directly from the circle entropy alone, as illustrated in figure 1. These results seems to be at odds with the arguments in the above discussion. However, we believe that the resolution is that the square lattice regularization does not allow the contribution of the entropy of the strip to be written as a geometric integral. More precisely, even if this contribution is local in sizes of the order ε along the strip, the quantity that is integrated will break rotational symmetry and also is not invariant under translations. The contribution will fluctuate according to the precise position of the strip. Of course, the same holds for the circle entropies but there is a cancellation when these quantities are combined in the mutual information.

Still this might suggest that the circle entropies can still be used to get the correct result for c_0 with a square lattice, if we follow a slightly more elaborate prescription: for a circle of radius R in the plane, let $S(R)$ be the entropy of the lattice points inside the circle. The entropy has an area term $2\pi aR/\delta$ proportional to R, where the latter can be determined with increasing precision in the lattice for large R. Then let us subtract this area term (as was done in figure 1 for a circle) but then average the result over different R to eliminate the fluctuations. For a discretization of the radial variable, $R_i = R_0 + i\Delta R$ with $i = 0, 1, \cdots$ with some small interval ΔR, we can construct the following function

$$\bar{S}(R_j) = \frac{1}{j} \sum_{i<j} \left(S(R_i) - 2\pi a \frac{R_i}{\delta} \right). \tag{3.11}$$

The idea would be that $\bar{S}(R_j)$ could converge to $-2\pi c_0$ for large j. This procedure does indeed give results which are near the actual value of c_0. For the points shown in figure 1 involving circles with radii up to 35 (and where we took equally spaced radii[22] with steps of 0.2), we found $2\pi c_0 = 0.0622$. The correct value is 0.0638, and so the error is only roughly 2.5%. However, it seems that the "noise" in figure 1 is not as random as one could have expected. In particular, after some critical radius, further averaging does not seem to improve the result — see figure 5. Further, changing the step ΔR in averaging the radius,

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[22]Our procedure was to define the disk for which we evaluate $S(R)$ by draw an imaginary circle on the square lattice and including all points within the circle as comprising the disk. This allows the circles to have an arbitrary radius and allows the step size to be less than the actual lattice spacing.
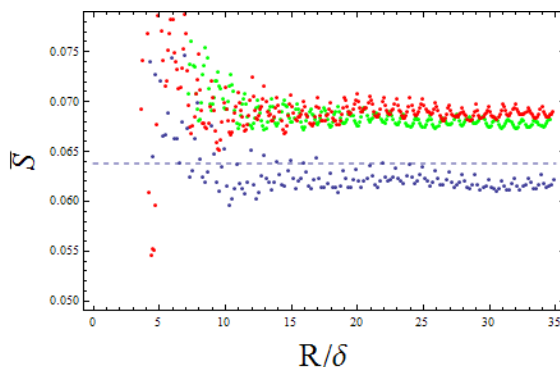
Figure 5. The averaged constant term in the entropy of circles $\bar{S}(R)$ in a square lattice as a function of the radius in lattice units, R/δ. The blue points are for a radius step $\Delta R = 0.2$ (as in figure 1). The red points are for $\Delta R = (3\pi)^{-1} \sim 0.106$. The green points are for the same ΔR but the circles are centered at the point $(1/3, 1/2)$ in lattice units, rather than at the origin. The correct value of $2\pi c_0$ is shown with the dashed line.

we obtain results that do not seem to coverge to the same value (and it is again unclear whether the sequence converges at all) — see figure 5. Changing the point where the circle is centered also changes the result, pointing to the lack of translational invariance in this averaging procedure.

Summarizing our discussion in this section, we have shown the universal constant $\tilde{c}_0$ appearing in the expansion of the mutual information (2.7) matches c_0 appearing in the entanglement entropy of a circle (2.1) under appropriate conditions. A more robust result was that $\tilde{c}_0$ can be computed by evaluating the constant term in the sum of the entropies of a disk and of the complementary exterior region. However, the latter equality still requires a geometric regulator that is applied in the same way in calculating both entropies. The latter ensures that no contribution to the constant comes from the annular region. Being able to evaluate the universal constant from the entanglement entropy of disk with an appropriate regulator is very useful from a practical point of view since the calculation of the mutual information function is generally much more difficult. In the conformal case, summing the two entropies also ensures that the resulting coefficient is invariant under conformal inversions. In this case, the inversion symmetry guarantees that the calculation of $\tilde{c}_0$ (or more generally, the universal contribution in higher dimensions) respects the naïve symmetry $S(V) \leftrightarrow S(\bar{V})$ of the entanglement entropy. However, for general lattice models, there is certainly no simple procedure which allows us to extract the correct universal coefficient c_0 from data for the circle entropies alone. Rather, it seems that an actual computation of mutual information is mandatory to produce the correct result for the constant term.

4 Mutual information and proof of the F-theorem

The proof [32] of the F-theorem in $d = 3$ uses strong subadditivity (SSA) of the entanglement entropy of boosted circles located on a common null cone, as depicted in figure 6(a).

(a)   (b)
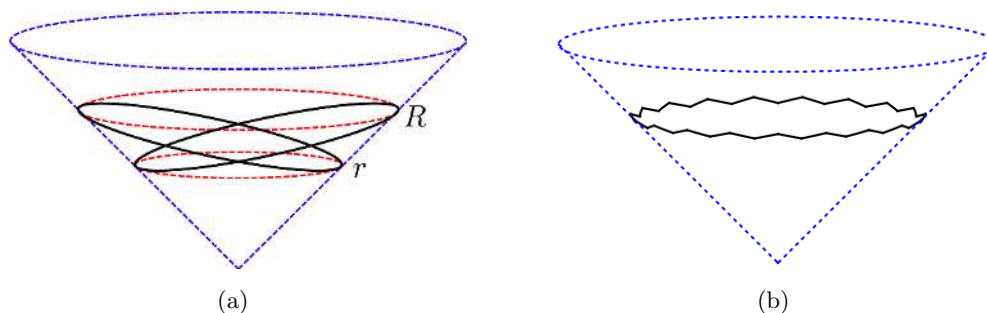
Figure 6. (a) Two (black) boosted circles lying on a common light cone cut across each other. The (red) constant time circles on the cone, of radius R and r, are tangent to the boosted circles. (b) An example of a wiggly circle appearing in the right-hand side of the SSA relation between N boosted circles rotated by angles $(2\pi/N)i$ with $i = 0, 1, \cdots, N-1$, around the axis of the null cone. The resulting wiggly circle is composed of $2N$ arcs.

Strong subadditivity is the inequality

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B),\tag{4.1}$$

for any spatial regions A and B which lie in a common Cauchy surface. Using many circles rotated around the axis of the light cone, one arrives at an inequality involving "wiggly circles", as shown in figure 6(b). To be more precise, the proof is based on the following assumptions:

(a) There is a Lorentz invariant regularization of entanglement entropy which is defined for regions with smooth boundaries except for a finite number of corners lying on null planes.

(b) This regulated entanglement entropy satisfies strong subadditivity at least for sets which cut each other producing new corners on null planes — such as the circles in figure 6(a).

(c) The wiggly circles have an entanglement entropy which approaches that of a circle with the same perimeter, as the number N of null corners goes to infinity.

These assumptions all refer to an essential new geometric feature that appears in the proof, i.e., the entangling surfaces contain corners lying on a null plane which arise where the circles described above cross each other — for simplicity, we refer to these corners as 'null cusps'. In a more conventional situation, where the entangling surface includes corners lying in a spatial plane, the entanglement entropy acquires additional logarithmic contributions, e.g., see [69–74]. The appearance of such terms in the entanglement entropy of the wiggly circles, illustrated in figure 6(b), would eliminate the power of SSA in proving the F-theorem. The second and third assumptions above are based on the idea that there is no intrinsic local measure of the 'angle' for a null cusp [32]. Hence, no local contribution from these features is expected to appear in the corresponding entanglement entropy, and in particular, no logarithmically divergent terms should appear. This feature was explained

in [32], however, as it is a key element in the present proof, we elaborate on this point in appendix B.

With these assumptions in mind, let R and r be the radius of the circles at constant time which are tangent to the boosted circles — see figure 6. Then the SSA inequality for N rotated boosted circles can be written as (see [32] for details)

$$S(\sqrt{Rr}) \geq \frac{1}{N} \sum_{i=1}^{N} \tilde{S} \left(\frac{2rR}{R + r - (R - r) \cos(\frac{\pi i}{N})} \right) . \qquad (4.2)$$

Here $S(x)$ is the entanglement entropy of a circle of radius x and $\tilde{S}(x)$ denotes the entropy of a wiggly circle with the same perimeter as a circle of radius x. By assumption (c), taking the limit $N \to \infty$ converts $\tilde{S}$ into S and this inequality becomes one involving circle entropies alone. The latter turns out to be equivalent to

$$S''(R) \leq 0 . \qquad (4.3)$$

Then, using the c-function

$$C(R) = \frac{1}{2\pi} \Big[R\, S'(R) - S(R) \Big] , \qquad (4.4)$$

we have

$$C'(R) = \frac{R}{2\pi} S''(R) \leq 0 . \qquad (4.5)$$

Now that according to eq. (2.1), $C(R)$ coincides with the constant c_0 at conformal fixed points. Hence the fact that $C(R)$ decreases with increasing R establishes the desired monotonicity property: $c_0^{UV} > c_0^{IR}$.

Now, we have shown mutual information can be used to provide a covariant regularization of entanglement entropy and so it is natural to revisit the assumptions of the above proof to see how they may be accommodated in this regularization. There is a complication, however, since mutual information gives a regularization introducing a cut-off ε, which is actually a distance in space. Hence we can regard it as a regularization depending on a new element, the 'framing' of the separation between the two entangling surfaces. We might note that analogous framing regularizations have been used in the literature studying Wilson loops [75]. In spite of the complications introduced in the geometry, we find it valuable to go through the following arguments, since it will help to clarify the correctness of the assumptions (a)−(c) of the F-theorem.

Eliminating frame dependence. Suppose we have a generic region A bounded by a smooth entangling curve $\Gamma(s)$, parametrized by distance s along the curve from some reference point. We would like to regulate the entanglement entropy across Γ using mutual information, however, in order to obtain a finite result depending only on region A, we must first eliminate the frame dependence noted above. We will then also demonstrate that the resulting entropy also satisfies assumptions (a)−(c) above. We begin regulating the entanglement entropy of Γ by evaluating the mutual information between regions defined by

$$\Gamma_\pm \equiv \Gamma(s) \pm \frac{1}{2}\, \varepsilon(s)\, \hat{\mathbf{n}}(s) \qquad (4.6)$$
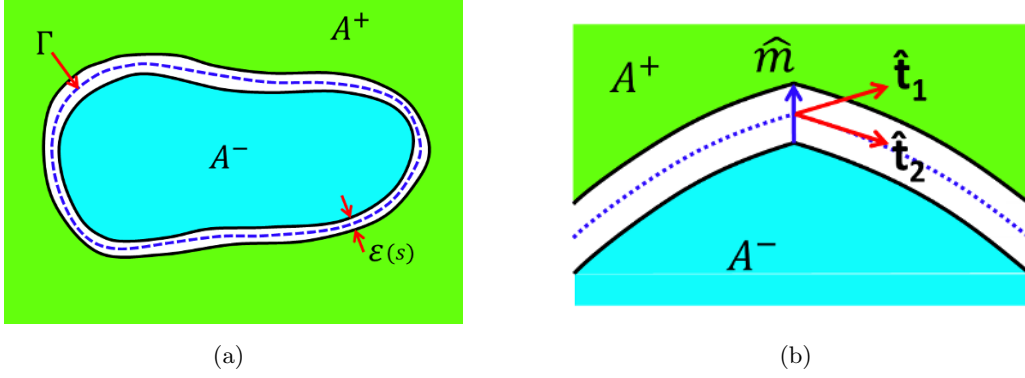
Figure 7. (a) Regularizing the entropy of A with boundary Γ (blue dashed line) using the mutual information $I(A^+, A^-)$. (b) Detail of a null cusp, i.e., a corner on a null plane, with the size of ε the same on both sides of the corner point.

where Γ_+ corresponds to the boundary of exterior region A^+ and Γ_- corresponding to the boundary of interior region A^-, as illustrated in figure 7(a). Further $\hat{\mathbf{n}}(s)$ is a spatial unit vector normal to Γ and as in section 2.2, $\varepsilon(s)$ is a macroscopic (i.e., $\varepsilon(s) \gg \delta$) but small distance which serves as the cut-off in the mutual information. In our general discussion here, we allow $\varepsilon(s)$ to vary smoothly along the curve.[23] Then, we expect

$$I(A^+, A^-) = \tilde{a} \int_\Gamma \frac{ds}{\varepsilon(s)} + I_0(A) + \mathcal{O}(\varepsilon), \tag{4.7}$$

where $I_0(A)$ is a physical quantity which does not depend on the framing $\varepsilon(s)$.[24] In the following, it is this quantity $I_0(A)$ which will play the role of the regulated entropy in the original proof of the F-theorem discussed above. Comparing the above with eq. (2.7) for a circular entangling curve (in a fixed point and with constant ε), we would have $I_0(A) = 2\pi R \tilde{b} - 4\pi \tilde{c}_0$. Further let us note that, as in eq. (2.7), $\tilde{a}$ is again the coefficient of the universal area term for generic strips in the limit of small width — see the discussion around eq. (2.15).

The reason for the structure in eq. (4.7) is the locality of the UV terms which depend on the framing, similar to the discussion in section 2.3. A local correction to eq. (4.7) depending on the derivatives of $\varepsilon(s)$ such as $\int ds\, \varepsilon'(s)/\varepsilon(s)$ is a total derivative and hence it does not contribute. Other local contributions involving higher derivatives of $\varepsilon(s)$, or curvature corrections, must be accompanied by extra powers of ε to compensate for the dimension of the derivatives and hence such terms would vanish with some power of ε as $\varepsilon \to 0$.[25] With eq. (4.6), we focused on symmetric framings or in the language of

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[23]We also ask $\varepsilon'(s)$ to be of the same order of ε, such that $\varepsilon'(s)/\varepsilon(s)$ converges to a finite number as the limit $\varepsilon \to 0$ is taken.

[24]In principle, there may be additional extensive contributions involving unusual (negative) powers of $\varepsilon(s)$, as discussed in appendix A.1. However, the following arguments still carry through essentially unchanged in such a situtation.

[25]In making this argument, one should recall that the width is set such that $\varepsilon \ll 1/m_i$ where m_i may be any scale appearing in defining the RG flow between the corresponding fixed points. Hence we can only expect the appearance of coupling constants for relevant operators in these higher order contributions. But again such couplings must be accompanied by extra (positive) powers of ε which would cause these terms to vanish in the limit $\varepsilon \to 0$.

sections 2.2 and 2.3, we chose $\alpha = 0$.[26] However, let us consider nonsymmetric framings for a moment, e.g., $\Gamma_+ = \Gamma(s) + \varepsilon_1(s)\hat{\mathbf{n}}(s)$ and $\Gamma_- = \Gamma(s) - \varepsilon_2(s)\hat{\mathbf{n}}(s)$ with $\varepsilon_1(s) \neq \varepsilon_2(s)$ in general. Then we might find additional finite frame-dependent terms in eq. (4.7) involving the difference of the two framings, e.g.,

$$\int ds\ \hat{\mathbf{n}} \cdot \partial_s \hat{\mathbf{t}} \ \frac{\varepsilon_1(s) - \varepsilon_2(s)}{\varepsilon_1(s) + \varepsilon_2(s)} \tag{4.8}$$

— compare to eq. (2.14). The α-dependent term in eq. (2.9) implicitly has its origin in a contribution of this form. Of course, such a contribution vanishes with the symmetric framing in eq. (4.6). The symmetric choice also means that $I_0(A)$ is the same when A^+ and A^- interchanged in eq. (4.7) and hence gives rise to the symmetry $I_0(A) = I_0(\bar{A})$ for the finite part.

The proof of the F-theorem requires that we consider boundaries $\Gamma(s)$ with null cusps, i.e., corners on null planes. Hence we must determine if such local features introduce additional frame-dependent terms in the mutual information and if so, how to eliminate them. Let us decompose the mutual information as

$$I(A^+, A^-) = S(A^+) + S(A^-) - S(A^s)\,, \tag{4.9}$$

where A^s is the strip, i.e., $A^s = \overline{A^+ \cup A^-}$.[27] Recall that while the entropies appearing in this decomposition are regulator dependent, the combination appearing in mutual information is universal. Further, we expect that any frame dependence from the null cusp will appear in a purely geometric contribution, whose form will depend on the details of the UV fixed point. As noted above, since the corner lies in a null plane, there is no intrinsic local measure of the 'angle' for a null cusp [32]. Hence the corresponding null cusps in Γ_+ and Γ_- will not produce any extra local contributions in $S(A^+)$ or $S(A^-)$, respectively.

However, the situation is different for the strip A^s since we can now identify three distinct vectors at the cusps, as illustrated in figure 7(b). First, for the two lines crossing each other on the null plane, we have the unit tangent vectors $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ (taken with the same orientation such that $\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2$ is null[28]). Lastly, there is a unit vector $\hat{\mathbf{m}}$ directed from the cusp on the inner boundary Γ_- to that on the outer boundary Γ_+.[29] The local geometry of the null cusp in the strip yields two new dimensionless quantities, which are both local and Lorentz invariant, i.e.,

$$q_1 \equiv \hat{\mathbf{m}} \cdot \hat{\mathbf{t}}_1\,, \qquad q_2 = \hat{\mathbf{m}} \cdot \hat{\mathbf{t}}_2\,. \tag{4.10}$$

From the requirement that the boundaries be spatially separated to each other we have $q_1^2 \leq 1$, $q_2^2 \leq 1$. Giving these two variables the geometry of the cusp is determined except

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[26]This choice further rules out a term of the form $\int ds\, \varepsilon'(s)/\varepsilon(s)$ since locally the integrand is odd under reflections in the plane orthogonal to Γ.

[27]Implicitly, eq. (4.9) assumes that the regulated entanglement entropies respect $S(V) = S(\bar{V})$ for general regions V in the ground state of the QFT. However, we do not really need to make this assumption here or in the following discussion. Rather it is a notational convenience to introduce $S(A^s)$ rather than using $S(A^+ \cup A^-)$. For example, in eq. (4.16), we would need to replace $S(A_1^s \cup B_1^s)$ with $S((A^+ \cap B^+) \cup A^- \cup B^-)$.

[28]Again, we note that $\hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 = 1$ follows from the latter. Hence this inner product carries no information about the 'angle' at the null cusp.

[29]We emphasize that this unit vector is not the same as $\hat{\mathbf{n}}$ in eq. (4.6), e.g., see appendix C.1.

for an overall scaling. In particular, the ratio of the framing sizes on each side of the cusp, ε_1 and ε_2, is given in terms of these variables,

$$\frac{\varepsilon_1}{\varepsilon_2} = \sqrt{\frac{1 - q_1^2}{1 - q_2^2}} \,. \tag{4.11}$$

Now the appearance of $\hat{\mathbf{m}}$ in the above definition (4.10) implicitly means that q_1 and q_2 depend on the framing at the null cusp. However, we argue that only a restricted amount of frame dependence can appear in $S(A^s)$ and hence in $I(A^+, A^-)$. In particular, the discussion in section 3 indicated that the strip entropy $S(A^s)$ is a purely local UV contribution. However, the only scales appearing in this entanglement entropy which are local to the cusp would be $\varepsilon_{1,2}$ and the cut-off δ and hence the contribution of the corner may be a function of the ratios $\varepsilon_{1,2}/\delta$. However the mutual information will not contain contributions depending on the UV cut-off.[30] Hence only the ratio $\varepsilon_1/\varepsilon_2$ is relevant but the latter is completely determined by q_1 and q_2 in eq. (4.11). Hence, the frame dependence of the null cusp can be expressed entirely in terms of q_1 and q_2 and hence the mutual information can be written as

$$I(A^+, A^-) = \tilde{a} \int_\Gamma \frac{ds}{\varepsilon(s)} + I_0(A) + f(q_1, q_2) \,, \tag{4.12}$$

where $f(q_1, q_2)$ is a local frame-dependent term, depending exclusively on q_1 and q_2, and $I_0(A)$ is a function of A, independent of framing.

Since $f(q_1, q_2)$ is the cusp dependent term in the mutual information, it must be a universal quantity. However, eq. (4.12) still leaves the definition of $f(q_1, q_2)$, and the frame dependence of $I(A^+, A^-)$, open to certain ambiguities. We remove these uncertainties with the following choices: first, we need to be careful with the integral in the first term on the right-hand side. We define it by simply integrating all along the curve Γ up to either side of the null cusp — see figure 7(b) — where the value of $\varepsilon(s)$ is defined there as the limit of the function $\varepsilon(s)$ from either side of the corner. This choice is simply made for convenience, however, other choices would produce finite redefinitions of the function $f(q_1, q_2)$. Finally, we fix $f(q_1, q_2)$ by demanding that $f(q_1, q_1) = 0$, i.e., this contribution vanishes when $\hat{\mathbf{t}}_1 = \hat{\mathbf{t}}_2$ (and there is no null cusp). This requirement eliminates the possible ambiguity in $f(q_1, q_2)$ (and hence on I_0) of redefining this contribution by the addition of a constant.

Our expectations for the frame dependence of null cusps can be checked in the model of extensive mutual information, discussed in appendix C. The result for this model is evaluated in section C.1 as

$$f(q_1, q_2) = \frac{\kappa_{UV}}{2} \left(2 + \frac{(q_1(q_1 + q_2) - 2)\arcsin(q_1)}{(q_1 - q_2)\sqrt{1 - q_1^2}} + \frac{(q_2(q_1 + q_2) - 2)\arcsin(q_2)}{(q_2 - q_1)\sqrt{1 - q_2^2}} \right) , \tag{4.13}$$

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[30]Of course, the situation where Γ contains a corner lying in a spatial plane would be different. In this case, we expect a local contribution proportional to $\log(\varepsilon/\delta)$ for the strip. The δ dependence of this term would be compensated by terms with $\log(\delta/R)$, where R some macroscopic scale, in the entropies $S(A^+)$ and $S(A^-)$. The mutual information will then have an additional finite (δ independent) term proportional to $\log(\varepsilon/R)$.
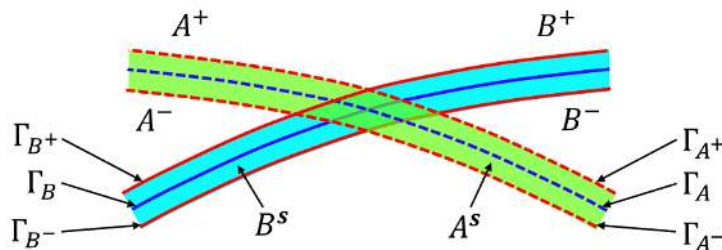
Figure 8. Two entangling surfaces, Γ_A and Γ_B, which intersect to form a null cusp. These surfaces are 'expanded' according to eq. (4.6) so that Γ_{A+} and Γ_{B+} still cross on a null plane and Γ_{A-} and Γ_{B-} cross on a nearby parallel null plane. The thin strips, A^s (light blue) and B^s (green), are complementary to the regions $A^+ \cup A^-$ and $B^+ \cup B^-$, respectively.

where κ_{UV} is a dimensionless constant characterizing the UV fixed point. As expected, $f(q_1, q_2)$ is a symmetric function of q_1 and q_2 since eq. (4.12) is invariant when A^+ and A^- are exchanged while $q_1 \leftrightarrow q_2$ under this interchange. Further it is straightforward to show that with $q_2 = q_1 + \epsilon$, $f(q_1, q_2)$ is vanishes as ϵ^2 in the limit $\epsilon \to 0$. Again, this limit, i.e., $q_2 \to q_1$, corresponds to the vanishing of the null cusp, i.e., the entangling curves become smooth with $\hat{\mathbf{t}}_1 = \hat{\mathbf{t}}_2$ in figure 7(b). On the other hand, $f(q_1, q_2)$ diverges in the limit $q_1 \to \pm 1$ or $q_2 \to \pm 1$ because in this limit, the entangling curves Γ_+ and Γ_- cease to be spatially displaced with respect to each other.

For general curves, we eliminate all frame-dependent terms from our definition of renormalized mutual information with

$$I_0(A) = I(A^+, A^-) - \tilde{a} \int_\Gamma \frac{ds}{\varepsilon(s)} - \sum_i f(q_1^i, q_2^i) \,, \qquad (4.14)$$

where the last sum is over all of the null cusps. The $I_0(A)$ defined by eq. (4.14) is both universal (i.e., independent of the UV regulator) and frame independent. Hence the result is a finite Lorentz invariant function of A alone. Hence we can use $I_0(A)$ as the Lorentz invariant regularization of the entanglement entropy required for the proof of the F-theorem, i.e., this quantity will satisfy the first of the three assumptions outlined at the beginning of this section.

Strong subadditivity. Let us now turn to the second of the assumptions required to prove the F-theorem. In particular, we ask: is there a sense in which the mutual information satisfies SSA? Consider the geometry illustrated in figure (8). We have two regions A and B with boundaries Γ_A and Γ_B, respectively. These boundaries intersect at some point to form a corner in a null plane, as occurs in the proof of the F-theorem.[31] Now following our prescription above, a strip is constructed around each of these boundaries so that we can regulate the entanglement entropy using the mutual information. The construction leads to two exterior regions, A^+ and B^+, and two interior regions, A^- and B^-, as well as the two strips $A^s = \overline{A^+ \cup A^-}$ and $B^s = \overline{B^+ \cup B^-}$. The goal is to obtain an inequality for I_0

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[31]In the following, we only consider a single null cusp but the discussion is easily extended to account for the contributions from any number of null cusps, as appear in eq. (4.14).
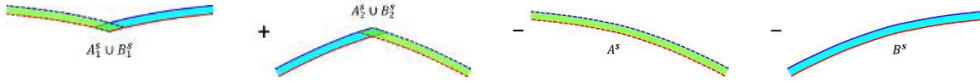
Figure 9. The quantity $J(A^s, B^s)$ is defined in eq. (4.16) by the combination of entanglement entropies for the thin regions shown above.

which is framing independent. Hence we are free to choose the framing of Γ_A and Γ_B such that Γ_{A+} intersects Γ_{B+} on a null plane and that these curves are spatial to one another, and similarly, the same for Γ_{A-} and Γ_{B-}.

Now, using the definitions of mutual information (4.9) and strong subadditivity (4.1), we find[32]

$$I(A^+, A^-) + I(B^+, B^-) = S(A^+) + S(A^-) - S(A^s) + S(B^+) + S(B^-) - S(B^s)$$
$$\geq S(A^+ \cup B^+) + S(A^+ \cap B^+) + S(A^- \cup B^-) + S(A^- \cap B^-) - S(A^s) - S(B^s)$$
$$= I(A^+ \cap B^+, A^- \cup B^-) + I(A^+ \cup B^+, A^- \cap B^-) + J(A^s, B^s). \qquad (4.15)$$

In the last line above, we have introduced

$$J(A^s, B^s) = S(A_1^s \cup B_1^s) + S(A_2^s \cup B_2^s) - S(A^s) - S(B^s), \qquad (4.16)$$

where the strips have been decomposed as $A^s = A_1^s \cup A_2^s$ and $B^s = B_1^s \cup B_2^s$, according to the intersections at the null cusp shown in figure 9.[33] The inequality (4.15) has (almost) the same form of the SSA inequality (4.1) but now written in terms of the mutual information. However, there is one additional term, i.e., $J(A^s, B^s)$, on the right-hand side of the inequality. The goal, however, is to write a SSA-like inequality for the regulated entropies, e.g., $I_0(A)$, rather than the mutual information. Hence we proceed by noting that $I(A^+ \cap B^+, A^- \cup B^-)$ and $I(A^+ \cup B^+, A^- \cap B^-)$ have the right form to provide the mutual information regularization of the entanglement entropies for $A \cap B$ and $A \cup B$, respectively. That is, if we take out the area terms and the frame-dependent terms coming from the null cusp, as in eq. (4.14), the previous inequality becomes

$$I_0(A) + I_0(B) \geq I_0(A \cup B) + I_0(A \cap B) + 2f(q_1, q_2) + J(A^s, B^s). \qquad (4.17)$$

Note that the area terms can be removed since they are precisely the same on the right- and left-hand sides of eq. (4.15).

Let us now turn our attention on the extra term $J(A^s, B^s)$ appearing in the inequalities above. Here it is important that the corners in the strips $A_1^s \cup B_1^s$ and $A_2^s \cup B_2^s$ are actually null cusps. If instead the corners in these strips were to lie in a spatial plane, the nontrivial angle would induce logarithmically divergent terms proportional to $\log \delta$

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[32]Note that the two spatial regions A^+ and B^+ will generally lie on different Cauchy slices and hence the strict definition of $A^+ \cap B^+$ would yield a linear region, i.e., a codimension two surface. However, our intention here is that $A^+ \cap B^+$ is a two-dimensional spatial region that lies within the intersection of the causal domains of A^+ and B^+. Similar definitions apply for the other intersections and unions of regions appearing in eq. (4.15) and in the subsequent discussion.

[33]The definition of the regions can be made more precise by writing expressions such as $A_1^s = A^s \cap \bar{B}^-$.

in the corresponding entanglement entropies appearing in $J(A^s, B^s)$. These log terms would not be compensated by the rest of terms in eq. (4.15), which are δ independent, and therefore this inequality would be insufficient for proving the F-theorem. However, since the entangling curves in the proof are all placed on a common null cone, the resulting corners are null cusps and hence we do not have to worry about $\log(\delta)$ terms here. Further, the area law terms proportional to $1/\delta$ also cancel in eq. (4.16) and hence J is a finite (δ independent) quantity, as much as the mutual information is. Therefore we must find the same universal result when J is computed with any regularization for the entropies in eq. (4.16). But further in section 3, we have argued the entanglement entropy of thin regions must be extensive (in any regularization). Thus, in the combination appearing in eq. (4.16), most of the contributions in the various strip entropies will cancel out and the only remaining contribution which could survive will be a strictly local (and UV) term associated with the corners in the first two strips. However, J can not depend on ε since there are no other dimensionful parameters in the geometry. Again the strip entropy is probing correlations beyond the scales of the UV CFT and so there are no dimensionful couplings available in the field theory to appear in the characterization of this local feature. For example, no finite terms involving $\log \varepsilon$ can arise for this reason.[34]

Hence, we conclude that J must be a universal function of q_1, q_2, the only geometric parameter describing the null cusp, which is local and dimensionless. From the definition in eq. (4.16), it is also clear that source of this q dependence must come from $S(A_1^s \cup B_1^s)$ and $S(A_2^s \cup B_2^s)$, the entanglement entropies of the two 'bent' strips of figure 9. However, it was precisely these two entropies which are added and subtracted in going from the second to the third line of eq. (4.15). Hence, the same two entanglement entropies appear in the mutual informations $I(A^+ \cap B^+, A^- \cup B^-)$ and $I(A^+ \cup B^+, A^- \cap B^-)$ but with a negative sign. Therefore it must be that[35] $J(q_1, q_2) = -2f(q_1, q_2)$ and the two q-dependent terms cancel on the right-hand side of eq. (4.17). With this cancellation, our regulated entropies satisfy an inequality like strong subadditivity (4.1), i.e.,

$$I_0(A) + I_0(B) \geq I_0(A \cup B) + I_0(A \cap B), \tag{4.18}$$

for A, B, cutting each other on a null plane. Hence we have shown that these regulated entropies satisfy assumption (b).

Limit of wiggly circles. It remains to see if the regulated entropies (4.14) satisfy the third assumption set out at the beginning of the section. That is, whether the regulated entropy of the wiggly circles $I_{0,w}(R)$ converges in the limit $N \to \infty$ to that of a smooth circle $I_{0,s}(R)$ with the same perimeter.

Here again, we will invoke our favorite arguments involving locality. The difference between these two quantities in the limit of large N must be given by the integration

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[34] Again, the situation would be different if the boundary corners lay in a spatial plane. In this case, the contribution from J would still be localized on the cusp but we expect a logarithmic contribution $\log(\varepsilon/\delta)$ where the UV cut-off appears as the second scale.

[35] It is important for this step that we have defined the additive constant on the function $f(q_1, q_2)$ by $f(q_1, q_1) = 0$, since it is evident that for the limit of no cusp we also have $J = 0$.

of some local term around the circumference. Further, the local term only depends on the UV CFT which, in particular, does not have any dimensionful couplings to introduce into this integral. Neither will the frame-dependent parameters ε or q appear since I_0 does not depend on these parameters. Thus, in order to eliminate the dimensions of the integral, we must divide by some local geometric quantity with dimensions of length. There are only two such quantities available: the first is the curvature of the circle, i.e., $k \equiv \hat{\mathbf{n}} \cdot \partial_s \hat{\mathbf{t}} \propto 1/R$ as described in section 2.3. However, the construction of I_0 was chosen to respect the reflection symmetry discussed there and so by the same arguments as given in that section, we cannot have a contribution involving an integral of the curvature. The second possible geometric quantity would be the length of the arcs between the null cusps, i.e., $\ell \equiv 2\pi R/2N$. Hence let us consider whether $I_{0,w}(R)$ might contain a contribution proportional to $\int ds/\ell = 2N$.[36] This 'area' term for the wiggly circle would also be present in the case where the boundary of the wiggly region lies on a null plane rather than on a null cone. But in considering I_0, we have regained Lorentz covariance. Thus by making a boost which keeps the null plane fixed, we can vary arbitrarily the amount by which the arcs extend in the null direction. In particular, a sufficiently large boost will bring the wiggly 'line' arbitrarily close to a straight line. Hence, this area term cannot be present since it is not present for the linear case. That is, the nonvanishing contribution to $I_{0,w}(R)$ must be independent of N in the limit of large N. Hence we must have

$$\lim_{N \to \infty} I_{0,w}(R) = I_{0,s}(R), \tag{4.19}$$

which establishes the third and final key ingredient required in the proof of the F-theorem.

We can summarize the above arguments by saying that the mutual information, stripped of the frame dependent terms in a universal way, can be defined given any region A with a spatial piecewise smooth boundary and where any corners lie in a null plane. This universal part of the mutual information provides a regulated entanglement entropy which satisfiesl the three assumptions (a)–(c) which are required for the proof [32] of the F-theorem. For a smooth circular entangling curve, the construction here coincides with that introduced in section 2.2 and hence an immediate consequence is that the constant term in $I_0(A)$ coincides with the one in $I(R+\varepsilon/2, R-\varepsilon/2)$, which we discussed in previous sections. Hence, combining our construction of the regulated entanglement entropy using the mutual information with the proof of [32], we have established the three-dimensional F-theorem as a proper c-theorem. In particular then, the new c-function becomes

$$C(R) = \frac{1}{4\pi}\Big[R\,\partial_R I_0(R) - I_0(R)\Big], \tag{4.20}$$

and the proof of [32] establishes $C'(R) \le 0$.

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[36]Let us note that other contributions with the right dimensions would have to take the form $\int ds\, k^{2n}\ell^{2n-1} \sim 1/N^{2n-1}$ which vanish in the limit $N \to \infty$. Contributions proportional to higher powers of N, e.g., $\int ds/(k^2\ell^3) \sim N^3$, can not arise because they are singular for vanishing curvature. Note that in both of the examples here, we are demanding that k only appear with even powers in order to respect the reflection symmetry.

5 Discussion

Our investigation began with the issue that entanglement entropy in QFT cannot be defined without explicit reference to a UV regulator. As we discussed, this introduces various ambiguities in trying to define the universal contribution to entanglement entropy for odd dimensions, in particular, for $d = 3$ — unless one restricts calculations to the class of allowed regulators. Implicitly the problem is that the divergences in entanglement entropy can not be renormalized away. For example, while we do may add a counterterm for the area term divergence in $d = 3$, we do not have a physical experiment which would allow us to fix the renormalized entanglement entropy against some physical result at a certain renormalization scale.

We were able to resolve the above problem by we showing that the mutual information effectively provides a 'universal' geometric regularization scheme for the entanglement entropy, which can be applied with any QFT. In this sense, mutual information is analogous to the framing regularization used in the literature of Wilson loops. Further, mutual information allows us to identify the physical renormalized 'central charge' c_0. In particular this means that, as for any other renormalized quantity, it is possible to compute this charge using any regularization that defines the continuum theory. In particular, our approach allows c_0 to be computed with a lattice regulator. We will also showed the proof of the F-theorem [32] can be written in terms of universal quantities derived from the mutual information.

Expressing c_0 in terms of the mutual information, instead of the entanglement entropy, has both practical and theoretical advantages. The main theoretical advantages are that it shows this quantity is universal, well defined and intrinsic to the CFT at the fixed points. Let us add that the mutual information is also a gauge invariant quantity [76, 77] and so evades the subtleties arising in defining entanglement entropy for theories involving gauge fields. Our approach also clarifies the correctness of the assumptions in the proof of the F-theorem [32].

From a more practical perspective, it was not clear from previous discussions in the literature how a sufficiently geometric cut-off could be defined for general theories in order to allow the extraction of a universal quantity from the entanglement entropy of a circle in odd dimensions. Zeta function or heat kernel regularizations can be defined only for free fields, where the relevant partition functions are given in terms of a differential operator. In practice, the same can be said of the radial lattice. We have shown a universal way to introduce a covariant universal short-distance cut-off is to consider the mutual information as a regulated version of the entanglement entropy.

The mutual information is a renormalized quantity in the sense that it can be computed under any regularization, specially for models defined in arbitrary type of lattices. For these, it is not even necessary to know the expression of the theory of the continuum. A lattice cut-off serves simultaneously to define the theory in terms of the microscopic degrees of freedom and to produce a finite entanglement entropy. However, the error in R, which is of the order of the lattice spacing, will spoil the definition of the constant term in the expansion, as discussed in section 2.1. Alternatively, an unambiguous definition of the

constant term in the entanglement entropy can be achieved if we introduce a second cut-off with the sole purpose of regulating the entropy, when the continuum theory is already known, as in section 3.

However, in its present form, the F-theorem in $d = 3$ cannot be considered as having the same degree of mathematical rigor as the c-theorem in $d = 2$ or the a-theorem in $d = 4$. The latter are based on well established mathematical properties of correlation functions or the S-matrix in QFT. Our understanding of entanglement entropy and related entanglement measures are not so well developed for QFT's. However, our general and simple arguments establish a solid case. The natural context for a proof in the mathematical sense seems to be the algebraic approach to QFT, where the mutual information is well defined in terms of states acting on local operator algebras.

F-theorem and topological theories. An example requiring further discussion is the case of topologically ordered phases or topological quantum field theories. In this case, the constant term in the entanglement entropy of a circle coincides with the topological entanglement entropy [64, 65]: $s_{\text{topo}} = -\gamma$. This result is somewhat surprising since one finds a nonzero (and possibly very large) central charge for a theory with no local degrees of freedom. On the other hand, the mutual information for a topological theory *vanishes*, at least if the distances in the relevant geometry are all longer than the scale of the onset of topological order. The latter may be some macroscopic scale in the continuum theory, but in many instances, it will correspond to the scale of the microscopic cut-off. A key question will be how ε compares to this 'topological scale', which we denote $\varepsilon_{\text{topo}}$. Specifically, if we can set $\varepsilon \ll \varepsilon_{\text{topo}}$ then the constant contribution of the annular strip will vanish and mutual information would yield the same charge as the circle entropy, i.e., the constant term in the mutual information would simply be twice the topological entropy (when we set $R \gg \varepsilon_{\text{topo}}$). However, if $\varepsilon \gg \varepsilon_{\text{topo}}$ then the entanglement entropy of the annulus will have a constant topological contribution and there will be a discrepancy between the charges defined by the mutual information and the circle entropy.

It may appear that the latter discrepancy has the potential to produce contradictions with F-theorem, however, as we will now explain, it simply produces a reorganization of the fixed point contributions.[37] Let us consider an RG flow between two fixed points, both of which have a topological sector. We imagine that the topological contribution to the entanglement entropy can be separated from that of the local or gapless field degrees of freedom, i.e., we can write $c_0 = c_{0,\text{local}} + c_{0,\text{topo}}$. Then evaluating the circle entropy with a geometric regulator and comparing the results for the two fixed points, we would find

$$\text{UV fixed point}: \quad c_0^{UV} = c_{0,\text{local}}^{UV} + c_{0,\text{topo}}^{UV}$$

$$\text{IR fixed point}: \quad c_0^{IR} = c_{0,\text{local}}^{IR} + c_{0,\text{topo}}^{IR}$$

$$c_0^{UV} - c_0^{IR} = \left(c_{0,\text{local}}^{UV} - c_{0,\text{local}}^{IR} \right) + \left(c_{0,\text{topo}}^{UV} - c_{0,\text{topo}}^{IR} \right) \tag{5.1}$$

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[37]We are grateful to Tarun Grover for discussions on this point.

Now let us make the same comparison using the mutual information. Further we imagine that the topological scale $\varepsilon_{\text{topo}}$ for the UV fixed point corresponds to the cut-off scale, i.e., the topological order is an inherent property of the continuum CFT describing the UV fixed point. Hence even though the annulus entropy probes the theory at some very short distance (but still macroscopic) scale ε, it still acquires a topological contribution characteristic of the UV theory. The RG flow is then described by the mutual information as

$$\text{UV fixed point}: \quad \tilde{c}_0^{UV} = c_{0,\text{local}}^{UV}$$

$$\text{IR fixed point}: \quad \tilde{c}_0^{IR} = c_{0,\text{local}}^{IR} + c_{0,\text{topo}}^{IR} - c_{0,\text{topo}}^{UV}$$

$$\tilde{c}_0^{UV} - \tilde{c}_0^{IR} = \left(c_{0,\text{local}}^{UV} - c_{0,\text{local}}^{IR} \right) + \left(c_{0,\text{topo}}^{UV} - c_{0,\text{topo}}^{IR} \right) \tag{5.2}$$

Hence both approaches agree on the change in the universal coefficient resulting from the RG flow. The key point is that the topological contribution in the entropy of the annular strip is proportional to $c_{0,\text{topo}}^{UV}$ at both fixed points. That is, the scale determining this contribution is the strip width ε, which is held fixed while the radius R varies between the UV and the IR scales.

We can extend the above discussion to consider the case where $\varepsilon_{\text{topo}}^{UV}$ corresponds to some macroscopic scale, which is still much shorter than the scales where the physics is described by the UV fixed point. As alluded to above, the mutual information description of the RG flow would correspond to that in eq. (5.2) as long as $\varepsilon \gg \varepsilon_{\text{topo}}^{UV}$. On the other hand, if we chose $\varepsilon \ll \varepsilon_{\text{topo}}^{UV}$, the mutual information results would coincide to those in eq. (5.1) because the entanglement entropy of the annulus would no longer include a topological contribution. Again, both choices would agree on the running of the universal coefficient relevant for the F-theorem.

Hence the above discussion seems to clear away the potential tension between the F-theorem and the discrepancy between the universal charges defined by the mutual information and the entanglement entropy. However, it calls into question whether or not our construction strictly satisfies the second requirement which we set out for a legitimate c-theorem. That is, does the mutual information actually provide a physical quantity that is intrinsic to the fixed point of interest, i.e., in eq. (5.2), we see that $\tilde{c}_0^{IR}$ appears to depend on $c_{0,\text{topo}}^{UV}$, a property of the UV fixed point.[38] However, we are not particularly concerned with this issue and instead suggest that the mutual information presents a perspective

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[38]A possible resolution of this tension would be if the topological sector on the UV does not contribute to the nontrivial running of $\tilde{c}_0$ in the RG flow. That is, if the topological terms are the same at the UV and IR ends of the RG flow, i.e., $c_{0,\text{topo}}^{IR} = c_{0,\text{topo}}^{UV}$, then the topological contributions would be eliminated in $\tilde{c}_0^{IR}$. An interesting example of such a flow was discussed in [31]. In that case, the UV fixed point corresponds to a Dirac fermion coupled to a $\mathbb{Z}_2$ gauge field and while in the IR, the theory flows to an purely topological gapped phase. At both of these fixed points, $c_{0,\text{topo}} = \gamma/2\pi = \log(2)/2\pi$. Note, however, that the RG flow does affect the topological part of the theory in a nontrivial way. In particular, the topological sectors in the UV and the IR are distinct, e.g., with Abelian and non-Abelian anyonic excitations, respectively. Unfortunately, it is not likely that the cancellation found in this example will be a generic property of general flows.

where the topological contribution to the entanglement should be thought as a 'topological remnant' of the RG flow.

For example, in the above discussion where $\varepsilon_{\text{topo}}$ was considered a macroscopic scale, the essential difference between $\varepsilon \gg \varepsilon_{\text{topo}}$ and $\varepsilon \ll \varepsilon_{\text{topo}}$ is that in the latter situation, the high energy contribution to the mutual information, e.g., eq. (2.13), probes the physical short-distance correlations that build up the long-range topological entanglement. We observe that the same is true of standard probes of the topological order: for example, in order to identify the topological phase through its anyonic excitations, we need to probe the theory at the energy scale of these excitations which corresponds to the scale of the gap, i.e., $1/\varepsilon_{\text{topo}}$ in the current discussion. Similarly, we might consider the standard approaches for evaluating the topological entanglement entropy [64, 65].[39] In order for these constructions to yield unambiguous results, the cut-off scale and the distances between regions must be much smaller than the correlation length of the model, i.e., $\varepsilon_{\text{topo}}$ [76, 77]. For example, the Kitaev-Preskill construction [64] gives the topological entanglement entropy as

$$\begin{aligned} -\gamma &= S(A) + S(B) + S(C) - S(AB) - S(AC) - S(BC) + S(ABC) \\ &= I(A,B) + I(A,C) - I(A,BC) \end{aligned} \tag{5.3}$$

for three adjacent regions A, B and C with a common vertex. Hence, the topological entropy can also be expressed in terms of the mutual information for regions which are close enough, i.e., their separation is less than the correlation length. Again this restriction is natural since the only physical correlations capable of building up entanglement are over distances shorter than this scale.

In our construction of the mutual information of circles, this restriction corresponds to having $\varepsilon \ll \varepsilon_{\text{topo}}$, the scale for the onset of topological order. Hence detecting the topological constant term with the mutual information requires that $\varepsilon_{\text{topo}}$ is a macroscopic scale, different than the UV cut-off scale. In other words, there should be actual local degrees of freedom at high energies giving rise to the topological gapped sector at lower energies. If there is a topological sector at the UV fixed point of the RG flow, this scale $\varepsilon_{\text{topo}}$ could in principle be arbitrarily small and unrelated to the RG flow under study, which only occurs at much longer scales. While the $\varepsilon \to 0$ prescription should always restore the correct value of the topological entropy in the mutual information, the peculiarity here is made clear by contrasting with the case of a regular CFT with gapless degrees of freedom, e.g., a massless free scalar. In this case, producing the correct constant term only requires choosing $\varepsilon \ll R$. That is, ε can be set independently of the possible physics at much shorter scales.

Of course, if we do not restrict ourselves to working in Minkowski space, the topological sector at the fixed point can be identified by other means, e.g., putting the theory on a topologically nontrivial manifold. This could in principle be used to set the correct topological entropy of the fixed point without going beyond the physics of the fixed point itself.

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[39]To simplify the discussion here, we consider the case of a purely topological theory. With additional local degrees of freedom, the result of the standard constructions will be the sum of the topological entropy plus a contribution from the additional degrees of freedom. Further, the latter will depend on the precise details of the geometry.

Alternate c-functions. As well as decreasing monotonically along RG flows, Zamolodchikov's c-function in two dimensions is also stationary at fixed points [2]. It was pointed out in [66] that the entropic c-function (4.4) in three dimensions fails to be stationary in the particular case of a free scalar field. In fact, stationarity is also lost for the free scalar when using the entropic c-function in two dimensions. This lack of stationarity is typically seen as a deficiency but we would like to advocate that, in fact, it is an advantage of using the entropic c-function. In particular, Zamolodchikov's original c-function treats the scalar theories with a tachyonic m^2 on an equal footing as those with positive m^2. In contrast, the entropic c-function seems to recognize that the theories with a tachyonic mass are unphysical and only allows for flows towards positive m^2.

In any event, the above discussion highlights the fact that the c-function is not uniquely defined. Rather given a particular c-function, we can in principle construct infinitely many other c-functions which are also monotonically decreasing and interpolate between the same fixed point values. Hence one may wonder whether a new version of the entropic c-function can be found to ameliorate the situation with respect to stationarity.

To begin, one might first consider the flow of our new c-function defined using the mutual information. In fact, we argue that these flows are unchanged in comparison to those for eq. (4.4) as follows: the entanglement entropy of the annular strip is a new ingredient in defining eq. (4.20) and so one might hope that this is enough to change the behaviour near the free scalar fixed point. However, the arguments in section 3 indicated that the entropy of this annular region is purely a UV quantity. Hence it will be unaffected by the introduction of a mass scale (with $\varepsilon \ll 1/m$) and so it will not modify the flow of the c-function near the fixed point of the massless free scalar (or any other fixed point).

At RG fixed points, the conformal symmetry of the underlying field theory was used to show that the charge c_0 can also be identified with the universal contribution in the free energy of the three-sphere [13].[40] We can also think of applying the conformal symmetry to map the geometry introduced in section 2.2 for the mutual information to various new geometries (either in R^3 or S^3). While the mutual information of these new configurations would still identify the same charge c_0 at the fixed points, it would define new c-functions on the RG flows that interpolate between fixed points since the conformal symmetry is lost along the flows. While one can easily test if some new c-function defined in this way is stationary at the free scalar fixed point, the challenge would be to prove that in general, it decreases monotonically along any RG flow.

The geometry introduced in section 2.2 for the mutual information is natural to consider because it implements a 'point-splitting' of the circular entangling surface in the original entanglement entropy. However, another interesting direction to investigate would be to replace the mutual information with some measure of quantum entanglement $E(A^+, A^-)$ (e.g., entanglement of formation, distillable entanglement or relative entropy of entanglement [78]) as the regulator of entanglement entropy. It can be expected that in the limit of

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[40]Recall that that in principle, the free energy has regularization ambiguities similar to the entanglement entropy. It would be interesting to investigate if a conformal mapping of the mutual information geometry might produce a practical new scheme in which the resulting 'free energy' which is independent of the UV regulator.

coinciding entangling curves (i.e., $\varepsilon \to 0$), the result should reproduce the behavior of the entropy to some extent. This is because entanglement measures for complementary regions in a global pure state coincide with twice the entanglement entropy, exactly as for the mutual information. However, this is likely a rather impractical approach due to the notorious difficulties in computing entanglement measures. However, we naturally wonder how much the expansion in ε/R for entanglement measures differ from the one of the mutual information. In particular, it would be interesting to see if the terms in the ε expansion coincide all the way down to the same constant term (in odd dimensions or the logarithmic term in even dimensions).

What is c_0? The original c-theorem in two dimensions points to the central charge c at the RG fixed points. However, c plays many different roles in the physics of the underlying conformal field theories. Cardy's conjecture [6] suggests that a key role was as the coefficient of the trace anomaly. This conjecture gives a unified perspective for c-theorems in any even number of spacetime dimensions, by suggesting that they all refer to the coefficient of the A-type anomaly. Of course, the more recent perspective is to relate c-theorems in general dimensions to the universal contribution in the entanglement entropy of a sphere [11, 12]. Our arguments in section 3 can be extended to higher dimensions and so our approach using the mutual information also produces the same 'central charge' in general dimensions. While the new entanglement proposal chooses the same charge as Cardy's conjecture for even dimensions, it also provides a unified framework to consider c-theorems in both even and odd numbers of dimensions.

This unified view can be carried further for free fields and we will show that the central charges defined by the entanglement entropy in odd dimensions can actually be obtained by analytic continuation of those in even dimensions. Similar ideas were considered in [23, 79]. The idea is that in order to compare between different dimensions, in each dimension we normalize the central charges by that in a particular 'reference' CFT, for example, the free massless scalar field. The central charges can be identified as the universal contribution to the free energy on a d-dimensional sphere, i.e., the coefficient of the logarithmic term for even dimensions and the constant term for odd dimensions [13]. For free field theories, we may use zeta-function regularization, which completely removes all of the power law divergences leaving only the desired universal contribution. In this case, the free energy is given by [80]:

$$F = -\log(Z(S^d)) = \frac{1}{2} \lim_{s \to 0} \left[\mu^{2s} \zeta'(s, d) + \zeta(s, d) \log \mu^2 \right], \tag{5.4}$$

where μ is a cut-off energy scale and $\zeta(s, d)$ is the appropriate zeta-function on the sphere S^d. Then the ratios of the charges for two free models will be $\zeta_1(0, d)/\zeta_2(0, d)$ for even dimensions and $\zeta_1'(0, d)/\zeta_2'(0, d)$ for odd dimensions, since $\zeta(0, d) = 0$ for odd dimensions. However, the latter also implies that $\zeta(s, d) = s\, \zeta'(0, d) + \ldots$ for small s in odd dimensions and so, in fact, in both cases we can write the ratios of central charges as

$$\lim_{s \to 0} \frac{\zeta_1(s, d)}{\zeta_2(s, d)} . \tag{5.5}$$

One might say that the anomaly coefficients (i.e., the coefficients of the logarithmic terms) vanish in odd dimensions but the ratios of anomalies do not. As an example, let us consider the ratio of the central charges for a Dirac fermion and a scalar in even dimensions. We also divide the ratio by the number of degrees of freedom of the fermion, i.e., $2^{d/2}$ and expand for large d [81]

$$\frac{C_{\text{Dirac}}}{2^{d/2}\, C_{\text{scalar}}} = (d-2) + k_0 + \frac{k_1}{d} + \frac{k_2}{d^2} + \dots \tag{5.6}$$

Now the exercise is to fit the coefficients k_i using the known values of the even-dimensional central charges [81] and then continue the resulting formula to odd dimensions. Surprisingly, this analytic continuation does a good job at predicting the corresponding ratios for the odd-dimensional charges (also divided by the number of degree of freedom of the Dirac field), even for $d = 3$. In particular, fitting the first 20 k's over the first 100 even dimensions, we find for $d = 3$

$$\frac{C_{\text{Dirac}}}{2C_{\text{scalar}}} = 1.7157936606\cdots , \tag{5.7}$$

which is extremely close the exact value: $1.7157936649\cdots$ [60].

Eq. (5.6) shows how different the entropic central charge can be from the standard counting of numbers of degrees of freedom [81]. The contribution of topological entanglement entropy to the charge in $d = 3$ is an extreme case of mismatch between c_0 and number of degrees of freedom. This also calls into question the standard idea that the c-theorem describes the loss of degrees of freedom along the RG flow. The relation of charge with entanglement entropy (or mutual information) suggests that a more fruitful approach might be to attribute its physical meaning to some measure of entanglement — see for example [31, 82].

One complication that obscures the physical meaning of this entropic central charge is that the c_0 term in eq. (1.1) is overwhelmed by the power-law divergent contributions. Even in our mutual information approach, the universal contribution is still a small subdominant contribution compared to, e.g., the area law term in eq. (2.7). From this perspective, it is interesting to observe that this same coefficient appears as the coefficient of the entropy density of a CFT on a curved background $R \times H^{d-1}$ where the curvature and temperature are related by $\mathcal{R} = -4\pi^2(d-1)(d-2)T^2$ [11–13].[41] It is natural to think that by varying the temperature that the thermal entropy is probing the theory at different scales and in its role of specifying the entropy density, the central charge appears in the leading contribution. Formulating a c-theorem in terms of the thermal entropy density (or free energy) in flat space was examined before but it was found that corresponding charge does not decrease in certain RG flows [83–85]. Here the conformal mapping of the entanglement entropy calculation suggests that we should instead consider the thermal entropy density in a particular curved space background. Certainly, this suggestion faces a number of challenges. First of all, the entropy density at fixed temperature may not be very well defined in a hyperbolic geometry, where the volume and the area grow in the same way

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[41]We would like to thank Kris Jensen for discussing this point.

at large radii. Hence it would be difficult to isolate a volume contribution to the total thermal entropy from possible boundary terms. In particular, it is insufficient to define the entropy density as the total entropy divided by the volume for large regions in the hyperbolic geometry. Of course, a second challenge is to show that the (properly defined) entropy density defines a proper *c*-function which decreases monotonically with decreasing temperature (and curvature). An additional complication in this regard is the following: we know how to uniquely define the theory on the curved background only at the conformal fixed points. When mass scales are introduced, however, we do not know how to fix the nonminimal couplings with the curvature. Hence without a specific prescription, there will be many possible interpolating theories for a given RG flow in flat space.

The F-theorem was originally discovered in a holographic framework [11, 12]. In this context, one is considering a strongly coupled three-dimensional boundary theory with a large number of degrees of freedom, which is dual to some classical gravity theory in a four-dimensional bulk. In the gravity description, the F-theorem corresponds to the monotonic behaviour of the corresponding entropy functional evaluated on some extremal surfaces (provided that the null energy condition holds for the matter fields in the bulk). One might note that this has some resemblance to the black hole area theorem. However, since the F-theorem is valid for general quantum field theories in three dimensions, an interesting question is what the dual gravitational interpretation of the F-theorem would be for theories which are not strongly coupled and/or do not contain a large number of degrees of freedom. It seems that some interesting monotonic behaviour continues to survive even in the realm of quantum gravity in the bulk.

Acknowledgments

A Holographic calculation of mutual information

In the framework of gauge/gravity duality, the entanglement entropy in the boundary QFT can be evaluated with a geometric calculation in the bulk gravity theory [53, 54]. In particular, the entanglement entropy between a (spatial) region V and its complement in the boundary is computed with the following prescription [53, 54]:

$$S(V) = \operatorname*{ext}_{v \sim V} \frac{\mathcal{A}(v)}{4\, G_{\mathrm{N}}}\,. \tag{A.1}$$
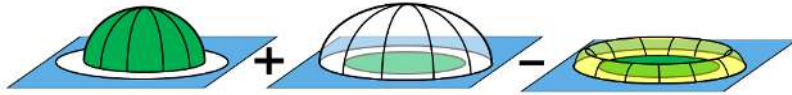
Figure 10. The holographic mutual information, for the boundary configuration illustrated in figure 2, requires evaluating the area of three minimal surfaces in the bulk. First, there are two hemi-spheres on the left which yield $S(A^+)$ and $S(A^-)$. Second, there is the annular surface on the right yielding $S(A^+ \cup A^-)$.

That is, one evaluates the Bekenstein-Hawking formula on all surfaces v in the bulk which are homologous to the boundary region V and the entanglement entropy is given by the extremal result. This prescription was found to pass wide range of consistency tests, e.g., see [13, 53, 54, 86–88], however, quite recently a derivation of eq. (A.1) was also presented in [89]. We might add that eq. (A.1) assumes that the bulk theory corresponds to Einstein gravity and this expression has been extended to certain higher curvature theories as well [88, 90] — see also [91–99]. Let us also note here that mutual information was discussed in a holographic context by [87, 100–102].

Now we wish apply this holographic formalism to evaluate the constant $\tilde{c}_0$ for the three-dimensional boundary CFT, following the approach given in section 2. That is, we apply the holographic techniques to calculate the mutual information between a disc of radius $R_- = R - \frac{1}{2}\varepsilon$ and the exterior of a circle of radius $R_+ = R + \frac{1}{2}\varepsilon$, as pictured in figure 2. To compute this mutual information from eq. (1.2), we must evaluate eq. (A.1) for two kinds of minimal surface in four-dimensional anti-de Sitter (AdS) space: first, minimal surfaces ending on a circle in the boundary and second, an extremal bulk surface homologous to a two-dimensional annulus, as illustrated in figure 10. The first has already been studied in detail [13, 53, 54] and we will review the main ideas of this calculation here, as it serves as a simple introduction to the more complicated calculations for the second class of surfaces.

Hence to begin, we determine the holographic entanglement entropy for a circular entangling surface in the three-dimensional boundary CFT. The vacuum state is represented by four-dimensional AdS space in the dual gravity theory and the bulk metric may be written as

$$ds^2 = \frac{L^2}{z^2} \left[dz^2 + dt_{\rm E}^2 + dr^2 + r^2 d\theta^2 \right] , \tag{A.2}$$

where L is the AdS curvature scale. We are working in Euclidean signature with $t_{\rm E}$ being the Euclidean time direction. The two remaining boundary directions are written in polar coordinates, with r and θ. We choose the entangling surface in the boundary to be the circle $r = R$ in the constant time slice $t_{\rm E} = 0$. If we describe the bulk surface with a profile $z(r)$, the induced metric on this surface is

$$ds^2 = \frac{L^2}{z^2} \left[(1 + z'^2)dr^2 + r^2 d\theta^2 \right] , \tag{A.3}$$

where the prime denotes differentiation with respect to r. The corresponding area is

given by

$$\mathcal{A} = 2\pi L^2 \int dr \frac{r}{z^2} \sqrt{1 + z'^2} \,. \qquad (A.4)$$

According to eq. (A.1), we must identify the surface with extremal area and so extremizing the 'action' (A.4) yields the following equation:

$$rzz'' + z(z')^3 + zz' + 2r(z')^2 + 2r = 0 \,. \qquad (A.5)$$

Now we must find a solution satisfying the boundary conditions that it reaches the AdS boundary at $r = R$, i.e., $z(r = R) = 0$ and that the profile is smooth at $r = 0$, i.e., $z'(r = 0) = 0$. It turns out there is a simple solution, namely the half-sphere [53, 54]

$$z^2 + r^2 = R^2 \,, \qquad (A.6)$$

shown on the left of figure 10. The minimal area may then be written as

$$\mathcal{A} = 2\pi L^2 R \int_\delta^R \frac{dz}{z^2} \,, \qquad (A.7)$$

where we have rewritten the result in terms of an integral over the bulk radius z. We have also introduced a UV regulator surface $z = \delta$, which cuts off this integral near the AdS boundary. In the boundary CFT, δ plays the role of short distance cut-off. Hence the holographic entanglement entropy becomes

$$S = \frac{\mathcal{A}}{4G_{\mathrm{N}}} = \frac{\pi}{2} \frac{L^2}{G_{\mathrm{N}}} \left(\frac{R}{\delta} - 1 \right) \,. \qquad (A.8)$$

Comparing this result to eq. (2.1), we would identify the fixed-point value of the entropic c-function as [11, 12]

$$c_0 = \frac{L^2}{4\,G_{\mathrm{N}}} \,. \qquad (A.9)$$

That is (up to a factor of -2π), this is the constant contribution in eq. (A.8) if we simply ignore the divergent term proportional to R/δ. Of course, the holographic regulator is naturally a geometric one, e.g., [58] and so this same c_0 also appears if we apply the formula of [34] to eq. (A.8), i.e., $2\pi c_0 = R\,\partial_R S(R) - S(R)$. The advantage of the mutual information, as discussed in section 2, is that it yields a finite result from which we may determine c_0. Hence in the present holographic framework, all dependence on δ will vanish after we subtract the area for the annular surface from that of the spheres.

Hence we move to remaining ingredient in our holographic calculation of the mutual information (1.2), i.e., $S(A^+ \cup A^-)$. Of course, this also corresponds to the entanglement entropy of the annular region between the two circles, i.e., $\overline{A^+ \cup A^-}$. Given this boundary region, there are two possible extremal surfaces in the bulk. The first would simply be the combination of the two hemispheres appearing in the previous calculation of $S(A^\pm)$. The second possible extremal surface has the shape of a two-dimensional annulus in the bulk, as illustrated in the right-most diagram of figure 10. Since we are interested in the limit $\varepsilon/R \ll 1$, this second one is the minimal area surface and hence it yields $S(A^+ \cup A^-)$ in

eq. (A.1). Now we can again describe this new annular surface with a profile $z(r)$, as in the previous calculation, and the area and equation of motion would be identical to the expressions in eqs. (A.4) and (A.5), respectively. The only change is that the boundary conditions are now chosen as $z(r = R_A) = 0$ and $z(r = R_B) = 0$. Here we note that finding the extremal bulk surface with these boundary conditions has already been studied in the context of studying correlation functions of circular Wilson loops [103–106].[42] These surfaces have also been considered in the context of holographic entanglement entropy by [71, 101, 107, 108]. Hence we only need to review the salient steps of these calculations here.

As a first step, we introduce 'improved' coordinates u and v with [105, 106]

$$\frac{r}{L} = \frac{e^v}{\sqrt{1 + u^2}}, \qquad \frac{z}{L} = \frac{u\, e^v}{\sqrt{1 + u^2}}. \qquad (A.10)$$

We note that both u and v are dimensionless and further that $u = z/r$. We can think of u as an 'improved' radial coordinate in the bulk AdS_4 spacetime. In particular, we have $u = 0$ at $z = 0$. Similarly, v plays the role of an 'improved' radial coordinate on the boundary. We will describe the extremal surface with a profile $v(u)$ and then the corresponding area becomes

$$\mathcal{A} = 2\pi L^2 \int \frac{du}{u^2} \sqrt{\frac{1 + (1 + u^2)^2 (\partial_u v)^2}{1 + u^2}}. \qquad (A.11)$$

The advantage of the new coordinate choice becomes apparent here since the integrand above does not contain any explicit dependence on v. Hence we have a constant of the motion, which we denote as

$$\frac{\delta \mathcal{A}}{\delta\, \partial_u v} = \frac{2\pi L^2}{q^2}. \qquad (A.12)$$

In the following, it turns out that we will be primarily interested in the regime $q \ll 1$, which will correspond to $\varepsilon \ll R$. It will be convenient to define a rescaled coordinate $u = q\, y$. With this choice, the area becomes

$$\mathcal{A} = \frac{2\pi L^2}{q^2} \int \frac{dy}{y^2} \sqrt{\frac{q^2 + (1 + q^2 y^2)^2 (\partial_y v)^2}{1 + q^2 y^2}}. \qquad (A.13)$$

Now using eq. (A.12), we find a first order equation to determine $v(y)$

$$\partial_y v = \frac{q\, y^2}{1 + q^2 y^2} \frac{1}{\sqrt{1 + q^2 y^2 - y^4}}. \qquad (A.14)$$

where we have implicitly assumed that $\partial_y v > 0$ here. At this point, we note that our description of the minimal surface will have two branches, as shown in figure 11. The

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[42]While the Wilson loops were studied in these reference for a four-dimensional boundary theory, the extremal surface lies within an AdS_3 slice of the five-dimensional bulk geometry. In the present case, the extremal surface is restricted to the AdS_3 slice defined by setting $t_\mathrm{E} = 0$ in eq. (A.2). Hence precisely, the same extremal surface appears with the same boundary conditions in both calculations.
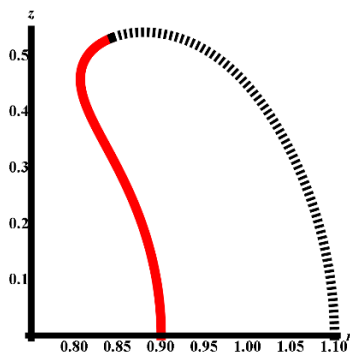
Figure 11. Shape of the minimal surface, with $R = 1$ and $\varepsilon = 0.2$ ($\alpha = 0.1$), which ends on the annulus. The red continuous curve corresponds to the left branch, which sees u start at $u = 0$ at the bottom before increasing to $u = u_+$. The black dashed curve corresponds to the right branch, with $u = u_+$ at the top decreasing to $u = 0$ at the bottom. Notice that the minimal surface sags inward: this is because the outer ring has a larger circumference than the inner one.

parameter y increases from $y = 0$ (and $v = \log(R_-/L)$) to the turning point at $y = y_+$ on the left branch, then decreases from $y = y_+$ to $y = 0$ (and $v = \log(R_+/L)$) on the right branch. At the turning point, $\partial_v y = 0$ or $\partial_y v$ diverges and so examining eq. (A.14), we identify y_+ as the positive root of $1 + q^2 y_+^2 - y_+^4 = 0$, i.e.,

$$y_+^2 = \sqrt{1 + \frac{q^4}{4}} + \frac{q^2}{2} \, . \tag{A.15}$$

The profile of the extremal surface is illustrated in figure 11 in terms of the original z and r coordinates. Note that this profile is asymmetric, as is natural since the outer circle on the boundary has a larger radius than the inner circle. Further note that the profile $z(r)$ is multi-valued. That is, the left branch extends in to region $r < R_-$ before bending back to meet the right branch at $y = y_+$. This inward 'sagging' of the the minimal surface arises because of the negative extrinsic curvature of the inner component of the entangling surface on the boundary [109]. The same properties will also true for the corresponding extremal surfaces in an analogous construction in higher dimensions. In contrast, in terms of the improved coordinates, the extremal profile $v(y)$ is symmetric about $y = y_+$, as well as being single-valued.

Now substituting eq. (A.14) into the integral (A.13) yields the extremal value of the area and the holographic entanglement entropy becomes

$$S = \frac{\mathcal{A}}{4G_{\rm N}} = \pi \frac{L^2}{G_{\rm N}} \frac{1}{q} \int_0^{y_+} \frac{dy}{y^2} \frac{1}{\sqrt{1 + q^2 y^2 - y^4}} \, , \tag{A.16}$$

where an additional factor of 2 was included here to account for integrating over both branches, as described above. We also account for the boundary conditions by integrating eq. (A.14) over the extremal surface to find

$$\log \frac{R_+}{R_-} = 2\,q \int_0^{y_+} dy \, \frac{y^2}{1 + q^2 y^2} \frac{1}{\sqrt{1 + q^2 y^2 - y^4}} \, . \tag{A.17}$$

Now the above integral can be given in terms of elliptic integrals, however, the resulting expression is not very enlightening.

It is clear that the integral in eq. (A.16) diverges at the lower limit $y = 0$ but, of course, we expected such a UV divergence to appear in the entanglement entropy. By subtracting the divergent behavior, i.e., $\int dy/y^2$, we can isolate a finite contribution to the entanglement entropy as

$$S_{\text{fin}} = \pi \frac{L^2}{G_{\text{N}}} \frac{1}{q} \left[\int_0^{y_+} \frac{dy}{y^2} \left(\frac{1}{\sqrt{1 + q^2 y^2 - y^4}} - 1 \right) - \frac{1}{y_+} \right], \qquad (A.18)$$

which can be expressed as

$$S_{\text{fin}} = \pi \frac{L^2}{G_{\text{N}}} \frac{1}{q \, y_+} \left[K(iy_+^2) - E(iy_+^2) \right], \qquad (A.19)$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind, respectively. Note that this contribution is completely independent of the short distance cut-off. The remaining cut-off dependent term takes the simple form

$$S_{\text{div}} = \frac{\pi}{2} \frac{L^2}{G_{\text{N}}} \left[\frac{1}{u_{+,min}} + \frac{1}{u_{-,min}} \right] \qquad (A.20)$$

where $u_{\pm,min} = q \, y_{\pm,min}$ is the minimum bulk 'radius' reached at the position of the regulator surface $z = \delta$ as v approaches $\log(R_\pm/L)$. Above, we have separated the two boundary contributions as they are not identical. Using eqs. (A.10) and (A.14), we find

$$\frac{1}{u_{\pm,min}} = \frac{R_\pm}{\delta} \left(1 - \frac{1}{2} \frac{\delta^2}{R_\pm^2} + \frac{1}{3} \frac{\delta^3}{R_\pm^3} + \cdots \right). \qquad (A.21)$$

An important point for the discussion below is that from this expansion, we can see that eq. (A.20) does not contain a cut-off independent contribution, i.e., it will not contribute to c_0 in our calculation of the mutual information. Rather we can rewrite S_{div} as

$$S_{\text{div}} = \frac{\pi}{2} \frac{L^2}{G_{\text{N}}} \left[\frac{R_+}{\delta} + \frac{R_-}{\delta} \right] + O(\delta). \qquad (A.22)$$

Hence we see that S_{div} contains the sum of the area law divergences appearing in eq. (A.8) for the two entangling circles. Therefore, as expected, these divergences will cancel in calculating the mutual information. Of course, eq. (A.22) also contains higher order contributions, which vanish in the limit $\delta \to 0$, but again no constant term appears.

Before proceeding with a detailed analysis of the finite contribution S_{fin}, let us make an important (qualitative) observation. First from eq. (A.17), it is clear that the limit $q \to 0$ coincides with the limit $R_+ \to R_-$. Now working in this regime, let us write the radii as eq. (2.5), i.e.,

$$R_- = R - \left(\frac{1}{2} - \alpha \right) \varepsilon, \qquad R_+ = R + \left(\frac{1}{2} + \alpha \right) \varepsilon. \qquad (A.23)$$

That is, as in eq. (2.6), we are allowing R to be shifted slightly from the average of the two circle radii. Substituting eq. (A.23) on the left-hand side of eq. (A.17), we have

$$\log \frac{R_+}{R_-} \simeq \frac{\varepsilon}{R} - \alpha \left(\frac{\varepsilon}{R}\right)^2 + \left(\frac{1}{12} + \alpha^2\right)\left(\frac{\varepsilon}{R}\right)^3 - \alpha\left(\frac{1}{4} + \alpha^2\right)\left(\frac{\varepsilon}{R}\right)^4 + \cdots. \quad (A.24)$$

Hence we see that introducing a nonvanishing α changes the nature of the above expansion, in that it includes both odd and even powers of ε/R. It is straightforward to show that if $\alpha = 0$ the expansion only contains odd powers.[43] These observations make clear that $\alpha = 0$ is a distinguished case in our analysis but we now examine the implications of this behavior for the mutual information.

In the regime of small q, we may rewrite eq. (A.17) using eq. (A.24) and making a Taylor series expansion[44] on the right-hand side in terms of q

$$\frac{\varepsilon}{R} - \alpha\left(\frac{\varepsilon}{R}\right)^2 + \cdots = q\left(\beta_0 + \beta_1 q^2 + \cdots\right), \quad (A.25)$$

which yields

$$q \simeq \frac{1}{\beta_0}\frac{\varepsilon}{R} - \frac{\alpha}{\beta_0}\left(\frac{\varepsilon}{R}\right)^2 + O\left[\left(\frac{\varepsilon}{R}\right)^3\right]. \quad (A.26)$$

We can make a similar small-q expansion in eq. (A.18) and it is straightforward to see that this finite contribution to the entanglement entropy takes the form

$$\begin{aligned}S_{\text{fin}} &= -\pi\frac{L^2}{G_{\text{N}}}\frac{1}{q}\left(\gamma_0 + \gamma_1 q^2 + \cdots\right) \\ &= -\pi\frac{L^2}{G_{\text{N}}}\beta_0\gamma_0\left(\frac{R}{\varepsilon} + \alpha + O(\varepsilon/R)\right).\end{aligned} \quad (A.27)$$

Hence we see that a constant term (i.e., independent of ε/R) appears in S_{fin} only when $\alpha \neq 0$ or rather this contribution can be eliminated by setting $\alpha = 0$. Combining this with the observation that no constant term appears in eq. (A.22) for S_{div}, we see that in our holographic calculation, $S(A^+ \cup A^-)$ will not contribute to $\tilde{c}_0$ in eq. (2.7) as long as we choose $\alpha = 0$ or rather $R = (R_+ + R_-)/2$. Of course, $\alpha = 0$ is precisely the choice which our general arguments in section 2.3 indicated was needed to eliminate the constant contribution coming from $S(A^+ \cup A^-)$ and so here we have a confirmation of this general reasoning in our holographic calculations. Hence we complete the calculation of the mutual information in the following with $\alpha = 0$.

We now evaluate the above expressions in the limit $q \to 0$, which we saw above corresponds to taking $\varepsilon \ll R$. From eq. (A.15), we have $y_+ \to 1$ in this limit and so

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[43]With $\alpha = 0$ in eq. (A.23), $\varepsilon \to -\varepsilon$ is equivalent to interchanging $R_+ \leftrightarrow R_-$, which changes the sign of the logarithm on the right-hand side of eq. (A.24).

[44]The specific values of the coefficients in the small-q expansion here and in eq. (A.26) are not important for the present argument. However, in the following, we will explicitly calculate the leading terms and find that $\beta_0 = 2\sqrt{\pi}\frac{\Gamma(3/4)}{\Gamma(1/4)}$ and $\gamma_0 = \pi^{1/2}\frac{\Gamma(3/4)}{\Gamma(1/4)}$.

eq. (A.19) reduces to

$$S_{\mathrm{fin}} = \pi \frac{L^2}{G_{\mathrm{N}}} \frac{1}{q} \left[K(i) - E(i) \right] + O(q)$$

$$= -\pi^{3/2} \frac{\Gamma(3/4)}{\Gamma(1/4)} \frac{L^2}{G_{\mathrm{N}}} \frac{1}{q} + O(q) \,. \tag{A.28}$$

Similarly, evaluating eq. (A.17) in this limit, we find

$$\frac{\varepsilon}{R} + O\left[\left(\frac{\varepsilon}{R} \right)^3 \right] = 2\,q \int_0^1 dy \, \frac{y^2}{\sqrt{1-y^4}} + O(q^3)$$

$$= 2\pi^{1/2} q \, \frac{\Gamma(3/4)}{\Gamma(1/4)} + O(q^3) \,. \tag{A.29}$$

Combining these expressions then yields

$$S_{\mathrm{fin}} = -\pi \frac{\Gamma(3/4)^2}{\Gamma(1/4)^2} \frac{L^2}{G_{\mathrm{N}}} \frac{2\pi R}{\varepsilon} + O(\varepsilon/R) \,. \tag{A.30}$$

Note with $R = (R_+ + R_-)/2$, eq. (A.22) can be written as

$$S_{\mathrm{div}} = \frac{L^2}{2\,G_{\mathrm{N}}} \frac{2\pi R}{\delta} \; + \; O(\delta) \,. \tag{A.31}$$

Now let us compare the leading results in eqs. (A.30) and (A.22) to the holographic entanglement entropy of a strip of width ε and length H [53, 54][45]

$$S_{\mathrm{strip}} \; = \; \frac{L^2}{2\,G_{\mathrm{N}}} \frac{H}{\delta} - \pi \frac{\Gamma(3/4)^2}{\Gamma(1/4)^2} \frac{L^2}{G_{\mathrm{N}}} \frac{H}{\varepsilon} \,. \tag{A.32}$$

Hence to leading order in the limit $\varepsilon \ll R$, the present entanglement entropy matches that for the strip above. In particular, with $H = 2\pi R$, S_{div} in eq. (A.31) precisely matches the area law term while S_{fin} in eq. (A.30) reproduces the finite contribution proportional to $1/\varepsilon$.

Combining our holographic results for the individual entanglement entropies, the mutual information (1.2) becomes

$$I(A^+, A^-) = 2\pi^2 \frac{\Gamma(3/4)^2}{\Gamma(1/4)^2} \frac{L^2}{G_{\mathrm{N}}} \frac{R}{\varepsilon} - \pi \frac{L^2}{G_{\mathrm{N}}} + O(\varepsilon/R) \,. \tag{A.33}$$

We see that this result takes precisely the expected form, given in eq. (2.7). In particular from the constant term, i.e., the term independent of ε/R, we identify the fixed-point value of the c-function (4.20) in our holographic CFT as

$$\tilde{c}_0 = \frac{L^2}{4\,G_{\mathrm{N}}} \,. \tag{A.34}$$

Further we have a precise match $\tilde{c}_0 = c_0$, ie, the above result agrees with the value identified in eq. (A.9) from the entanglement entropy of a circle. We should not be surprised by this

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[45]The reader may find it easier to compare with the strip entropy in [35], which uses the same notation and conventions as in the present calculation.

agreement here since holography naturally provides covariant regulator in our calculation of the entanglement entropy. Of course, this match occurred because only the circle entropies contributed to the constant term in eq. (A.33). That is, as was stressed above, with the choice $\alpha = 0$, the entanglement entropy of the annular region did not have a constant term. It would interesting to verify the analogous result in a holographic framework where the bulk theory involved a higher curvature interactions, e.g., [88, 90] or where the bulk solution describes a holographic RG flow, e.g., [34, 35, 59].

A.1 Strip entanglement entropy in RG flows

In this section, we wish to illustrate with a holographic calculation that if the RG flow is driven by an operator which is close to being marginal, one may find additional 'area law' terms in the mutual information (2.7) which diverge in the limit $\varepsilon \to 0$. Our calculations here are a simple extension of the holographic analysis of [58]. There the focus was on identifying situations where the RG flow would generate additional universal terms accompanied by logarithmic divergences $\log(\mu\delta)$. However, as we will see below additional 'unconventional' power law divergences may also be generated and in the appropriate limit, the mutual information is also modified. For example, with $d = 3$, if the RG flow is initiated by a relevant operator with conformal dimension in the range $3 > \Delta > 5/2$, then the entanglement entropy (2.1) will contain an additional contribution of the form $2\pi R\, a'\, \lambda^2/\delta^{2\Delta-5}$ where λ is the coupling constant for the new operator and a' is a dimensionless coefficient. An analogous term of the form $2\pi R\, \tilde{a}'\, \lambda^2/\varepsilon^{2\Delta-5}$ also appears in the mutual information (2.7) if $\lambda\, \varepsilon^{3-\Delta} \ll 1$. We note that similar contributions are also generated in appropriate circumstances with the extensive model studied below in appendix C — see footnote 50.

The following calculations follow quite closely the analysis presented in [58]: for simplicity, we focus on the case of Einstein gravity coupled to a scalar field in four bulk dimensions, with an action of the form

$$I = \frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^4x \, \sqrt{-g} \left[\frac{6}{L^2} + R - \frac{1}{2}(\partial\Phi)^2 - \frac{m^2}{2}\,\Phi^2 - U(\Phi) \right] , \qquad (A.35)$$

where the potential $U(\Phi)$ contains interactions which are cubic or higher order in Φ. According to the standard AdS/CFT dictionary [116] (in the vacuum with $\Phi = 0$), the scalar field is dual to a scalar operator with conformal dimension

$$\Delta = \frac{3}{2} + \sqrt{\frac{9}{4} + m^2 L^2} \qquad (A.36)$$

in the boundary theory. We write the asymptotically AdS$_4$ bulk metric with a flat boundary as

$$ds^2 = \frac{L^2}{z^2} \left(dz^2 + f(z)\, \eta_{ij} dx^i dx^j \right) , \qquad (A.37)$$

where $f(z) \to 1$ as $z \to 0$. Now in a holographic RG flow, the leading asymptotic behaviour of the fields will be

$$\begin{aligned} f(z) &= 1 - \frac{1}{8} \lambda^2\, z^{2(3-\Delta)} + \cdots , \\ \Phi(z) &= \lambda\, z^{3-\Delta} + \cdots , \end{aligned} \qquad (A.38)$$

where λ is the (dimensionful) coupling constant in the boundary theory which initiates the RG flow. Both of the ellipses above represent terms involving higher powers of the combination $\lambda z^{3-\Delta}$.

Now let us consider a strip where the entangling surface consists of two straight parallel lines separated by a distance ε. With some work (see [58] for details), the holographic entanglement entropy becomes[46]

$$ S_{\text{strip}} = \frac{L^2}{2G_{\text{N}}} H \int_\delta^{z_*} dz \, \frac{f(z)^{1/2}}{z^2} \left[1 - \frac{f(z_*)^2 \, z^4}{f(z)^2 \, z_*^4} \right]^{-1/2} , \qquad (A.39) $$

where $z = \delta$ is the UV regulator surface in the bulk and H is an IR scale introduced to regulate the length of the strip — we note that the area of the entangling surface is then $2H$. Further z_* is the maximum value of z which the extremal surface reaches in the bulk, and it is determined in terms of the width of the strip by

$$ \varepsilon = 2 \int_0^{z_*} dz \, \frac{f(z_*) \, z^2}{f(z)^{3/2} \, z_*^2} \left[1 - \frac{f(z_*)^2 \, z^4}{f(z)^2 \, z_*^4} \right]^{-1/2} . \qquad (A.40) $$

Given the asymptotic behaviour of the metric function $f(z)$ in eq. (A.38) and that $z \leq z_*$ in both of the previous integrals, we can evaluate eqs. (A.39) and (A.40) perturbatively in λ if we have $\lambda z_*^{3-\Delta} \ll 1$. It is straightforward to see from eq. (A.40) that ε and z_* are linearly related (to leading order) and therefore we are actually evaluating the entanglement entropy (A.39) as a perturbative expansion in dimensionless combination $\lambda \varepsilon^{3-\Delta}$. Let us begin with eq. (A.40), which yields

$$ \varepsilon = 2z_* \int_0^1 \frac{u^2 \, du}{\sqrt{1-u^4}} \left[1 + \frac{(\lambda z_*^{3-\Delta})^2}{8} \left(\frac{u^{6-2\Delta}}{2} - \frac{1 - u^{6-2\Delta}}{1-u^4} \right) + \cdots \right] $$
$$ = \gamma z_* \left[1 + \frac{(\lambda z_*^{3-\Delta})^2}{16} \left(1 - \frac{3-\Delta}{2} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \frac{\Gamma\left(\frac{9-2\Delta}{4}\right)}{\Gamma\left(\frac{11-2\Delta}{4}\right)} \right) + \cdots \right] , \qquad (A.41) $$

where

$$ \gamma = 2\sqrt{\pi} \, \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} . \qquad (A.42) $$

Alternatively, it will be convenient to write

$$ \frac{1}{z_*} = \frac{\gamma}{\varepsilon} \left[1 + \frac{1}{16} \left(\frac{\lambda \varepsilon^{3-\Delta}}{\gamma^{3-\Delta}} \right)^2 \left(1 - 2 \frac{3-\Delta}{7-2\Delta} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \frac{\Gamma\left(\frac{9-2\Delta}{4}\right)}{\Gamma\left(\frac{7-2\Delta}{4}\right)} \right) + \cdots \right] . \qquad (A.43) $$

Similarly, we can write eq. (A.39) as

$$ S_{\text{strip}} = \frac{L^2}{2G_{\text{N}}} \frac{H}{z_*} \int_{\delta/z_*}^1 \frac{du}{u^2 \sqrt{1-u^4}} \left[1 - \frac{(\lambda z_*^{3-\Delta})^2}{8} u^4 \left(\frac{u^{2-2\Delta}}{2} + \frac{1 - u^{6-2\Delta}}{1-u^4} \right) + \cdots \right] . (A.44) $$

We evaluate the above expression in the limit $\delta/z_* \to 0$ but also with the assumption that we are considering a relevant operator with conformal dimension in the range $3 > \Delta > 5/2$. The final expression for the entanglement entropy then takes the form

$$ S_{\text{strip}} \simeq c_1 \frac{2H}{\delta} - c_1' \frac{2H \lambda^2}{\delta^{2\Delta-5}} - \tilde{a} \frac{H}{\varepsilon} - \tilde{a}' \frac{H \lambda^2}{\varepsilon^{2\Delta-5}} + \cdots , \qquad (A.45) $$

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[46]Note there is a typo in the corresponding eq. (3.15) in [58]!

with

$$c_1 = \frac{L^2}{4G_N}, \qquad c_1' = \frac{L^2}{64G_N}\frac{1}{2\Delta - 5}, \qquad \tilde{a} = \frac{L^2}{4G_N}\gamma^2, \tag{A.46}$$

$$\tilde{a}' = -\frac{\sqrt{\pi}\,L^2}{32\,G_N}\gamma^{2\Delta - 5}\frac{3 - \Delta}{2\Delta - 5}\frac{\Gamma\left(\frac{9-2\Delta}{4}\right)}{\Gamma\left(\frac{11-2\Delta}{4}\right)}.$$

Examining this result, we see that the two terms proportional to c_1 and $\tilde{a}$ reproduce the entanglement entropy (A.32) of the strip calculated in the AdS vacuum. The term proportional to c_1' is a new area law divergence which will generally appear in the entanglement entropy with any geometry for the entangling surface. Hence in calculating the mutual information, each of the separate entropies in eq. (2.2) would contain such a term and so they would cancel amongst each other to yield a finite result — see below. Finally, the above expression is calculated in the regime where the width of the strip was smaller than the scale set by the corresponding coupling, i.e., $\lambda\,\varepsilon^{3-\Delta} \ll 1$, and this limit yields the $\tilde{a}'$ term, which is also proportional to the area of the entangling surface. In our construction, this would produce an extra contribution of the form $2\pi R\,\tilde{a}'\,\lambda^2/\varepsilon^{2\Delta-5}$ in the mutual information (2.7).

The appearance of these unusual area law terms in the mutual information does not effect the discussion in the main text in an essential way. For example, in section 4, the definition of $I_0(A)$ in eq. (4.7) or eq. (4.12) would simply involve removing an additional extensive contribution, i.e., $\tilde{a}'\,\lambda^2\int_\Gamma ds/\varepsilon(s)^{2\Delta-5}$. In general, defining $I_0(A)$ would require removing all of contributions in $I(A^+, A^-)$ which are divergent in the limit $\varepsilon \to 0$. However, the subsequent discussion of the properties of $I_0(A)$ would remain unchanged and hence the proof of the F-theorem remains intact.

Again an essential assumption in the above calculation for $d = 3$ was that the dimension of the relevant operator lay in the range $3 > \Delta > 5/2$. We note that in special case $\Delta = 5/2$, the additional power law contributions above would be replaced by a logarithmic divergence, as discussed in [58]. Further, the above analysis is easily extended to higher dimensions using the results of [58] and one finds additional unconventional power law contributions for $d > \Delta > (d + 2)/2$.

It is interesting to also apply the above analysis to evaluate the holographic entanglement entropy for a circle in $d = 3$. Again, this is a straightforward extension of the calculations in [58] and we only present the results here. In particular, in the regime where the conformal dimension satisfies $3 > \Delta > 5/2$ and the radius of the circle is small, i.e., $\lambda\,R^{3-\Delta} \ll 1$, we find

$$S_{\text{circle}} \simeq c_1\frac{2\pi R}{\delta} - c_1'\frac{2\pi R\,\lambda^2}{\delta^{2\Delta-5}} - 2\pi\,c_0 - 2\pi\,b'\,R^{2(3-\Delta)}\,\lambda^2 + \cdots, \tag{A.47}$$

where c_1 and c_1' are given by the same expressions as in eq. (A.46), $c_0 = L^2/(4G_N)$ as in eq. (A.9) and

$$b' = \frac{L^2}{16G_N}\frac{3 - \Delta}{(7 - 2\Delta)(2\Delta - 5)}. \tag{A.48}$$

Hence the mutual information for these small circles becomes

$$I(A^+, A^-) = \tilde{a}\frac{2\pi R}{\varepsilon} + \tilde{a}'\frac{2\pi R\,\lambda^2}{\varepsilon^{2\Delta-5}} - 4\pi\,c_0 - 4\pi\,b'\,R^{2(3-\Delta)}\,\lambda^2 + \cdots, \tag{A.49}$$

where above we must also have $\varepsilon \ll R$, and the finite part becomes

$$I_0(A) = -4\pi\, c_0 - 4\pi\, b'\, R^{2(3-\Delta)}\, \lambda^2 + \cdots . \tag{A.50}$$

Note that the latter now includes 'nonlocal' contribution that depends on the radius R in a sublinear way.[47] As a result, this contribution is not eliminated in the c-function (4.20). Rather we find

$$C(R) = c_0 + (2\Delta - 5)\, b'\, R^{2(3-\Delta)}\, \lambda^2 + \cdots , \tag{A.51}$$

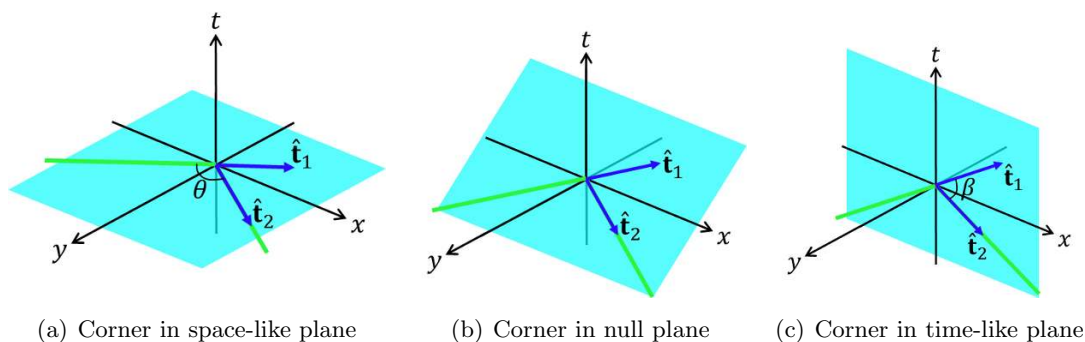however, we recover $C(R \to 0) = c_0$, as expected.

B Null cusps

An important aspect of the proof of the F-theorem, discussed in section 4, is that the various entangling surfaces contain corners but these are all 'null cusps', i.e., each of the corners lies on a null plane, as illustrated in figure 12(b). A key element in the proof is then that these null cusps do not introduce any new local contributions in the corresponding entanglement entropy. While this point was already explained in [32], we would like to explain it in more detail here because it is central to the discussion of section 4.

The absence of new local contributions for null cusps may be contrasted with the more conventional situation, where the entangling surface includes a corner lying in a spatial plane, as illustrated in figure 12(a). In this situation (e.g., see [69–74]), the entanglement entropy acquires an additional logarithmic contribution, $S_{\text{corner}} = -a(\theta) \log(\ell/\delta)$ where θ is the opening angle of the corner. Further, as the spatial corner becomes smooth, the coefficient of this logarithm vanishes quadratically, i.e., $a(\theta) \sim \sigma\, (\pi - \theta)^2$ in the limit $\theta \to \pi$. Adopting the notation of section 4, we may describe the corner in terms of $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$, the two unit tangent vectors to the entangling surface at (on either side of) the corner point — see figure 12(a). For the spatial corner, we have $\hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 = \cos(\pi - \theta) < 1$ and so it is straightforward to show that in the nearly smooth limit, the vanishing of the corner contribution takes the form $a(\theta) \sim \sigma\, |\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2|^2$.

The spatial corner in figure 12(a) and the null cusp in figure 12(b) can be distinguished by the difference of the tangent vectors. In particular, $\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2$ is a spatial vector in the first case and a null vector in the second case. Now the corner on the null plane can be approached by a corner on a boosted spatial plane, in the limit of a large boost, i.e., a large boost along the y-axis will bring the plane in figure 12(a) arbitrarily close to that in figure 12(b) (as seen in the original reference frame). However, simply boosting the initial tangent vectors will produce new tangent vectors with arbitrarily large t and y components — in particular, such a procedure would leave the inner product $|\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2|^2$ fixed. Hence we must adjust the tangent vectors while boosting to bring the initial entangling surface with a spatial corner in figure 12(a) to that shown in figure 12(b). The details of this adjustment procedure are not important but it is clear that with such an approach, $|\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2|^2$ shrinks

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[47]We note that this nonlocal contribution persists for any relevant operator. However, in the regime $\Delta < 5/2$, the behaviour becomes faster than linear. As usual, a logarithm appears at precisely $\Delta = 5/2$, i.e., the above power is replaced by $R \log(R\lambda^2)$.

(a) Corner in space-like plane   (b) Corner in null plane   (c) Corner in time-like plane

Figure 12. The possible types of corners which may arise in an entangling surface (green). The corners are characterized by $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$, the two unit tangent vectors on either side of the corner point. The three cases can then be distinguished with: a) $|\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2|^2 > 0$; b) $|\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2|^2 = 0$; and c) $|\hat{\mathbf{t}}_1 - \hat{\mathbf{t}}_2|^2 < 0$.

and approaches zero in the limit of an infinite boost. Hence given the expression above, we see that the corner contribution in the entanglement entropy also vanishes in this limit.[48]

Alternatively, one can formulate the argument above beginning with an entangling surface with a corner in a time-like plane, as illustrated in figure 12(c). In this case, the two components of the entangling surface are space-like separated but boosted relative to one another, by some boost-angle β. The subsequent discussion of the corner contribution to the entanglement entropy follows essentially that above after analytic continuing the angle $\beta = i(\pi - \theta)$. The conclusion is again that if one approaches the corner on the null plane by boosting the time-like corner, the corresponding corner contribution vanishes in the limit of an infinite boost.

A more general perspective comes from the idea that there is no intrinsic local measure of an 'angle' for a null cusp. In particular, for a spatial corner, the geometry is encoded in $\hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 = \cos(\pi - \theta) < 1$ while in the time-like case, we have $\hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 = \cosh\beta > 1$. However, for a null cusp, $\hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 = 1$ and there is no additional parameter, which is intrinsic to the entangling surface itself, that can be used to describe the corner geometry. Hence the corner coefficient could at best be a fixed constant for all null cusps. However, SSA requires the coefficient of the logarithmic contribution to be negative for a spatial corner (i.e., $a(\theta) \geq 0$ for all θ), but in contrast, the coefficient of log term for a time-like corner has a positive sign (for all β). It naturally follows that the constant for the null cusps must be zero. Alternatively, one can imagine approaching a smooth entangling surface through a sequence of null cusps, which again argue that the fixed constant must vanish.

C The extensive mutual information model

In general, it is difficult to find explicit examples where the mutual information can be computed exactly. The holographic model in the previous appendix provides one such ex-

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[48]Describing the adjustment procedure in terms of the opening angle θ, we would say that θ approaches π in the infinite boost limit.

ample. In this appendix, we consider a model [47] for the mutual information in relativistic QFT which follows by imposing an additional property of extensivity

$$I(A, B) + I(A, C) = I(A, B \cup C) \,. \tag{C.1}$$

With this constraint, the general form of the mutual information satisfying the combined requirements of strong subadditivity and Lorentz covariance is known. While the resulting expression can be shown to provide the mutual information of a free fermion in two dimensions, it is not known at present if this form corresponds to the actual mutual information of any other specific QFT's. We use this model here to investigate the new c-function proposed in section 2 since it has the advantage of having a simple analytical expression. We also note that by construction, this expression satisfies all of the constraints imposed by strong subadditivity, which are rather nontrivial in the Minkowski space geometry and largely determines the essential features of the entanglement entropy. In particular, the extensive mutual information model already involves a c-theorem for a specific c-function, a fact which will help us to test our conjectures by relating this property with the putative c-theorem for eq. (2.8), which yields $\tilde{c}_0$ at fixed points.

Explicitly, for the extensive mutual information in $d = 3$, we have [47]

$$I(A^+, A^-) = -\int_{\partial A^-} ds_x \int_{\partial A^+} ds_y \, H(|x - y|) \, \hat{\mathbf{t}}_x \cdot \hat{\mathbf{t}}_y \,, \tag{C.2}$$

with $\hat{\mathbf{t}}_x$ the unit vector tangent to the boundary at the point x and $H(\ell)$ a function of the distance between the points — see figure 13. The direction of tangent vectors $\hat{\mathbf{t}}$ is chosen such that $\varepsilon_{abc} \, \hat{\mathbf{u}}^a \, \hat{\mathbf{n}}^b \, \hat{\mathbf{t}}^c > 0$ where $\hat{\mathbf{u}}$ is a future-directed time-like vector and $\hat{\mathbf{n}}$ is the unit normal to the boundary pointing into the region on the Cauchy surface on which A^+ and A^- are defined.[49] Further we have

$$H(\ell) = -\int^\ell dy \, \frac{\kappa(y)}{y^3} \,. \tag{C.3}$$

An integration constant is left undetermined here but it is irrelevant since adding a constant to $H(\ell)$ does not change the result for the mutual information in eq. (C.2). The function $\kappa(\ell)$ is a c-function in the sense that it must be positive, dimensionless, decreasing with increasing ℓ, and constant at the conformal fixed points.

Let us first calculate the mutual information for the annular geometry in figure 13 at a conformal point with $\kappa(\ell) = \kappa$ is some fixed constant. Eq. (C.3) then yields $H(\ell) = \kappa/(2\ell^2)$ and we have
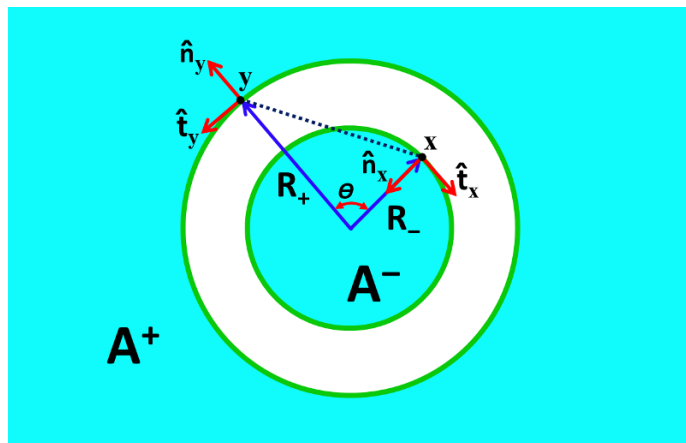
$$I(A^+, A^-) = \pi\kappa \, R_+ \, R_- \int_0^{2\pi} d\theta \, \frac{\cos\theta}{R_+^2 + R_-^2 - 2R_+R_- \, \cos\theta} = \pi^2\kappa \left(\frac{R_+^2 + R_-^2}{R_+^2 - R_-^2} - 1 \right) \,. \tag{C.4}$$

Substituting for R_+ and R_- with eq. (2.5), i.e., $R_- = R - (1/2 - \alpha)\varepsilon$ and $R_+ = R + (1/2 + \alpha)\varepsilon$, and then expanding for small ε/R, we find

$$I(A^+, A^-) \equiv I(R, \varepsilon, \alpha) = \pi^2\kappa \, \frac{R}{\varepsilon} + \pi^2\kappa \, (\alpha - 1) + O(\varepsilon/R) \,. \tag{C.5}$$

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[49]Implicitly we are assuming that $\varepsilon_{txy} = +1$ here. However, we also note the result for $I(A^+, A^-)$ would not change if another convention was chosen to determine the direction of $\hat{\mathbf{t}}$.

Figure 13. Mutual information in the extensive model is given directly by a double integral of a certain function over the points x and y on the boundaries of the regions $A+$ and A^-.

Hence the mutual information has the expected form given in eq. (2.7) with $\tilde{a} = \pi\kappa/2$, $\tilde{b} = 0$ and $\tilde{c}_0 = \pi\kappa\,(1-\alpha)/4$. We also see that if we change α, then the constant $\tilde{c}_0$ is shifted as in eqs. (2.9) and (2.10).

Our next task is to calculate the mutual information in the case $\kappa(\ell)$ describes a RG flow running between a UV fixed point with $\kappa(\ell \to 0) = \kappa_{UV}$ and an infrared point where $\kappa(\ell \to \infty) = \kappa_{IR}$. For simplicity, we will assume the derivative of this c-function vanishes at the origin, i.e., $\kappa'(0) = 0$.[50] To simplify the calculations, we will fix α and specifically we choose $\alpha = 0$ in accord with the prescription discussed in section 2.3. The mutual information then becomes

$$I(R,\varepsilon) = -2\pi(R^2 - \varepsilon^2/4) \int_0^{2\pi} d\theta \, \cos\theta \int^{q(\theta)} dy \, \frac{\kappa(y)}{y^3} \,, \qquad (C.6)$$

with

$$q(\theta) = \sqrt{2R^2 + \varepsilon^2/2 - (2R^2 - \varepsilon^2/2)\,\cos\theta} \,. \qquad (C.7)$$

In order to analyze the running of the constant term, we calculate the c-function in eq. (2.8) here but we do not yet the limit $\varepsilon \to 0$, i.e.,

$$C_\varepsilon(R) = \frac{1}{4\pi} \left(R \frac{\partial I(R,\varepsilon)}{\partial R} - I(R,\varepsilon) \right) \,. \qquad (C.8)$$

Now using eqs. (C.6) and (C.7) and integrating by parts once, the final expression can be written as

$$C_\varepsilon(R) = R^4 \int_0^{2\pi} d\theta \, \frac{\kappa(q(\theta))}{q(\theta)^4} \left[\frac{1}{2} \left(1 - \frac{\varepsilon^4}{16R^4} \right) \sin^2\theta - \left(1 - \frac{\varepsilon^2}{4R^2} \right) \cos\theta\,(1 - \cos\theta) \right] \,. \quad (C.9)$$

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[50]Note that if the small$-\ell$ expansion of the c-function takes the form $\kappa = \kappa_{UV} + \lambda\,\ell^a + \cdots$ with $0 < a < 1$, then the mutual information will contain subleading 'area law' contributions of the form $R\,\lambda/\varepsilon^{1-a}$. (A contribution proportional to $\log(\varepsilon)$ appears with $a = 1$.) These unusual 'area law' contributions indicate that the RG flow is generated by perturbing the UV conformal point with a relevant, but close to marginal, operator, as discussed in appendix A.1.

One can readily confirm that the integrand above is everywhere nonsingular in the limit $\varepsilon \to 0$ and hence in the model with extensive mutual information, our c-function (2.8) becomes

$$C(R) = \lim_{\varepsilon \to 0} C_\varepsilon(R) = \frac{1}{8} \int_0^{2\pi} d\theta \; \kappa(2R\sin(\theta/2)) = \frac{1}{2} \int_0^{2R} \frac{dy \; \kappa(y)}{\sqrt{4R^2 - y^2}} \,. \qquad \text{(C.10)}$$

Either of the above expressions shows that $C(R)$ averages the c-function appearing in the extensive model over all scales from zero out to the diameter of the annulus geometry used to define our new c-function. In fact, if we observe that $2R\sin(\theta/2)$ is the chord length between two points separated by an angle θ on the circle of radius R, then the averaging in the angular integral is over the distance between all pairs of points on the circle. But we also observe that the measure in the y integration in the second expression gives the greatest weight to scales near $y \simeq 2R$, i.e., near the infrared scale defined by the circle. In the limits of large and small circles, we find that the c-function interpolates between $C(R \to 0) = \pi \kappa_{UV}/4$ and $C(R \to \infty) = \pi \kappa_{IR}/4$. Further, it is evident that $C(R)$ decreases monotonically with increasing R as a consequence of the decreasing character of $\kappa(y)$. Incidentally, this simple example shows c-functions are by no means unique, i.e., given a c-function, we can construct infinitely many other c-functions which are also monotonically decreasing and interpolate between the same fixed point values.

It is also interesting to examine the running of the area term in the mutual information, i.e., the term proportional to $2\pi R$ in eq. (2.7). First we can integrate by parts to re-express the mutual information (C.6) as

$$I(R, \varepsilon) = 2\pi R^4 \left(1 - \frac{\varepsilon^2}{4R^2}\right)^2 \int_0^{2\pi} d\theta \, \frac{\kappa(q(\theta))}{q(\theta)^4} \, \sin^2 \theta \,. \qquad \text{(C.11)}$$

Now the limit $\varepsilon \to 0$ produces a singularity in the integrand at $\theta = 0$ (and 2π) since eq. (C.7) yields $q(\theta \simeq 0) \simeq \sqrt{R^2\theta^2 + \varepsilon^2}$. This potential singularity is the source of the contribution $2\pi R\tilde{a}/\varepsilon$ and we can approximate the result $\theta \simeq 0$ portion of the integral by replacing $\kappa(q(\theta)) \simeq \kappa(0) = \kappa_{UV}$. The result is then the same as found above in eq. (C.5) for a conformal fixed point with $\tilde{a} = \pi \kappa_{UV}/2$. We note that as this result applies for all values of the radius R and as expected, the coefficient depends only on information from the UV portion of the RG flow.

Now for small R, i.e., smaller than any scales appearing in the flow defined by $\kappa(\ell)$, $\kappa(0) = \kappa_{UV}$ is a good approximation for the entire integration in eq. (C.11) and so the entire result will coincide with that in eq. (C.5) (with $\alpha = 0$). In particular, then $\tilde{b} = 0$ for small R but this could again be anticipated because we are only probing the UV fixed point and there are no dimensionful parameters from which $\tilde{b}$ could be constructed. In contrast, while large R definitely probes the IR fixed point, eq. (C.11) actually samples the interpolating function $\kappa(\ell)$ over all scales from $\ell = 0$ to $2R$. Hence we can expect a nonvanishing $\tilde{b}$ will appear in the large radius limit. In fact, considering eqs. (2.7) and (C.6), we find

$$\tilde{b} = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left[\left. \frac{\partial I(R, \varepsilon)}{\partial R} \right|_{R \to \infty} - \left. \frac{\partial I(R, \varepsilon)}{\partial R} \right|_{R \to 0} \right] = -2 \int_0^\infty \frac{dy}{y^2} \left[\kappa_{UV} - \kappa(y) \right] \,. \qquad \text{(C.12)}$$

Notice that our assumption that $\kappa'(0) = 0$ ensures that the $y = 0$ end of the integration yields a finite result. Further, because $\kappa(\ell)$ decreases with growing ℓ, we note that the integrand is everywhere positive and hence $\tilde{b} < 0$. That is, the area term in eq. (2.7) picks up a finite negative contribution in the infrared. This result is in accordance with the results of [32, 52] on the running of the area term for the entanglement entropy of a circle.[51]

We conclude here by summarizing our results for the extensive mutual information model for small and large circles and a generic value of α:

$$I(R,\varepsilon,\alpha) = \frac{\pi^2 \kappa_{UV}}{\varepsilon} R - \pi^2 \kappa_{UV} + \alpha \pi^2 \kappa_{UV} + O(\varepsilon) \qquad \text{for } R \to 0\,, \quad \text{(C.14)}$$

$$I(R,\varepsilon,\alpha) = \left(\frac{\pi^2 \kappa_{UV}}{\varepsilon} + 2\pi \tilde{b} \right) R - \pi^2 \kappa_{IR} + \alpha \pi^2 \kappa_{UV} + O(\varepsilon) \quad \text{for } R \to \infty\,.$$

where $\tilde{b}$ is given by eq. (C.12). We see the constant term, i.e., the term independent of R, at the infrared fixed point is intrinsic to the infrared theory alone only with the choice $\alpha = 0$, as argued in section 2.3. With our analysis here, it may seem quite remarkable that any choice of α yields a $\tilde{c}_0$ which is only proportional to the IR quantity κ_{IR}, and is independent of the UV value κ_{UV} or any of the details of RG flow incorporated in the interpolating function $\kappa(\ell)$.

C.1 Contribution of a strip with a null cusp

For the discussion in section 4, it is interesting to evaluate the contributions coming from null cusps within extensive mutual information model. In particular, we would like to evaluate the quantities $f(q_1, q_2)$ and $J(q_1, q_2)$ defined in eqs. (4.12) and (4.16). Since these are purely local contributions, we can simplify the calculation by evaluating these quantities for null cusp formed by the intersection of two straight lines lying in a null plane. Using standard Minkowski coordinates, we can choose without generality the null plane such that its tangent space is spanned by the unit vector $\hat{\mathbf{x}}$ and the null vector $\hat{\mathbf{v}} = \hat{\mathbf{t}} + \hat{\mathbf{y}}$. Now a framing must be constructed for each of the boundary lines as in eq. (4.6) but again for simplicity, we choose $\varepsilon(s)$ to be a (different) fixed constant for each of the lines. The geometry under consideration will then consist of two straight strips with edges lying in two parallel null planes. However, there is still a certain freedom in choosing the precise framing. We fix the latter[52] by choosing the unit vector directed between the two intersection points of the strip edges to be $\hat{\mathbf{m}} = \hat{\mathbf{y}} = (0,0,1)$. At the same time, we may write the tangent vectors to the two lines as

$$\hat{\mathbf{t}}_1 = \hat{\mathbf{x}} + q_1 \hat{\mathbf{v}} = (q_1, 1, q_1)\,, \qquad \text{and} \qquad \hat{\mathbf{t}}_2 = \hat{\mathbf{x}} + q_2 \hat{\mathbf{v}} = (q_2, 1, q_2)\,. \qquad \text{(C.15)}$$

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[51]We can extend the comparison with [32, 52] by considering

$$\frac{1}{2\pi} \lim_{\varepsilon \to 0} \left[\left. \frac{\partial I(R,\varepsilon)}{\partial R} \right|_R - \left. \frac{\partial I(R,\varepsilon)}{\partial R} \right|_{R \to 0} \right] = -2 \int_0^{2R} \frac{dy}{y^2} \frac{\kappa_{UV} - \kappa(y)}{\sqrt{1 - (y/2R)^2}}\,, \qquad \text{(C.13)}$$

which decreases with increasing R, i.e., from the UV to the IR.

[52]One can verify that given any framing as described above, $\hat{\mathbf{m}}$ can always be brought to the desired form by applying an SO(1,2) transformation Λ, which satisfies $\hat{\mathbf{v}}^T \cdot \Lambda \cdot \hat{\mathbf{v}} = 0$.

The tangent vectors are parametrized above according to the definitions in eq. (4.10), i.e.,

$$q_1 = \widehat{\mathbf{m}} \cdot \hat{\mathbf{t}}_1 \qquad \text{and} \qquad q_2 = \widehat{\mathbf{m}} \cdot \hat{\mathbf{t}}_2 \,. \tag{C.16}$$

Recall that the limit $q_2 \to q_1$ corresponds the case where the tangent vectors become equal and hence the cusp vanishes.

Our following discussion is adapted to null cusp illustrated in figure 8, with $\hat{\mathbf{t}}_1$ being the tangent to Γ_B while $\hat{\mathbf{t}}_2$ is tangent to Γ_A. Further note that in the configuration shown there, we would have $q_1 > 0$ and $q_2 < 0$. Now let us take the first pair of edges, i.e., the boundaries of A^- and B^-, to lie in the null plane passing through the origin with[53]

$$\Gamma_{A^-} : \ (q_2\,x, x, q_2\,x) \qquad \text{and} \qquad \Gamma_{B^-} : \ (q_1\,x, x, q_1\,x) \,, \tag{C.17}$$

where the curves were chosen to intersect at $(0,0,0)$. The second pair of edges, i.e., the boundaries of A^+ and B^+, are then placed in the second null plane with

$$\Gamma_{A^+} : \ (q_2\,x, x, q_2\,x + \tilde{\varepsilon}) \qquad \text{and} \qquad \Gamma_{B^-} : \ (q_1\,x, x, q_1\,x + \tilde{\varepsilon}) \,, \tag{C.18}$$

which then intersect at $(0, 0, \tilde{\varepsilon})$. We note that with the above parameterization, it may appear that, e.g., Γ_{A^+} and Γ_{A^-} are separated by the distance $\tilde{\varepsilon}$. However, this is the displacement along the vector $\widehat{\mathbf{m}}$ rather than the normal vector between the two curves, as specified in eq. (4.6). It is straightforward to show that the appropriate unit normal vectors for the two strips are

$$\hat{\mathbf{n}}_A = (-q_2^2, -q_2, 1 - q_2^2)/\sqrt{1 - q_2^2} \quad \text{and} \quad \hat{\mathbf{n}}_B = (-q_1^2, -q_1, 1 - q_1^2)/\sqrt{1 - q_1^2} \,. \tag{C.19}$$

Then let us specify the widths across the A and B strips measured as the proper distance along these normals as ε_A and ε_B. We find that these widths are related to $\tilde{\varepsilon}$ by

$$\tilde{\varepsilon} = \frac{\varepsilon_A}{\sqrt{1 - q_2^2}} = \frac{\varepsilon_B}{\sqrt{1 - q_1^2}} \,. \tag{C.20}$$

In the main text, we used the notation $\varepsilon_A = \varepsilon_2$ and $\varepsilon_B = \varepsilon_1$, and hence eq. (4.11) follows directly from these relations.

We can now determine $f(q_1, q_2)$ by evaluating the mutual information $I(A^+ \cup B^+, A^- \cap B^-)$, however, we must introduce an infrared cut-off to produce finite integrals. On Γ_{A^-} and Γ_{B^-}, we limit the range of x to $-L/2 < x < L/2$. The range of the x integration along Γ_{A^+} is chosen so that the ends of the corresponding strip A^s lies along the normal $\hat{\mathbf{n}}_A$, given in eq. (C.19). Hence we choose $-L/2 - q_2\tilde{\varepsilon} < x < L/2 - q_2\tilde{\varepsilon}$ for Γ_{A^+}. Similarly, we choose $-L/2 - q_1\tilde{\varepsilon} < x < L/2 - q_1\tilde{\varepsilon}$ on Γ_{B^+} so that the ends of B^s lie parallel to $\hat{\mathbf{n}}_B$.

---

[53]As before, we use standard Cartesian coordinates (t, x, y) in describing the geometry here.

Now the corresponding mutual information is given by

$$
\begin{aligned}
I&(q_2, q_1, \varepsilon, L) \\
&= \int_0^{L/2-q_2\tilde{\varepsilon}} dx_1 \int_0^{L/2} dx_2 \frac{\kappa/2}{(x_1-x_2)^2 + (q_2 x_1 + \tilde{\varepsilon} - q_2 x_2)^2 - (q_2 x_1 - q_2 x_2)^2} \qquad \text{(C.21)} \\
&+ \int_0^{L/2-q_2\tilde{\varepsilon}} dx_1 \int_{-L/2}^0 dx_2 \frac{\kappa/2}{(x_1-x_2)^2 + (q_2 x_1 + \tilde{\varepsilon} - q_1 x_2)^2 - (q_2 x_1 - q_1 x_2)^2} \\
&+ \int_{-L/2-q_1\tilde{\varepsilon}}^0 dx_1 \int_{-L/2}^0 dx_2 \frac{\kappa/2}{(x_1-x_2)^2 + (q_1 x_1 + \tilde{\varepsilon} - q_1 x_2)^2 - (q_1 x_1 - q_1 x_2)^2} \\
&+ \int_{-L/2-q_1\tilde{\varepsilon}}^0 dx_1 \int_0^{L/2} dx_2 \frac{\kappa/2}{(x_1-x_2)^2 + (q_1 x_1 + \tilde{\varepsilon} - q_2 x_2)^2 - (q_1 x_1 - q_2 x_2)^2} \, .
\end{aligned}
$$

As in eq. (4.12), we must subtract $\tilde{a} \int ds/\varepsilon(s)$, the area term along the curve running along the midpoint line of the cusp. Recall that in eq. (C.5), we found $\tilde{a} = \pi\kappa/2$ for this model and hence we have

$$
\begin{aligned}
\tilde{a} \int \frac{ds}{\varepsilon(s)} &= \frac{\pi\kappa}{2} \int_{-(L+q_1\tilde{\varepsilon})/2}^0 \frac{ds}{\varepsilon_B} + \frac{\pi\kappa}{2} \int_0^{(L-q_2\tilde{\varepsilon})/2} \frac{ds}{\varepsilon_A} \\
&= \frac{\pi\kappa}{4} \left(\frac{L}{\varepsilon_A} + \frac{L}{\varepsilon_B} + \frac{q_1}{\sqrt{1-q_1^2}} - \frac{q_2}{\sqrt{1-q_2^2}} \right) , \qquad \text{(C.22)}
\end{aligned}
$$

where we have simplified the final expression using eq. (C.20). Here we have ended the curves abruptly at the infrared cut-off and the corresponding end-points will introduce another frame dependent contribution. The latter can be independently evaluated for a single interval with no cusp and is given by $-\kappa\left[\log(L/\sqrt{\varepsilon_A\varepsilon_B}) + 1\right]$. Subtracting this contribution and the area term (C.22), we find

$$
\begin{aligned}
f(q_1, q_2) &= \lim_{\tilde{\varepsilon}\to 0} \Bigg(I(q_2, q_1, \varepsilon, L) + \kappa\left[\log(L/\sqrt{\varepsilon_A\varepsilon_B}) + 1\right] \qquad \text{(C.23)} \\
&\qquad\qquad\qquad\qquad - \frac{\pi\kappa}{4}\left[\frac{L}{\varepsilon_A} + \frac{L}{\varepsilon_B} + \frac{q_1}{\sqrt{1-q_1^2}} - \frac{q_2}{\sqrt{1-q_2^2}} \right] \Bigg) \\
&= \frac{\kappa}{2}\left(2 + \frac{(q_1(q_1+q_2)-2)\arcsin(q_1)}{(q_1-q_2)\sqrt{1-q_1^2}} + \frac{(q_2(q_1+q_2)-2)\arcsin(q_2)}{(q_2-q_1)\sqrt{1-q_2^2}} \right) ,
\end{aligned}
$$

as was given in eq. (4.13).

Note that the infrared regulator L is arbitrary and does not enter in the final result. As a result, the width $\tilde{\varepsilon}$ also drops out of the final expression which must be dimensionless. Hence the result depends only on q_1 and q_2 and in fact, it is symmetric in exchanging $q_1 \leftrightarrow q_2$. Further, it is straightforward to verify that the expression above for $f(q_1, q_2)$ vanishes in the limit $q_2 \to q_1$.

Implicitly, we have also presented the calculation for a constant κ, however, one can readily verify that in the case of a nontrivial RG flow with $\kappa(\ell)$, the quantity that enters the final expression (C.23) is $\kappa_{UV} = \kappa(0)$. One may also repeat the calculation choosing

the bent strip in the positive y region, i.e., evaluating $I(A^+ \cap B^+, A^- \cup B^-)$, and exactly the same result is found for $f(q_1, q_2)$.

Finally we would like to determine $J(q_1, q_2)$ as defined in eq. (4.16), however, we need the entanglement entropy to evaluate this expression. The natural definition of $S(A)$ in this model is given modifying the double integral (C.2) so that x and y run along the boundary of the region A, i.e., along the same boundary,[54]

$$S(A) = -\int_{\partial A} ds_x \int_{\partial A} ds_y \, H(|x-y|) \, \hat{\mathbf{t}}_x \cdot \hat{\mathbf{t}}_y . \tag{C.24}$$

Substituting this expression into the definition of the mutual information (1.2) then reproduces the expected expression in eq. (C.2). Of course, the above expression for $S(A)$ is divergent but a short-distance cut-off can introduced by only integrating over regions where $|x-y| > \delta$. However, it is not difficult to see that the integrals in eq. (4.16) involving x and y on the same null plane, which are the only ones which depend on δ, cancel in the combination $J(q_1, q_2)$. Of course, this was to be expected since in the main text, we argued that $J(q_1, q_2)$ was a finite regulator-independent quantity. Hence it is straightforward to see that evaluating the expression (4.16) for $J(q_1, q_2)$ involves the same integrals as above and we find

$$J(q_1, q_2) = -2f(q_1, q_2), \tag{C.25}$$

as argued in section 4.

D Circle entropy for a scalar field

In this appendix, we review two computations of the constant term in the entanglement entropy (2.1) of a circle for a free massless scalar. First, we use the conformal mapping from the causal domain of the disk to $R \times H^2$, where H^2 is the two-dimensional hyperbolic plane. The second calculation involves a similar mapping to (the static patch of) three-dimensional de Sitter space. Our purpose here is to highlight the ambiguity in the definition of this constant term, even when using "covariant" cut-offs. In the section 3, we show how this ambiguity can be eliminated with a prescription emerging from what we learned from the mutual information.

D.1 Mapping to $R \times H^2$

We calculate the entanglement entropy of a circle for a conformally coupled massless scalar field in three space-time dimensions as the thermodynamic entropy of thermal gas of scalar particles in hyperbolic space [13–16].

The causal domain of dependence of a disk $r < R$ (and $t = 0$) in flat space

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 \tag{D.1}$$

---
[54]Essentially the same expression for the entanglement entropy emerges from the heuristic model in [48].

is simply given by $r < R - |t|$. This region is conformally mapped to $R \times H^2$ by the change of coordinates

$$t = R \frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)} \,, \tag{D.2}$$

$$r = R \frac{\sinh u}{\cosh u + \cosh(\tau/R)} \,. \tag{D.3}$$

After eliminating a Weyl rescaling, the metric in these new coordinates is

$$ds^2 = -d\tau^2 + R^2 (du^2 + \sinh u^2 d\theta^2) \,, \tag{D.4}$$

where we recognize the spatial metric describes a hyperbolic space with scalar curvature

$$\mathcal{R} = -\frac{2}{R^2} \,. \tag{D.5}$$

As explained in [13], the density matrix transforms unitarily under conformal transformations, and in the new hyperbolic space, corresponds to the thermal equilibrium state with temperature $T = 1/(2\pi R)$. Hence, the entanglement entropy of the circle becomes the thermodynamic entropy $S(T)$ in $R \times H^2$ and can be written as

$$S(T) = (1 + T \, \partial_T) \log Z(T) \,. \tag{D.6}$$

In the present calculation, $\log Z(T)$ is the effective action of a massless conformally coupled scalar field ϕ,

$$\left(\partial_\tau^2 - \Delta_{H^2} + \frac{1}{8}\mathcal{R} \right) \phi = 0 \,. \tag{D.7}$$

We refer the interested reader to [13–16] for further details.

We are interested in the constant term of the expansion of the entropy. The time coordinate is compactified to a circle of size $1/T$ after passing to Euclidean coordinates. To calculate $\log Z$, we use the heat kernel method which gives

$$\log Z = \frac{1}{2} \int_0^\infty \frac{dt}{t} \, K(t) \,, \tag{D.8}$$

where $K(t)$ factorizes as

$$K(t) = K_{H^2}(t) \, K_{S^1}(t) \, e^{-t\mathcal{R}/8} \,. \tag{D.9}$$

The explicit expressions for the heat kernels of the Laplace operator in hyperbolic space $K_{H^2}(t)$ and the circle $K_{S^1}(t)$ are well known [110–112],

$$K_{S^1}(t) = \frac{2}{\sqrt{4\pi t}\, T} \sum_{n=1}^\infty e^{-\frac{n^2}{4T^2 t}} \,, \tag{D.10}$$

$$K_{H^2}(t) = f(t) \int d^2x \, \sqrt{g} \,. \tag{D.11}$$

In the second expression, the integral is over the volume of the hyperbolic space and

$$f(t) = \lim_{\rho \to 0} \frac{R^3 e^{-\frac{t}{4R^2}}}{2^{5/2}\pi^{3/2} t^{3/2}} \int_{\rho/R}^\infty ds \, \frac{s e^{-s^2 R^2/4t}}{(\cosh s - \cosh \rho/R)^{\frac{1}{2}}} \,. \tag{D.12}$$

Then the calculation of $\log Z$ involves solving the integrals in t and s, as well as an infinite sum in n. After some algebra, we obtain

$$S = \text{vol}(H^2) \, \frac{\sqrt{2}}{8\pi R^2} \int_0^\infty ds \, \frac{2\coth(s/2) + s\,\text{csch}(s/2)^2 - s^2 \coth(s/2)\,\text{csch}(s/2)^2}{s^2(\cosh(s)-1)^{1/2}} \,. \quad \text{(D.13)}$$

The result of this integral gives

$$S = \frac{\text{vol}(H^2)}{16\pi R^2} \left[\log 2 - \frac{3\zeta(3)}{2\pi^2} \right] \,. \quad \text{(D.14)}$$

Here, we can recognize the factor in the brackets as also appearing in the final result for c_0^{scalar} in eq. (2.16).

Hence the constant term in the entropy comes from the product of a numerical constant times the volume of the hyperbolic space in eq. (D.4). Of course, the latter is divergent and so we must introduce a cut-off $u_{\max}$ in the radial integral

$$\text{vol}(H^2) = 2\pi R^2 \int_0^{u_{\max}} du \, \sinh(u) \,. \quad \text{(D.15)}$$

The desired universal contribution to the entropy is then determined by taking the constant term in an expansion of the regulated volume with respect to the short-distance cut-off in the entanglement entropy calculation. With this aim, let us define the short-distance cut-off δ in the original radial coordinates in flat space by summing the entropy contributions up to $r_{\max} = R - \delta$. Then, taking into account the mapping of coordinates (D.3), we find

$$u_{\max} = -\log\left(\frac{\delta/R}{2 - \delta/R} \right) \,. \quad \text{(D.16)}$$

Thus expanding in δ/R, we have

$$\text{vol}(H^2) = 2\pi R^2 \left(\frac{R}{\delta} - \frac{3}{2} + O(\delta/R) \right) \,. \quad \text{(D.17)}$$

Substituting this result back in eq. (D.14), the constant term in the entropy (2.1) is

$$2\pi c_0 = \frac{3}{2} 2\pi c_0^{scalar} \,, \quad \text{(D.18)}$$

where the coefficient c_0^{scalar} is given in eq. (2.16). Of course, our result here differs from the expected constant term by a factor $3/2$.

Evidently, the result above depends on the details of our cut-off prescription. For example, let us consider slightly modifying the short-distance cut-off and integrating up to $r_{\max} = R - \hat{\delta}$ with

$$\hat{\delta} = \delta \left(1 + a_1 \frac{\delta}{R} + a_2 \frac{\delta^2}{R^2} + \cdots \right).. \quad \text{(D.19)}$$

This small change in the prescription makes a small change in the total regulated volume but it has a finite effect on the constant term in the δ expansion, with eq. (D.17) becoming

$$\text{vol}(H^2) = 2\pi R^2 \left(\frac{R}{\delta} - \left(\frac{3}{2} + a_1 \right) + O(\delta/R) \right) \,. \quad \text{(D.20)}$$

Hence we may reproduce the desired coefficient (2.16) in this calculation by choosing $a_1 = -1/2$. However, this seems a rather mysterious and arbitrary choice as presented here.

As an example, the correct coefficient can be obtained if one defines the radial cut-off with $\delta = 2R \exp[-u_{\max}]$ rather than eq. (D.16). This prescription, used in [60], is motivated by the corresponding holographic calculations [11–13]. This choice yields to an expansion as in eq. (D.19) for the effective cut-off $\hat{\delta}$ in the flat space integration in terms of δ/R

$$\hat{\delta} = \delta \left(1 - \frac{1}{2} \frac{\delta}{R} + \frac{1}{4} \frac{\delta^2}{R^2} + \cdots \right) . \tag{D.21}$$

As explained in section 3, the success of this prescription has its origin in respecting a certain inversion symmetry. Of course, the correct coefficient will also be produced by any other prescription where the corresponding expansion (D.19) yields $a_1 = -1/2$. However, at this point, in this specific calculation, and without using the results from zeta function regularization of the free energy of the sphere, we do not have in principle any reason to prefer one expansion from the other. This is a manifestation of the ambiguities in the evaluating constant term in the entanglement entropy (2.1).

D.2 Mapping to de Sitter space

This section is based on Dowker's calculation of the entanglement entropy for spheres in even dimensions [113], using a conformal mapping to de Sitter space. Here we apply the same approach to calculate the universal contribution in $d = 3$.

In this case, we apply the following coordinate transformation,

$$t = R \frac{\sqrt{1 - \hat{r}^2/R^2} \, \sinh(\tau/R)}{1 + \sqrt{1 - \hat{r}^2/R^2} \, \cosh(\tau/R)} , \tag{D.22}$$

$$r = \frac{\hat{r}}{1 + \sqrt{1 - \hat{r}^2/R^2} \, \cosh(\tau/R)} , \tag{D.23}$$

to the flat space metric (D.1). After eliminating of the Weyl scaling, the causal domain of the disk $r < R$ has been mapped to the static patch of de Sitter space with metric

$$ds^2 = - \left(1 - \frac{\hat{r}^2}{R^2} \right) d\tau^2 + \frac{d\hat{r}^2}{1 - \frac{\hat{r}^2}{R^2}} + \hat{r}^2 d\theta^2 . \tag{D.24}$$

As happens with the mapping to the hyperbolic geometry, in this new conformal frame, the entanglement entropy of the circle can be thought as the thermal entropy in de Sitter space. Here, the temperature is associated with the cosmological horizon of the static patch (D.24) and the static observer measures $T = 1/(2\pi R)$, where R is now also the curvature scale of the de Sitter geometry. Hence the one route to evaluating the entropy is first to compute the energy density at a finite temperature T and then to integrate this density over the static patch to obtain the total energy $E(T)$. The entropy is determined with the thermodynamic formula

$$S(T) = \int_0^T \frac{dT}{T} \frac{dE(T)}{dT} . \tag{D.25}$$

The final step is to fix T to be the equilibrium temperature of the horizon, i.e., $T = 1/(2\pi R)$.

The energy density for a conformally coupled free scalar at finite temperature in de Sitter space was calculated in [114] by conformally transforming the known expressions in Rindler space at finite temperature with respect to Rindler time. Following [113], for convenience we change radial coordinates in de Sitter metric as

$$z^2 = \frac{R - \hat{r}}{R + \hat{r}},\tag{D.26}$$

with $z \in (0, 1)$. With this radial coordinate, the horzion becomes $z = 0$ while the origin sits at $z = 1$. The de Sitter metric (D.24) now becomes

$$ds^2 = \frac{4}{(1 + z^2)^2}\left(-z^2 d\tau^2 + R^2 dz^2\right) + R^2 \left(\frac{1 - z^2}{1 + z^2}\right)^2 d\theta^2.\tag{D.27}$$

In these coordinates, the energy density in thermal de Sitter space is given by [114]

$$\langle\, T_\tau{}^\tau\,\rangle = \frac{T(1 + z^2)^3}{64\sqrt{2}\,\pi\, R^2\, z^3}\left[W_3 - \frac{1}{4}W_1\right]\tag{D.28}$$

where

$$W_n \equiv -2^{\frac{4-n}{2}}\int_0^\infty \frac{dy}{\cosh(y/2)^n}\frac{\sin(2\pi^2 R T)}{\cosh(2\pi R T\, y) - \cos(2\pi^2 R T)}\,.\tag{D.29}$$

The entropy is now obtained from eq. (D.28) by integrating the energy density over the volume of the static patch and using eq. (D.25). Solving the integrals, we find

$$\begin{aligned}
S &= \frac{1}{2^{7/2}}\int_0^{1/(2\pi R)}\frac{dT}{T}\frac{d(T\,[W_3 - \frac{1}{4}W_1])}{dT}\int_{z_{\min}}^1 \frac{(1 - z^2)}{z^2}dz \\
&= \frac{1}{2}\left(z_{\min} - 2 + \frac{1}{z_{\min}}\right)(2\pi\,c_0^{scalar})\,.
\end{aligned}\tag{D.30}$$

Here the volume of the static patch is finite but the result is divergent due to the singular contribution of the energy density for $T < 1/(2\pi R)$ as one approaches the horizon. To give a meaning to the regularization on the coordinate $z > z_{\min}$ used here, let us translate it to planar coordinates by the transformations in eqs. (D.23) and (D.26). Choosing $r_{\max} = R - \hat{\delta}$ with $\hat{\delta} = \delta\left(1 + a_1\delta/R + \cdots\right)$ as in eq. (D.19), we find the coefficient of the constant term is

$$c_0 = \left(\frac{3}{2} - a_1\right) c_0^{scalar}\,.\tag{D.31}$$

Hence we recognize the same coefficients and ambiguities as in the previous calculations with the hyperbolic geometry.

References

[1] G. 't Hooft, *Recent developments in gauge theories*, Plenum Press, U.S.A. (1980), reprinted in *Unity of Forces in the Universe Vol. II*, A. Zee ed., World Scientific, Singapore (1982).

[2] A.B. Zamolodchikov, *Irreversibility of the flux of the renormalization group in a* 2D *field theory*, *JETP Lett.* **43** (1986) 730 [INSPIRE].

[3] Z. Komargodski and A. Schwimmer, *On renormalization group flows in four dimensions*, *JHEP* **12** (2011) 099 [arXiv:1107.3987] [INSPIRE].

[4] Z. Komargodski, *The constraints of conformal symmetry on RG flows*, *JHEP* **07** (2012) 069 [arXiv:1112.4538] [INSPIRE].

[5] M.A. Luty, J. Polchinski and R. Rattazzi, *The a-theorem and the asymptotics of* 4D *quantum field theory*, *JHEP* **01** (2013) 152 [arXiv:1204.5221] [INSPIRE].

[6] J.L. Cardy, *Is there a c theorem in four-dimensions?*, *Phys. Lett.* **B 215** (1988) 749 [INSPIRE].

[7] S. Deser and A. Schwimmer, *Geometric classification of conformal anomalies in arbitrary dimensions*, *Phys. Lett.* **B 309** (1993) 279 [hep-th/9302047] [INSPIRE].

[8] H. Elvang, D.Z. Freedman, L.-Y. Hung, M. Kiermaier, R.C. Myers and S. Theisen, *On renormalization group flows and the a-theorem in* 6d, *JHEP* **10** (2012) 011 [arXiv:1205.3994] [INSPIRE].

[9] H. Elvang and T.M. Olson, *RG flows in d dimensions, the dilaton effective action and the a-theorem*, *JHEP* **03** (2013) 034 [arXiv:1209.3424] [INSPIRE].

[10] A. Bhattacharyya, L.-Y. Hung, K. Sen and A. Sinha, *On c-theorems in arbitrary dimensions*, *Phys. Rev.* **D 86** (2012) 106006 [arXiv:1207.2333] [INSPIRE].

[11] R.C. Myers and A. Sinha, *Seeing a c-theorem with holography*, *Phys. Rev.* **D 82** (2010) 046006 [arXiv:1006.1263] [INSPIRE].

[12] R.C. Myers and A. Sinha, *Holographic c-theorems in arbitrary dimensions*, *JHEP* **01** (2011) 125 [arXiv:1011.5819] [INSPIRE].

[13] H. Casini, M. Huerta and R.C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036 [arXiv:1102.0440] [INSPIRE].

[14] H. Casini and M. Huerta, *Entanglement entropy for the n-sphere*, *Phys. Lett.* **B 694** (2010) 167 [arXiv:1007.1813] [INSPIRE].

[15] S.N. Solodukhin, *Entanglement entropy, conformal invariance and extrinsic geometry*, *Phys. Lett.* **B 665** (2008) 305 [arXiv:0802.3117] [INSPIRE].

[16] C. Holzhey, F. Larsen and F. Wilczek, *Geometric and renormalized entropy in conformal field theory*, *Nucl. Phys.* **B 424** (1994) 443 [hep-th/9403108] [INSPIRE].

[17] I. Affleck and A.W.W. Ludwig, *Universal noninteger 'ground state degeneracy' in critical quantum systems*, *Phys. Rev. Lett.* **67** (1991) 161 [INSPIRE].

[18] D. Friedan and A. Konechny, *On the boundary entropy of one-dimensional quantum systems at low temperature*, *Phys. Rev. Lett.* **93** (2004) 030402 [hep-th/0312197] [INSPIRE].

[19] K. Jensen and A. O'Bannon, *Holography, entanglement entropy and conformal field theories with boundaries or defects*, *Phys. Rev.* **D 88** (2013) 106006 [arXiv:1309.4523] [INSPIRE].

[20] J. Estes, K. Jensen, A. O'Bannon, E. Tsatis and T. Wrase, *On holographic defect entropy*, *JHEP* **05** (2014) 084 [arXiv:1403.6475] [INSPIRE].

[21] D. Gaiotto, *Boundary F-maximization*, arXiv:1403.8052 [INSPIRE].

[22] D.L. Jafferis, I.R. Klebanov, S.S. Pufu and B.R. Safdi, *Towards the F-theorem: $N = 2$ field theories on the three-sphere*, *JHEP* **06** (2011) 102 [arXiv:1103.1181] [INSPIRE].

[23] I.R. Klebanov, S.S. Pufu and B.R. Safdi, *F-theorem without supersymmetry*, *JHEP* **10** (2011) 038 [arXiv:1105.4598] [INSPIRE].

[24] I.R. Klebanov, S.S. Pufu, S. Sachdev and B.R. Safdi, *Entanglement entropy of $3D$ conformal gauge theories with many flavors*, *JHEP* **05** (2012) 036 [arXiv:1112.5342] [INSPIRE].

[25] A. Amariti and M. Siani, *Z-extremization and F-theorem in Chern-Simons matter theories*, *JHEP* **10** (2011) 016 [arXiv:1105.0933] [INSPIRE].

[26] A. Amariti and M. Siani, *F-maximization along the RG flows: a proposal*, *JHEP* **11** (2011) 056 [arXiv:1105.3979] [INSPIRE].

[27] T. Morita and V. Niarchos, *F-theorem, duality and SUSY breaking in one-adjoint Chern-Simons-Matter theories*, *Nucl. Phys.* **B 858** (2012) 84 [arXiv:1108.4963] [INSPIRE].

[28] D.R. Gulotta, C.P. Herzog and S.S. Pufu, *From necklace quivers to the F-theorem, operator counting and $T(\mathrm{U}(N))$*, *JHEP* **12** (2011) 077 [arXiv:1105.2817] [INSPIRE].

[29] F. Benini, C. Closset and S. Cremonesi, *Comments on $3D$ Seiberg-like dualities*, *JHEP* **10** (2011) 075 [arXiv:1108.5373] [INSPIRE].

[30] J.S. Dowker, *Entanglement entropy for odd spheres*, arXiv:1012.1548 [INSPIRE].

[31] T. Grover, *Entanglement monotonicity and the stability of gauge theories in three spacetime dimensions*, *Phys. Rev. Lett.* **112** (2014) 151601 [arXiv:1211.1392] [INSPIRE].

[32] H. Casini and M. Huerta, *On the RG running of the entanglement entropy of a circle*, *Phys. Rev.* **D 85** (2012) 125016 [arXiv:1202.5650] [INSPIRE].

[33] E.H. Lieb and M.B. Ruskai, *Proof of the strong subadditivity of quantum-mechanical entropy*, *J. Math. Phys.* **14** (1973) 1938 [INSPIRE].

[34] H. Liu and M. Mezei, *A refinement of entanglement entropy and the number of degrees of freedom*, *JHEP* **04** (2013) 162 [arXiv:1202.2070] [INSPIRE].

[35] R.C. Myers and A. Singh, *Comments on holographic entanglement entropy and RG flows*, *JHEP* **04** (2012) 122 [arXiv:1202.2068] [INSPIRE].

[36] A. Lewkowycz, R.C. Myers and M. Smolkin, *Observations on entanglement entropy in massive QFT's*, *JHEP* **04** (2013) 017 [arXiv:1210.6858] [INSPIRE].

[37] R.D. Sorkin, *On the entropy of the vacuum outside a horizon*, in *General relativity and gravitation. Volume 1*, B. Bertotti et al. eds., Springer, Germany (1983).

[38] L. Bombelli, R.K. Koul, J. Lee and R.D. Sorkin, *A quantum source of entropy for black holes*, *Phys. Rev.* **D 34** (1986) 373 [INSPIRE].

[39] M. Srednicki, *Entropy and area*, *Phys. Rev. Lett.* **71** (1993) 666 [hep-th/9303048] [INSPIRE].

[40] T. Grover, A.M. Turner and A. Vishwanath, *Entanglement entropy of gapped phases and topological order in three dimensions*, *Phys. Rev.* **B 84** (2011) 195120 [arXiv:1108.4038] [INSPIRE].

[41] C. Closset, T.T. Dumitrescu, G. Festuccia, Z. Komargodski and N. Seiberg, *Contact terms, unitarity and F-maximization in three-dimensional superconformal theories*, *JHEP* **10** (2012) 053 [arXiv:1205.4142] [INSPIRE].

[42] E. Gerchkovitz, J. Gomis and Z. Komargodski, *Sphere partition functions and the Zamolodchikov metric*, *JHEP* **11** (2014) 001 [arXiv:1405.7271] [INSPIRE].

[43] M.M. Wolf, F. Verstraete, M.B. Hastings and J.I. Cirac, *Area laws in quantum systems: mutual information and correlations*, *Phys. Rev. Lett.* **100** (2008) 070502 [arXiv:0704.3906].

[44] H. Casini and M. Huerta, *A finite entanglement entropy and the c-theorem*, *Phys. Lett.* **B 600** (2004) 142 [hep-th/0405111] [INSPIRE].

[45] H. Casini, *Mutual information challenges entropy bounds*, *Class. Quant. Grav.* **24** (2007) 1293 [gr-qc/0609126] [INSPIRE].

[46] H. Casini, *Entropy localization and extensivity in the semiclassical black hole evaporation*, *Phys. Rev.* **D 79** (2009) 024015 [arXiv:0712.0403] [INSPIRE].

[47] H. Casini and M. Huerta, *Remarks on the entanglement entropy for disconnected regions*, *JHEP* **03** (2009) 048 [arXiv:0812.1773] [INSPIRE].

[48] B. Swingle, *Mutual information and the structure of entanglement in quantum field theory*, arXiv:1010.4038 [INSPIRE].

[49] J. Cardy, *Some results on the mutual information of disjoint regions in higher dimensions*, *J. Phys.* **A 46** (2013) 285402 [arXiv:1304.7985] [INSPIRE].

[50] H. Araki, *Relative entropy of states of von Neumann algebras*, *Publ. Res. Inst. Math. Sci. Kyoto* **1976** (1976) 809 [INSPIRE].

[51] H. Narnhofer, *Entropy for quantum fields in curved space time*, *Class. Quant. Grav.* **28** (2011) 145016 [INSPIRE].

[52] H. Casini, F.D. Mazzitelli and E. Testé, *Area terms in entanglement entropy*, *Phys. Rev.* **D 91** (2015) 104035 [arXiv:1412.6522] [INSPIRE].

[53] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [hep-th/0603001] [INSPIRE].

[54] S. Ryu and T. Takayanagi, *Aspects of holographic entanglement entropy*, *JHEP* **08** (2006) 045 [hep-th/0605073] [INSPIRE].

[55] H. Casini and M. Huerta, *Entanglement entropy in free quantum field theory*, *J. Phys.* **A 42** (2009) 504007 [arXiv:0905.2562] [INSPIRE].

[56] M.P. Hertzberg and F. Wilczek, *Some calculable contributions to entanglement entropy*, *Phys. Rev. Lett.* **106** (2011) 050404 [arXiv:1007.0993] [INSPIRE].

[57] M.P. Hertzberg, *Entanglement entropy in scalar field theory*, *J. Phys.* **A 46** (2013) 015402 [arXiv:1209.4646] [INSPIRE].

[58] L.-Y. Hung, R.C. Myers and M. Smolkin, *Some calculable contributions to holographic entanglement entropy*, *JHEP* **08** (2011) 039 [arXiv:1105.6055] [INSPIRE].

[59] H. Liu and M. Mezei, *Probing renormalization group flows using entanglement entropy*, *JHEP* **01** (2014) 098 [arXiv:1309.6935] [INSPIRE].

[60] I.R. Klebanov, S.S. Pufu, S. Sachdev and B.R. Safdi, *Renyi entropies for free field theories*, *JHEP* **04** (2012) 074 [arXiv:1111.6290] [ɪɴSPIRE].

[61] P. Calabrese and J.L. Cardy, *Entanglement entropy and quantum field theory*, *J. Stat. Mech.* (2004) P06002 [hep-th/0405152] [ɪɴSPIRE].

[62] L.-Y. Hung, R.C. Myers and M. Smolkin, *Twist operators in higher dimensions*, *JHEP* **10** (2014) 178 [arXiv:1407.6429] [ɪɴSPIRE].

[63] X.G. Wen, *Quantum field theory of many-body systems: from the origin of sound to an origin of light and electrons*, Oxford University Press, Oxford U.K. (2004).

[64] A. Kitaev and J. Preskill, *Topological entanglement entropy*, *Phys. Rev. Lett.* **96** (2006) 110404 [hep-th/0510092] [ɪɴSPIRE].

[65] M. Levin and X.G. Wen, *Detecting topological order in a ground state wave function*, *Phys. Rev. Lett.* **96** (2006) 110405 [cond-mat/0510613].

[66] I.R. Klebanov, T. Nishioka, S.S. Pufu and B.R. Safdi, *Is renormalized entanglement entropy stationary at RG fixed points?*, *JHEP* **10** (2012) 058 [arXiv:1207.3360] [ɪɴSPIRE].

[67] B.R. Safdi, *Exact and numerical results on entanglement entropy in $(5+1)$-dimensional CFT*, *JHEP* **12** (2012) 005 [arXiv:1206.5025] [ɪɴSPIRE].

[68] L.-Y. Hung, R.C. Myers, M. Smolkin and A. Yale, *Holographic calculations of Renyi entropy*, *JHEP* **12** (2011) 047 [arXiv:1110.1084] [ɪɴSPIRE].

[69] H. Casini and M. Huerta, *Universal terms for the entanglement entropy in $2+1$ dimensions*, *Nucl. Phys.* **B 764** (2007) 183 [hep-th/0606256] [ɪɴSPIRE].

[70] H. Casini, M. Huerta and L. Leitao, *Entanglement entropy for a Dirac fermion in three dimensions: Vertex contribution*, *Nucl. Phys.* **B 814** (2009) 594 [arXiv:0811.1968] [ɪɴSPIRE].

[71] T. Hirata and T. Takayanagi, *AdS/CFT and strong subadditivity of entanglement entropy*, *JHEP* **02** (2007) 042 [hep-th/0608213] [ɪɴSPIRE].

[72] R.C. Myers and A. Singh, *Entanglement entropy for singular surfaces*, *JHEP* **09** (2012) 013 [arXiv:1206.5225] [ɪɴSPIRE].

[73] P. Bueno, R.C. Myers and W. Witczak-Krempa, *Universality of corner entanglement in conformal field theories*, *Phys. Rev. Lett.* **115** (2015) 021602 [arXiv:1505.04804] [ɪɴSPIRE].

[74] P. Bueno and R.C. Myers, *Corner contributions to holographic entanglement entropy*, *JHEP* **08** (2015) 068 [arXiv:1505.07842] [ɪɴSPIRE].

[75] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [ɪɴSPIRE].

[76] H. Casini, M. Huerta and J.A. Rosabal, *Remarks on entanglement entropy for gauge fields*, *Phys. Rev.* **D 89** (2014) 085012 [arXiv:1312.1183] [ɪɴSPIRE].

[77] H. Casini and M. Huerta, *Entanglement entropy for a Maxwell field: Numerical calculation on a two dimensional lattice*, *Phys. Rev.* **D 90** (2014) 105013 [arXiv:1406.2991] [ɪɴSPIRE].

[78] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Quantum entanglement*, *Rev. Mod. Phys.* **81** (2009) 865 [quant-ph/0702225] [ɪɴSPIRE].

[79] S. Giombi and I.R. Klebanov, *Interpolating between a and F*, *JHEP* **03** (2015) 117 [arXiv:1409.1937] [ɪɴSPIRE].

[80] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge U.K. (1982).

[81] A. Cappelli and G. D'Appollonio, *On the trace anomaly as a measure of degrees of freedom*, *Phys. Lett.* **B 487** (2000) 87 [hep-th/0005115] [INSPIRE].

[82] J.I. Latorre, C.A. Lütken, E. Rico and G. Vidal, *Fine grained entanglement loss along renormalization group flows*, *Phys. Rev.* **A 71** (2005) 034301 [quant-ph/0404120] [INSPIRE].

[83] A.H. Castro Neto and E.H. Fradkin, *The thermodynamics of quantum systems and generalizations of Zamolodchikov's c-theorem*, *Nucl. Phys.* **B 400** (1993) 525 [cond-mat/9301009] [INSPIRE].

[84] T. Appelquist, A.G. Cohen, M. Schmaltz and R. Shrock, *New constraints on chiral gauge theories*, *Phys. Lett.* **B 459** (1999) 235 [hep-th/9904172] [INSPIRE].

[85] T. Appelquist, A.G. Cohen and M. Schmaltz, *A new constraint on strongly coupled gauge theories*, *Phys. Rev.* **D 60** (1999) 045003 [hep-th/9901109] [INSPIRE].

[86] M. Headrick and T. Takayanagi, *A holographic proof of the strong subadditivity of entanglement entropy*, *Phys. Rev.* **D 76** (2007) 106013 [arXiv:0704.3719] [INSPIRE].

[87] M. Headrick, *Entanglement Renyi entropies in holographic theories*, *Phys. Rev.* **D 82** (2010) 126010 [arXiv:1006.0047] [INSPIRE].

[88] L.-Y. Hung, R.C. Myers and M. Smolkin, *On holographic entanglement entropy and higher curvature gravity*, *JHEP* **04** (2011) 025 [arXiv:1101.5813] [INSPIRE].

[89] A. Lewkowycz and J. Maldacena, *Generalized gravitational entropy*, *JHEP* **08** (2013) 090 [arXiv:1304.4926] [INSPIRE].

[90] J. de Boer, M. Kulaxizi and A. Parnachev, *Holographic entanglement entropy in Lovelock gravities*, *JHEP* **07** (2011) 109 [arXiv:1101.5781] [INSPIRE].

[91] X. Dong, *Holographic entanglement entropy for general higher derivative gravity*, *JHEP* **01** (2014) 044 [arXiv:1310.5713] [INSPIRE].

[92] J. Camps, *Generalized entropy and higher derivative Gravity*, *JHEP* **03** (2014) 070 [arXiv:1310.6659] [INSPIRE].

[93] A. Bhattacharyya, A. Kaviraj and A. Sinha, *Entanglement entropy in higher derivative holography*, *JHEP* **08** (2013) 012 [arXiv:1305.6694] [INSPIRE].

[94] A. Bhattacharyya, M. Sharma and A. Sinha, *On generalized gravitational entropy, squashed cones and holography*, *JHEP* **01** (2014) 021 [arXiv:1308.5748] [INSPIRE].

[95] A. Bhattacharyya and M. Sharma, *On entanglement entropy functionals in higher derivative gravity theories*, *JHEP* **10** (2014) 130 [arXiv:1405.3511] [INSPIRE].

[96] R.X. Miao and W.Z. Guo, *Holographic entanglement entropy for the most general higher derivative gravity*, *JHEP* **08** (2015) 031 [arXiv:1411.5579] [INSPIRE].

[97] A.F. Astaneh, A. Patrushev and S.N. Solodukhin, *Entropy discrepancy and total derivatives in trace anomaly*, arXiv:1412.0452 [INSPIRE].

[98] R.-X. Miao, *Universal terms of entanglement entropy for 6D CFTs*, arXiv:1503.05538 [INSPIRE].

[99] Y. Huang and R.-X. Miao, *A note on the resolution of the entropy discrepancy*, *Phys. Lett.* **B 749** (2015) 489 [arXiv:1504.02301] [INSPIRE].

[100] P. Hayden, M. Headrick and A. Maloney, *Holographic mutual information is monogamous*, *Phys. Rev.* **D 87** (2013) 046003 [arXiv:1107.2940] [INSPIRE].

[101] P. Fonda, L. Giomi, A. Salvio and E. Tonni, *On shape dependence of holographic mutual information in AdS$_4$*, *JHEP* **02** (2015) 005 [arXiv:1411.3608] [INSPIRE].

[102] E. Tonni, *Holographic entanglement entropy: near horizon geometry and disconnected regions*, *JHEP* **05** (2011) 004 [arXiv:1011.0166] [INSPIRE].

[103] K. Zarembo, *Wilson loop correlator in the AdS/CFT correspondence*, *Phys. Lett.* **B 459** (1999) 527 [hep-th/9904149] [INSPIRE].

[104] P. Olesen and K. Zarembo, *Phase transition in Wilson loop correlator from AdS/CFT correspondence*, hep-th/0009210 [INSPIRE].

[105] N. Drukker and B. Fiol, *On the integrability of Wilson loops in AdS$_5 \times S^5$: some periodic ansatze*, *JHEP* **01** (2006) 056 [hep-th/0506058] [INSPIRE].

[106] L.F. Alday and A.A. Tseytlin, *On strong-coupling correlation functions of circular Wilson loops and local operators*, *J. Phys.* **A 44** (2011) 395401 [arXiv:1105.1537] [INSPIRE].

[107] P. Krtous and A. Zelnikov, *Entanglement entropy of spherical domains in Anti-de Sitter space*, *Phys. Rev.* **D 89** (2014) 104058 [arXiv:1311.1685] [INSPIRE].

[108] P. Krtous and A. Zelnikov, *Minimal surfaces and entanglement entropy in Anti-de Sitter space*, *JHEP* **10** (2014) 077 [arXiv:1406.7659] [INSPIRE].

[109] D.D. Blanco, H. Casini, L.-Y. Hung and R.C. Myers, *Relative entropy and holography*, *JHEP* **08** (2013) 060 [arXiv:1305.3182] [INSPIRE].

[110] A. Grigoraryan and M. Noguchi, *The heat kernel on hyperbolic space*, *Bull. Lond. Math. Soc.* **30** (1998) 643.

[111] A. Grigoraryan, *Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold*, *J. Funct. Anal.* **127** (1995) 363.

[112] A. Debiard, B. Gaveau and E. Mazet, *Théorèmes de comparison en géométrie riemannienne*, *Publ. Kyoto Univ.* **12** (1976) 391.

[113] J.S. Dowker, *Entanglement entropy for even spheres*, arXiv:1009.3854 [INSPIRE].

[114] J.S. Dowker, *Casimir effect around a cone*, *Phys. Rev.* **D 36** (1987) 3095 [INSPIRE].

[115] R. Lohmayer, H. Neuberger, A. Schwimmer and S. Theisen, *Numerical determination of entanglement entropy for a sphere*, *Phys. Lett.* **B 685** (2010) 222 [arXiv:0911.4283] [INSPIRE].

[116] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [INSPIRE].